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Cartan's Structure Theory of Symmetry Pseudo-Groups, Coverings and Multi-Valued Solutions for the Khokhlov–Zabolotskaya Equation

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Abstract We derive two non-equivalent coverings for the modified Khokhlov–Zabolotskaya equation from Maurer–Cartan forms of its symmetry pseudo-group. Also we find Bäcklund transformations between the obtained covering equations. We apply these results to constructing multi-valued solutions for the Khokhlov–Zabolotskaya equation.

Keywords Lie pseudo-groups · Maurer–Cartan forms · Symmetries of differential equations · Coverings of differential equations

Mathematics Subject Classification (2000) 58H05 · 58J70 · 35A30

1 Introduction

In this paper we continue the research of [26], where it was shown that the known coverings of Liouville's equation, the Khokhlov–Zabolotskaya equation, and the Boyer–Finley equation can be derived from Maurer–Cartan (MC) forms of their contact symmetry pseudo-groups.

Coverings [16–19] (or prolongation structures [32], or zero-curvature representations [34], or integrable extensions [2]) are of great importance in geometry of differential equations. They are a starting point for inverse scattering transformations, Bäcklund transformations, recursion operators, nonlocal symmetries and nonlocal conservation laws. Different techniques are developed for constructing coverings of partial differential equations (PDEs) in two independent variables, [5, 6, 11, 12, 22, 23, 30, 32], while in the case of more than two independent variables the problem is more difficult, see, e.g., [9, 10, 22, 27, 28, 31, 35]. In the pioneering work [20], Cartan's method of equivalence was applied to the covering problem for equations in three independent variables. One of the

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results of [20] is a deduction of the system

$$q_t = (q^2 - u)\, q_x - u_y - q\, u_x, \tag{1}$$

$$q_y = q\, q_x - u_x, \tag{2}$$

whose integrability conditions coincide with the Khokhlov–Zabolot skaya equation (KZ) [33]

$$u_{yy} = u_{tx} + u\, u_{xx} + u_x^2. \tag{3}$$

In terms of [16–19], (1) and (2) define an infinite-dimensional covering over KZ. From (2) it follows that there exists a function v such that

$$v_x = q, \qquad v_y = \frac{1}{2}\, q^2 - u. \tag{4}$$

Then (1) entails that v satisfies the modified Khokhlov–Zabolotskaya equation (mKZ)

$$v_{yy} = v_{tx} + \left(\frac{1}{2}\, v_x^2 - v_y\right) v_{xx}. \tag{5}$$

Eliminating q from (4) provides a Miura transformation from mKZ to KZ:

$$u = \frac{1}{2}\, v_x^2 - v_y. \tag{6}$$

In [26] it is shown that (1), (2) can be obtained from MC forms of the contact symmetry pseudo-group of KZ. In present paper, we apply the same technique to mKZ. We use Élie Cartan's method of equivalence, [3, 7, 8, 15, 29], to compute MC forms for the pseudo-group of contact symmetries of (5). Then we find two linear combinations of these forms which provide non-equivalent coverings over mKZ. One of these coverings was derived in [4] by means of another technique. We obtain a Bäcklund transformation between the covering equations. Combining the coverings with the Miura transformation (6) yields Miura transformations from the covering equations to KZ. Finally, we apply these results to construct three families of multi-valued solutions of KZ. Each of these families depend on two arbitrary functions of one variable. Previously multi-valued solutions of KZ were studied by means of the theory of symmetries of PDEs in [17, § 8.3.4], see also [13, 14, 21].

2 Preliminaries

2.1 Coverings of PDEs

Let $\pi_\infty : J^\infty(\pi) \to \mathbb{R}^n$ be the infinite jet bundle of local sections of the bundle $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. The coordinates on $J^\infty(\pi)$ are (x^i, u_I), where $I = (i_1, \dots, i_k)$ are symmetric multi-indices, $i_1, \dots, i_k \in \{1, \dots, n\}$, $u_\emptyset = u$, and for any local section f of π there exists a section $j_\infty(f) : \mathbb{R}^n \to J^\infty(\pi)$ such that $u_I(j_\infty(f)) = \partial^{\#I}(f)/\partial x^{i_1} \dots \partial x^{i_k}$, $\#I = \#(i_1, \dots, i_k) = k$. The *total derivatives* on $J^\infty(\pi)$ are defined in the local coordinates as

