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Coverings of Differential Equations and Cartan's Structure Theory of Lie Pseudo-Groups

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Abstract We establish relations between Maurer–Cartan forms of symmetry pseudo-groups and coverings of differential equations. Examples include Liouville's equation, the Khokhlov–Zabolotskaya equation, and the Boyer–Finley equation.

Keywords Lie pseudo-groups · Maurer–Cartan forms · Symmetries of differential equations · Coverings of differential equations

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1 Introduction

Élie Cartan's structure theory of Lie pseudo-groups, [6–9, 16, 21, 40], provide a convenient and powerful tool to study geometry of differential equations (DEs), [3–5, 10, 11, 13–15, 19, 21–24, 30–33, 35, 41, 42]. The theory is based on a possibility to characterize transformations from a Lie pseudo-group in terms of a finite number of invariant differential 1-forms called *Maurer–Cartan forms* of the pseudo-group. Unlike computational technique of Sophus Lie's infinitesimal method, [1, 20, 27, 29, 39, 43], Cartan's approach does not use analysis and integration of over-determined systems of partial DEs and allows one to compute Maurer–Cartan forms for symmetry pseudo-groups of DEs by means of operations of linear algebra and differentiation. The Maurer–Cartan forms contain full information about their pseudo-group. In particular, they give all differential invariants of the pseudo-group, thus providing a complete and efficient algorithm for solving equivalence and symmetry classification problems for DEs of physical and mathematical significance, [10, 11, 32–36].

In this paper, we establish a relation between the technique of Maurer–Cartan forms and the theory of coverings of DEs, [25–27].

Coverings (or Wahlquist–Estabrook prolongation structures, [44], or zero-curvature representations, [47], or integrable extensions, [4], etc.) are of great importance in geometry of

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DEs. The theory of coverings is an adequate universum for dealing with nonlocal symmetries and conservation laws, inverse scattering constructions for soliton equations, Bäcklund transformations, recursion operators, and deformations of nonlinear DEs, [25–27]. A standard approach to finding coverings is developed by Wahlquist and Estabrook, [44]. It was designed to apply to equations with two independent variables. Extending the method to DEs with three or more independent variables is a difficult problem, see, e.g., [17, 18, 37, 38].

Our paper presents the following observation: for some DEs covering equations can be deduced from invariant combinations of Maurer–Cartan forms of their symmetry pseudo-groups. Our examples include well-known coverings of Liouville's equation, the Khokhlov–Zabolotskaya equation, and the Boyer–Finley equation.

2 Cartan's Structure Theory of Symmetry Pseudo-Groups of DEs

In this section, we outline the algorithm of computing Maurer–Cartan forms for pseudo-groups of contact symmetries for DEs of the second order with one dependent variable, see details in [32, 35, 36]. All considerations are of local nature, and all mappings are real analytic. Let $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a vector bundle with the local base coordinates $(x^1, \dots, x^n)$ and the local fibre coordinate u; then by $J^2(\pi)$ denote the bundle of the second-order jets of sections of π, with the local coordinates (x^i, u, u_i, u_{ij}), $i, j \in \{1, \dots, n\}$, $i \le j$. For every local section $(x^i, f(x))$ of π, denote by $j_2(f)$ the corresponding 2-jet $(x^i, f(x), \partial f(x)/\partial x^i, \partial^2 f(x)/\partial x^i \partial x^j)$. A differential 1-form ϑ on $J^2(\pi)$ is called a *contact form* if it is annihilated by all 2-jets of local sections: $j_2(f)^* \vartheta = 0$. In the local coordinates every contact 1-form is a linear combination of the forms $\vartheta_0 = du - u_i\, dx^i$, $\vartheta_i = du_i - u_{ij}\, dx^j$, $i, j \in \{1, \dots, n\}$, $u_{ji} = u_{ij}$ (here and later we use the Einstein summation convention, so $u_i\, dx^i = \sum_{i=1}^n u_i\, dx^i$, etc.). A local diffeomorphism $\Delta : J^2(\pi) \to J^2(\pi)$, $\Delta : (x^i, u, u_i, u_{ij}) \mapsto (\bar{x}^i, \bar{u}, \bar{u}_i, \bar{u}_{ij})$, is called a *contact transformation* if for every contact 1-form $\bar{\vartheta}$ the form $\Delta^* \bar{\vartheta}$ is also contact. We denote by $\mathrm{Cont}(J^2(\pi))$ the pseudo-group of contact transformations on $J^2(\pi)$.

