Contact Equivalence Problem for Nonlinear Wave Equations

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Abstract. The moving coframe method is applied to solve the local equivalence problem for the class of nonlinear wave equations in two independent variables under an action of the pseudo-group of contact transformations. The structure equations and the complete sets of differential invariants for symmetry groups are found. The solution of the equivalence problem is given in terms of these invariants.

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Introduction

In this article we consider a local equivalence problem for the class of nonlinear second order wave equations

$$w_{tt} = f(x, w_x)\, w_{xx} + g(x, w_x) \tag{1}$$

under a contact transformation pseudo-group. Two equations are said to be equivalent if there exists a contact transformation mapping one equation to the other. Élie Cartan developed a general method for solving equivalence problems for submanifolds under an action of a Lie pseudo-group, [1] - [5]. The method provides an effective means of computing complete systems of differential invariants and associated invariant differential operators. The necessary and sufficient condition for equivalence of two submanifolds is formulated in terms of the differential invariants. The invariants parameterize the classifying manifold associated with given submanifolds. Cartan's solution to the equivalence problem states that two submanifolds are (locally) equivalent if and only if their classifying manifolds (locally) overlap. The symmetry classification problem for classes of differential equations is closely related to the problem of local equivalence: symmetry groups and their Lie algebras of two equations are necessarily isomorphic if these equations are equivalent, while the converse statement is not true in general. The preliminary symmetry group classification for the class (1) is given in [9]. In [10], it was proposed to transform equation (1) to the equivalent quasi-linear system of the first order

$$u_t = a(x, u)\, v_x, \qquad v_t = b(x, u)\, u_x, \tag{2}$$

and the symmetry classification for non-linearizable cases of this system is given. In [15] several cases of infinite symmetry algebras for equation (1) are found, and one linearizable case is given.

In the present paper, we apply Cartan's equivalence method, [1] - [5], [8], [13], in its form developed by Fels and Olver, [6, 7], to find all differential invariants of symmetry groups and to solve the local contact equivalence problem for equations from the class (2) in terms of their coefficients. Unlike Lie's infinitesimal method, Cartan's approach allows us to find differential invariants and invariant differential operators without analyzing over-determined systems of PDEs at all, and requires differentiation and linear algebra operations only.

The paper is organized as follows. In Section 1, we begin with some notation, and briefly describe the approach to computing symmetry groups of differential equations via the moving coframe method of [6]. In Section 2, the method is applied to the class of nonlinear wave equations (2). Finally, we make some concluding remarks.

1. Pseudo-group of contact transformations and symmetries of differential equations

In this paper, all considerations are of local nature, and all mappings are real analytic. Suppose $\mathcal{E} = \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a trivial bundle with the local base coordinates $(x^1, ..., x^n)$ and the local fibre coordinates $(u^1, ..., u^m)$; then by $J^1(\mathcal{E})$ denote the bundle of the first-order jets of sections of $\mathcal{E}$, with the local coordinates $(x^i, u^\alpha, p_i^\alpha)$, $i \in \{1, ..., n\}$, $\alpha \in \{1, ..., m\}$. For every local section $(x^i, f^\alpha(x))$ of $\mathcal{E}$, the corresponding 1-jet $(x^i, f^\alpha(x), \partial f^\alpha(x)/\partial x^i)$ is denoted by $j_1(f)$. A differential 1-form ϑ on $J^1(\mathcal{E})$ is called a *contact form*, if it is annihilated by all 1-jets of local sections: $j_1(f)^*\vartheta = 0$. In the local coordinates every contact 1-form is a linear combination of the forms $\vartheta^\alpha = du^\alpha - p_i^\alpha \, dx^i$, $\alpha \in \{1, ..., m\}$ (here and later we use the Einstein summation convention, so $p_i^\alpha \, dx^i = \sum_{i=1}^n p_i^\alpha \, dx^i$, etc.) A local diffeomorphism

$$\Delta : J^1(\mathcal{E}) \rightarrow J^1(\mathcal{E}), \qquad \Delta : (x^i, u^\alpha, p_i^\alpha) \mapsto (\overline{x}^i, \overline{u}^\alpha, \overline{p}_i^\alpha), \tag{3}$$

is called a *contact transformation*, if for every contact 1-form ϑ, the form $\Delta^*\overline{\vartheta}$ is also contact, in other words, if $\Delta^*\overline{\vartheta}^\alpha = d\overline{u}^\alpha - \overline{p}_i^\alpha \, d\overline{x}^i = \zeta_\beta^\alpha(x, u, p) \, \vartheta^\beta$ for some functions ζ_β^α on $J^1(\mathcal{E})$.

Cartan's method of equivalence, [2, 5, 13], allows us to compute invariant 1-forms which define the pseudo-group of contact transformations. The result of its application is the following (see [11]). Consider the lifted coframe

$$\begin{aligned} \Theta^\alpha &= a_\beta^\alpha \, (du^\beta - p_j^\beta \, dx^j), \\ \Xi^i &= c_\beta^i \, \Theta^\beta + b_j^i \, dx^j, \\ \Sigma_i^\alpha &= f_{i\beta}^\alpha \, \Theta^\beta + g_{ij}^\alpha \, \Xi^j + a_\beta^\alpha \, B_i^j \; dp_j^\beta \end{aligned} \tag{4}$$

on $J^1(\mathcal{E}) \times \mathcal{H}$, where $\mathcal{H}$ is the Lie group of block lower triangular matrices of the form

$$\begin{pmatrix} a^\alpha_\beta & 0 & 0 \\ c^i_\gamma \, a^\gamma_\beta & b^i_j & 0 \\ (f^\alpha_{i\gamma} + g^\alpha_{ik}\, c^k_\gamma)\, a^\gamma_\beta & g^\alpha_{ik}\, b^k_j & a^\alpha_\beta \, B^j_i \end{pmatrix},$$

and the parameters a^α_β, b^i_j, c^i_β, $f^\alpha_{i\beta}$, and g^α_{ij} obey the requirements $\det\left(a^\alpha_\beta\right) \neq 0$, $\det\left(b^i_j\right) \neq 0$, $b^i_k\, B^k_j = \delta^i_j$, and $g^\alpha_{ij} = g^\alpha_{ji}$. Then a transformation $\Upsilon : J^1(\mathcal{E}) \times \mathcal{H} \to J^1(\mathcal{E}) \times \mathcal{H}$ satisfies the conditions

$$\Upsilon^* \, \overline{\Theta}^\alpha = \Theta^\alpha, \qquad \Upsilon^* \, \overline{\Xi}^i = \Xi^i, \qquad \Upsilon^* \, \overline{\Sigma}^\alpha_i = \Sigma^\alpha_i$$

if and only if it is projectable on $J^1(\mathcal{E})$ and its projection $\Delta : J^1(\mathcal{E}) \to J^1(\mathcal{E})$ is a contact transformation. The lifted coframe has the structure equations

$$\begin{aligned} d\Theta^\alpha &= \Phi^\alpha_\beta \wedge \Theta^\beta + \Xi^k \wedge \Sigma^\alpha_k, \\ d\Xi^i &= \Psi^i_k \wedge \Xi^k + \Pi^i_\gamma \wedge \Theta^\gamma, \\ d\Sigma^\alpha_i &= \Phi^\alpha_\gamma \wedge \Sigma^\gamma_i - \Psi^k_i \wedge \Sigma^\alpha_k + \Lambda^\alpha_{i\beta} \wedge \Theta^\beta + \Omega^\alpha_{ij} \wedge \Xi^j, \end{aligned} \tag{5}$$

where Φ^α_β, Ψ^i_k, Π^i_γ, $\Lambda^\alpha_{i\beta}$, and Ω^α_{ij} are 1-forms on $J^1(\mathcal{E}) \times \mathcal{H}$, and, as it is shown in [11], the coframe is involutive.

