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CONTACT-EQUIVALENCE PROBLEM FOR LINEAR HYPERBOLIC EQUATIONS

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ABSTRACT. We consider the local equivalence problem for the class of linear second-order hyperbolic equations in two independent variables under an action of the pseudo-group of contact transformations. É. Cartan's method is used for finding the Maurer–Cartan forms for symmetry groups of equations from the class and computing structure equations and complete sets of differential invariants for these groups. The solution of the equivalence problem is formulated in terms of these differential invariants.

Introduction

In the present paper, we find necessary and sufficient conditions for two equations from the class of linear second-order hyperbolic equations

$$u_{tx} = T(t,x)u_t + X(t,x)u_x + U(t,x)u \tag{1}$$

to be equivalent under an action of the contact transformation pseudo-group. We use Élie Cartan's method of equivalence [1–5] in its form developed by Fels and Olver [6, 7] to compute the Maurer–Cartan forms, the structure equations, the basic invariants, and the invariant derivatives for symmetry groups of equations from the class. All differential invariants are functions of the basic invariants and their invariant derivatives. The differential invariants parametrize classifying manifolds associated with given equations. Cartan's solution to the equivalence problem states that two equations are (locally) equivalent if and only if their classifying manifolds (locally) overlap.

The symmetry classification problem for classes of differential equations is closely related to the problem of local equivalence: symmetry groups of two equations are necessarily isomorphic if these equations are equivalent, while the converse statement is not true in general. The symmetry analysis of linear second-order hyperbolic equations (1) was done by Lie [16, Vol. 3, pp. 492–523]. Two semi-invariants, $H = -T_t + TX + U$ and $K = -X_x + TX + U$, were discovered by Laplace [15]. These functions are unaltered under an action of the pseudo-groups of linear transformations $\bar{u} = c(t,x)u$. In [22], Ovsiannikov found the invariants $P = KH^{-1}$ and $Q = (\ln|H|)_{tx}H^{-1}$ and used them to classify Eqs. (1) with nontrivial symmetry groups. In [9, Theorem 2.3] and [10, Sec. 10.4.2], it was claimed that the invariants P and Q form a basis of differential invariants for Eqs. (1), while all the other invariants are functions of P and Q and their invariant derivatives. In [14], a basis of five invariants and operators of invariant differentiation are found in the case $P_x \neq 0$. In the case $P_t \neq 0$ and $P_x \neq 0$, two bases of four invariants are computed in [12].

In [18], the invariant version of Lie's infinitesimal method was developed and applied to the symmetry classification of the class (1).

The symmetry classification problem and invariants for the class of linear parabolic equations $u_{xx} = T(t,x)u_t + X(t,x)u_x + U(t,x)u$ are studied in [11, 13, 16, 23] by Lie's infinitesimal method. In [20, 21], Cartan's method is applied to solve the contact equivalence problem for this class.

The paper is organized as follows. In Sec. 1, we begin with some notation and use Cartan's equivalence method to find the invariant 1-forms and the structure equations for the pseudo-group of contact transformations on the bundle of second-order jets. In Sec. 2, we briefly describe the approach to computing Maurer–Cartan forms and structure equations for symmetry groups of differential equations via

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the moving coframe method of Fels and Olver. In Sec. 3, the method is applied to the class of hyperbolic equations (1). Finally, we make some concluding remarks.

1. Pseudo-Group of Contact Transformations

In this paper, all considerations are of a local nature and all mappings are real analytic. Let $\mathcal{E} = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a trivial bundle with the local base coordinates $(x^1, \dots, x^n)$ and the local fibre coordinate u; then by $J^2(\mathcal{E})$ denote the bundle of second-order jets of sections of $\mathcal{E}$, with the local coordinates (x^i, u, p_i, p_{ij}), $i, j \in \{1, \dots, n\}$, $i \le j$. For every local section $(x^i, f(x))$ of $\mathcal{E}$, the corresponding 2-jet $(x^i, f(x), \partial f(x)/\partial x^i, \partial^2 f(x)/\partial x^i \partial x^j)$ is denoted by $j_2(f)$. A differential 1-form ϑ on $J^2(\mathcal{E})$ is called a *contact form* if it is annihilated by all 2-jets of local sections: $j_2(f)^*\vartheta = 0$. In the local coordinates every contact 1-form is a linear combination of the forms $\vartheta_0 = du - p_i dx^i$, $\vartheta_i = dp_i - p_{ij} dx^j$, $i, j \in \{1, \dots, n\}$, $p_{ji} = p_{ij}$ (here and later we use the Einstein summation convention, so $p_i dx^i = \sum\limits_{i=1}^{n} p_i dx^i$, etc.). A local diffeomorphism

$$\Delta\colon J^2(\mathcal{E}) \to J^2(\mathcal{E}), \quad \Delta\colon (x^i, u, p_i, p_{ij}) \mapsto (\bar{x}^i, \bar{u}, \bar{p}_i, \bar{p}_{ij}), \tag{2}$$

is called a *contact transformation* if for every contact 1-form ϑ the form $\Delta^*\bar{\vartheta}$ is also contact. We use Cartan's method of equivalence [5, 24] to obtain a collection of invariant 1-forms for the pseudo-group of contact transformations on $J^2(\mathcal{E})$. For this, take the coframe

$$\{\vartheta_0, \vartheta_i, dx^i, dp_{ij} \mid i, j \in \{1, \dots, n\},\ i \le j\}$$

on $J^2(\mathcal{E})$. A contact transformation (2) acts on this coframe in the following manner:

$$\Delta^* \begin{pmatrix} \bar{\vartheta}_0 \\ \bar{\vartheta}_i \\ d\bar{x}^i \\ d\bar{p}_{ij} \end{pmatrix} = S \begin{pmatrix} \vartheta_0 \\ \vartheta_k \\ dx^k \\ dp_{kl} \end{pmatrix},$$

where $S\colon J^2(\mathcal{E}) \to \mathcal{G}$ is an analytic function and $\mathcal{G}$ is the Lie group of nondegenerate block matrices of the form

$$\begin{pmatrix} a & \tilde{a}^k & 0 & 0 \\ \tilde{g}_i & h_i^k & 0 & 0 \\ \tilde{c}^i & \tilde{f}^{ik} & b_k^i & r^{ikl} \\ \tilde{s}_{ij} & \tilde{w}_{ij}^k & \tilde{z}_{ijk} & \tilde{q}_{ij}^{kl} \end{pmatrix}.$$

In these matrices, $i, j, k, l \in \{1, \dots, n\}$, r^{ikl} are defined for $k \le l$, $\tilde{s}_{ij}$, $\tilde{w}_{ij}^k$, and $\tilde{z}_{ijk}$ are defined for $i \le j$, and $\tilde{q}_{ij}^{kl}$ are defined for $i \le j$, $k \le l$.

Let us show that $\tilde{a}^k = 0$. Indeed, the exterior (nonclosed!) ideal

$$\mathcal{I} = \operatorname{span}\{\vartheta_0, \vartheta_i\}$$

has the derived ideal

$$\delta\mathcal{I} = \{\omega \in \mathcal{I} \mid d\omega \in \mathcal{I}\} = \operatorname{span}\{\vartheta_0\}.$$

Since $\Delta^*\bar{\mathcal{I}} \subset \mathcal{I}$ implies $\Delta^*(\delta\bar{\mathcal{I}}) \subset \delta(\Delta^*\bar{\mathcal{I}}) \subset \delta\mathcal{I}$, we obtain $\Delta^*\bar{\vartheta}_0 = a\vartheta_0$.

For convenience in the following computations, we denote by (B_i^j) the inverse matrix for (b_i^j), so

$$b_i^j B_j^k = \delta_i^k,$$

by (H_i^j) denote the inverse matrix for (h_i^j), so

$$h_i^j H_j^k = \delta_i^k,$$

make the change of variables on $\mathcal{G}$ such that

$$g_i = \tilde{g}_i a^{-1}, \quad f^{ij} = \tilde{f}^{ik} H_k^j, \quad c^i = \tilde{c}^i a^{-1} - f^{ik} g_k, \quad s_{ij} = \tilde{s}_{ij} a^{-1} - \tilde{w}_{ij}^k H_k^m g_m - \tilde{z}_{ijm} B_k^m c^k,$$

$$w_{ij}^k = \tilde{w}_{ij}^m H_m^k - \tilde{z}_{ijm} B_l^m f^{lk}, \quad z_{ijk} = \tilde{z}_{ijm} B_k^m, \quad q_{ij}^{kl} = \tilde{q}_{ij}^{kl} - \tilde{z}_{ijm} B_{m'}^m r^{m'kl},$$

and define $Q^{kl}_{k'l'}$ by

$$Q^{kl}_{k'l'} q^{k'l'}_{ij} = \delta^k_i \delta^l_j.$$

In accordance with Cartan's method of equivalence, we take the lifted coframe

$$\begin{pmatrix} \Theta_0 \\ \Theta_i \\ \Xi^i \\ \Sigma_{ij} \end{pmatrix} = S \begin{pmatrix} \vartheta_0 \\ \vartheta_k \\ dx^k \\ dp_{kl} \end{pmatrix} = \begin{pmatrix} a\vartheta_0 \\ g_i\Theta_0 + h^k_i \vartheta_k \\ c^i\Theta_0 + f^{ik}\Theta_k + b^i_k dx^k + r^{ikl} dp_{kl} \\ s_{ij}\Theta_0 + w^k_{ij}\Theta_k + z_{ijk}\Xi^k + q^{kl}_{ij} dp_{kl} \end{pmatrix} \tag{3}$$

on $J^2(\mathcal{E}) \times \mathcal{G}$. Expressing du, dx^k, dp_k, and dp_{kl} from (3) and substituting them into $d\Theta_0$, we have

$$\begin{aligned} d\Theta_0 &= da \wedge \vartheta_0 + ad\vartheta_0 = daa^{-1} \wedge \Theta_0 + adx^i \wedge dp_i = daa^{-1} \wedge \Theta_0 + adx^i \wedge \vartheta_i \\ &= \Phi^0_0 \wedge \Theta_0 + aB^i_k H^m_i \Xi^k \wedge \Theta_m + aH^m_i R^{ikl} \Sigma_{kl} \wedge \Theta_m + aH^m_i (B^i_k f^{kj} + R^{ikl} w^j_{kl}) \Theta_j \wedge \Theta_m, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \Phi^0_0 = daa^{-1} + aH^{m'}_i (B^i_k (c^k + R^{ikl} s_{kl}) \Theta_{m'} - g_{m'} B^i_k (\Xi^k - c^k\Theta_0 - f^{kj}\Theta_j) \\ - g_{m'} R^{ikl} (\Sigma_{kl} - s_{kl}\Theta_0 - w^m_{kl}\Theta_m - z_{klm}\Xi^m)) \end{aligned}$$

and $R^{jkl} = -r^{ik'l'} B^j_i Q^{kl}_{k'l'}$.