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\#I \ge 0} u_{Ii}\, \frac{\partial}{\partial u_I}.$$

We have $[D_i, D_j] = 0$ for $i, j \in \{1, \dots, n\}$. A differential equation $F(x^i, u_K) = 0$ defines a submanifold $\mathcal{E}^\infty = \{D_I(F) = 0 | \#I \geq 0\} \subset J^\infty(\pi)$, where $D_I = D_{i_1} \circ \ldots \circ D_{i_k}$ for $I = (i_1, \dots, i_k)$. We denote restrictions of D_i on $\mathcal{E}^\infty$ as $\overline{D}_i$.

In local coordinates, a *covering* over $\mathcal{E}^\infty$ is a bundle $\widetilde{\mathcal{E}}^\infty = \mathcal{E}^\infty \times \mathcal{Q} \to \mathcal{E}^\infty$ with fibre coordinates q^κ, $\kappa \in \{1, \dots, N\}$ or $\kappa \in \mathbb{N}$, equipped with *extended total derivatives*

$$\widetilde{D}_i = \overline{D}_i + \sum_\kappa T_i^\kappa(x^j, u_I, q^\tau) \frac{\partial}{\partial q^\kappa}$$

such that $[\widetilde{D}_i, \widetilde{D}_j] = 0$ whenever $(x^i, u_I) \in \mathcal{E}^\infty$.

Example 1 System (1), (2) provides an infinite-dimensional covering over KZ with the fibre coordinates $q_0 = q$, $q_k = \partial^k q / \partial x^k$, $k \in \mathbb{N}$, and the extended total derivatives

$$\widetilde{D}_t = \overline{D}_t + \sum_{k=0}^\infty \widetilde{D}_x^k((q_0^2 - u)\, q_1 - u_y - q_0\, u_x) \frac{\partial}{\partial q_k},$$

$$\widetilde{D}_x = \overline{D}_x + \sum_{k=0}^\infty q_{k+1} \frac{\partial}{\partial q_k},$$

$$\widetilde{D}_y = \overline{D}_y + \sum_{k=0}^\infty \widetilde{D}_x^k(q_0\, q_1 - u_x) \frac{\partial}{\partial q_k}.$$

2.2 Cartan's Structure Theory of Contact Symmetry Pseudo-Groups of PDEs

A *pseudo-group* on a manifold M is a collection of local diffeomorphisms of M, which is closed under composition *when defined*, contains an identity and is closed under inverse. A *Lie pseudo-group* is a pseudo-group whose diffeomorphisms are local analytic solutions of an involutive system of partial differential equations. Élie Cartan's approach to Lie pseudo-groups is based on a possibility to characterize transformations from a pseudo-group in terms of a set of invariant differential 1-forms called *Maurer–Cartan forms*. The MC forms for a Lie pseudo-group can be computed by means of algebraic operations and differentiation. Expressions of differentials of the MC forms in terms of themselves give *structure equations* of the pseudo-group. The structure equations contain the full information about their pseudo-group.

Example 2 Consider the bundle $J^2(\pi)$ of jets of the second order of the bundle π. A differential 1-form ϑ on $J^2(\pi)$ is called a *contact form* if it is annihilated by all 2-jets of local sections: $j_2(f)^*\vartheta = 0$. In the local coordinates every contact 1-form is a linear combination of the forms $\vartheta_0 = du - u_i\, dx^i$, $\vartheta_i = du_i - u_{ij}\, dx^j$, $i, j \in \{1, \dots, n\}$, $u_{ji} = u_{ij}$. A local diffeomorphism $\Delta : J^2(\pi) \to J^2(\pi)$, $\Delta : (x^i, u, u_i, u_{ij}) \mapsto (\overline{x}^i, \overline{u}, \overline{u}_i, \overline{u}_{ij})$, is called a *contact transformation* if for every contact 1-form $\overline{\vartheta}$ the form $\Delta^*\overline{\vartheta}$ is also contact. We denote by $\mathrm{Cont}(J^2(\pi))$ the pseudo-group of contact transformations on $J^2(\pi)$. Consider the following 1-forms