The structure equations (5) remain unchanged if we make the following change of the modified Maurer - Cartan forms Φ^α_β, Ψ^i_k, Π^i_γ, $\Lambda^\alpha_{i\beta}$, and Ω^α_{ij} :

$$\begin{array}{lcl} \Phi^\alpha_\beta & \mapsto & \Phi^\alpha_\beta + K^\alpha_{\beta\gamma}\, \Theta^\gamma, \\ \Psi^i_k & \mapsto & \Psi^i_k + L^i_{kj}\, \Xi^j + M^i_{k\gamma}\, \Theta^\gamma, \\ \Pi^i_\gamma & \mapsto & \Pi^i_\gamma + M^i_{k\gamma}\, \Xi^k + N^i_{\gamma\epsilon}\, \Theta^\epsilon, \\ \Lambda^\alpha_{i\beta} & \mapsto & \Lambda^\alpha_{i\beta} + P^\alpha_{i\beta\gamma}\, \Theta^\gamma + Q^\alpha_{i\beta k}\, \Xi^k + K^\alpha_{\gamma\beta}\, \Sigma^\gamma_i - M^k_{i\beta}\, \Sigma^\alpha_k, \\ \Omega^\alpha_{ij} & \mapsto & \Omega^\alpha_{ij} + Q^\alpha_{i\beta j}\, \Theta^\beta + R^\alpha_{ijk}\, \Xi^k - L^k_{ij}\, \Sigma^\alpha_k, \end{array} \tag{6}$$

where $K^\alpha_{\gamma\epsilon}$, L^i_{kj}, $M^i_{k\gamma}$, $N^i_{\gamma\epsilon}$, $P^\alpha_{i\beta\gamma}$, $Q^\alpha_{i\beta k}$, and R^α_{ijk} are arbitrary functions on $J^1(\mathcal{E}) \times \mathcal{H}$ satisfying the following symmetry conditions:

$$\begin{array}{l} K^\alpha_{\gamma\epsilon} = K^\alpha_{\epsilon\gamma}, \quad L^i_{kj} = L^i_{jk}, \quad N^i_{\gamma\epsilon} = N^i_{\epsilon\gamma}, \\ P^\alpha_{i\beta\gamma} = P^\alpha_{i\gamma\beta}, \quad Q^\alpha_{i\beta k} = Q^\alpha_{k\beta i}, \quad R^\alpha_{ijk} = R^\alpha_{ikj} = R^\alpha_{jik}. \end{array}$$

Another approach to construct 1-forms characterizing contact transformations is presented in [14].

Suppose $\mathcal{R}$ is a first-order differential equation in m dependent and n independent variables. We consider $\mathcal{R}$ as a sub-bundle in $J^1(\mathcal{E})$. Let $Cont(\mathcal{R})$ be the group of contact symmetries for $\mathcal{R}$. It consists of all the contact transformations on $J^1(\mathcal{E})$ mapping $\mathcal{R}$ to itself. The moving coframe method, [6, 7], is applicable to find invariant 1-forms characterizing $Cont(\mathcal{R})$ is the same way, as the lifted coframe (4) to $J^1(\mathcal{E}) \times \mathcal{H}$ characterizes $Cont(J^1(\mathcal{E}))$. We briefly outline this approach.

Let $\iota : \mathcal{R} \to J^1(\mathcal{E})$ be an embedding. The invariant 1-forms of $Cont(\mathcal{R})$ are restrictions of the coframe (4) on $\mathcal{R}$: $\theta^\alpha = \iota^* \Theta^\alpha$, $\xi^i = \iota^* \Xi^i$, and $\sigma^\alpha_i = \iota^* \Sigma^\alpha_i$ (for brevity we identify the map $\iota \times id : \mathcal{R} \times \mathcal{H} \to J^1(\mathcal{E}) \times \mathcal{H}$ with $\iota : \mathcal{R} \to J^1(\mathcal{E})$). The forms θ^α, ξ^i, and σ^α_i have some linear dependencies, i.e., there exists a non-trivial set of functions

S_α, T_i, and U^i_α on $\mathcal{R}\times\mathcal{H}$ such that $S_\alpha\,\theta^\alpha + T_i\,\xi^i + U^i_\alpha\,\sigma^\alpha_i \equiv 0$. These functions are lifted invariants of $Cont(\mathcal{R})$. Setting them equal to appropriate constants allows us to specify some parameters a^α_β, b^i_j, c^i_β, $f^\alpha_{i\beta}$, and g^α_{ij} of the group $\mathcal{H}$ as functions of the coordinates on $\mathcal{R}$ and the other group parameters.

After these normalizations, some restrictions of the forms $\phi^\alpha_\beta = \iota^*\Phi^\alpha_\beta$, $\psi^i_k = \iota^*\Psi^i_k$, $\pi^i_\beta = \iota^*\Pi^i_\beta$, $\lambda^\alpha_{i\beta} = \iota^*\Lambda^\alpha_{i\beta}$, and $\omega^\alpha_{ij} = \iota^*\Omega^\alpha_{ij}$, or some their linear combinations, become semi-basic, i.e., they do not include the differentials of the parameters of $\mathcal{H}$. From (6), we have the following statements: (i) if ϕ^α_β is semi-basic, then its coefficients at σ^γ_j and ξ^j are lifted invariants of $Cont(\mathcal{R})$; (ii) if ψ^i_k or π^i_β are semi-basic, then their coefficients at σ^γ_j are lifted invariants of $Cont(\mathcal{R})$. Setting these invariants equal to some constants, we get specifications of some more parameters of $\mathcal{H}$ as functions of the coordinates on $\mathcal{R}$ and the other group parameters.

More lifted invariants can appear as essential torsion coefficients in the reduced structure equations

$$\begin{aligned} d\theta^\alpha &= \phi^\alpha_\beta \wedge \theta^\beta + \xi^k \wedge \sigma^\alpha_k, \\ d\xi^i &= \psi^i_k \wedge \xi^k + \pi^i_\gamma \wedge \theta^\gamma, \\ d\sigma^\alpha_i &= \phi^\alpha_\gamma \wedge \sigma^\gamma_i - \psi^k_i \wedge \sigma^\alpha_k + \lambda^\alpha_{i\beta} \wedge \theta^\beta + \omega^\alpha_{ij} \wedge \xi^j. \end{aligned}$$

After normalizing these invariants and repeating the process, two outputs are possible. In the first case, the reduced lifted coframe appears to be involutive. Then this coframe is the desired set of defining forms for $Cont(\mathcal{R})$. In the second case, when the reduced lifted coframe does not satisfy Cartan's test, we should use the procedure of prolongation, [13, ch 12].

2. Structure and invariants of symmetry groups for nonlinear wave equations

We apply the method described in the previous section to the class of nonlinear wave equations (2). Denote $x^1 = t$, $x^2 = x$, $u^1 = u$, $u^2 = v$, $p^1_1 = u_t$, $p^1_2 = u_x$, $p^2_1 = v_t$, and $p^2_2 = v_x$, The coordinates on $\mathcal{R}$ are $\{t, x, u, v, u_x, v_x\}$, and the embedding $\iota : \mathcal{R} \to J^1(\mathcal{E})$ is defined by (2). For simplicity in the following computations, we put $F(x,u) = (a(x,u)\,b(x,u))^{1/2}$ and $G(x,u) = (b(x,u)/a(x,u))^{1/2}$, so $a(x,u) = F(x,u)/G(x,u)$ and $b(x,u) = F(x,u)\,G(x,u)$.