The multipliers of $\Xi^k \wedge \Theta_m$, $\Sigma_{kl} \wedge \Theta_m$, and $\Theta_j \wedge \Theta_m$ in (4) are essential torsion coefficients. We normalize them by setting $aB^i_k H^m_i = \delta^m_k$, $R^{ikl} = 0$, and $f^{kj} = f^{jk}$. Therefore, the first normalization is

$$h^k_i = aB^k_i, \quad r^{ikl} = 0, \quad f^{kj} = f^{jk}. \tag{5}$$

Analyzing $d\Theta_i$, $d\Xi^i$, and $d\Sigma_{ij}$ in the same way, we obtain the following normalizations:

$$q^{kl}_{ij} = aB^k_i B^l_j, \quad s_{ij} = s_{ji}, \quad w^k_{ij} = w^k_{ji}, \quad z_{ijk} = z_{jik} = z_{ikj}. \tag{6}$$

After these reductions the structure equations for the lifted coframe have the form

$$\begin{aligned} d\Theta_0 &= \Phi^0_0 \wedge \Theta_0 + \Xi^i \wedge \Theta_i, \\ d\Theta_i &= \Phi^0_i \wedge \Theta_0 + \Phi^k_i \wedge \Theta_k + \Xi^k \wedge \Sigma_{ik}, \\ d\Xi^i &= \Phi^0_0 \wedge \Xi^i - \Phi^i_k \wedge \Xi^k + \Psi^{i0} \wedge \Theta_0 + \Psi^{ik} \wedge \Theta_k, \\ d\Sigma_{ij} &= \Phi^k_i \wedge \Sigma_{kj} - \Phi^0_0 \wedge \Sigma_{ij} + \Upsilon^0_{ij} \wedge \Theta_0 + \Upsilon^k_{ij} \wedge \Theta_k + \Lambda_{ijk} \wedge \Xi^k, \end{aligned}$$

where the forms Φ^0_0, Φ^0_i, Φ^k_i, Ψ^{i0}, Ψ^{ij}, Υ^0_{ij}, Υ^k_{ij}, and Λ_{ijk} are defined by the following equations:

$$\begin{aligned} \Phi^0_0 &= daa^{-1} - g_k\Xi^k + (c^k + f^{km} g_m)\Theta_k, \\ \Phi^0_i &= dg_i + g_k db^k_j B^j_i - (g_i g_k + s_{ik} + c^j z_{ijk})\Xi^k + c^k \Sigma_{ik} + (g_i c^k + g_i g_m f^{mk} - c^j w^k_{ij} + f^{mk} s_{im})\Theta_k, \\ \Phi^k_i &= \delta^k_i daa^{-1} - db^k_j B^j_i + (g_i \delta^k_j - w^k_{ij} - f^{km} z^i_{jm})\Xi^j + f^{km}\Sigma_{im} + f^{jm} w^k_{ij}\Theta_m, \\ \Psi^{i0} &= dc^i + f^{ij}\Phi^0_j + c^k\Phi^i_k + (c^i f^{mj} g_m - c^k f^{mj} w^i_{kj})\Theta_j - c^k f^{ij}\Sigma_{kj} + c^k(f^{im} z^{kmj} + w^i_{kj} - g_k\delta^i_j - g_j\delta^i_k)\Xi^j, \\ \Psi^{ij} &= df^{ij} + (f^{ik}\delta^j_m + f^{jk}\delta^i_m)\Phi^m_k + (c^i\delta^j_k + c^j\delta^i_k - f^{ij}, g_k + f^{im} f^{jl} z_{klm})\Xi^k \\ &\quad + f^{ij}(c^k + f^{km} g_m)\Theta_k - f^{ik} f^{jm}\Sigma_{km}, \end{aligned} \tag{7}$$

$$
\begin{aligned}
\Upsilon^0_{ij} &= ds_{ij} - s_{ij}\,daa^{-1} + s_{kj}\,db^k_m B^m_i + s_{ik}\,db^k_m B^m_j + s_{ij}\Phi^0_0 + w^k_{ij}\Phi^0_k + z_{ijk}\Psi^{k0},\\
\Upsilon^k_{ij} &= dw^k_{ij} - w^k_{ij}\,daa^{-1} + (w^k_{il}\delta^{m'}_j + w^k_{jl}\delta^{m'}_i)\,db^l_m B^m_{m'} + (s_{ij}\delta^k_m + z_{ijl}f^{m'k}w^l_{m'm})\Xi^m\\
&\quad + w^m_{ij}\Phi^k_m + f^{lk}(w^m_{il}\delta^{m'}_j + w^m_{jl}\delta^{m'}_i)\Sigma_{m'm} - (c^k + f^{mk}g_m)\Sigma_{ij},\\
\Lambda_{ijk} &= dz_{ijk} - 2z_{ijk}\,daa^{-1} + z_{ijl}\,db^l_m B^m_k + z_{ilk}\,db^l_m B^m_j + z_{ljk}\,db^l_m B^m_i + z_{ijk}\Phi^0_0 + z_{ijk}g_m\Xi^m\\
&\quad + g_i\Sigma_{jk} + g_j\Sigma_{ik} + g_k\Sigma_{ij} - w^l_{ij}\Sigma_{lk} - w^l_{ik}\Sigma_{lj} - w^l_{jk}\Sigma_{li} - f^{lm}(z_{imj}\Sigma_{kl} + z_{imk}\Sigma_{jl} + z_{jmk}\Sigma_{il}).
\end{aligned}
\tag{7}
$$

Let $\mathcal{H}$ be the subgroup of $\mathcal{G}$ defined by (5) and (6). We shall prove that the restriction of the lifted coframe (3) to $J^2(\mathcal{E}) \times \mathcal{H}$ satisfies Cartan's test of involutivity, [24, Definition 11.7]. The structure equations remain unchanged under the following transformation of the forms (7):

$$
\Phi^0_0 \mapsto \tilde\Phi^0_0,\quad \Phi^k_i \mapsto \tilde\Phi^k_i,\quad \Phi^0_i \mapsto \tilde\Phi^0_i,\quad \Psi^{ij} \mapsto \tilde\Psi^{ij},\quad \Psi^{i0} \mapsto \tilde\Psi^{i0},\quad \Upsilon^0_{ij} \mapsto \tilde\Upsilon^0_{ij},\quad \Upsilon^k_{ij} \mapsto \tilde\Upsilon^k_{ij},\quad \Lambda_{ijk} \mapsto \tilde\Lambda_{ijk},
$$

where

$$
\begin{gathered}
\tilde\Phi^0_0 = \Phi^0_0 + K\Theta_0,\quad \tilde\Phi^k_i = \Phi^k_i + L^{kl}_i\Theta_l + M^k_i\Theta_0,\quad \tilde\Phi^0_i = \Phi^0_i + M^k_i\Theta_k + N_i\Theta_0,\\
\tilde\Psi^{ij} = \Psi^{ij} + P^{ij}\Theta_0 + S^{ijk}\Theta_k - L^{ij}_k\Xi^k,\quad \tilde\Psi^{i0} = \Psi^{i0} + P^{ij}\Theta_j + T^i\Theta_0 + K\Xi^i - M^i_k\Xi^k,\\
\tilde\Upsilon^0_{ij} = \Upsilon^0_{ij} + U_{ij}\Theta_0 + V^k_{ij}\Theta_k + W_{ijk}\Xi^k + K\Sigma_{ij} + M^k_i\Sigma_{kj},\\
\tilde\Upsilon^k_{ij} = \Upsilon^k_{ij} + X^{kl}_{ij}\Theta_l + V^k_{ij}\Theta_0 + Y^k_{ijl}\Xi^l + L_i\Sigma_{lj},\quad \tilde\Lambda_{ijk} = \Lambda_{ijk} + Z_{ijkl}\Xi^l + Y^l_{ijk}\Theta_l + W_{ijk}\Theta_0
\end{gathered}
\tag{8}
$$

and K, L^{kl}_i, M^k_i, N_i, P^{ij}, S^{ijk}, T^i, U_{ij}, V^k_{ij}, W_{ijk}, X^{kl}_{ij}, Y^k_{ijl}, and Z_{ijkl} are arbitrary constants satisfying the following symmetry conditions:

$$
\begin{gathered}
L^{kl}_i = L^{lk}_i,\quad P^{ij} = P^{ji},\quad S^{ijk} = S^{jik} = S^{ikj},\quad U_{ij} = U_{ji},\quad V^k_{ij} = V^k_{ji},\\
W_{ijk} = W_{jik} = W_{ikj},\quad X^{kl}_{ij} = X^{kl}_{ji} = X^{lk}_{ij},\quad Y^k_{ijl} = Y^k_{jil} = Y^k_{ilj},\quad Z_{ijkl} = Z_{jikl} = Z_{ijlk} = Z_{ikjl}.
\end{gathered}
\tag{9}
$$

The number of such constants

$$
\begin{aligned}
r^{(1)} &= 1 + \frac{n^2(n+1)}{2} + n^2 + n + \frac{n(n+1)}{2} + \frac{n(n+1)(n+2)}{6} + n + \frac{n(n+1)}{2} + \frac{n^2(n+1)}{2}\\
&\quad + \frac{n(n+1)(n+2)}{6} + \frac{n^2(n+1)^2}{4} + \frac{n^2(n+1)(n+2)}{6} + \frac{n(n+1)(n+2)(n+3)}{24}\\
&= \frac{1}{24}(n+1)(n+2)(11n^2 + 29n + 12)
\end{aligned}
$$

is the degree of indeterminancy of the lifted coframe, [24, Definition 11.2]. The reduced characters of this coframe, [24, Definition 11.4], are easily found:

$$
\begin{aligned}
s'_i &= \frac{1}{2}(n+1)(n+4) - i &&\text{for } i \in \{1, \dots n+1\},\\
s'_{n+1+j} &= \frac{1}{2}(n+1-j)(n+2-j) &&\text{for } j \in \{1, \dots, n\}.
\end{aligned}
$$

A simple calculation shows that

$$
r^{(1)} = s'_1 + 2s'_2 + 3s'_3 + \cdots + (2n+1)s'_{2n+1}.
$$

Thus, the Cartan test is satisfied and the lifted coframe is involutive.