Let $\mathcal{H} \subset \mathbb{R}^{(2n+1)(n+3)(n+1)/3}$ be an open set with local coordinates $(a, b^i_k, c^i, f^{ik}, g_i, s_{ij}, w^k_{ij}, z_{ijk})$, $i, j, k \in \{1, \dots, n\}$, $i \le j$, such that $a \ne 0$, $\det(b^i_k) \ne 0$, $f^{ik} = f^{ki}$, $z_{ijk} = z_{jik} = z_{ikj}$. Let (B^i_k) be the inverse matrix for the matrix (b^k_l), so $B^i_k\, b^k_l = \delta^i_l$. We consider the *lifted coframe*

$$
\begin{aligned}
&\Theta_0 = a\, \vartheta_0, \qquad \Theta_i = g_i\, \Theta_0 + a\, B^k_i\, \vartheta_k, \qquad \Xi^i = c^i\, \Theta_0 + f^{ik}\, \Theta_k + b^i_k\, dx^k, \\
&\Sigma_{ij} = s_{ij}\, \Theta_0 + w^k_{ij}\, \Theta_k + z_{ijk}\, \Xi^k + a\, B^i_k\, B^j_l\, du_{kl},
\end{aligned} \tag{1}
$$

defined on $J^2(\pi) \times \mathcal{H}$. As it is shown in [35], the forms (1) are Maurer–Cartan forms for $\mathrm{Cont}(J^2(\pi))$, that is, a local diffeomorphism $\widehat{\Delta} : J^2(\pi) \times \mathcal{H} \to J^2(\pi) \times \mathcal{H}$ satisfies the conditions $\widehat{\Delta}^* \bar{\Theta}_0 = \Theta_0$, $\widehat{\Delta}^* \bar{\Theta}_i = \Theta_i$, $\widehat{\Delta}^* \bar{\Xi}^i = \Xi^i$, and $\widehat{\Delta}^* \bar{\Sigma}_{ij} = \Sigma_{ij}$ if and only if it is projectable on $J^2(\pi)$, and its projection $\Delta : J^2(\pi) \to J^2(\pi)$ is a contact transformation.

The structure equations for $\mathrm{Cont}(J^2(\pi))$ have the form

$$
\begin{aligned}
d\Theta_0 &= \Phi^0_0 \wedge \Theta_0 + \Xi^i \wedge \Theta_i, \\
d\Theta_i &= \Phi^0_i \wedge \Theta_0 + \Phi^k_i \wedge \Theta_k + \Xi^k \wedge \Sigma_{ik}, \\
d\Xi^i &= \Phi^0_0 \wedge \Xi^i - \Phi^i_k \wedge \Xi^k + \Psi^{i0} \wedge \Theta_0 + \Psi^{ik} \wedge \Theta_k, \\
d\Sigma_{ij} &= \Phi^k_i \wedge \Sigma_{kj} - \Phi^0_0 \wedge \Sigma_{ij} + \Upsilon^0_{ij} \wedge \Theta_0 + \Upsilon^k_{ij} \wedge \Theta_k + \Lambda_{ijk} \wedge \Xi^k,
\end{aligned}
$$

where the additional forms Φ_0^0, Φ_i^0, Φ_i^k, Ψ^{i0}, Ψ^{ij}, Υ_{ij}^0, Υ_{ij}^k, and Λ_{ijk} depend on differentials of the coordinates of $\mathcal{H}$.