$$\begin{aligned}
&\Theta_0 = a\, \vartheta_0, \qquad \Theta_i = g_i\, \Theta_0 + a\, B_i^k\, \vartheta_k, \qquad \Xi^i = c^i\, \Theta_0 + f^{ik}\, \Theta_k + b_k^i\, dx^k, \\
&\Sigma_{ij} = s_{ij}\, \Theta_0 + w_{ij}^k\, \Theta_k + z_{ijk}\, \Xi^k + a\, B_k^i\, B_l^j\, du_{kl},
\end{aligned} \tag{7}$$

defined on $J^2(\pi) \times \mathcal{H}$, where $\mathcal{H}$ is an open subset of $\mathbb{R}^{(2n+1)(n+3)(n+1)/3}$ with local coordinates $(a, b^i_k, c^i, f^{ik}, g_i, s_{ij}, w^k_{ij}, z_{ijk})$, $i, j, k \in \{1, \dots, n\}$, $i \le j$, such that $a \ne 0$, $\det(b^i_k) \ne 0$, $f^{ik} = f^{ki}$, $z_{ijk} = z_{ikj} = z_{jik}$, while (B^i_k) is the inverse matrix for the matrix (b^k_l). As it is shown in [25], the forms (7) are MC forms for $\mathrm{Cont}(J^2(\pi))$, that is, a local diffeomorphism $\widehat{\Delta} : J^2(\pi) \times \mathcal{H} \to J^2(\pi) \times \mathcal{H}$ satisfies the conditions $\widehat{\Delta}^* \overline{\Theta}_0 = \Theta_0$, $\widehat{\Delta}^* \overline{\Theta}_i = \Theta_i$, $\widehat{\Delta}^* \overline{\Xi}^i = \Xi^i$, and $\widehat{\Delta}^* \overline{\Sigma}_{ij} = \Sigma_{ij}$ if and only if it is projectable on $J^2(\pi)$, and its projection $\Delta : J^2(\pi) \to J^2(\pi)$ is a contact transformation. The structure equations for $\mathrm{Cont}(J^2(\pi))$ have the form

$$
\begin{aligned}
d\Theta_0 &= \Phi^0_0 \wedge \Theta_0 + \Xi^i \wedge \Theta_i,\\
d\Theta_i &= \Phi^0_i \wedge \Theta_0 + \Phi^k_i \wedge \Theta_k + \Xi^k \wedge \Sigma_{ik},\\
d\Xi^i &= \Phi^0_0 \wedge \Xi^i - \Phi^i_k \wedge \Xi^k + \Psi^{i0} \wedge \Theta_0 + \Psi^{ik} \wedge \Theta_k,\\
d\Sigma_{ij} &= \Phi^k_i \wedge \Sigma_{kj} - \Phi^0_0 \wedge \Sigma_{ij} + \Upsilon^0_{ij} \wedge \Theta_0 + \Upsilon^k_{ij} \wedge \Theta_k + \Lambda_{ijk} \wedge \Xi^k,
\end{aligned}
$$

where the additional forms Φ^0_0, Φ^0_i, Φ^k_i, Ψ^{i0}, Ψ^{ij}, Υ^0_{ij}, Υ^k_{ij}, and Λ_{ijk} depend on differentials of the coordinates of $\mathcal{H}$.

Example 3 Suppose $\mathcal{E}$ is a second-order differential equation in one dependent and n independent variables. We consider $\mathcal{E}$ as a submanifold in $J^2(\pi)$. Let $\mathrm{Cont}(\mathcal{E})$ be the group of contact symmetries for $\mathcal{E}$. It consists of all the contact transformations on $J^2(\pi)$ mapping $\mathcal{E}$ to itself. Let $\iota_0 : \mathcal{E} \to J^2(\pi)$ be an embedding, and $\iota = \iota_0 \times \mathrm{id} : \mathcal{E} \times \mathcal{H} \to \mathrm{J}^2(\pi) \times \mathcal{H}$. The MC forms of $\mathrm{Cont}(\mathcal{E})$ can be derived from the forms $\theta_0 = \iota^*\Theta_0$, $\theta_i = \iota^*\Theta_i$, $\xi^i = \iota^*\Xi^i$, and $\sigma_{ij} = \iota^*\Sigma_{ij}$ by means of Cartan's method of equivalence, see details and examples in [7, 24, 25].