It is easy to directly verify that a transformation

$$
\hat\Delta\colon J^2(\mathcal{E}) \times \mathcal{H} \to J^2(\mathcal{E}) \times \mathcal{H}
$$

satisfies the conditions

$$
\hat\Delta^*\bar\Theta_0 = \Theta_0,\quad \hat\Delta^*\bar\Theta_i = \Theta_i,\quad \hat\Delta^*\bar\Xi^i = \Xi^i,\quad \hat\Delta^*\bar\Sigma_{ij} = \Sigma_{ij}
\tag{10}
$$

if and only if it is projectable on $J^2(\mathcal{E})$, and its projection $\Delta\colon J^2(\mathcal{E}) \to J^2(\mathcal{E})$ is a contact transformation.

Since (10) imply

$$\hat{\Delta}^* d\bar{\Theta}_0 = d\Theta_0, \quad \hat{\Delta}^* d\bar{\Theta}_i = d\Theta_i, \quad \hat{\Delta}^* d\bar{\Xi}^i = d\Xi^i, \quad \hat{\Delta}^* d\bar{\Sigma}_{ij} = d\Sigma_{ij},$$

we have

$$\hat{\Delta}^*(\bar{\Phi}_0^0 \wedge \bar{\Theta}_0 + \bar{\Xi}^i \wedge \bar{\Theta}_i) = (\hat{\Delta}^* \bar{\Phi}_0^0) \wedge \Theta_0 + \Xi^i \wedge \Theta_i = \Phi_0^0 \wedge \Theta_0 + \Xi^i \wedge \Theta_i,$$

$$\hat{\Delta}^*(\bar{\Phi}_i^0 \wedge \bar{\Theta}_0 + \bar{\Phi}_i^k \wedge \bar{\Theta}_k + \bar{\Xi}^k \wedge \bar{\Sigma}_{ik}) = \hat{\Delta}^*(\bar{\Phi}_i^0) \wedge \Theta_0 + \hat{\Delta}^*(\bar{\Phi}_i^k) \wedge \Theta_k + \Xi^k \wedge \Sigma_{ik}$$
$$= \Phi_i^0 \wedge \Theta_0 + \Phi_i^k \wedge \Theta_k + \Xi^k \wedge \Sigma_{ik},$$

$$\hat{\Delta}^*(\bar{\Phi}_0^0 \wedge \bar{\Xi}^i - \bar{\Phi}_k^i \wedge \bar{\Xi}^k + \bar{\Psi}^{i0} \wedge \bar{\Theta}_0 + \bar{\Psi}^{ik} \wedge \bar{\Theta}_k)$$
$$= \hat{\Delta}^*(\bar{\Phi}_0^0) \wedge \Xi^i - \hat{\Delta}^*(\bar{\Phi}_k^i) \wedge \Xi^k + \hat{\Delta}^*(\bar{\Psi}^{i0}) \wedge \Theta_0 + \hat{\Delta}^*(\bar{\Psi}^{ik}) \wedge \Theta_k$$
$$= \Phi_0^0 \wedge \Xi^i - \Phi_k^i \wedge \Xi^k + \Psi^{i0} \wedge \Theta_0 + \Psi^{ik} \wedge \Theta_k,$$

$$\hat{\Delta}^*(\bar{\Phi}_i^k \wedge \bar{\Sigma}_{kj} - \bar{\Phi}_0^0 \wedge \bar{\Sigma}_{ij} + \bar{\Upsilon}_{ij}^0 \wedge \bar{\Theta}_0 + \bar{\Upsilon}_{ij}^k \wedge \bar{\Theta}_k + \bar{\Lambda}_{ijk} \wedge \bar{\Xi}^k)$$
$$= \hat{\Delta}^*(\bar{\Phi}_i^k) \wedge \Sigma_{kj} - \hat{\Delta}^*(\bar{\Phi}_0^0) \wedge \Sigma_{ij} + \hat{\Delta}^*(\bar{\Upsilon}_{ij}^0) \wedge \Theta_0 + \hat{\Delta}^*(\bar{\Upsilon}_{ij}^k) \wedge \Theta_k + \hat{\Delta}^*(\bar{\Lambda}_{ijk}) \wedge \Xi^k$$
$$= \Phi_i^k \wedge \Sigma_{kj} - \Phi_0^0 \wedge \Sigma_{ij} + \Upsilon_{ij}^0 \wedge \Theta_0 + \Upsilon_{ij}^k \wedge \Theta_k + \Lambda_{ijk} \wedge \Xi^k.$$

Therefore, we have the following transformation rules:

$$\begin{gathered} \hat{\Delta}^*(\bar{\Phi}_0^0) = \tilde{\Phi}_0^0, \quad \hat{\Delta}^*(\bar{\Phi}_i^k) = \tilde{\Phi}_i^k, \quad \hat{\Delta}^*(\bar{\Phi}_i^0) = \tilde{\Phi}_i^0, \\ \hat{\Delta}^*(\bar{\Psi}^{ij}) = \tilde{\Psi}^{ij}, \quad \hat{\Delta}^*(\bar{\Psi}^{i0}) = \tilde{\Psi}^{i0}, \quad \hat{\Delta}^*(\bar{\Upsilon}_{ij}^0) = \tilde{\Upsilon}_{ij}^0, \quad \hat{\Delta}^*(\bar{\Upsilon}_{ij}^k) = \tilde{\Upsilon}_{ij}^k, \quad \hat{\Delta}^*(\bar{\Lambda}_{ijk}) = \tilde{\Lambda}_{ijk}, \end{gathered} \tag{11}$$

where the constants $K, \ldots, Z_{ijkl}$ in (8) are replaced by arbitrary functions on $J^2(\mathcal{E}) \times \mathcal{H}$ such that the same symmetry conditions (9) are satisfied.

2. Contact Symmetries of Differential Equations

Suppose $\mathcal{R}$ is a second-order differential equation in one dependent and n independent variables. We consider $\mathcal{R}$ as a sub-bundle in $J^2(\mathcal{E})$. Let $\mathrm{Cont}(\mathcal{R})$ be the group of contact symmetries for $\mathcal{R}$. It consists of all the contact transformations on $J^2(\mathcal{E})$ mapping $\mathcal{R}$ to itself. The moving-coframe method [6, 7] is applicable to finding invariant 1-forms characterizing $\mathrm{Cont}(\mathcal{R})$ in the same way as the restriction of the lifted coframe (3) to $J^2(\mathcal{E}) \times \mathcal{H}$ characterizes $\mathrm{Cont}(J^2(\mathcal{E}))$. We briefly outline this approach.

Let $\iota \colon \mathcal{R} \to J^2(\mathcal{E})$ be an embedding. The invariant 1-forms of $\mathrm{Cont}(\mathcal{R})$ are restrictions of the coframe (3), (5), (6) to $\mathcal{R}$:

$$\theta_0 = \iota^* \Theta_0, \quad \theta_i = \iota^* \Theta_i, \quad \xi^i = \iota^* \Xi^i, \quad \sigma_{ij} = \iota^* \Sigma_{ij}$$

(for brevity we identify the map $\iota \times \mathrm{id} \colon \mathcal{R} \times \mathcal{H} \to J^2(\mathcal{E}) \times \mathcal{H}$ with $\iota \colon \mathcal{R} \to J^2(\mathcal{E})$). The forms θ_0, θ_i, ξ^i, and σ_{ij} have some linear dependencies, i.e., there exists a nontrivial set of functions E^0, E^i, F_i, and G^{ij} on $\mathcal{R} \times \mathcal{H}$ such that

$$E^0 \theta_0 + E^i \theta_i + F_i \xi^i + G^{ij} \sigma_{ij} \equiv 0.$$

These functions are lifted invariants of $\mathrm{Cont}(\mathcal{R})$. Setting them equal to some constants allows us to specify some parameters a, b_i^k, c_i, g_i, f^{ij}, s_{ij}, w_{ij}^k, and z_{ijk} of the group $\mathcal{H}$ as functions of the coordinates on $\mathcal{R}$ and the other group parameters.

After these normalizations, a part of the forms

$$\begin{gathered} \phi_0^0 = \iota^* \Phi_0^0, \quad \phi_i^k = \iota^* \Phi_i^k, \quad \phi_i^0 = \iota^* \Phi_i^0, \\ \psi^{ij} = \iota^* \Psi^{ij}, \quad \psi^{i0} = \iota^* \Psi^{i0}, \quad \upsilon_{ij}^0 = \iota^* \Upsilon_{ij}^0, \quad \upsilon_{ij}^k = \iota^* \Upsilon_{ij}^k, \quad \lambda_{ijk} = \iota^* \Lambda_{ijk}, \end{gathered}$$

or some of their linear combinations, become semi-basic, i.e., they do not include the differentials of the parameters of $\mathcal{H}$. From (11) and (8), we have the following statements:

(i) if ϕ_0^0 is semi-basic, then its coefficients at θ_k, ξ^k, and σ_{kl} are lifted invariants of $\mathrm{Cont}(\mathcal{R})$;

(ii) if ϕ_i^0 or ϕ_i^k is semi-basic, then their coefficients at ξ^k and σ_{kl} are lifted invariants of $\mathrm{Cont}(\mathcal{R})$;

(iii) if ψ^{i0}, ψ^{ij}, or λ_{ijk} is semi-basic, then their coefficients at σ_{kl} are lifted invariants of $\mathrm{Cont}(\mathcal{R})$. Setting these invariants equal to some constants, we get specifications of some more parameters of $\mathcal{H}$ as functions of the coordinates on $\mathcal{R}$ and the other group parameters.