Suppose $\mathcal{E}$ is a second-order differential equation in one dependent and n independent variables. We consider $\mathcal{E}$ as a submanifold in $J^2(\pi)$. Let Cont($\mathcal{E}$) be the group of contact symmetries for $\mathcal{E}$. It consists of all the contact transformations on $J^2(\pi)$ mapping $\mathcal{E}$ to itself. Let $\iota_0 : \mathcal{E} \to J^2(\pi)$ be an embedding, and $\iota = \iota_0 \times \mathrm{id} : \mathcal{E} \times \mathcal{H} \to \mathrm{J}^2(\pi) \times \mathcal{H}$. The invariant 1-forms of Cont($\mathcal{E}$) are restrictions of the forms (1) to $\mathcal{E} \times \mathcal{H}$: $\theta_0 = \iota^* \Theta_0$, $\theta_i = \iota^* \Theta_i$, $\xi^i = \iota^* \Xi^i$, and $\sigma_{ij} = \iota^* \Sigma_{ij}$. The forms θ_0, θ_i, ξ^i, and σ_{ij} have some linear dependencies, i.e., there exists a non-trivial set of functions E^0, E^i, F_i, and G^{ij} on $\mathcal{E} \times \mathcal{H}$ such that $E^0\,\theta_0 + E^i\,\theta_i + F_i\,\xi^i + G^{ij}\,\sigma_{ij} \equiv 0$. These functions are lifted invariants of Cont($\mathcal{E}$). Setting them equal to some constants allows us to specify some parameters a, b_i^k, c_i, g_i, f^{ij}, s_{ij}, w_{ij}^k, and z_{ijk} as functions of the coordinates on $\mathcal{E}$ and the other parameters.

After these normalizations, a part of the forms $\phi_0^0 = \iota^* \Phi_0^0$, $\phi_i^k = \iota^* \Phi_i^k$, $\phi_i^0 = \iota^* \Phi_i^0$, $\psi^{ij} = \iota^* \Psi^{ij}$, $\psi^{i0} = \iota^* \Psi^{i0}$, $\upsilon_{ij}^0 = \iota^* \Upsilon_{ij}^0$, $\upsilon_{ij}^k = \iota^* \Upsilon_{ij}^k$, and $\lambda_{ijk} = \iota^* \Lambda_{ijk}$, or some their linear combinations, become semi-basic, i.e., they do not include the differentials of the parameters of $\mathcal{H}$. Setting coefficients of the semi-basic forms equal to some constants, we get specifications of some more parameters of $\mathcal{H}$.

More lifted invariants can appear as essential torsion coefficients in the reduced structure equations

$$\begin{aligned}
d\theta_0 &= \phi_0^0 \wedge \theta_0 + \xi^i \wedge \theta_i,\\
d\theta_i &= \phi_i^0 \wedge \theta_0 + \phi_i^k \wedge \theta_k + \xi^k \wedge \sigma_{ik},\\
d\xi^i &= \phi_0^0 \wedge \xi^i - \phi_k^i \wedge \xi^k + \psi^{i0} \wedge \theta_0 + \psi^{ik} \wedge \theta_k,\\
d\sigma_{ij} &= \phi_i^k \wedge \sigma_{kj} - \phi_0^0 \wedge \sigma_{ij} + \upsilon_{ij}^0 \wedge \theta_0 + \upsilon_{ij}^k \wedge \theta_k + \lambda_{ijk} \wedge \xi^k.
\end{aligned}$$

After normalizing these invariants and repeating the process, two outputs are possible. In the first case, the reduced lifted coframe appears to be involutive. Then this coframe is the desired set of Maurer–Cartan forms for Cont($\mathcal{E}$). In the second case, when the reduced lifted coframe does not satisfy Cartan's test, we should use the procedure of prolongation, [40, Chap. 12].