3 Structure of Symmetry Pseudo-Group and Coverings of the Modified Khokhlov–Zabolotskaya Equation

By the method outlined in Example 3 we compute MC forms and structure equations for the pseudo-group of contact symmetries of (5). The structure equations read

$$
\begin{aligned}
d\theta_0 &= \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2 + \xi^1 \wedge \theta_2 + \xi_3 \wedge \theta_3,\\
d\theta_1 &= \left(\frac{1}{2}\theta_2 + \xi^2\right) \wedge \theta_0 + \left(\frac{3}{2}\eta_1 + \xi^3 - \frac{3}{2}\sigma_{22}\right) \wedge \theta_1\\
&\quad + 2\theta_3 \wedge \xi^2 + \left(\eta_1 + \theta_3 - \sigma_{22} + \xi^3\right) \wedge \theta_2\\
&\quad + \xi^1 \wedge \sigma_{11} + (\xi^1 + \xi^2) \wedge \sigma_{12} + \xi^3 \wedge \sigma_{13},\\
d\theta_2 &= \frac{1}{2}(\eta_1 - \sigma_{22}) \wedge \theta_2 + \xi^1 \wedge \sigma_{12} + (\xi^1 + \xi^2) \wedge \sigma_{22} + \xi^3 \wedge \sigma_{23},\\
d\theta_3 &= \frac{1}{2}\sigma_{22} \wedge \theta_0 - \xi^2 \wedge \theta_2 + \left(\eta_1 + \frac{1}{2}\xi^3 - \sigma_{22}\right) \wedge \theta_3 + \xi^1 \wedge \sigma_{13} + \xi^3 \wedge (\sigma_{12} + \sigma_{22})\\
&\quad + (\xi^1 + \xi^2) \wedge \sigma_{23},\\
d\xi^1 &= -\frac{1}{2}\left(\eta_1 + 2\xi^3 - 3\sigma_{22}\right) \wedge \xi^1,
\end{aligned}
$$