More lifted invariants can appear as essential torsion coefficients in the reduced structure equations

$$\begin{aligned}
d\theta_0 &= \phi_0^0 \wedge \theta_0 + \xi^i \wedge \theta_i, \\
d\theta_i &= \phi_i^0 \wedge \theta_0 + \phi_i^k \wedge \theta_k + \xi^k \wedge \sigma_{ik}, \\
d\xi^i &= \phi_0^0 \wedge \xi^i - \phi_k^i \wedge \xi^k + \psi^{i0} \wedge \theta_0 + \psi^{ik} \wedge \theta_k, \\
d\sigma_{ij} &= \phi_i^k \wedge \sigma_{kj} - \phi_0^0 \wedge \sigma_{ij} + \upsilon_{ij}^0 \wedge \theta_0 + \upsilon_{ij}^k \wedge \theta_k + \lambda_{ijk} \wedge \xi^k.
\end{aligned}$$

After normalizing these invariants and repeating the process, two outputs are possible. In the first case, the reduced lifted coframe appears to be involutive. Then this coframe is the desired set of defining forms for $\mathrm{Cont}(\mathcal{R})$. In the second case, when the reduced lifted coframe does not satisfy Cartan's test, we should use the procedure of prolongation [24, Chap. 12].

3. Structure and Invariants of Symmetry Groups for Linear Hyperbolic Equations

We apply the method described in the previous section to the class of linear hyperbolic equations (1). Denote $x^1 = t$, $x^2 = x$, $p_1 = u_t$, $p_2 = u_x$, $p_{11} = u_{tt}$, $p_{12} = u_{tx}$, and $p_{22} = u_{xx}$. The coordinates on $\mathcal{R}$ are $\{(t, x, u, u_t, u_x, u_{tt}, u_{xx})\}$, and the embedding $\iota\colon \mathcal{R} \to J^2(\mathcal{E})$ is defined by (1). At the first step, we analyze the linear dependence between the reduced forms θ_0, θ_i, ξ^i, and σ_{ij}. Without loss of generality, we suppose that $b_1^1 \neq 0$ and $b_2^2 \neq 0$; then we find

$$\sigma_{12} = E_1\sigma_{11} + E_2\sigma_{22} + E_3\theta_0 + E_4\theta_1 + E_5\theta_2 + E_6\xi^1 + E_7\xi^2,$$

where, for example,

$$E_1 = -(b_1^1 b_2^2 + b_2^1 b_1^2)^{-1} b_1^1 b_2^1, \quad E_2 = -(b_1^1 b_2^2 + b_2^1 b_1^2)^{-1} b_2^2 b_1^2.$$

Setting $E_1, E_2, \ldots, E_7$ equal to 0 sequentially, we have

$$\begin{aligned}
&E_1 = 0 \implies b_2^1 = 0, \\
&E_2 = 0 \implies b_1^2 = 0, \\
&E_3 = 0 \implies s_{12} = -z_{112}c^1 - z_{122}c^2 + g_1(b_2^2)^{-1}T + g_2(b_1^1)^{-1}X - (b_1^1 b_2^2)^{-1}U, \\
&E_4 = 0 \implies w_{12}^1 = -z_{112}f^{11} - z_{122}f^{12} - (b_2^2)^{-1}T, \\
&E_5 = 0 \implies w_{12}^2 = -z_{112}f^{12} - z_{122}f^{22} - (b_1^1)^{-1}X, \\
&E_6 = 0 \implies z_{112} = -a(b_1^1)^{-2}(b_2^2)^{-1}(Tu_{tt} + (2TX + 2U - H)u_t + (X_t + X^2)u_x + (U_t + XU)u), \\
&E_7 = 0 \implies z_{122} = -a(b_1^1)^{-1}(b_2^2)^{-2}(Xu_{xx} + (T_x + T^2)u_t + (2TX + 2U - K)u_x + (U_x + TU)u),
\end{aligned}$$

where $H = -T_t + TX + U$ and $K = -X_x + TX + U$ are the Laplace invariants [15], [23, Sec. 9].

At the second step, we analyze the semi-basic forms ϕ_j^i and ϕ_j^0. We have

$$\phi_1^2 \equiv f^{12}\sigma_{11} + (g_1 + (b_1^1)^{-1}X)\xi^2 \pmod{\theta_0, \theta_1, \theta_2, \xi^1};$$

therefore we take $f^{12} = 0$, $g_1 = -(b_1^1)^{-1}X$. This yields

$$\phi_1^2 \equiv (-w_{11}^2 + af^{22}(b_1^1)^{-2}(b_2^2)^{-1}(Tu_{tt} + (2TX + 2U - H)u_t + (X_t + X^2)u_x + (U_t + XU)u))\xi^1 \pmod{\theta_0, \theta_1, \theta_2};$$

therefore we set

$$w_{11}^2 = af^{22}(b_1^1)^{-2}(b_2^2)^{-1}(Tu_{tt} + (2TX + 2U - H)u_t + (X_t + X^2)u_x + (U_t + XU)u).$$

After that, we have

$$\phi^1_2 \equiv (g_2 + (b^2_2)^{-1}T)\xi^1 + (-w^1_{22} + af^{11}(b^1_1)^{-1}(b^2_2)^{-2}(Xu_{xx} + (T_x + T^2)u_t + (2TX + 2U - K)u_x + (U_x + TU)u))\xi^2 \pmod{\theta_0, \theta_1, \theta_2},$$

and so we set $g_2 = -(b^2_2)^{-1}T$ and

$$w^1_{22} = af^{11}(b^1_1)^{-1}(b^2_2)^{-2}(Xu_{xx} + (T_x + T^2)u_t + (2TX + 2U - K)u_x + (U_x + TU)u).$$

Then we have

$$\phi^0_1 \equiv c^1\sigma_{11} \pmod{\theta_0, \theta_1, \theta_2, \xi^1, \xi^2}, \quad \phi^0_2 \equiv c^2\sigma_{22} \pmod{\theta_0, \theta_1, \theta_2, \xi^1, \xi^2},$$

and so we set $c^1 = 0$ and $c^2 = 0$. Now we obtain

$$\phi^0_1 \equiv K(b^1_1)^{-1}(b^2_2)^{-1}\xi^2 \pmod{\theta_0, \theta_1, \theta_2}, \quad \phi^0_2 \equiv H(b^1_1)^{-1}(b^2_2)^{-1}\xi^1 \pmod{\theta_0, \theta_1, \theta_2}. \tag{12}$$

There are two possibilities now: $H \equiv K \equiv 0$ or at least one of the Laplace invariants is not identically equal to 0.

We denote by $\mathcal{S}_1$ the subclass of Eqs. (1) such that $H \equiv K \equiv 0$. For an equation from $\mathcal{S}_1$ we use the procedures of absorption and prolongation, [24], to compute the structure equations:

$$\begin{aligned}
d\theta_0 &= \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2,\\
d\theta_1 &= \eta_2 \wedge \theta_1 + \xi^1 \wedge \sigma_{11},\\
d\theta_2 &= \eta_3 \wedge \theta_2 + \xi^2 \wedge \sigma_{22},\\
d\xi^1 &= (\eta_1 - \eta_2) \wedge \xi^1 + \eta_4 \wedge \theta_1,\\
d\xi^2 &= (\eta_1 - \eta_3) \wedge \xi^2 + \eta_5 \wedge \theta_2,\\
d\sigma_{11} &= (2\eta_2 - \eta_1) \wedge \sigma_{11} + \eta_6 \wedge \theta_1 + \eta_7 \wedge \xi^1,\\
d\sigma_{22} &= (2\eta_3 - \eta_1) \wedge \sigma_{22} + \eta_8 \wedge \theta_2 + \eta_9 \wedge \xi^2,\\
d\eta_1 &= 0,\\
d\eta_2 &= \pi_1 \wedge \theta_1 + \eta_4 \wedge \sigma_{11} - \eta_6 \wedge \xi^1,\\
d\eta_3 &= \pi_2 \wedge \theta_2 + \eta_5 \wedge \sigma_{22} - \eta_8 \wedge \xi^2,\\
d\eta_4 &= -\pi_1 \wedge \xi^1 + \pi_3 \wedge \theta_1 + (\eta_1 - 2\eta_2) \wedge \eta_4,\\
d\eta_5 &= -\pi_2 \wedge \xi^2 + \pi_4 \wedge \theta_2 + (\eta_1 - 2\eta_3) \wedge \eta_5,\\
d\eta_6 &= 2\pi_1 \wedge \sigma_{11} + \pi_5 \wedge \theta_1 + \pi_6 \wedge \xi^1 + (\eta_2 - \eta_1) \wedge \eta_6 - \eta_4 \wedge \eta_7,\\
d\eta_7 &= \pi_6 \wedge \theta_1 + \pi_7 \wedge \xi^1 - 3\eta_6 \wedge \sigma_{11} + (3\eta_2 - 2\eta_1) \wedge \eta_7,\\
d\eta_8 &= 2\pi_2 \wedge \sigma_{22} + \pi_8 \wedge \theta_2 + \pi_9 \wedge \xi^2 + (\eta_3 - \eta_1) \wedge \eta_8 - \eta_5 \wedge \eta_9,\\
d\eta_9 &= \pi_9 \wedge \theta_2 + \pi_{10} \wedge \xi^2 - 3\eta_8 \wedge \sigma_{22} + (3\eta_3 - 2\eta_1) \wedge \eta_9.
\end{aligned}$$

In these equations, the forms $\eta_1, \ldots, \eta_9$ on $J^2(\mathcal{E}) \times \mathcal{H}$ depend on differentials of the parameters of $\mathcal{H}$, while the forms $\pi_1, \ldots, \pi_{10}$ depend on differentials of the prolongation variables. From the structure equations it follows that Cartan's test for the lifted coframe

$$\{\theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \sigma_{22}, \eta_1, \eta_2, \ldots, \eta_9\}$$

is satisfied; therefore, the coframe is involutive.