Example 1 For the symmetry pseudo-group of Liouville's equation

$$u_{xy} = e^u, \tag{2}$$

we obtain the Maurer–Cartan forms

$$\begin{aligned}
\theta_0 &= du - u_x dx - u_y dy,\\
\theta_1 &= q^{-1}\,(du_x - u_{xx} dx - e^u dy),\\
\theta_2 &= q\, e^{-u}\,(du_y - e^u dx - u_{yy} dy),\\
\xi^1 &= q\, dx,\\
\xi^2 &= q^{-1} e^u dy,\\
\sigma_{11} &= q^{-2}\big(du_{xx} - u_x du_x + (u_x u_{xx} + q^3 r_1)\, dx\big),\\
\sigma_{22} &= q^2\, e^{-2u}\big(du_{yy} - u_y du_y + (u_y u_{yy} + e^{3u} q^{-3} r_2) dy\big),\\
\eta_1 &= q^{-1}(dq - u_x\, \xi^1),\\
\eta_2 &= dr_1 - 3\, r_1\, \eta_1 + q^{-2}(u_{xx} + u_x^2)(\theta_1 + \xi^2) + 3\, q^{-1} u_x\, \sigma_{11} + r_3\, \xi^1,\\
\eta_3 &= dr_2 + 3\, r_2(\eta_1 + \theta_0) + q^2 e^{-2u}(u_{yy} + u_y^2)(\theta_2 + \xi^1) + 3\, q e^{-u} u_y \sigma_{22} + r_4\, \xi^2,
\end{aligned} \tag{3}$$

where $q=b^1_1$, $r_1=z_{111}$, $r_2=z_{222}$, r_3 and r_4 are arbitrary parameters, while $\sigma_{12}=0$. The forms satisfy the following structure equations:

$$
\begin{aligned}
d\theta_0 &= \xi^1\wedge\theta_1+\xi^2\wedge\theta_2,\\
d\theta_1 &= -\eta_1\wedge\theta_1-\theta_0\wedge\xi^2+\xi^1\wedge\sigma_{11},\\
d\theta_2 &= \eta_1\wedge\theta_2-\theta_0\wedge(\theta_2+\xi^1)+\xi^2\wedge\sigma_{22},\\
d\xi^1 &= \eta_1\wedge\xi^1,\\
d\xi^2 &= (\theta_0-\eta_1)\wedge\xi^2,\\
d\sigma_{11} &= \eta_2\wedge\xi^1-2\,\eta_1\wedge\sigma_{11},\\
d\sigma_{22} &= \eta_3\wedge\xi^2+2(\eta_1-\theta_0)\wedge\sigma_{22},\\
d\eta_1 &= -(\theta_1+\xi^2)\wedge\xi^1,\\
d\eta_2 &= \alpha_1\wedge\xi^1-3\,\eta_1\wedge\eta_2+2(\theta_1+\xi^2)\wedge\sigma_{11},\\
d\eta_3 &= \alpha_2\wedge\xi^2+3(\eta_1-\theta_0)\wedge\eta_3+2(\theta_2+\xi^1)\wedge\sigma_{22},
\end{aligned}
$$

where α_1 and α_2 depend on dr_3 and dr_3.

3 Coverings of DEs

Let $\pi_\infty : J^\infty(\pi)\to\mathbb{R}^n$ be the infinite jet bundle of local sections of the bundle π. The coordinates on $J^\infty(\pi)$ are (x^i,u_I), where $I=(i_1,\dots,i_k)$ are symmetric multi-indexes, $i_1,\dots,i_k\in\{1,\dots,n\}$, and for $I=\emptyset$ we put $u_\emptyset=u$. For any local section f of π there exists a section $j_\infty(f):\mathbb{R}^n\to J^\infty(\pi)$ such that $u_I(j_\infty(f))=\partial^{\#I}(f)/\partial x^{i_1}\dots\partial x^{i_k}$, where $\#I=\#(i_1,\dots,i_k)=k$ and $\#\emptyset=0$. Contact forms on $J^\infty(\pi)$ are defined by the requirement to satisfy $j_\infty(f)^*\,\vartheta=0$ for any f. They are linear combinations of the forms $\vartheta_0=\vartheta_\emptyset=du-u_i\,dx^i$ and $\vartheta_I=du_I-u_{Ii}\,dx^i$, $\#I>0$. The *total derivatives* on $J^\infty(\pi)$ are defined in the local coordinates as