$$
\begin{aligned}
d\xi^2 &= \left(\theta_3 - \frac{1}{2}\theta_0\right) \wedge \xi^1 + \frac{1}{2}(\eta_1 + \sigma_{22}) \wedge \xi^2 + (\theta_2 + \xi^2) \wedge \xi^3, \\
d\xi^3 &= 2(\theta_2 + \xi^2) \wedge \xi^1 + \sigma_{22} \wedge \xi^3, \\
d\sigma_{11} &= 2\eta_1 \wedge (\sigma_{11} + \sigma_{12}) + \eta_2 \wedge \xi^1 + \eta_3 \wedge (\xi^1 + \xi^2) + \eta_4 \wedge \xi^3 + 3(2\theta_2 - \sigma_{23}) \wedge \theta_1 \\
&\quad + (\theta_3 - \theta_0 + 3\sigma_{11} + 2\sigma_{12}) \wedge \sigma_{22} + (3\sigma_{13} - 2\sigma_{23}) \wedge \theta_2, \\
d\sigma_{12} &= \eta_1 \wedge (\sigma_{12} + \sigma_{22}) + \eta_3 \wedge \xi^1 + \eta_5 \wedge \xi^3 + \frac{1}{2}\theta_0 \wedge \sigma_{22} + \frac{13}{2}\theta_1 \wedge \xi^1 - (\theta_3 - 2\sigma_{12}) \wedge \sigma_{22} \\
&\quad + \frac{1}{2}\theta_2 \wedge (11\xi^1 + 3\xi^2 + 2\sigma_{23}) - 2(2\sigma_{13} + \sigma_{23}) \wedge \xi^1, \\
d\sigma_{13} &= \frac{1}{2}\eta_1 \wedge (3\sigma_{13} + \sigma_{23}) + \eta_3 \wedge \xi^3 + \eta_4 \wedge \xi^1 + \eta_5 \wedge (\xi^1 + \xi^2) - \sigma_{22} \wedge \left(\frac{5}{2}\sigma_{13} + \sigma_{23}\right) \\
&\quad + \frac{1}{2}\theta_0 \wedge (3\theta_2 + 2\xi^2 - \sigma_{23}) + \frac{1}{2}\theta_1 \wedge (13\xi^3 - \sigma_{22}) + (2\sigma_{11} + 3\sigma_{12} + \sigma_{22}) \wedge \xi^1 \\
&\quad + \frac{1}{2}\theta_2 \wedge (6\theta_3 + 13\xi^3 - 4\sigma_{12} - 2\sigma_{22}) - \theta_3 \wedge (4\xi^2 - \sigma_{23}) + \frac{1}{2}\xi^3 \wedge (11\sigma_{13} + 6\sigma_{23}) \\
&\quad + (4\sigma_{12} + 3\sigma_{22}) \wedge \xi^2, \\
d\sigma_{22} &= 2(2\theta_2 + 2\xi^2 - \sigma_{23}) \wedge \xi^1 + \frac{1}{2}\sigma_{22} \wedge \xi^3, \\
d\sigma_{23} &= \frac{1}{2}(\eta_1 + 5\xi^3 - 3\sigma_{22}) \wedge \sigma_{23} + \left(\eta_5 - \xi^3 - \sigma_{12} - \frac{3}{2}\sigma_{22}\right) \wedge \xi^1 \\
&\quad + \frac{3}{2}(\theta_2 + \xi^2) \wedge (2\xi^3 - \sigma_{22}), \\
d\eta_1 &= \xi^1 \wedge (\theta_2 + \xi^2) + \frac{1}{2}\xi^3 \wedge \sigma_{22}, \\
d\eta_2 &= \pi_1 \wedge \xi^1 + \pi_2 \wedge (\xi^1 + \xi^2) + \pi_3 \wedge \xi^3 + \left(\frac{9}{2}\eta_2 - 10\sigma_{13}\right) \wedge \sigma_{22} - 3(\sigma_{13} - 2\sigma_{23}) \wedge \sigma_{12} \\
&\quad + \frac{1}{2}\eta_1 \wedge (5\eta_2 + 6\eta_3 - 13\theta_1 + 16\theta_2 - 16\sigma_{13} - 8\sigma_{23}) + \frac{1}{2}\eta_3 \wedge (2\theta_3 - \theta_0 + 6\sigma_{22}) \\
&\quad + 5\eta_4 \wedge \theta_2 - \eta_5 \wedge (3\theta_1 + 2\theta_2) - \theta_0 \wedge (16\theta_2 - 5\sigma_{23}) + \theta_1 \wedge (9\sigma_{12} + 14\sigma_{22}) \\
&\quad + (32\theta_3 + 26\sigma_{11} + 5\sigma_{12} - 12\sigma_{22}) \wedge \theta_2 + \sigma_{23} \wedge (10\theta_3 + 9\sigma_{11} - 4\sigma_{22}), \\
d\eta_3 &= \pi_2 \wedge \xi^1 + \pi_4 \wedge \xi^3 + \frac{1}{2}\eta_1 \wedge (6\theta_2 + 8\xi^1 + 8\xi^2 + 3\eta_3) - 2\eta_4 \wedge \xi^1 - 3\sigma_{12} \wedge \sigma_{23} \\
&\quad + 3\eta_5 \wedge (\theta_2 + 2\xi^1 + 2\xi^2) - \theta_3 \wedge (41\xi^1 - 33\xi^2 - 6\sigma_{23}) + 10\sigma_{12} \wedge (2\xi^1 + 3\xi^2) \\
&\quad + \frac{1}{2}\theta_0 \wedge (24\theta_2 + 41\xi^1 + 33\xi^2 - 3\sigma_{23}) + 6\theta_2 \wedge (4\theta_3 - 3\sigma_{12}) + 14\xi^1 \wedge \sigma_{11} \\
&\quad + \left(\frac{7}{2}\eta_3 + \frac{21}{2}\theta_1 - 20(\xi^1 + \xi^2) - 3\sigma_{13}\right) \wedge \sigma_{22}, \\
d\eta_4 &= \pi_2 \wedge \xi^3 + \pi_3 \wedge \xi^1 + \pi_4 \wedge (\xi^1 + \xi^2) + 2\eta_1 \wedge (\eta_4 + \eta_5 + 2\xi^3) + \eta_2 \wedge \xi^1
\end{aligned}
$$