The same calculations show that the symmetry group of the linear wave equation $u_{tx} = 0$ has the same structure equations but with a different lifted coframe. All the essential torsion coefficients in the structure equations are constants. Thus, applying [24, Theorem 15.12], we obtain a well-known result, [23, Sec. 9]: every equation from $\mathcal{S}_1$ is contact equivalent to the wave equation.

Now we return to the case of $H \not\equiv 0$ or $K \not\equiv 0$. Since we can replace H and K by renaming the independent variables $t \mapsto x$, $x \mapsto t$, we put $H \not\equiv 0$ without loss of generality. Then we use (12) and take $b^2_2 = H(b^1_1)^{-1}$. After this, the form $\phi^1_1 + \phi^2_2 - 2\phi^0_0$ becomes semi-basic. Since

$$\phi^1_1 + \phi^2_2 - 2\phi^0_0 \equiv f^{11}\sigma_{11} + f^{22}\sigma_{22} \pmod{\theta_1, \theta_2, \xi^1, \xi^2},$$

we take $f^{11} = 0$ and $f^{22} = 0$. Then we have

$$\phi^1_1 + \phi^2_2 - 2\phi^0_0 \equiv -(w^1_{11} + H^{-1}(b^1_1)^{-1}(H_t + 2XH))\xi^1 - (w^2_{22} + H^{-2}b^1_1(H_x + 2TH))\xi^2 \pmod{\theta_1, \theta_2},$$

and so we take

$$w^1_{11} = -H^{-1}(b^1_1)^{-1}(H_t + 2XH), \quad w^2_{22} = -H^{-2}b^1_1(H_x + 2TH).$$

At the third step, we analyze the structure equations. After absorption of torsion they have the form

$$\begin{aligned}
d\theta_0 &= \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2,\\
d\theta_1 &= \eta_2 \wedge \theta_1 + \xi^1 \wedge \sigma_{11} - P\theta_0 \wedge \xi^2,\\
d\theta_2 &= (2\eta_1 - \eta_2) \wedge \theta_2 - \theta_0 \wedge \xi^1 + \xi^2 \wedge \sigma_{22},\\
d\xi^1 &= (\eta_1 - \eta_2) \wedge \xi^1,\\
d\xi^2 &= (\eta_2 - \eta_1) \wedge \xi^2,\\
d\sigma_{11} &= (2\eta_2 - \eta_1) \wedge \sigma_{11} + \eta_3 \wedge \xi^1 - P_t(b^1_1)^{-1}\theta_0 \wedge \xi^2 + (Q + 1 - 2P)\theta_1 \wedge \xi^2,\\
d\sigma_{22} &= (3\eta_1 - 2\eta_2) \wedge \sigma_{22} + \eta_4 \wedge \xi^2 + (P + Q - 2)\theta_2 \wedge \xi^1,
\end{aligned} \tag{13}$$

where the functions $P = KH^{-1}$ and $Q = (HH_{tx} - H_tH_x)H^{-3} = (\ln|H|)_{tx}H^{-1}$ are invariants of the symmetry group and the 1-forms $\eta_1, \ldots, \eta_4$ depend on differentials of parameters of the group $\mathcal{H}$ (these forms are not necessarily the same as in the case of an equation from $\mathcal{S}_1$).

We denote by $\mathcal{S}_2$ the subclass of Eqs. (1) such that $P_t \not\equiv 0$. This subclass is not empty, since, for example, the equation $u_{tx} = t^2x^2u_t + u$ belongs to $\mathcal{S}_2$. For an equation from $\mathcal{S}_2$ we can normalize $P_t(b^1_1)^{-1}$, the only essential torsion coefficient in the structure equations (13), to 1 by setting $b^1_1 = P_t$. Then, after prolongation, we have the involutive lifted coframe

$$\boldsymbol{\theta} = \{\theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \sigma_{22}, \eta_1, \eta_2, \eta_3\}$$

with the structure equations

$$\begin{aligned}
d\theta_0 &= \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2,\\
d\theta_1 &= \eta_1 \wedge \theta_1 - P\theta_0 \wedge \xi^2 - J_2\theta_1 \wedge \xi^1 - J_1\theta_1 \wedge \xi^2 + \xi^1 \wedge \sigma_{11},\\
d\theta_2 &= \eta_1 \wedge \theta_2 - \theta_0 \wedge \xi^1 + J_2\theta_2 \wedge \xi^1 + J_1\theta_2 \wedge \xi^2 + \xi^2 \wedge \sigma_{22},\\
d\xi^1 &= J_1\xi^1 \wedge \xi^2,\\
d\xi^2 &= J_2\xi^1 \wedge \xi^2,\\
d\sigma_{11} &= \eta_1 \wedge \sigma_{11} + \eta_2 \wedge \xi^1 - \theta_0 \wedge \xi^2 + (Q + 1 - 2P)\theta_1 \wedge \xi^2 + 2J_1\xi^2 \wedge \sigma_{11},\\
d\sigma_{22} &= \eta_1 \wedge \sigma_{22} + \eta_3 \wedge \xi^2 + (P - 2 + Q)\theta_2 \wedge \xi^1 - 2J_2\xi^1 \wedge \sigma_{22},\\
d\eta_1 &= (P - 1)\xi^1 \wedge \xi^2,\\
d\eta_2 &= \pi_1 \wedge \xi^1 + \eta_1 \wedge \eta_2 - 3J_1\eta_2 \wedge \xi^2 + J_2\theta_0 \wedge \xi^2\\
&\quad + (4PJ_2 - 2QJ_2 - \mathbb{D}_1(Q) - 2J_2 + 3)\theta_1 \wedge \xi^2 + (2J_1J_2 + 2 - 3P + 3Q - 2\mathbb{D}_2(J_2))\xi^2 \wedge \sigma_{11},\\
d\eta_3 &= \pi_2 \wedge \xi^2 + \eta_1 \wedge \eta_3 + 3J_2\eta_3 \wedge \xi^1 + (2J_1(P + Q - 2) - \mathbb{D}_2(Q) - \mathbb{D}_2(P))\theta_2 \wedge \xi^1\\
&\quad + (2P - 3 - 2J_1J_2 + 2\mathbb{D}_2(J_2) + Q)\xi^1 \wedge \sigma_{22},
\end{aligned} \tag{14}$$

where the functions $J_1 = -P_{tx}H^{-1}$ and $J_2 = (H_tP_t - HP_{tt})H^{-1}(P_t)^{-2}$ are invariants of the symmetry group of an equation from $\mathcal{S}_2$ and the operators

$$\mathbb{D}_1 = \frac{\partial}{\partial \xi^1} = (P_t)^{-1}D_t, \quad \mathbb{D}_2 = \frac{\partial}{\partial \xi^2} = P_tH^{-1}D_x$$

are invariant differentiations associated with ξ^1 and ξ^2. These operators are defined by the identity

$$dF = \mathbb{D}_1(F)\xi^1 + \mathbb{D}_2(F)\xi^2,$$

where $F = F(t,x)$ is an arbitrary function. The commutator identity for the invariant differentiations has the form

$$[\mathbb{D}_1, \mathbb{D}_2] = -J_1\mathbb{D}_1 - J_2\mathbb{D}_2. \tag{15}$$

We have $\mathbb{D}_1(P) = 1$, and, applying (15) to P, we obtain the *syzygy*

$$J_1 = -\mathbb{D}_1(\mathbb{D}_2(P)) - J_2\mathbb{D}_2(P). \tag{16}$$

If $\mathbb{D}_2(P)\mathbb{D}_1(Q) \neq \mathbb{D}_2(Q)$, i.e., if $P_tQ_x \neq P_xQ_t$, then, applying (15) to Q and using (16), we have

$$J_2 = ([\mathbb{D}_1, \mathbb{D}_2](Q) - \mathbb{D}_1(Q)\mathbb{D}_1(\mathbb{D}_2(P)))(\mathbb{D}_2(P)\mathbb{D}_1(Q) - \mathbb{D}_2(Q))^{-1}.$$

Therefore, in this case the functions P and Q are a basis of differential invariants of the symmetry group. But P and Q are not necessarily a basis in the case of their functional dependence (cf. [9, Theorem 2.3], [10, Sec. 10.4.2]). To prove this statement, we consider the equation

$$u_{tx} = u_t + \frac{2(p(t)-1)}{q(t)(t+x)}u_x + \frac{2}{q(t)(t+x)^2}(1-(p(t)-1)(t+x))u \tag{17}$$

with arbitrary functions $p(t)$ and $q(t)$ such that $p'(t) \neq 0$ and $q'(t) \neq 0$. For this equation we have

$$P = p(t), \quad Q = q(t),$$
$$J_2 = -2(q'(t))^{-1}(t+x)^{-1} - p''(t)(p'(t))^{-2} - q'(t)(p'(t)q(t))^{-1},$$
$$\mathbb{D}_1(P) = 1, \quad \mathbb{D}_2(P) = 0, \quad \mathbb{D}_1(Q) = q'(t)(p'(t))^{-1}, \quad \mathbb{D}_2(Q) = 0,$$

and by induction the only nontrivial higher-order differential invariants $\mathbb{D}_1^i(Q)$ depend on t. Since $J_{2,x} \neq 0$, the function J_2 is independent of P, Q, and all their invariant derivatives. Thus, for the whole subclass $\mathcal{S}_2$ we should take the functions P, Q, and J_2 as a basis for the set of differential invariants of the symmetry group. To construct all the other invariants, we apply $\mathbb{D}_1$ and $\mathbb{D}_2$ to P, Q, and J_2. The commutator identity (15) allows us to permute $\mathbb{D}_1$ and $\mathbb{D}_2$, so we need only deal with the invariants $P_{jk} = \mathbb{D}_1^j(\mathbb{D}_2^k(P))$, $Q_{jk} = \mathbb{D}_1^j(\mathbb{D}_2^k(Q))$, and $J_{2,jk} = \mathbb{D}_1^j(\mathbb{D}_2^k(J_2))$, where $j \geq 0$, $k \geq 0$.