$$
D_i=\frac{\partial}{\partial x^i}+\sum_{\#I\ge 0}u_{Ii}\frac{\partial}{\partial u_I}.
$$

We have $[D_i,D_j]=0$ for $i,j\in\{1,\dots,n\}$ and $\vartheta_I=D_I(\vartheta_0)$, where $D_I=D_{i_1}\circ\dots\circ D_{i_k}$ for $I=(i_1,\dots,i_k)$.

A differential equation $F(x^i,u_I)=0$, $\#I\le q$, defines a submanifold $\mathcal{E}^\infty=\{D_K(F)=0\mid \#K\ge 0\}\subset J^\infty(\pi)$. We denote restrictions of D_i and ϑ_I on $\mathcal{E}^\infty$ as $\bar{D}_i$ and $\bar{\vartheta}_I$, respectively.

In local coordinates, a *covering* over $\mathcal{E}^\infty$ is a bundle $\widetilde{\mathcal{E}}^\infty=\mathcal{E}^\infty\times\mathcal{V}\to\mathcal{E}^\infty$ with fibre coordinates v^κ, $\kappa\in\{1,\dots,N\}$ or $\kappa\in\mathbb{N}$, equipped with extended total derivatives

$$
\widetilde{D}_i=\bar{D}_i+\sum_\kappa T^\kappa_i(x^j,u_I,v^\tau)\frac{\partial}{\partial v^\kappa}
$$

such that $[\widetilde{D}_i,\widetilde{D}_j]=0$ whenever $(x^i,u_I)\in\mathcal{E}^\infty$.

In terms of differential forms, the covering is defined by the forms

$$
\widetilde{\vartheta}^\kappa=dv^\kappa-T^\kappa_i(x^j,u_I,v^\tau)\,dx^i
$$

such that $d\widetilde{\vartheta}^{\kappa} \equiv 0 \pmod{\widetilde{\vartheta}^{\tau}, \bar{\vartheta}_I}$ whenever $(x^i, u_I) \in \mathcal{E}^{\infty}$. Let us call $\widetilde{\vartheta}^{\kappa}$ *Wahlquist–Estabrook forms* (*WE forms*) of the covering.

Remark 1 Let $\widetilde{\vartheta}$ be a WE form of a covering $\widetilde{\mathcal{E}}^{\infty}$, and s be an arbitrary non-zero parameter such that ds is independent of the differentials of the coordinates of $\widetilde{\mathcal{E}}^{\infty}$. Then the form $s \cdot \widetilde{\vartheta}$ is also a WE form of the covering.

Example 2 Consider the system of DEs

$$v_x = u_x + \lambda\, e^v, \qquad v_y = -\frac{1}{2\lambda}\, e^{u-v}, \tag{4}$$

$\lambda = \text{const}$, $\lambda \neq 0$. We have $(v_x)_y = (v_y)_x$ if and only if u satisfies Liouville's equation (2). Therefore, system (4) defines one-dimensional covering over (2) with the extended total derivatives

$$\widetilde{D}_x = \bar{D}_x + (u_x + \lambda\, e^v)\frac{\partial}{\partial v}, \qquad \widetilde{D}_y = \bar{D}_y - \frac{1}{2\lambda}\, e^{u-v}\frac{\partial}{\partial v}$$

and the WE form

$$\widetilde{\vartheta} = dv - (u_x + \lambda\, e^v)\, dx + \frac{1}{2\lambda}\, e^{u-v}\, dy \tag{5}$$

such that

$$d\widetilde{\vartheta} \equiv \left(-\lambda\, u_x\, dx + \frac{1}{2\lambda}\, e^{u-v}\, dy\right) \wedge \widetilde{\vartheta} + (u_{xy} - e^u)\, dx \wedge dy \pmod{\theta_0, \theta_1, \theta_2}.$$

Excluding u from (4), we obtain Clairin's equation, [12],

$$v_{xy} + \lambda\, e^v\, v_y = 0. \tag{6}$$

Therefore, (4) is a Bäcklund transformation between (2) and (6).