There are three cases to be treated separately: Case A, when $F_u \neq 0$ and $G_x \neq 0$, Case B, when $G_x = 0$, and Case C, when $F_u = 0$.

In the case B system (2) has the form $u_t = F(x,u)\,(G(u))^{-1}\,v_x$, $v_t = F(x,u)\,G(u)\,u_x$, so the change of variables $\tilde{u} = H(u)$ provided $H'(u) = G(u)$ transforms this system into the system $\tilde{u}_t = \tilde{F}(x,\tilde{u})\,v_x$, $v_t = \tilde{F}(x,\tilde{u})\,\tilde{u}_x$ with $\tilde{F}(x,\tilde{u}) = F(x, H^{-1}(\tilde{u})) = F(x,u)$. Therefore we drop the tildes and conclude that in the case B system (2) is equivalent to the system

$$u_t = F(x,u)\,v_x, \qquad v_t = F(x,u)\,u_x. \tag{7}$$

In the case C system (2) has the form $u_t = F(x)\,(G(x,u))^{-1}\,v_x$, $v_t = F(x)\,G(x,u)\,u_x$, so the change of variables $\tilde{x} = H(x)$ provided $H'(x) = 1/F(x)$ transforms this system into the system $u_t = (\tilde{G}(\tilde{x},u))^{-1}\,v_{\tilde{x}}$, $v_t = \tilde{G}(\tilde{x},u)\,u_{\tilde{x}}$, with $\tilde{G}(\tilde{x},u) = G(H^{-1}(\tilde{x}),u) = G(x,u)$. Next, the contact transformation $\overline{t} = v$, $\overline{x} = u$, $\overline{u} = \tilde{x}$, and $\overline{v} = t$ maps the last system to the system in the form $\overline{u}_{\overline{t}} = \overline{F}(\overline{x},\overline{u})\,\overline{v}_{\overline{x}}$, $\overline{v}_{\overline{t}} = \overline{F}(\overline{x},\overline{u})\,\overline{u}_{\overline{x}}$. Thus in the case C system (2) is equivalent under a contact transformation to the system in the form (7) too.

Let us analyze the system

$$u_t = F(x,u)\,(G(x,u))^{-1}\,v_x, \qquad v_t = F(x,u)\,G(x,u)\,u_x \tag{8}$$

in the case A. For brevity we denote

$$P = \frac{G_x\,F^2}{F_u}. \tag{9}$$

Computing the linear dependence conditions for the reduced forms θ^α, ξ^i, and σ^α_i by means of MAPLE, we express the group parameters a^1_2, a^2_1, b^1_2, b^2_1, f^1_{11}, f^1_{12}, f^2_{21}, f^2_{22}, g^1_{11}, g^1_{12}, g^2_{12}, and g^2_{22}. Particularly, since

$$\sigma^1_1 \equiv \frac{F\,(a^1_1 - G\,a^1_2)\,(a^1_1 + G\,a^1_2)\,\det(a^\alpha_\beta)}{G\,(b^1_1 - G\,b^1_2)\,(b^1_1 + G\,b^1_2)\,\det(b^i_j)}\,\sigma^2_2 \quad (\mathrm{mod}\,\theta^1,\theta^2,\xi^1,\xi^2,\sigma^1_2),$$

we take $a^1_2 = G^{-1}\,a^1_1$. Then

$$\sigma^1_1 \equiv \frac{(b^2_1 - F\,b^2_2)}{(b^1_2 - F\,b^1_1)}\,\sigma^1_2 \quad (\mathrm{mod}\,\theta^1,\theta^2,\xi^1,\xi^2),$$

and we take $b^2_1 = F\,b^2_2$. Similarly, we set the coefficients of σ^1_1 at θ^1, θ^2, ξ^1, and ξ^2 equal to zero, and express f^1_{11}, f^1_{12}, g^1_{11}, and g^1_{12}, respectively.

Then we obtain

$$\sigma^2_1 \equiv \frac{F\,(a^2_1 + G\,a^2_2)\,b^2_2}{(b^1_2 + F\,b^1_1)\,a^1_1}\,\sigma^1_2 \quad (\mathrm{mod}\,\theta^1,\theta^2,\xi^1,\xi^2,\sigma^2_2),$$

so we take $a^1_2 = -G\,a^2_2$. Now we get

$$\sigma^2_1 \equiv -\frac{2\,F\,b^2_2}{(b^1_1 + F\,b^1_2)}\,\sigma^2_2 \quad (\mathrm{mod}\,\theta^1,\theta^2,\xi^1,\xi^2).$$

Since $b^2_2 \neq 0$ (otherwise $b^2_1 = 0$ and $\det(b^i_j) = 0$), we set the coefficient at σ^2_2 equal to 1, and obtain $b^1_2 = -(F^{-1}\,b^1_1 + 2\,b^2_2)$. After that, we set the coefficients of σ^2_1 at θ^1, θ^2, ξ^1, and ξ^2 equal to zero and find f^2_{21}, f^2_{22}, g^2_{12}, and g^2_{22}, respectively. This yields

$$\sigma^1_1 = 0, \qquad \sigma^2_1 = \sigma^2_2. \tag{10}$$

At the next step we analyze the forms $\phi^\alpha_\beta = \iota^*\,\Phi^\alpha_\beta$ and $\psi^i_j = \iota^*\,\Psi^i_j$ reduced by setting (10) and substituting the values of a^1_2, a^2_1, b^1_2, b^2_1, f^1_{11}, f^1_{12}, f^2_{21}, f^2_{22}, g^1_{11}, g^1_{12}, g^2_{12}, and g^2_{22} obtained at the previous step. The form ϕ^1_2 is semi-basic now, and $\phi^1_2 \equiv c^2_2\,\sigma^1_2(\mathrm{mod}\,\theta^1,\theta^2,\xi^1,\xi^2,\sigma^2_2)$. So we take $c^2_2 = 0$. For the semi-basic form ϕ^2_1 we have $\phi^2_1 \equiv (c^2_1 + c^1_1)\,\sigma^1_2(\mathrm{mod}\,\theta^1,\theta^2,\xi^1,\xi^2,\sigma^2_2)$, so we put $c^2_1 = -c^1_1$. Then

$$\phi^1_2 \equiv \frac{F_u\,(P - F\,G\,u_x - F\,v_x)\,a^1_1}{4\,F\,G^2\,a^2_2\,(b^1_1 + F\,b^2_2)}\,\xi^1 \quad (\mathrm{mod}\,\theta^1,\theta^2,\xi^2),$$

and we take $a^2_2 = F_u\, a^1_1\, (P - F\, G\, u_x - F\, v_x)\, F^{-1}\, G^{-2}\, (b^1_1 + F\, b^2_2)^{-1}$. Then we have the semi-basic linear combination $\psi^1_1 - \phi^1_1 + \phi^2_2$ with

$$\psi^1_1 - \phi^1_1 + \phi^2_2 \equiv \frac{(P - F\, G\, u_x - F\, v_x)\, a^1_1\, c^1_1 - F\, G\, b^2_2}{(P - F\, G\, u_x - F\, v_x)\, a^1_1}\, \sigma^1_2 \quad (\mathrm{mod}\, \theta^1, \theta^2, \xi^1, \xi^2),$$