For $s \geq 0$, the sth-order *classifying manifold* associated with the lifted coframe $\boldsymbol{\theta}$ and an open subset $U \subset \mathbb{R}^2$ is

$$\mathcal{C}^{(s)}(\boldsymbol{\theta}, U) = \{(P_{jk}(t,x), Q_{jk}(t,x), J_{2,jk}(t,x)) \mid 0 \leq j+k \leq s,\ (t,x) \in U\}. \tag{18}$$

Since all the functions P_{jk}, Q_{jk}, and $J_{2,jk}$ depend on two variables t and x, it follows that $\rho_s = \dim \mathcal{C}^{(s)}(\boldsymbol{\theta}, U) \leq 2$ for all $s \geq 0$. Let

$$r = \min\{s \mid \rho_s = \rho_{s+1} = \rho_{s+2} = \dots\}$$

be the *order of the coframe* $\boldsymbol{\theta}$. Since $P_t \neq 0$, we have

$$1 \leq \rho_0 \leq \rho_1 \leq \rho_2 \leq \dots \leq 2.$$

In any case, $r + 1 \leq 2$. Hence from [24, Theorem 15.12] we see that two linear hyperbolic equations (1) from the subclass $\mathcal{S}_2$ are locally equivalent under a contact transformation if and only if their second-order classifying manifolds (18) locally overlap.

Remark 1. A Lie pseudo-group is called structurally intransitive [17] if it is not isomorphic to any transitive Lie pseudo-group. In [4], Cartan proved that a Lie pseudo-group is structurally intransitive whenever it has essential invariants. An invariant of a Lie pseudo-group with the structure equations

$$d\omega^i = A^i_{\beta k}\pi^\beta \wedge \omega^k + T^i_{jk}\omega^j \wedge \omega^k$$

is called *essential* if it is a first integral of the *systatic system* $A^i_{\beta k}\omega^k$. From the structure equations (14) it follows that the systatic system for the symmetry pseudo-group of an equation from $\mathcal{S}_2$ is generated by the forms ξ^1 and ξ^2. First integrals of these forms are arbitrary functions of t and x. Therefore, the invariants P, Q, J_2, and all the nonconstant derived invariants are essential. Thus, the symmetry pseudo-group of Eq. (1) from the subclass $\mathcal{S}_2$ is structurally intransitive, and the moving-coframe method is applicable to finding Maurer–Cartan forms for differential equations with structurally intransitive symmetry pseudo-groups (cf. [17]).

Remark 2. In [14, Theorem 1], the following basis of invariants for the symmetry group of Eq. (1) is found: $\{P, Q, J_3^1, J_3^2, J_3^3\}$, where

$$\begin{aligned}
J_3^1 &= H^{-3}(KH_{tx} + HK_{tx} - H_tK_x - H_xK_t),\\
J_3^2 &= H^{-9}(HK_x - KH_x)^2(HKH_{tt} - H^2K_{tt} - 3KH_t^2 + 3HH_tK_t),\\
J_3^3 &= H^{-9}(HK_t - KH_t)^2(HKH_{xx} - H^2K_{xx} - 3KH_x^2 + 3HH_xK_x).
\end{aligned}$$

Using (16), we have the following expressions for invariants J_3^1, J_3^2, and J_3^3 in terms of P, Q, J_2, and their invariant derivatives:

$$\begin{aligned}
J_3^1 &= 2PQ + \mathbb{D}_1(\mathbb{D}_2(P)) + J_2\mathbb{D}_2(P),\\
J_3^2 &= J_2(\mathbb{D}_2(P))^2,\\
J_3^3 &= \mathbb{D}_2(P)(\mathbb{D}_1(\mathbb{D}_2(P)) + J_2\mathbb{D}_2(P)) - \mathbb{D}_2(\mathbb{D}_2(P)).
\end{aligned}$$

The following operators of invariant differentiation are found in [14]:

$$\tilde{X}_1 = H^{-3}(HK_x - KH_x)D_t, \quad \tilde{X}_2 = H^2(HK_x - KH_x)^{-1}D_x.$$

We have $\tilde{X}_1 = \mathbb{D}_2(P)\mathbb{D}_1$ and $\tilde{X}_2 = \mathbb{D}_2(P)^{-1}\mathbb{D}_2$. Then in the case $\mathbb{D}_2(P) \equiv 0 \equiv P_x$ the operator $\tilde{X}_2$ is not defined, while $\tilde{X}_1$ is trivial, $J_3^1 = 2PQ$, $J_3^2 = 0$, and $J_3^3 = 0$. Therefore, the functions P, Q, J_3^1, J_3^2, and J_3^3 are not a basis of invariants of the symmetry group for Eq. (17).

Remark 3. In the theorem of [12], two sets of functions are stated to be bases for invariants of symmetry groups of Eqs. (1): the first set consists of functions P, Q, $I = P_tP_xH^{-1}$, and $\tilde{Q} = (\ln|K|)_{tx}K^{-1}$, and the second set consists of functions P, Q, I, and $-J_2$. The operators of invariant differentiation are taken in the form $\mathcal{D}_1 = P_t^{-1}D_t$ and $\mathcal{D}_2 = P_x^{-1}D_x$. We have $I = \mathbb{D}_2(P)$; therefore the function I can be excluded from both sets. Also we have $\tilde{Q} = QP^{-1} + J_2\mathbb{D}_2(P)P^{-2} + \mathbb{D}_1(\mathbb{D}_2(P))P^{-2} - \mathbb{D}_2(P)P^{-3}$, $\mathcal{D}_1 = \mathbb{D}_1$, and $\mathcal{D}_2 = (\mathbb{D}_2(P))^{-1}\mathbb{D}_2$. Therefore, in the case $P_x = 0 = \mathbb{D}_2(P)$ we have $I = 0$ and $\tilde{Q} = QP^{-1}$, and so the functions P, Q, I, and $\tilde{Q}$ are not a basis of invariants for the symmetry group of Eq. (17).

The function J_2 and the operator $\mathcal{D}_1$ are not defined when $P_t \equiv 0$ (for example of this case we take the Moutard equation $u_{tx} = U(t,x)u$. So the second set of functions is not a basis of invariants of symmetry groups for the *whole* class (1).

Now we return to the case $P_t \equiv 0$. Then the torsion coefficients in the structure equations (13) are independent of the group parameters, while $dP = P_xb_1^1H^{-1}\xi^2$. We denote by $\mathcal{S}_3$ the subclass of Eqs. (1) such that $P_t \equiv 0$, $P_x \neq 0$. This subclass is not empty, since, for example, the equation $u_{tx} = x^2u_x + u$ belongs to $\mathcal{S}_3$. For an equation from $\mathcal{S}_3$ we normalize $b_1^1 = HP_x^{-1}$. After absorption of torsion and prolongation, we obtain the involutive lifted coframe

$$\boldsymbol{\theta} = \{\theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \sigma_{22}, \eta_1, \eta_2, \eta_3\}$$

with the structure equations

$$
\begin{aligned}
d\theta_0 &= \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2,\\
d\theta_1 &= \eta_1 \wedge \theta_1 - P\theta_0 \wedge \xi^2 - L\theta_1 \wedge \xi^2 + \xi^1 \wedge \sigma_{11},\\
d\theta_2 &= \eta_1 \wedge \theta_2 - \theta_0 \wedge \xi^1 + L\theta_2 \wedge \xi^2 + \xi^2 \wedge \sigma_{22},\\
d\xi^1 &= L\xi^1 \wedge \xi^2,\\
d\xi^2 &= 0,\\
d\sigma_{11} &= \eta_1 \wedge \sigma_{11} + \eta_2 \wedge \xi^1 + (Q+1-2P)\theta_1 \wedge \xi^2 + 2L\xi^2 \wedge \sigma_{11},\\
d\sigma_{22} &= \eta_1 \wedge \sigma_{22} + \eta_3 \wedge \xi^2 + (P-2+Q)\theta_2 \wedge \xi^1,\\
d\eta_1 &= (P-1)\xi^1 \wedge \xi^2,\\
d\eta_2 &= \pi_1 \wedge \xi^1 - \eta_1 \wedge \eta_2 - 3L\eta_2 \wedge \xi^2 - \mathbb{D}_1(Q)\theta_1 \wedge \xi^2 + (3Q-3P+2)\xi^2 \wedge \sigma_{11},\\
d\eta_3 &= \pi_2 \wedge \xi^2 + \eta_1 \wedge \eta_3 - (4L+1-2PL-2QL+\mathbb{D}_2(Q))\theta_2 \wedge \xi^1 + (Q-3+2P)\xi^1 \wedge \sigma_{22},
\end{aligned}
$$

where the function $L = (HP_{xx} - H_x P_x)(P_x)^{-2}H^{-1}$ is an invariant of the symmetry group and the operators of invariant differentiation are $\mathbb{D}_1 = P_x H^{-1} D_t$ and $\mathbb{D}_2 = (P_x)^{-1} D_x$. We have $\mathbb{D}_1(P) = 0$, $\mathbb{D}_2(P) = 1$, and

$$[\mathbb{D}_1, \mathbb{D}_2] = L\mathbb{D}_1. \tag{19}$$

In the case $\mathbb{D}_1(Q) \neq 0$, we apply (19) to Q and obtain $L = [\mathbb{D}_1, \mathbb{D}_2](Q)(\mathbb{D}_1(Q))^{-1}$. Therefore, in this case the functions P and Q are a basis for the set of differential invariants of the symmetry group. But if $\mathbb{D}_1(Q) = 0$, then the functions P and Q are not necessarily a basis. For example, consider the equation

$$u_{tx} = -\frac{2(p(x)-1)}{q(x)(t+x)}u_t + u_x + \frac{2}{q(x)(t+x)^2}(p(x) + (p(x)-1)(t+x))u,$$

where $p(x)$ and $q(x)$ are arbitrary functions such that $p'(x) \neq 0$ and $q'(x) \neq 0$. This equation has the following invariants:

$$P = p(x), \quad Q = q(x), \quad L = 2(p'(x))^{-1}(t+x)^{-1} + p''(x)(p'(x))^{-2} + q'(x)(p'(x)q(x))^{-1}.$$