$$+\eta_3\wedge(4\theta_2+\xi^1+\xi^2)+\frac{1}{2}\eta_4\wedge(8\sigma_{22}-21\xi^3)+2\eta_5\wedge(\sigma_{22}-\xi_3)+\frac{5}{2}\sigma_{11}\wedge\sigma_{22}$$
$$+\left(\frac{1}{2}\theta_0-\theta_3\right)\wedge(43\xi^3-\sigma_{22})+\frac{1}{2}\theta_1\wedge(69\theta_2-3\xi^1+9\xi^2-9\sigma_{23})-6\sigma_{13}\wedge\sigma_{23}$$
$$+\frac{1}{2}\theta_2\wedge(4\xi^1+12\xi^2+21\sigma_{13}+\sigma_{23})+\frac{1}{2}\xi^3\wedge(67\sigma_{11}+18\sigma_{12}-28\sigma_{22}),$$
$$d\eta_5=\pi_4\wedge\xi^1+\frac{1}{2}\eta_1\wedge(2\eta_5+2\xi^3+\sigma_{22})-3\eta_3\wedge\xi^1+3\eta_5\wedge(\sigma_{22}-\xi^3)+\frac{5}{2}\sigma_{12}\wedge\sigma_{22}$$
$$-\frac{1}{4}(\theta_0-2\theta_3)\wedge(6\xi^3-\sigma_{22})+\frac{1}{2}\theta_2\wedge(5\sigma_{23}-64\xi^1-3\xi^2)+\frac{13}{2}\xi^3\wedge(\sigma_{12}+2\sigma_{22})$$
$$+\frac{1}{2}\xi^1\wedge(52\theta_1+12\xi^2-23\sigma_{13}-19\sigma_{23})+\frac{3}{2}\xi^2\wedge\sigma_{23}.$$

The forms $\eta_1,\dots,\eta_5$ appear in the step of absorption of torsion in the structure equations. We have

$$\xi^1=r\,dt,\qquad \xi^2=v_{xx}^2\,r^{-1}\left(dx+v_x\,dy+\left(\frac{1}{2}v_x^2+v_y\right)dt\right),$$
$$\xi^3=v_{xx}(2\,v_x\,dt+dy),$$
$$\eta_1=3\,v_{xx}^{-1}\,dv_{xx}-2\,r^{-1}\,dr-\frac{1}{2}\,v_{xx}\,dy-v_x\,v_{xx}\,dt,$$

where $r=b_1^1\neq 0$. We need not explicit expressions for the other MC forms in the sequel. We take the following linear combination

$$\eta_1-k_1\,\xi^1-k_2\,\xi^2-k_3\,\xi^3=3\,\frac{dv_{xx}}{v_{xx}}-2\frac{dr}{r}-\frac{k_2\,v_{xx}^2}{r}\,dx$$
$$-\frac{v_{xx}}{2\,r}\left((2\,k_3+1)r+2k_2v_xv_{xx}\right)dy$$
$$-\frac{1}{r}\left(k_2\left(\frac{1}{2}v_x^2+v_y\right)v_{xx}^2+(2\,k_3+1)\,v_xv_{xx}r+k_1r^2\right)dt,$$

where k_1, $k_2\neq 0$, k_3 are constants, and substitute

$$v_{xx}=-\frac{w_1^2}{k_2^2\,w},\qquad r=-\frac{w_1^3}{k_2^3\,w}.$$

Then we have

$$\eta_1-k_1\xi^1-k_2\xi^2-k_3\xi^3=-\left(\frac{dw}{w}-\frac{w_1}{w}\,dx-\frac{w_1}{2k_2^2w}\left((2k_3+1)w_1+2k_2^2v_xw^2\right)dy\right.$$
$$\left.-\frac{w_1}{k_2^3w}\left(k_1w_1^2+k_2(2k_3+1)v_xw_1+k_2^3\left(\frac{1}{2}v_x^2+v_y\right)\right)dt\right).$$