so we take $c^1_1 = F\, G\, b^2_2\, (P - F\, G\, u_x - F\, v_x)^{-1}\, (a^1_1)^{-1}$. Similarly, we set the coefficients of ϕ^1_2 and ϕ^2_1 at ξ^2 equal to zero, and find f^1_{22} and f^2_{11}, respectively. Then

$$\phi^2_1 \equiv \frac{F^2_u\, (P - F\, G\, u_x - F\, v_x)\, (P - F\, G\, u_x + F\, v_x)}{4\, F^3\, G^2\, b^2_2\, (b^1_1 + F\, b^2_2)}\, \xi^1 \quad (\mathrm{mod}\, \theta^1, \theta^2),$$

so we set the coefficient at ξ^1 equal to 1 and find

$$b^1_1 = \frac{F^2_u\, (P - F\, G\, u_x - F\, v_x)\, (P - F\, G\, u_x + F\, v_x) - 4\, F^4\, G^2\, (b^2_2)^2}{4\, F^3\, G^2\, b^2_2}.$$

Then the semi-basic linear combination $\psi^1_2 + 2\, (\phi^2_2 - \phi^1_1)$ gives

$$\psi^1_2 + 2(\phi^2_2 - \phi^1_1) \equiv \frac{16F^5 G^2 (b^2_2)^2 c^1_2 - F^3_u((P - F\, G\, u_x)^2 - F^2\, v^2_x)}{16\, F^5\, G^2\, a^1_1\, (b^2_2)^2}\, \sigma^2_2 \quad (\mathrm{mod}\, \theta^1, \theta^2, \xi^1, \xi^2),$$

therefore we put

$$c^1_2 = -\frac{F^3_u\, ((P - F\, G\, u_x)^2 - F\, v^2_x)}{16\, F^5\, G^2\, a^1_1\, (b^2_2)^2}.$$

At the third step, we analyze the reduced structure equations. After absorption, we have an essential torsion coefficient at $\xi^1 \wedge \sigma^1_2$ in $d\sigma^1_2$. This coefficient depends on f^2_{12}; we set the coefficient equal to zero and express f^2_{12}, while the expression is too long to be written in full. Similarly, we express f^2_{21} from the essential torsion coefficient at $\xi^2 \wedge \sigma^2_2$ in $d\sigma^2_2$. Then after absorption of torsion in all the structure equations we have

$$d\sigma^2_2 = \zeta_1 \wedge (2\, \theta^1 + \sigma^2_2) + \zeta_2 \wedge (\theta^1 + \sigma^2_2) + \zeta_3 \wedge \theta^2 + \zeta_4 \wedge (\xi^1 + \xi^2) - \xi^2 \wedge \sigma^2_2$$
$$- \frac{2\, F^5\, G^2\, (b^2_2)^2\, \left(G\, P_x + (G\, u_x - v_x)\, P_u\right)}{F^2_u\, a^1_1\, (P - FGu_x + Fv_x)\, (P - FGu_x - Fv_x)^3}\, \theta^2 \wedge \sigma^2_2,$$

where ζ_1, ζ_2, ζ_3, and ζ_4 are 1-forms on $\mathcal{R} \times \mathcal{H}$, and the last torsion coefficient is essential. There are two possibilities now: $P \neq const$ and $P = const$. Denote by $\mathcal{P}_1$ the subclass of systems (8) such that $G_x \neq 0$, $F_u \neq 0$, and $P \neq const$. For a system from $\mathcal{P}_1$ we set the coefficient at $\theta^2 \wedge \sigma^2_2$ in $d\sigma^2_2$ equal to 1 and obtain

$$a^1_1 = -\frac{2\, F^5\, G^2\, (b^2_2)^2\, \left(G\, P_x + (G\, u_x - v_x)\, P_u\right)}{F^2_u\, (P - FGu_x + Fv_x)\, (P - FGu_x - Fv_x)^3}.$$

Similarly, we set the essential torsion coefficient at $\theta^1 \wedge \theta^2$ in $d\xi^1$ equal to zero and find

$$b^2_2 = -\frac{F_u\, (P - FGu_x - Fv_x)}{2\, F^2 G}.$$

Next, we express g^1_{22} from the essential torsion coefficient at $\theta^1 \wedge \xi^1$ in $d\sigma^1_2$. Now the essential torsion coefficient at $\theta^1 \wedge \sigma^1_2$ in $d\theta^1$ has the form

$$R = \frac{F_u\, (FGP_x + PP_u)\, (P - FGu_x + Fv_x)^2}{2\, F^3\, \left(G\, P_x + (G\, u_x - v_x)\, P_u\right)^2}.$$

This function is an invariant of the symmetry group for a system from $\mathcal{P}_1$, together with its invariant derivatives $\mathcal{D}_i(R)$, $i \in \{1, ..., 6\}$, defined by $dR = \mathcal{D}_1(R)\,\theta^1 + \mathcal{D}_2(R)\,\theta^2 + \mathcal{D}_3(R)\,\xi^1 + \mathcal{D}_4(R)\,\xi^2 + \mathcal{D}_5(R)\,\sigma^1_2 + \mathcal{D}_6(R)\,\sigma^2_2$. The invariant $\mathcal{D}_3(R)$ depends on g^2_{11}; we set $\mathcal{D}_3(R) = 0$ and express g^2_{11}.

Now all the parameters of the group $\mathcal{H}$ are expressed as functions of x, u, u_x, and v_x. The structure equations of the symmetry group for a system from $\mathcal{P}_1$ have the form