We have $\mathbb{D}_1(Q) = 0$, $\mathbb{D}_2(Q) = q'(x)(p'(x))^{-1}$, and by induction the only nontrivial higher-order differential invariants $\mathbb{D}_2^i(Q)$ depend on x. Since $L_t \neq 0$, the function L is independent of P, Q, and all their invariant derivatives. Thus, for the whole subclass $\mathcal{S}_3$ we should take the functions P, Q, and L as a basis for the set of differential invariants of the symmetry group. The sth order classifying manifold associated with the coframe $\boldsymbol{\theta}$ and an open subset $U \in \mathbb{R}^2$ can be taken in the form

$$\mathcal{C}^{(s)}(\boldsymbol{\theta}, U) = \{(P(x), Q_{jk}(t,x), L_{jk}(t,x)) \mid 0 \leq j+k \leq s,\ (t,x) \in U\}, \tag{20}$$

with $Q_{jk} = \mathbb{D}_1^j(\mathbb{D}_2^k(Q))$ and $L_{jk} = \mathbb{D}_1^j(\mathbb{D}_2^k(L))$. Then two equations from $\mathcal{S}_3$ are equivalent under a contact transformation if and only if their second-order classifying manifolds (20) (locally) overlap.

Now we consider the case $P \equiv \text{const}$. Then we have

$$dQ = Q_t(b_1^1)^{-1}\xi^1 + Q_x b_1^1 H^{-1}\xi^2.$$

We denote by $\mathcal{S}_4$ the subclass of Eqs. (1) such that $P \equiv \text{const}$, $Q_t \neq 0$. This subclass is not empty, since, for example, the equation $u_{tx} = (t-x)^3 u_x + (t-x)^2 u$ belongs to $\mathcal{S}_4$. For an equation from $\mathcal{S}_4$ we normalize $b_1^1 = Q_t$. Then, after absorption of torsion and prolongation, we have the involutive lifted coframe

$$\boldsymbol{\theta} = \{\theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \sigma_{22}, \eta_1, \eta_2, \eta_3\}$$

with the structure equations

$$
\begin{aligned}
d\theta_0 &= \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2,\\
d\theta_1 &= \eta_1 \wedge \theta_1 - P\theta_0 \wedge \xi^2 - M_2\theta_1 \wedge \xi^1 - M_1\theta_1 \wedge \xi^2 + \xi^1 \wedge \sigma_{11},
\end{aligned}
$$

$$
\begin{aligned}
d\theta_2 &= \eta_1 \wedge \theta_2 - \theta_0 \wedge \xi^1 + M_2\theta_2 \wedge \xi^1 + M_1\theta_2 \wedge \xi^2 + \xi^2 \wedge \sigma_{22},\\
d\xi^1 &= M_1\xi^1 \wedge \xi^2,\\
d\xi^2 &= M_2\xi^1 \wedge \xi^2,\\
d\sigma_{11} &= \eta_1 \wedge \sigma_{11} + \eta_2 \wedge \xi^1 + (Q+1-2P)\theta_1 \wedge \xi^2 + 2M_1\xi^2 \wedge \sigma_{11},\\
d\sigma_{22} &= \eta_1 \wedge \sigma_{22} + \eta_3 \wedge \xi^2 + (P-2+Q)\theta_2 \wedge \xi^1 - 2M_2\xi^1 \wedge \sigma_{22},\\
d\eta_1 &= (P-1)\xi^1 \wedge \xi^2,\\
d\eta_2 &= \pi_1 \wedge \xi^1 + \eta_1 \wedge \eta_2 - 3M_1\eta_2 \wedge \xi^2 - (1+2M_2+2QM_2-4PM_2)\theta_1 \wedge \xi^2\\
&\quad + (Q-2M_1M_2-3P-2\mathbb{D}_1(M_1)+2)\xi^2 \wedge \sigma_{11},\\
d\eta_3 &= \pi_2 \wedge \xi^2 + \eta_1 \wedge \eta_3 + 3M_2\eta_3 \wedge \xi^1 - (4M_1-2M_1P-2M_1Q+\mathbb{D}_2(Q))\theta_2 \wedge \xi^1\\
&\quad + (2M_1M_2+2P-3+2\mathbb{D}_1(M_1)+3Q)\xi^1 \wedge \sigma_{22},
\end{aligned}
$$

where the functions $M_1 = -Q_{tx}H^{-1}$ and $M_2 = (H_tQ_t - HQ_{tt})H^{-1}Q_t^{-2}$ are invariants of the symmetry group and the operators of invariant differentiation are $\mathbb{D}_1 = Q_t^{-1}D_t$ and $\mathbb{D}_2 = Q_tH^{-1}D_x$. We have $[\mathbb{D}_1, \mathbb{D}_2] = -M_1\mathbb{D}_1 - M_2\mathbb{D}_2$. Since $\mathbb{D}_1(Q) = 1$, applying the commutator identity to Q, we have the syzygy $M_1 = -\mathbb{D}_1(\mathbb{D}_2(Q)) - M_2\mathbb{D}_2(Q)$. The functions Q and M_2 are a basis for the set of all invariants of the symmetry group of an equation from $\mathcal{S}_4$. We take the sth-order classifying manifold associated with the coframe $\boldsymbol{\theta}$ and an open subset $U \in \mathbb{R}^2$ in the form

$$
\mathcal{C}^{(s)}(\boldsymbol{\theta}, U) = \{(P, Q_{jk}(t,x), M_{2,jk}(t,x)) \mid 0 \le j+k \le s,\ (t,x) \in U\} \tag{21}
$$

with $Q_{jk} = \mathbb{D}_1^j(\mathbb{D}_2^k(Q))$ and $M_{2,jk} = \mathbb{D}_1^j(\mathbb{D}_2^k(M_2))$. Then two equations from $\mathcal{S}_4$ are equivalent under a contact transformation if and only if their second-order classifying manifolds (21) (locally) overlap.

Next we denote by $\mathcal{S}_5$ the subclass of Eqs. (1) such that $P \equiv \text{const}$, $Q_t \equiv 0$, and $Q_x \neq 0$. This subclass is not empty, since, for example, the equation

$$
u_{tx} = -\frac{2(\lambda-1)}{q(x)(t+x)}u_t + u_x + \frac{2(\lambda+(\lambda-1)(t+x))}{q(x)(t+x)^2}u
$$

has invariants $P = \lambda \equiv \text{const}$ and $Q = q(x)$ and belongs to $\mathcal{S}_5$. For an equation from $\mathcal{S}_5$ we normalize $b_1^1 = HQ_x^{-1}$. Then, after absorption of torsion and prolongation, we have the involutive lifted coframe

$$
\boldsymbol{\theta} = \{\theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \sigma_{22}, \eta_1, \eta_2, \eta_3\}
$$

with the structure equations

$$
\begin{aligned}
d\theta_0 &= \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2,\\
d\theta_1 &= \eta_1 \wedge \theta_1 - P\theta_0 \wedge \xi^2 - N\theta_1 \wedge \xi^2 + \xi^1 \wedge \sigma_{11},\\
d\theta_2 &= \eta_1 \wedge \theta_2 - \theta_0 \wedge \xi^1 + N\theta_2 \wedge \xi^2 + \xi^2 \wedge \sigma_{22},\\
d\xi^1 &= N\xi^1 \wedge \xi^2,\\
d\xi^2 &= 0,\\
d\sigma_{11} &= \eta_1 \wedge \sigma_{11} + \eta_2 \wedge \xi^1 + (Q+1-2P)\theta_1 \wedge \xi^2 + 2N\xi^2 \wedge \sigma_{11},\\
d\sigma_{22} &= \eta_1 \wedge \sigma_{22} + \eta_3 \wedge \xi^2 + (P-2+Q)\theta_2 \wedge \xi^1,\\
d\eta_1 &= (P-1)\xi^1 \wedge \xi^2,
\end{aligned}
$$

$$d\eta_2 = \pi_1 \wedge \xi^1 + \eta_1 \wedge \eta_2 - 3N\eta_2 \wedge \xi^2 + (2 - 3P + 3Q)\xi^2 \wedge \sigma_{11},$$

$$d\eta_3 = \pi_2 \wedge \xi^2 + \eta_1 \wedge \eta_3 + (2N(P + Q - 2) - 1)\theta_2 \wedge \xi^1 + (2P + Q - 3)\xi^1 \wedge \sigma_{22},$$

where the function $N = (HQ_{xx} - H_x Q_x)H^{-1}Q_x^{-2}$ is an invariant of the symmetry group and the operators of invariant differentiation are $\mathbb{D}_1 = Q_x H^{-1} D_t$ and $\mathbb{D}_2 = Q_x^{-1} D_x$. We have $[\mathbb{D}_1, \mathbb{D}_2] = -N\mathbb{D}_1$, $\mathbb{D}_1(Q) = 0$, and $\mathbb{D}_2(Q) = 1$. The functions Q and N are a basis for the set of all invariants of the symmetry group of an equation from $\mathcal{S}_5$. We take the sth-order classifying manifold associated with the coframe $\boldsymbol{\theta}$ and an open subset $U \in \mathbb{R}^2$ in the form

$$\mathcal{C}^{(s)}(\boldsymbol{\theta}, U) = \{(P, Q(x), \mathbb{D}_1^j(\mathbb{D}_2^k(N))(t, x)) \mid 0 \le j + k \le s,\ (t, x) \in U\}. \tag{22}$$

Then two equations from $\mathcal{S}_5$ are equivalent under a contact transformation if and only if their second-order classifying manifolds (22) (locally) overlap.