This form is equal to zero whenever $w_1=w_x$ and w satisfies the following system of PDEs:

$$w_t=\frac{k_1}{k_2^3}w_x^3+\frac{2k_3+1}{k_2^2}v_xw_x^2+\left(\frac{1}{2}v_x^2+v_y\right)w_x,\qquad w_y=\frac{2k_3+1}{2k_2^3}w_x^2+v_xw_x.\tag{8}$$

This system is integrable iff

$$\left(v_{yy} - v_{tx} - \left(\frac{1}{2}v_x^2 - v_y\right)v_{xx}\right)w_x + \frac{3k_1k_2 - (2k_3+1)^2}{k_2^4}v_{xx}w_x^3 = 0.$$

To enforce this condition to coincide with mKZ we put $k_1 = \frac{(2k_3+1)^2}{3k_2}$. For simplicity of notation we also put $k_4 = \frac{2k_3+1}{2k_2}$. Then (8) reads

$$w_t = \frac{4}{3}k_4^2 w_x^3 + 2k_4 v_x w_x^2 + \left(\frac{1}{2}v_x^2 + v_y\right)w_x, \qquad w_y = k_4 w_x^2 + v_x w_x. \tag{9}$$

When $k_4 = 0$, we have the following system:

$$w_t = \left(\frac{1}{2}v_x^2 + v_y\right)w_x, \qquad w_y = v_x w_x. \tag{10}$$

It implies that

$$v_x = \frac{w_y}{w_x}, \qquad v_y = \frac{w_t}{w_x} - \frac{w_y^2}{2w_x^2}. \tag{11}$$

The integrability condition of this system is the following equation:

$$w_{yy} = w_{tx} + \frac{w_y^2 - w_t w_x}{w_x^2}w_{xx}. \tag{12}$$

When $k_4 \neq 0$, we put $w = s/k_4$ in (9) and obtain

$$s_t = \frac{4}{3}s_x^3 + 2v_x s_x^2 + \left(\frac{1}{2}v_x^2 + v_y\right)s_x, \qquad s_y = s_x^2 + v_x s_x. \tag{13}$$

This system was found in [4] via another technique. From (13) we have

$$v_x = \frac{s_y}{s_x} - s_x, \qquad v_y = \frac{s_t}{s_x} - \frac{s_y^2}{2s_x^2} - s_y + \frac{s_x^2}{6}. \tag{14}$$

This system is integrable iff the function s satisfies the following equation:

$$s_{yy} = s_{tx} + \left(\frac{s_y^2 - s_t s_x}{s_x^2} + \frac{s_x^2}{3}\right)s_{xx}. \tag{15}$$

Evidently, we can rewrite (10) and (13) in terms of infinite-dimensional coverings over mKZ in the same way as (1), (2) are rewritten in Example 1.

Combining (11) with (6), we have a Miu ra transformation

$$u = \frac{w_y^2}{w_x^2} - \frac{w_t}{w_x} \tag{16}$$

from (12) to (3). Also, (14) and (6) provide a Miura transformation

$$u = \frac{s_y^2}{s_x^2} - \frac{s_t}{s_x} + \frac{s_x^2}{3} \tag{17}$$

from (15) to (3).

Applying Cartan's method of equivalence, it is easy to show that the symmetry pseudo-groups of (12) and (15) are not isomorphic, so these equations are not contact-equivalent. Equations (10), (11), (13), and (14) entail the following Bäcklund transformation from (15) to (12)

$$w_t = \left(\frac{s_t}{s_x} + \frac{2s_x^2}{3} - 2s_y\right) w_x, \qquad w_y = \left(\frac{s_y}{s_x} - s_x\right) w_x,$$

with the inverse transformation

$$s_t = \frac{4s_x^3}{3} + \frac{s_x(2w_y s_x + w_t)}{w_x}, \qquad s_y = s_x^2 + \frac{w_y s_x}{w_x}.$$