$$\begin{aligned}
d\theta^1 &= \tfrac{1}{6}\,(6\,K_3 + 1 - 4\,K_2K_3K_5 - 2\,K_4)\,\theta^1\wedge\theta^2\\
&\quad +\tfrac{1}{6}\,(K_4 + 1 + 2\,K_2K_3K_5 - 6\,K_3)\,K_2^{-1}K_3^{-1}\,\theta^1\wedge\xi^1\\
&\quad +\tfrac{1}{3}\,(K_4 + 1 + 3\,K_1K_2K_3 + 3\,K_2K_3K_6 - 6\,K_3 - 4\,K_2K_3K_5)\,K_2^{-1}K_3^{-1}\,\theta^1\wedge\xi^2\\
&\quad +K_2K_3\,\theta^1\wedge\sigma^2_2 - \tfrac{1}{4}\,\theta^2\wedge\xi^1 + \xi^2\wedge\sigma^1_2,\\
d\theta^2 &= K_4\,\theta^1\wedge\theta^2 - \theta^1\wedge\xi^1 + K_5\,\theta^2\wedge\xi^1 + K_6\,\theta^2\wedge\xi^2 + K_2K_3\,\theta^2\wedge\sigma^2_2 + (\xi^1+\xi^2)\wedge\sigma^2_2,\\
d\xi^1 &= K_1K_2\,\theta^1\wedge\xi^1 + \tfrac{1}{2}\,K_1K_2\,\theta^2\wedge\xi^1 + K_1\,\xi^1\wedge\xi^2,\\
d\xi^2 &= K_2K_3\,\theta^1\wedge\theta^2 - \theta^1\wedge\xi^1 + (K_1K_2 - 1)\,\theta^1\wedge\xi^2 + \tfrac{1}{2}\,(2\,K_3 - 1 + K_1K_2)\,\theta^2\wedge\xi^2\\
&\quad +\tfrac{1}{6}\,(K_4 + 1 - 4\,K_2K_3K_5 - 6\,K_3)\,K_2^{-1}K_3^{-1}\,\xi^1\wedge\xi^2,\\
d\sigma^1_2 &= -K_{10}\,\theta^1\wedge\theta^2 - K_{11}\,\theta^1\wedge\xi^2 + (K_4 - 1)\,\theta^1\wedge\sigma^1_2\\
&\quad +\tfrac{1}{12}\,(6\,K_3 - 1 + 4\,K_2K_3K_5 - K_4)\,K_2^{-1}K_3^{-1}\,\theta^2\wedge\xi^1\\
&\quad -K_9\,\theta^2\wedge\xi^2 + \tfrac{1}{6}\,(2\,K_4 - 3\,K_1K_2 + 4\,K_2K_3K_5 - 12\,K_3 + 2)\,\theta^2\wedge\sigma^1_2\\
&\quad +K_8\,\xi^1\wedge\xi^2 - \tfrac{1}{3}\,(K_4 - 6\,K_3 + 1 - K_2K_3K_5)\,K_2^{-1}K_3^{-1}\,\xi^1\wedge\sigma^1_2 + \tfrac{1}{4}\,\xi^1\wedge\sigma^2_2\\
&\quad +K_7\,\xi^2\wedge\sigma^1_2 - K_{12}\,\xi^2\wedge\sigma^2_2 + K_2K_3\,\sigma^1_2\wedge\sigma^2_2,\\
d\sigma^2_2 &= K_{19}\,\theta^1\wedge\theta^2 + K_{17}\wedge\theta^1\wedge\xi^1 + K_{14}\,\theta^1\wedge\xi^2 + (K_4 - K_1K_2 + 1)\,\theta^1\wedge\sigma^2_2\\
&\quad +K_{18}\,\theta^2\wedge\xi^1 + K_{15}\,\theta^2\wedge\xi^2 + \tfrac{1}{3}\,(2\,K_2K_3K_5 + K_4 - 2)\,\theta^2\wedge\sigma^2_2 - K_{16}\,\xi^1\wedge\xi^2\\
&\quad +K_{13}\,\xi^1\wedge\sigma^2_2 - \xi^2\wedge\sigma^1_2 - K_{20}\,\xi^2\wedge\sigma^2_2,
\end{aligned}$$

where

$$K_1 = \frac{2Fv_x\,(2\,F_u^2GP + F_uG_uFP - F_{uu}FGP - G^2F^2F_{xu} - F_uFGP_u + G^2F_xF_uF)}{(P - u_xGF + Fv_x)\,F_u^2G\,(-P + u_xGF + Fv_x)},$$

$$K_2 = \frac{F_u\,(-P + u_xGF + Fv_x)\,(-P + u_xGF - Fv_x)^2}{2F^3v_x\,(u_xP_uG + P_xG - P_uv_x)},$$

$$K_3 = \frac{v_x\,(PP_u + FGP_x)}{(FGu_x + Fv_x - P)\,(GP_x + GP_uu_x - P_uv_x)},$$

while the expressions for K_4, ..., K_{20} are too long to be written in full.

The functions K_1, ..., K_{20} are differential invariants of the symmetry group $Cont(\mathcal{R})$ for system (8) from $\mathcal{P}_1$. All the other differential invariants of $Cont(\mathcal{R})$ are functions of K_j and their invariant derivatives $K_{j,I} = \mathcal{D}_I(K_j)$, where for a multi-index $I = (i_1, i_2, ..., i_l)$ of length $\#I = l$ we denote $\mathcal{D}_I = \mathcal{D}_{i_1} \circ \mathcal{D}_{i_2} \circ ... \circ \mathcal{D}_{i_l}$, $i_k \in \{1, ..., 6\}$ for $k \in \{1, ..., l\}$. For $s \geq 0$ the s^{th} order *classifying manifold* associated with the coframe $\boldsymbol{\theta} = \{\theta^1, \theta^2, \xi^1, \xi^2, \sigma^1_2, \sigma^2_2\}$ and an open subset V in space $\mathbb{R}^4$ with the coordinates (x, u, u_x, v_x) is

$$\mathcal{C}^{(s)}(\boldsymbol{\theta}, V) = \{(K_{j,I}(x, u, u_x, v_x)) \mid j \in \{1, ..., 20\},\ \#I \leq s,\ (x, u, u_x, v_x) \in V\}. \qquad (11)$$

Since all the functions $K_{j,I}$ depend on four variables x, u, u_x, and v_x, it follows that $\rho_s = \dim \mathcal{C}^{(s)}(\boldsymbol{\theta}, V) \leq 4$ for all $s \geq 0$. Let $r = \min\{s \mid \rho_s = \rho_{s+1} = \rho_{s+2} = ...\}$ be the *order of the coframe* $\boldsymbol{\theta}$. We have $0 \leq \rho_0 \leq \rho_1 \leq \rho_2 \leq ... \leq 4$. In any case, $r + 1 \leq 4$. Hence from Theorem 8.19 of [13] we see that two systems (8) from the subclass $\mathcal{P}_1$ are locally equivalent under a contact transformation if and only if their fourth order classifying manifolds (11) locally overlap. The dimension of $Cont(\mathcal{R})$ is equal to $6 - \dim \mathcal{C}^{(4)}(\boldsymbol{\theta}, V)$. Therefore $\dim Cont(\mathcal{R}) \geq 2$, as it should be, since every system (8) is invariant under the symmetries with infinitesimal generators $\partial/\partial t$ and $\partial/\partial v$.

Now we consider the case $G_x \neq 0$, $F_u \neq 0$, and $P = m = const$. From (9) it follows that the system $H_x = -m\, F^{-1}$, $H_u = G$ is compatible, therefore there exists a function $H(x, u)$ such that $dH = -m\, F^{-1}\, dx + G\, du$. Then the change of variables $\tilde{u} = H(x, u)$, $\tilde{v} = v - m\, t$ maps system (8) to the system $\tilde{u}_t = \tilde{F}(x, \tilde{u})\, \tilde{v}_x$, $\tilde{v}_t = \tilde{F}(x, \tilde{u})\, \tilde{u}_x$ with $\tilde{F}(x, \tilde{u}) = F(x, u)$. Dropping tildes, we obtain system (7). Thus in the case $P = const$ system (8) is equivalent to system (7).

Let us consider system (7). The computations are similar, so we omit them and present the results. The structure of the symmetry group for system (7) is different in the cases of $(\ln F)_{xu} \neq 0$ and $(\ln F)_{xu} = 0$. We denote by $\mathcal{P}_2$ the subclass of systems (7) such that $(\ln F)_{xu} \neq 0$. For a system from $\mathcal{P}_2$ all the parameters of the group $\mathcal{H}$ are functions of x, u, u_x, and v_x. The structure equations for the coframe $\boldsymbol{\theta}$ have the form

$$d\theta^1 = (L_3\, \theta^1 + \xi^1) \wedge \theta^2 + \tfrac{1}{3}\, (3\, L_2 L_3 + L_2 - L_4 + L_1 - L_1 L_3)\, \theta^1 \wedge \xi^1 + (L_4\, \theta^1 - \sigma^1_2) \wedge \xi^2,$$

$$d\theta^2 = \tfrac{1}{2}\, L_3\, \theta^1 \wedge \theta^2 + \theta^1 \wedge \xi^2 + \tfrac{1}{3}\, (3\, L_2 L_3 - 2\, L_2 - L_4 + L_1 - L_1 L_3)\, \theta^2 \wedge \xi^1$$
$$-(2\, L_2 + L_1 - L_4)\, \theta^2 \wedge \xi^2 + (\xi^1 + \xi^2) \wedge \sigma^2_2,$$