4 Wahlquist–Estabrook Forms and Maurer–Cartan Forms

A very interesting feature is that for some DEs WE forms of coverings appear to be invariant combinations of Maurer–Cartan forms for symmetry pseudo-groups of these equations. We present three examples of this phenomenon.

4.1 Liouville's Equation

Comparing (5) with (3), we have the following observation: the linear combination

$$\mu_1 = \eta_1 - \lambda\, \xi^1 + \frac{1}{2\lambda}\, \xi^2 = \frac{dq}{q} - (u_x + \lambda\, q)\, dx + \frac{1}{2\lambda} \cdot \frac{e^u}{q}\, dy$$

of forms (3) after the change of variable $q = e^v$ becomes the WE form (5) of the covering (4): $\mu_1 = \widetilde{\vartheta}$. Another covering for (2) can be obtained from the linear combination

$$\mu_2 = \eta_1 - \lambda\, \xi^1 + \frac{1}{2\lambda}\, \xi^2 - \frac{1}{2}\, \theta_0$$

by means of the substitution $q = e^{v+u/2}$:

$$\mu_2 = dv - \left(\frac{1}{2}\,u_x + \lambda\,e^{v+u/2}\right)dx + \frac{1}{2}\left(u_y + \frac{1}{\lambda}\,e^{-v+u/2}\right)dy.$$

This form corresponds to the one-dimensional covering of (2) with the extended total derivatives

$$\widetilde{D}_x = \bar{D}_x + \left(\frac{1}{2}\,u_x + \lambda\,e^{v+u/2}\right)\frac{\partial}{\partial v},$$

$$\widetilde{D}_y = \bar{D}_y - \frac{1}{2}\left(u_y + \frac{1}{\lambda}\,e^{-v+u/2}\right)\frac{\partial}{\partial v}.$$

The corresponding equations

$$v_x = \frac{1}{2}\,u_x + \lambda\,e^{v+u/2}, \qquad v_y = -\frac{1}{2}\left(u_y + \frac{1}{\lambda}\,e^{-v+u/2}\right)$$

define a Bäcklund transformation between Liouville's equation and d'Alembert's equation $v_{xy} = 0$.

4.2 Khokhlov–Zabolotskaya Equation

The Khokhlov–Zabolotskaya equation

$$u_{zz} = u_{xy} + u\,u_{yy} + u_y^2 \tag{7}$$

describes dynamics of nonlinear acoustic waves, [45]. In [28] this equation appears in geometric study of classes of the first order systems of partial DEs reducible to a single equation of the second order. Namely, (7) follows from the system

$$v_x = (v^2 - u)v_y - u_z - v\,u_y, \qquad v_z = v\,v_y - u_y \tag{8}$$

as its integrability conditions: $(v_x)_z = (v_z)_x$ whenever (7) is satisfied. System (8) provide an infinite-dimensional covering for (7) with fibre coordinates $v_0 = v$, $v_k = \partial^k v/\partial y^k$, $k \in \mathbb{N}$, the extended total derivatives

$$\widetilde{D}_x = \bar{D}_x + \sum_{j=0}^{\infty} \widetilde{D}_y^j\big((v_0^2 - u)\,v_1 - u_z - v_0\,u_y\big)\frac{\partial}{\partial v_j},$$