Finally, we denote by $\mathcal{S}_6$ the subclass of Eqs. (1) such that $P \equiv \text{const}$, $Q \equiv \text{const}$. This subclass is not empty, since, for example, the equation

$$u_{tx} = -tu_t - \lambda x u_x - \lambda t x u \tag{23}$$

has the invariants $P = \lambda$ and $Q = 0$, while the Euler–Poisson equation

$$u_{tx} = 2\mu^{-1}(t + x)^{-1}u_t + 2\lambda\mu^{-1}(t + x)^{-1}u_x - 4\lambda\mu^{-2}(t + x)^{-2}u \tag{24}$$

has the invariants $P = \lambda$ and $Q = \mu$, [23, Sec. 9.2]. For an equation from $\mathcal{S}_6$, after absorption of torsion and prolongation we have the involutive lifted coframe

$$\boldsymbol{\theta} = \{\theta_0, \theta_1, \theta_2, \xi^1, \xi^2, \sigma_{11}, \sigma_{22}, \eta_1, \eta_2, \eta_3, \eta_4\}$$

with the structure equations

$$\begin{aligned}
d\theta_0 &= \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2,\\
d\theta_1 &= \eta_2 \wedge \theta_1 - P\theta_0 \wedge \xi^2 + \xi^1 \wedge \sigma_{11},\\
d\theta_2 &= (2\eta_1 - \eta_2) \wedge \theta_2 - \theta_0 \wedge \xi^1 + \xi^2 \wedge \sigma_{22},\\
d\xi^1 &= (\eta_1 - \eta_2) \wedge \xi^1,\\
d\xi^2 &= (\eta_2 - \eta_1) \wedge \xi^2,\\
d\sigma_{11} &= (2\eta_2 - \eta_1) \wedge \sigma_{11} + \eta_3 \wedge \xi^1 + (Q + 1 - 2P)\theta_1 \wedge \xi^2,\\
d\sigma_{22} &= (3\eta_1 - 2\eta_2) \wedge \sigma_{22} + \eta_4 \wedge \xi^2 + (P - 2 + Q)\theta_2 \wedge \xi^1,\\
d\eta_1 &= (P - 1)\xi^1 \wedge \xi^2,\\
d\eta_2 &= (P - Q - 1)\xi^1 \wedge \xi^2,\\
d\eta_3 &= \pi_1 \wedge \xi^1 - (2\eta_1 - 3\eta_2) \wedge \eta_3 + (3(Q - P) + 2)\xi^2 \wedge \sigma_{11},\\
d\eta_4 &= \pi_2 \wedge \xi^2 + (4\eta_1 - 3\eta_2) \wedge \eta_4 + (3(Q - 1) + 2P)\xi^1 \wedge \sigma_{22}.
\end{aligned}$$

All the invariants of the symmetry group for an equation from $\mathcal{S}_6$ are constants, and the classifying manifold is a point. Thus, an equation from $\mathcal{S}_6$ is equivalent to one of the equations (23) or (24) with the same values of P and Q, [23, Sec. 9.2].

The results of the above calculations are summarized in the following statement.

Theorem. *The class of linear hyperbolic equations* (1) *is divided into the six subclasses* $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_6$ *invariant under an action of the pseudo-group of contact transformations*:

$\mathcal{S}_1$ *consists of all equations* (1) *such that* $H \equiv 0$ *and* $K \equiv 0$;

$\mathcal{S}_2$ *consists of all equations* (1) *such that* $P_t \ne 0$;

$\mathcal{S}_3$ *consists of all equations* (1) *such that* $P_t \equiv 0$ *and* $P_x \ne 0$;

$\mathcal{S}_4$ *consists of all equations* (1) *such that* $P \equiv \text{const}$ *and* $Q_t \ne 0$;

$\mathcal{S}_5$ consists of all equations (1) *such that* $P \equiv \text{const}$, $Q_t \equiv 0$, *and* $Q_x \neq 0$;

$\mathcal{S}_6$ consists of all equations (1) *such that* $P \equiv \text{const}$ *and* $Q \equiv \text{const}$.

Every equation from the subclass $\mathcal{S}_1$ is locally equivalent to the linear wave equation $u_{tx} = 0$.

Every equation from the subclass $\mathcal{S}_6$ is locally equivalent to either Eq. (23) *when $Q = 0$ or to Eq.* (24) *when $Q \neq 0$.*

For the subclass $\mathcal{S}_2$, the basic invariants are P, Q, and J_2, and the operators of invariant differentiation are $\mathbb{D}_1 = P_t^{-1} D_t$ and $\mathbb{D}_2 = P_t H^{-1} D_x$.

For the subclass $\mathcal{S}_3$, the basic invariants are P, Q, and L, and the operators of invariant differentiation are $\mathbb{D}_1 = P_x H^{-1} D_t$ and $\mathbb{D}_2 = P_x^{-1} D_x$.

For the subclass $\mathcal{S}_4$, the basic invariants are Q, M_1, and M_2, and the operators of invariant differentiation are $\mathbb{D}_1 = Q_t^{-1} D_t$ and $\mathbb{D}_2 = Q_t H^{-1} D_x$.

For the subclass $\mathcal{S}_5$, the basic invariants are Q and N, and the operators of invariant differentiation are $\mathbb{D}_1 = Q_x H^{-1} D_t$ and $\mathbb{D}_2 = Q_x^{-1} D_x$.

Two equations from one of the subclasses $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_4$, or $\mathcal{S}_5$ are locally equivalent to each other if and only if the classifying manifolds (18), (20), (21), *or* (22) *for these equations locally overlap.*

Conclusions

In this paper, the moving-coframe method of [6] is applied to the local equivalence problem for the class of linear second-order hyperbolic equations in two independent variables under an action of the pseudo-group of contact transformations. The class is divided into six invariant subclasses. For all of the subclasses, the Maurer–Cartan forms for symmetry groups, the bases of differential invariants, and the invariant differentiation operators are found. This allowed us to solve the equivalence problem for the whole class of linear hyperbolic equations. It is shown that the moving-coframe method is applicable to structurally intransitive symmetry groups. The method uses linear algebra and differentiation operations only and does not require analyzing overdetermined systems of partial differential equations or using procedures of integration.

REFERENCES

1. É. Cartan, "Sur la structure des groupes infinis de transformations," in: *Œuvres Complètes*, Pt. II, Vol. 2, Gauthier-Villars, Paris (1953), pp. 571–714.
2. É. Cartan, "Les sous-groupes des groupes continus de transformations," ibid., pp. 719–856.
3. É. Cartan, "Les groupes de transformations continus, infinis, simples," ibid., pp. 857–925.
4. É. Cartan, "La structure des groupes infinis," ibid., pp. 1335–1384.
5. É. Cartan, "Les problèmes d'équivalence," ibid., pp. 1311–1334.
6. M. Fels and P. J. Olver, "Moving coframes, I. A practical algorithm," *Acta Appl. Math.*, **51**, 161–213 (1998)
7. M. Fels and P. J. Olver, "Moving coframes, II. Regularization and theoretical foundations," *Acta Appl. Math.*, **55**, 127–208 (1999).
8. R. B. Gardner, *The Method of Equivalence and Its Applications*, SIAM, Philadelphia (1989).
9. N. H. Ibragimov, "Infinitesimal method in the theory of invariants of algebraic and differential equations," *Notices South African Math. Soc.*, **29**, 61–70 (1997).
10. N. H. Ibragimov, *Elementary Lie Group Analysis and Ordinary Differential Equations*, John Wiley and Sons, New York (1999).
11. N. H. Ibragimov, "Laplace type invariants for parabolic equations," *Nonlinear Dynam.*, **28**, 125–133 (2002).
12. N. H. Ibragimov, "Invariants of hyperbolic equations: Solution to Laplace's problem," *Prikl. Mekh. Tekh. Fiz.*, **45**, No. 2, 11–21 (2004).
13. I. K. Johnpillai and F. M. Mahomed, "Singular invariant equation for the (1+1) Fokker–Planck equation," *J. Phys. A*, **34**, 11033–11051 (2001).

14. I. K. Johnpillai, F. M. Mahomed, and C. Wafo Soh, "Basis of joint invariants for (1+1) linear hyperbolic equations," *J. Nonlinear Math. Phys.*, **9**, Supplement 2, 49–59 (2002).
15. P. S. Laplace, "Recherches sur le calcul intégral aux différences partielles," *Mémoires de l'Académie Royale de Sciences de Paris*, 341–401 (1773–1777), reprinted in *Œuvres Complètes*, **IX**, Gauthier-Villars, Paris (1893), pp. 3–68. English translation: New York (1966).
16. S. Lie, *Gesammelte Abhandlungen*, B. 1–6, BG Teubner, Leipzig (1922–1937).
17. I. G. Lisle and G. J. Reid, "Geometry and structure of Lie pseudogroups from infinitesimal defining equations," *J. Symbolic Comput.*, **26** 355–379 (1998).
18. I. G. Lisle and G. J. Reid, *Symmetry Classification Using Invariant Moving Frames*, `http://www.apmaths.uwo.ca/~reid` (1999).
19. O. I. Morozov, "Moving coframes and symmetries of differential equations," *J. Phys. A*, **35**, 2965–2977 (2002).
20. O. I. Morozov, *Contact Equivalence Problem for Linear Parabolic Equations*, `arXiv:math.ph/0304045` (2003).
21. O. I. Morozov, "Symmetries of differential equations and Cartan's equivalence method," in: *Proc. Fifth Int. Conf. "Symmetry in Nonlinear Mathematical Physics," Kyiv, Ukraine, 23–29 June 2003*, Pt. 1, pp. 196–203 (2004).
22. L. V. Ovsiannikov, "Group properties of the Chaplygin equation," *Prikl. Mekh. Tekh. Fiz.*, No. 3, 126–145 (1960).
23. L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York (1982).
24. P. J. Olver, *Equivalence, Invariants, and Symmetry*, Cambridge Univ. Press, Cambridge (1995).

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