$$d\xi^1 = -\tfrac{1}{2}\, \theta^1 \wedge \xi^1 + \theta^2 \wedge \xi^1 + L_1\, \xi^1 \wedge \xi^2,$$

$$d\xi^2 = -\tfrac{1}{2}\, \theta^1 \wedge \xi^2 + \theta^2 \wedge \xi^2 + L_2\, \xi^1 \wedge \xi^2,$$

$$d\sigma^1_2 = \tfrac{1}{3}\, (4\, L_2 + 3\, L_2 L_3 - L_3 L_1 - 4\, L_4 + 6 + 4\, L_1)\, (\theta^1 \wedge \theta^2 - \xi^2 \wedge \sigma^1_2) + L_7\, \theta^1 \wedge \xi^1$$
$$+(2\, L_2 L_3 - \tfrac{14}{3}\, L_2^2 - 2\, L_2^2 L_3 + \tfrac{1}{3}\, L_2 L_4 + \tfrac{1}{13}\, L_2 - \tfrac{1}{3}\, L_1 L_2 L_3 - 7\, L_2 L_1 - \tfrac{25}{6}\, L_4 + \tfrac{1}{3}\, {L_1}^2 L_3$$
$$+\tfrac{7}{3}\, L_1 L_4 - \tfrac{2}{3}\, L_3 L_1 + 1 + \tfrac{25}{6}\, L_1 - \tfrac{1}{2}\, L_6 - \tfrac{7}{3}\, L_1^2)\, \theta^1 \wedge \xi^2 + \tfrac{1}{2}\, L_3\, \theta^1 \wedge \sigma^1_2 - 2\, L_2\, \theta^2 \wedge \xi^1$$
$$+\theta^2 \wedge (L_6\, \xi^2 - (L_3 + 1)\, \sigma^1_2) + \tfrac{1}{3}\, (14\, L_2 + 6\, L_2 L_3 - 2\, L_3 L_1 - 8\, L_4 + 3 + 14\, L_1)\, \xi^2 \wedge \sigma^2_2$$
$$+L_5\, \xi^1 \wedge \xi^2 + \tfrac{1}{3}\, (L_4 - 3\, L_2 L_3 - 4\, L_2 - L_1 + L_1 L_3)\, \xi^1 \wedge \sigma^1_2 + \xi^1 \wedge \sigma^2_2,$$

$$
\begin{aligned}
d\sigma_2^2 = &-\tfrac{1}{6}\,(3\,L_2L_3 + L_2 + \tfrac{3}{2} + L_1 - L_1L_3 - L_4)\,\theta^1\wedge\theta^2 + \tfrac{1}{12}\,(2\,L_1^2 - 2\,L_1^2L_3 + 4\,L_1L_3 \\
&+2\,L_1L_2L_3 - 2\,L_1L_4 - L_1 + 6\,L_1L_2 + 6\,L_8 + 2\,L_2L_4 + L_4 - 6 - 12\,L_2L_3 + 4\,L_2^2 \\
&+12\,L_2^2L_3 + 2\,L_2)\,\theta^1\wedge(\xi^1 - \xi^2) + \tfrac{1}{2}\,(L_3+1)\,\theta^1\wedge\sigma_2^2 - (L_1 - L_8 + L_7 - 3\,L_2)\,\theta^2\wedge\xi^1 \\
&+L_8\,\theta^2\wedge\xi^2 - L_3\,\theta^2\wedge\sigma_2^2 - \tfrac{1}{6}\,(3\,L_2L_3 + 4\,L_2 - L_4 + L_1 - L_3L_1)\,\xi^1\wedge\sigma_2^1 \\
&+\tfrac{1}{2}\,(2\,L_4 - 4\,L_2 + 1 - 4\,L_1)\,\xi^1\wedge\sigma_2^2 + \tfrac{1}{6}\,(L_4 - 3\,L_2L_3 - 4\,L_2 - L_1 + L_1L_3)\,\xi^2\wedge\sigma_2^1 \\
&+\tfrac{1}{2}\,L_5\,\xi^1\wedge\xi^2 + \tfrac{1}{6}\,(6\,L_2L_3 + 2_2 + 3 + 2\,L_1 - 2\,L_1L_3 - 2\,L_4)\,\xi^2\wedge\sigma_2^2,
\end{aligned}
$$

where

$$
\begin{aligned}
L_1 = &\,(3\,v_x^2F_u^2 - 3\,F_{uu}v_x^2F - 5\,v_xF_{xu}F + 5\,v_xF_xF_u - 3\,u_xF_xF_u + 3\,F_{uu}u_x^2F - 3\,u_x^2F_u^2 \\
&+3\,u_xF_{xu}F)\,(u_x^2 - v_x^2)^{-1}\,F_u^{-2}, \\
L_2 = &\,(F_u^2L_1u_x + 8\,v_xF_u^2 - 8\,v_xFF_{uu} + F_u^2L_1v_x)\,F_u^{-2}\,(3\,u_x - 5\,v_x)^{-1}, \\
L_3 = &\,\tfrac{1}{64}\,F_u^3\,(u_x^2 - v_x^2)\,(6\,u_x^2F_uL_2L_1 - u_x^2F_uL_1^2 - 9\,u_x^2F_uL_2^2 + 8\,F_xL_1v_x - 24\,Fv_xL_{2,x} \\
&+8\,FL_{1,x}v_x - 6\,F_uv_x^2L_2L_1 + F_uL_1^2v_x^2 - 24\,F_xL_2v_x + 9\,F_uv_x^2L_2^2)\,v_x^{-2}(FF_{xu} - F_xF_u)^{-2}, \\
L_4 = &-\tfrac{1}{16}\,F_u\,(u_x^2 - v_x^2)\,(6\,u_x^2F_uL_2^2 + 9\,u_x^2F_uL_2 - 3\,u_x^2F_uL_1 - 4\,u_x^2F_uL_1^2 + 10\,u_x^2F_uL_2L_1 \\
&+18\,u_xF_xL_2 - 6\,u_xF_xL_1 - 6\,F_uv_x^2{L_2}^2 + 10\,F_xL_1v_x + 4\,F_uL_1^2v_x^2 + 16\,FL_{1,x}v_x \\
&-9\,F_uv_x^2L_2 + 3\,F_uL_1v_x^2 - 30\,F_xL_2v_x - 10\,F_uv_x^2L_2L_1)\,(FF_{xu} - F_xF_u)^{-1},
\end{aligned}
$$

while L_5, ..., L_8 are too long to be written in full. All the differential invariants of $Cont(\mathcal{R})$ are functions of L_j and their invariant derivatives $L_{j,I} = \mathcal{D}_I(L_j) = \mathcal{D}_{i_1}\circ\mathcal{D}_{i_2}\circ ... \circ\mathcal{D}_{i_l}(L_j)$ (the operators $\mathcal{D}_i$ are not the same as in the case $\mathcal{P}_1$!) The s^{th} order classifying manifold associated with the coframe $\boldsymbol{\theta}$ and an open subset V is

$$
\mathcal{C}^{(s)}(\boldsymbol{\theta}, V) = \{(L_{j,I}(x,u,u_x,v_x)) \mid j\in\{1,...,8\},\ \#I\le s,\ (x,u,u_x,v_x)\in V\}. \qquad (12)
$$

Since all the functions $L_{j,I}$ depend on four variables x, u, u_x, and v_x, it follows that $\rho_s = \dim\mathcal{C}^{(s)}(\boldsymbol{\theta}, V) \le 4$ for all $s\ge 0$, and the order r of the coframe $\boldsymbol{\theta}$ satisfies $r+1\le 4$ again. Two systems (7) from the subclass $\mathcal{P}_2$ are locally equivalent under a contact transformation if and only if their fourth order classifying manifolds (12) locally overlap, and $\dim Cont(\mathcal{R}) = 6 - \dim\mathcal{C}^{(4)}(\boldsymbol{\theta}, V)\ge 2$.