$$\widetilde{D}_y = \bar{D}_y + \sum_{j=0}^{\infty} v_{j+1}\frac{\partial}{\partial v_j},$$

$$\widetilde{D}_z = \bar{D}_z + \sum_{j=0}^{\infty} \widetilde{D}_y^j(v_0\,v_1 - u_y)\frac{\partial}{\partial v_j},$$

and the WE forms

$$\widetilde{\vartheta}_0 = dv_0 - \big((v_0^2 - u)\,v_1 - u_z - v_0\,u_y\big)\,dx - v_1\,dy - (v_0\,v_1 - u_y)\,dz,$$

$$\widetilde{\vartheta}_k = \widetilde{D}_y^k(\widetilde{\vartheta}_0), \quad k \in \mathbb{N}.$$

Applying the method described in Sect. 2 to the symmetry pseudo-group of (7), we obtain the structure equations

$$
\begin{aligned}
d\theta_0 &= \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2 + \xi^3 \wedge \theta_3,\\
d\theta_1 &= \eta_1 \wedge \left(\theta_2 + \frac{3}{2}\theta_1\right) + (\theta_2 + 2\xi^2) \wedge \theta_0 + \xi^1 \wedge \sigma_{11} + \xi^2 \wedge \sigma_{12}\\
&\quad + \xi^3 \wedge \sigma_{13} - \frac{1}{2}\sigma_{22} \wedge \theta_1 - 2\sigma_{23} \wedge \theta_3,\\
d\theta_2 &= \frac{1}{2}\eta_1 \wedge \theta_2 + \xi^1 \wedge \sigma_{12} + \left(\xi^2 + \frac{1}{2}\theta_2\right) \wedge \sigma_{22} + \xi^3 \wedge \sigma_{23},\\
d\theta_3 &= \eta_1 \wedge \theta_3 + \xi^3 \wedge \theta_0 + \xi^3 \wedge \sigma_{12} + \xi^1 \wedge \sigma_{13} + \left(\frac{1}{2}\theta_3 + \xi^3\right) \wedge \sigma_{22}\\
&\quad + (\theta_2 + \xi^2) \wedge \sigma_{23},\\
d\xi^1 &= -\left(\frac{1}{2}\eta_1 - \sigma_{22}\right) \wedge \xi^1,\\
d\xi^2 &= -(\eta_1 + \theta_0) \wedge \xi^1 + \frac{1}{2}(\eta_1 + \sigma_{22}) \wedge \xi^2 + \sigma_{23} \wedge \xi^3,\\
d\xi^3 &= 2\sigma_{23} \wedge \xi^1 + \frac{1}{2}\sigma_{22} \wedge \xi^3,\\
d\sigma_{11} &= 2\eta_1 \wedge (\theta_0 + \sigma_{11} + \sigma_{12}) + \eta_2 \wedge \xi^1 + \eta_3 \wedge \xi^2 + \eta_4 \wedge \xi^3\\
&\quad - 6\theta_1 \wedge \theta_2 + \sigma_{11} \wedge \sigma_{22} + 4\sigma_{13} \wedge \sigma_{23},\\
d\sigma_{12} &= \eta_1 \wedge (\sigma_{12} + \sigma_{22}) + \eta_3 \wedge \xi^1 + \theta_0 \wedge \sigma_{22} + 7\theta_1 \wedge \xi^1\\
&\quad + \theta_2 \wedge (2\xi^1 + \xi^2) + \theta_3 \wedge \xi^3 + \sigma_{12} \wedge \sigma_{22},\\
d\sigma_{13} &= \eta_1 \wedge \left(\frac{3}{2}\sigma_{13} + \sigma_{23}\right) + \eta_3 \wedge \xi^3 + \eta_4 \wedge \xi^1\\
&\quad + (6\theta_1 + 5\theta_2 + 3\xi^2) \wedge \xi^3 + (3\theta_2 - 2\xi^1 + 3\xi^2) \wedge \theta_3\\
&\quad + \sigma_{13} \wedge \sigma_{22} + (3\theta_0 + 3\sigma_{12} + 2\sigma_{22}) \wedge \sigma_{23},\\
d\sigma_{22} &= 4(\theta_2 + \xi^2) \wedge \xi^1,\\
d\sigma_{23} &= \frac{1}{2}\eta_1 \wedge \sigma_{23} + (\theta_3 + \xi^3) \wedge \xi^1 + (\theta_2 + \xi^2) \wedge \xi^3,\\
d\eta_1 &= -2(\theta_2 + \xi^2) \wedge \xi^1,\\
d\eta_2 &= \alpha_1 \wedge \xi^1 + \alpha_2 \wedge \xi^2 + \alpha_3 \wedge \xi^3 + \frac{3}{2}\eta_2 \wedge \sigma_{22} - \eta_3 \wedge \theta_0\\
&\quad + \eta_1 \wedge \left(12\theta_1 + 4\theta_2 + \frac{5}{2}\eta_2 + 3\eta_3\right) + 6\eta_4 \wedge \sigma_{23}\\
&\quad + 2(2\theta_0 + 7\sigma_{11} + 2\sigma_{12}) \wedge \theta_2 + 6\theta_1 \wedge \sigma_{12}\\
&\quad + 4(\sigma_{13} - 2\sigma_{23}) \wedge \theta_3,
\end{aligned}
$$