4 Multi-Valued Solutions of the Khokhlov–Zabolotskaya Equation

Now we apply the above results to finding multi-valued solutions for KZ. We use the following ansatz, [1, Chap. VIII, § 5.IV]:

$$v_t = F(v_x), \qquad v_y = G(v_x). \tag{18}$$

This system is compatible for arbitrary (smooth) functions F and G. Substituting for (18) in (5), we obtain

$$\left(\left(G'(v_x)\right)^2 - F'(v_x) + G(v_x) - \frac{1}{2}v_x^2\right) v_{xx} = 0. \tag{19}$$

We denote $v_x = z$ and put

$$F(z) = -\frac{1}{6}z^3 + \int \left(\left(G'(z)\right)^2 + G(z)\right) dz. \tag{20}$$

Then (19) is satisfied for an arbitrary function G. System (18) implies that the function z satisfies the following quasi-linear system of PDEs:

$$z_t = F'(z) z_x, \qquad z_y = G'(z) z_x. \tag{21}$$

The general solution of this system in implicit form reads

$$Q(x + F'(z)t + G'(z)y) = z. \tag{22}$$

where $Q(x) = z(0, x, 0)$ is an arbitrary (smooth) function. When (18) is satisfied, the Miura transformation (6) has the form

$$u = \frac{1}{2}z^2 - G(z). \tag{23}$$

Thus (23), (22), and (20) define a family of (multi-valued) solutions of KZ depending on two arbitrary functions G and Q. Figure 1 shows the graph of this solution with $G(z) = \frac{1}{5}z^2$ and $Q(x) = (1 + x^2)^{-1}$ at $t = \frac{1}{2}$.

The same computations provide multi-valued solutions of KZ corresponding to (12) and (15). Namely, substituting for

$$w_t = F(w_x), \qquad w_y = G(w_x) \tag{24}$$

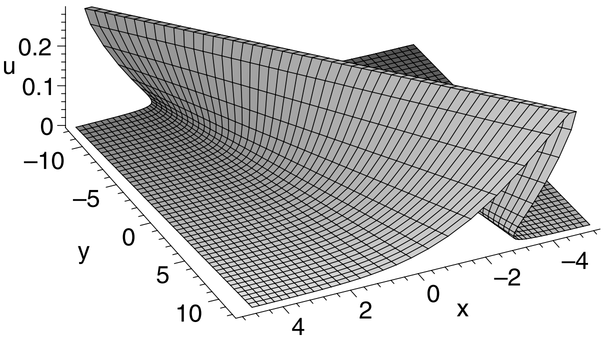

Fig. 1 The graph of solution (23), (22), (20) of (3) with $G(z) = \frac{1}{5}z^2$ and $Q(x) = (1+x^2)^{-1}$ at $t = \frac{1}{2}$

in (12) yields

$$\left((G'(w_x))^2 - F'(w_x) - \left(\frac{G(w_x)}{w_x}\right)^2 + \frac{F(w_x)}{w_x}\right) w_{xx} = 0. \tag{25}$$

So for $z = w_x$ we put

$$F(z) = z \int \left(\left(\frac{G(z)}{z}\right)^2 - (G'(z))^2\right) \frac{dz}{z}. \tag{26}$$

The function z satisfies the same system (21) with the same general solution (22). From (24) and (16) we have now

$$u = \left(\frac{G(z)}{z}\right)^2 - \frac{F(z)}{z}. \tag{27}$$

Thus we obtain a family of solutions of (3) defined in implicit form by (27), (22), and (26) with arbitrary functions G and Q.

Likewise, we put

$$s_t = F(s_x), \qquad s_y = G(s_x) \tag{28}$$

in (15) and get

$$\left((G'(s_x))^2 - F'(s_x) - \left(\frac{G(s_x)}{s_x}\right)^2 + \frac{F(s_x)}{s_x} + \frac{s_x^2}{3}\right) s_{xx} = 0.$$

Then for $z = s_x$ we have

$$F(z) = z \int \left(\left(\frac{G(z)}{z}\right)^2 - (G'(z))^2\right) \frac{dz}{z} - \frac{z^3}{6}. \tag{29}$$

From (28) and (17) we have

$$u = \left(\frac{G(z)}{z}\right)^2 - \frac{F(z)}{z} + \frac{z^2}{3}. \tag{30}$$

This gives a family of solutions of (3) defined in implicit form by (30), (22), and (29) with arbitrary functions G and Q.

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