If $(\ln F)_{xu} = 0$, then $F(x,u) = S(x)\,\tilde{F}(u)$ for arbitrary functions S and $\tilde{F}$. Then the change of variables $\tilde{x} = H(x)$ provided $H'(x) = (S(x))^{-1}$ maps the system $u_t = S(x)\,\tilde{F}(u)\,v_x$, $v_t = S(x)\,\tilde{F}(u)\,u_x$, to the system $u_t = \tilde{F}(u)\,v_{\tilde{x}}$, $v_t = \tilde{F}(u)\,u_{\tilde{x}}$. We drop the tildes for simplicity of notation and consider the system

$$
u_t = F(u)\,v_x, \qquad v_t = F(u)\,u_x. \qquad (13)
$$

The computations show that there are three non-equivalent types of systems (13): denote by $\mathcal{P}_3$ the subclass of systems (13) such that $F_u\ne 0$ and

$$
M_1 = \frac{4FF_u^2F_{uu} + 4F^2F_{uu}^2 - 4F^2F_uF_{uuu} - 3F_u^4}{F_u^4} \ne const,
$$

by $\mathcal{P}_4$ denote the subclass of systems (13) such that $F_u \neq 0$ and $M_1 = const$, finally, by $\mathcal{P}_5$ denote the subclass of systems (13) such that $F_u = 0$.

The subclass $\mathcal{P}_3$ is not empty; for example, system (13) with $F(u) = (1 + u^2)^{-1}$ belongs to $\mathcal{P}_3$. For a system from $\mathcal{P}_3$ the structure equations of the symmetry pseudo-group after a prolongation have the form

$$d\theta^1 = \eta_1 \wedge \theta^1 - \theta^2 \wedge \xi^1 + \xi^2 \wedge \sigma^1_2,$$

$$d\theta^2 = \eta_1 \wedge \theta^2 + \theta^1 \wedge \xi^2 - M_2\, \theta^2 \wedge \xi^1 - (2\, M_2 + M_3)\, \theta^2 \wedge \xi^2 + (\xi^1 + \xi^2) \wedge \sigma^2_2,$$

$$d\xi^1 = M_3\, \xi^1 \wedge \xi^2,$$

$$d\xi^2 = M_2\, \xi^1 \wedge \xi^2,$$

$$d\sigma^1_2 = \eta_1 \wedge \sigma^1_2 + \eta_2 \wedge \xi^2 + M_1\, \theta^1 \wedge \xi^1 - 2\, M_2\, \theta^2 \wedge \xi^1 - M_2\, \xi^1 \wedge \sigma^1_2 + \xi^1 \wedge \sigma^2_2,$$

$$d\sigma^2_2 = \eta_1 \wedge \sigma^2_2 + \eta_3 \wedge (\xi^1 + \xi^2) + M_1\, \theta^2 \wedge \xi^2 - 2\,(M_2 + M_3)\, \xi^2 \wedge \sigma^2_2,$$

$$d\eta_1 = (M_1 - 1)\, \xi^1 \wedge \xi^2,$$

$$\begin{aligned} d\eta_2 = {} & \mu_1 \wedge \xi^2 + \eta_1 \wedge \eta_2 + 2\, M_2\, \eta_2 \wedge \xi^1 - \eta_3 \wedge \xi^1 + (\mathcal{D}_4(M_1) + 2M_2 - M_1 M_3)\ \theta^1 \wedge \xi^1 \\ & - \Big(2\mathcal{D}_4(M_2) + 4M_2^2 - M_1\Big)\ \theta^2 \wedge \xi^1 + (\mathcal{D}_4(M_2) + 2M_1 - M_2 M_3 - 1)\ \xi^1 \wedge \sigma^1_2 \\ & -(4M_2 + M_3)\, \xi^1 \wedge \sigma^2_2, \end{aligned}$$

$$\begin{aligned} d\eta_3 = {} & \mu_2 \wedge (\xi^1 + \xi^2) + \eta_1 \wedge \eta_3 - 3\,(M_2 + M_3)\, \eta_3 \wedge \xi^2 - (2M_1 M_2 + 1)\, \theta^2 \wedge \xi^2 \\ & + (4M_1 - 2\mathcal{D}_3(M_2)(\mathcal{D}_4(M_1) - 1) + 2M_2(M_2 + M_3) - 3)\ \xi^2 \wedge \sigma^2_2, \end{aligned}$$

where η_1, η_2, η_3, μ_1, and μ_2 are 1-forms on $\mathcal{R} \times \mathcal{H}$. The only non-zero *reduced character*, [13, def 11.4], is $s_1' = 2$, therefore the symmetry pseudo-group for system (13) from $\mathcal{P}_3$ depends on two arbitrary functions of one variable. The invariants M_2 and M_3 are defined by $M_2 = (2FF_{uu}M_{1,u} - FF_uM_{1,uu} - 2F_u^2 M_{1,u})F^{-1}F_u^{-1}M_{1,u}^{-2}$ and $M_3 = -(M_2\, \mathcal{D}_4(M_1) + \mathcal{D}_{(3,4)}(M_1))$, where for an arbitrary function $R(u)$ we have $dR = \mathcal{D}_3(R)\, \xi^1 + \mathcal{D}_4(R)\, \xi^2$ with the invariant derivatives $\mathcal{D}_3 = M_{1,u}^{-1}\, \partial/\partial u$ and $\mathcal{D}_4 = (1 - 4F^2 M_{1,u}^2 F_u^{-2})\, M_{1,u}^{-1}\, \partial/\partial u$. We have $\mathcal{D}_3(M_1) = 1$ and $\mathcal{D}_4(M_1) = 1 - 4F^2 M_{1,u}^2 F_u^{-2}$. Since $M_1 \neq const$, then M_2 and $\mathcal{D}_4(M_1)$ depend on M_1 functionally: $M_2 = H_1(M_1)$ and $\mathcal{D}_4(M_1) = H_2(M_1)$. All the other differential invariants can be expressed as functions of M_1. For example, we have $\mathcal{D}_3(M_2) = H_1'(M_1)\, \mathcal{D}_3(M_1) = H_1'(M_1)$ and $\mathcal{D}_4(M_2) = \mathcal{D}_4(M_1)\, \mathcal{D}_3(M_2) = H_2(M_1)\, H_1'(M_1)$.

The first order classifying manifold associated with the coframe $\boldsymbol{\theta} = \{\theta^1, \theta^2, \xi^1, \xi^2, \sigma^1_2, \sigma^2_2, \eta_1, \eta_2, \eta_3\}$ and an open subset $W \subset \mathbb{R}$ can be parameterized by M_1, M_2, and $\mathcal{D}_4(M_1)$:

$$\mathcal{C}^{(1)}(\boldsymbol{\theta}, W) = \{(M_1(u), M_2(u), \mathcal{D}_4(M_1)(u)) \mid \ u \in W\}. \tag{14}$$

Two systems (13) from $\mathcal{P}_3$ are equivalent under a contact transformation iff their classifying manifolds (14) (locally) overlap, [13, Th 15.22], i.e., they have the same functions H_1 and H_2.

The subclass $\mathcal{P}_4$ is not empty; for example, systems (13) with $F(u) = \exp(C\arctan(\sinh(\lambda\,u)))$, $F(u) = \mathrm{e}^u$, or $F(u) = u^m$, $m \neq 0$, belong to $\mathcal{P}_4$. For a system from $\mathcal{P}_4$ the structure equations of symmetry pseudo-group after a prolongation have the form

$$d\theta^1 = \eta_1 \wedge \theta^1 - \theta^2 \wedge \xi^1 + \xi^2 \wedge \sigma^1_2,$$

$$d\theta^2 = \eta_2 \wedge \theta^2 - \theta^1 \wedge \xi^2 + (\xi^1 + \xi^2) \wedge \sigma^2_2,$$

$$d\xi^1 = (\eta_1 - \eta_2) \wedge (\xi^1 + 2\,\xi^2),$$

$$d\xi^2 = (\eta_2 - \eta_1) \wedge \xi^2,$$

$$d\sigma^1_2 = -2\,\eta_1 \wedge (\theta^2 - \sigma^1_2) + \eta_2 \wedge (2\,\theta^2 - \sigma^1_2) + \eta_3 \wedge \xi^2 + M_1\,\theta^1 \wedge \xi^1 + \xi^1 \wedge \sigma^2_2,$$

$$d\sigma^2_2 = (2\,\eta_2 - \eta_1) \wedge \sigma^2_2 + \eta_4 \wedge (\xi^1 + \xi^2) + M_1\,\theta^2 \wedge \xi^2,$$

$$d\eta_1 = (M_1 - 1)\,\xi^1 \wedge \xi^2,$$

$$d\eta_2 = -(M_1 - 1)\,\xi^1 \wedge \xi^2,$$

$$d\eta_3 = \mu_1 \wedge \xi^2 + (3\,\eta_1 - 2\,\eta_2) \wedge \eta_3 - 2\,(M_1 + 1)\,(\eta_1 - \eta_2) \wedge \theta^1 + 4\,(\eta_1 + \eta_2) \wedge \sigma^2_2 - \eta_4 \wedge \xi^1$$
$$+(3\,M_1 - 4)\,\theta^2 \wedge \xi^1 + (4\,M_1 - 3)\,\xi^1 \wedge \sigma^1_2,$$

$$d\eta_4 = \mu_2 \wedge (\xi^1 + \xi^2) - (2\,\eta_1 - 3\,\eta_2) \wedge \eta_4 + (4\,M_1 - 3)\,\xi^2 \wedge \sigma^2_2,$$

where η_1, η_2, η_3, η_4, μ_1, and μ_2 are 1-forms on $\mathcal{R} \times \mathcal{H}$. The only non-zero reduced character is $s'_1 = 2$, therefore the symmetry pseudo-group for system (13) from $\mathcal{P}_3$ depends on two arbitrary functions of one variable. Since $M_1 = \mathit{const}$, all the other differential invariants are equal to zero, and the classifying manifold is a point. Two systems from $\mathcal{P}_4$ are equivalent under a contact transformation iff they have the same values of M_1.

A system from $\mathcal{P}_5$ with $F(u) \equiv m = \mathit{const}$ can be transformed to the system

$$u_t = v_x, \qquad v_t = u_x \tag{15}$$

by the change of variables $t \mapsto m^{-1}\,t$. The symmetry pseudo-group for system (15) has the structure equations

$$d\theta^1 = \eta_1 \wedge \theta^1 + \xi^1 \wedge \sigma^1_2,$$

$$d\theta^2 = \eta_2 \wedge \theta^2 + \xi^2 \wedge \sigma^2_2,$$

$$d\xi^1 = \eta_3 \wedge \xi^1 + \eta_4 \wedge \theta^1,$$

$$d\xi^2 = \eta_5 \wedge \xi^2 + \eta_6 \wedge \theta^2,$$

$$d\sigma^1_2 = (\eta_1 - \eta_3) \wedge \sigma^1_2 + \eta_7 \wedge \theta^1 + \eta_8 \wedge \xi^1,$$

$$d\sigma^2_2 = (\eta_2 - \eta_5) \wedge \sigma^2_2 + \eta_9 \wedge \theta^2 + \eta_{10} \wedge \xi^2,$$

where η_1, ..., η_{10} are 1-forms on $\mathcal{R} \times \mathcal{H}$. The non-zero reduced characters are $s'_1 = 6$ and $s'_2 = 4$, therefore the pseudo-group depends on 4 arbitrary functions of two variables.

The subclasses $\mathcal{P}_3$ and $\mathcal{P}_4$ are linearizable: the contact transform $\tilde{t} = v$, $\tilde{x} = u$, $\tilde{u} = x$, and $\tilde{v} = t$ maps system (13) to the system $\tilde{u}_{\tilde{t}} = F(\tilde{x})\, \tilde{v}_{\tilde{x}}$, $\tilde{v}_{\tilde{t}} = (F(\tilde{x}))^{-1}\, \tilde{u}_{\tilde{x}}$. Therefore all systems (8) with infinite-dimensional symmetry pseudo-groups are linearizable, cf. [15].

The results of the computations are summarized in the following

Theorem : *Every system from the class of nonlinear wave equations (8) is equivalent under a contact transformation to a system from one of the five invariant subclasses* $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, and $\mathcal{P}_5$: $\mathcal{P}_1$ *consists of all systems (8) such that* $G_x \neq 0$, $F_u \neq 0$, *and* $G_x\, F^2\, F_u^{-1} \neq$ *const*, $\mathcal{P}_2$ *consists of all systems (7) such that* $(\ln F)_{xu} \neq 0$, $\mathcal{P}_3$ *consists of all systems (13) such that* $M_1 = (4FF_u^2F_{uu} + 4F^2F_{uu}^2 - 4F^2F_uF_{uuu} - 3F_u^4)\, F_u^{-4} \neq$ *const*, $\mathcal{P}_4$ *consists of all systems (13) such that* $M_1 =$ *const*, and $\mathcal{P}_5$ *consists of system (15).*

Systems from $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *have finite-dimensional symmetry groups, while systems from* $\mathcal{P}_3$, $\mathcal{P}_4$, *and* $\mathcal{P}_5$ *are linearizable and have infinite-dimensional symmetry pseudo-groups.*

Two systems from one of the subclasses $\mathcal{P}_1$, $\mathcal{P}_2$, *or* $\mathcal{P}_3$ *are equivalent to each other under a contact transformation if and only if the classifying manifolds (11), (12), or (14) for these systems locally overlap. Two systems from the subclass* $\mathcal{P}_4$ *are equivalent if and only if they have the same constant value of the invariant* M_1.

Conclusion

In this paper, the moving coframe method of [6] is applied to the local equivalence problem for a class of systems of nonlinear wave equations under an action of the pseudo-group of contact transformations. We have found five invariant subclasses and shown that every system of nonlinear wave equations can be transformed to a system from one of these subclasses. The structure equations and the differential invariants for all the subclasses are found. The solution of the equivalence problem is given in terms of the differential invariants. Three of the invariant subclasses consist of linearizable systems with infinite-dimensional symmetry pseudo-groups. Therefore all the linearizable cases for non-linear wave equations are classified.

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