$$d\eta_3 = \alpha_2 \wedge \xi^1 + \eta_1 \wedge \left(\frac{3}{2}\eta_3 + 4\xi^2\right) + 2\theta_0 \wedge (6\theta_2 - 2\xi^1 + 9\xi^2)$$
$$+ 8\xi^1 \wedge \sigma_{11} - 6\xi^3 \wedge \sigma_{13} - 2(3\theta_2 - 2\xi^1 + 6\xi^2) \wedge \sigma_{12}$$
$$+ \left(\frac{3}{2}\eta_3 + 6\theta_1 - 4\xi^2\right) \wedge \sigma_{22} - 2(6\theta_3 + \xi^3) \wedge \sigma_{23},$$

$$d\eta_4 = \alpha_2 \wedge \xi^3 + \alpha_3 \wedge \xi^1 + 2(\eta_4 + \wedge\xi^3) \wedge \eta_1 + \left(\frac{3}{2}\eta_4 - 4\xi^3\right) \wedge \sigma_{22}$$
$$- 2(\theta_0 + 7\sigma_{11} + 2\sigma_{12}) \wedge \xi^3 - 2(5\theta_2 - 2\xi^1 + 5\xi^2) \wedge \sigma_{13}$$
$$+ 5(\eta_3 + 6\theta_1 + 4\theta_2 + 2\xi^2) \wedge \sigma_{23}.$$

The Maurer–Cartan forms of interest are

$$\xi^1 = s\,dx, \qquad \xi^2 = \frac{u_{yz}^2 + u_{yy}\,s^2 - u\,u_{yy}^2}{s\,u_{yy}}\,dx + \frac{u_{yy}}{s}\,dy + \frac{u_{yz}}{s}\,dz,$$

$$\sigma_{23} = \frac{du_{yz} + (u_y\,u_{yz} + u_z\,u_{yy})\,dx}{s\,(u_{yy})^{1/2}} - \frac{u_{yz}}{s\,(u_{yy})^{3/2}}\,du_{yy} + \frac{u_y\,(u_{yy})^{1/2}}{s}\,dz,$$

where s is a non-zero parameter. We take the linear combination

$$\mu = \sigma_{23} + K\,(\xi^1 - \xi^2)$$

where $K = (u_{yy}u_{yyz} - u_{yz}u_{yyy})\,u_{yy}^{-5/2}$ is an invariant of the symmetry pseudo-group of equation (7), and make the following change of variables:

$$u_{yz} = v_0\,v_1^2\,K^{-2}, \qquad u_{yy} = v_1^2\,K^{-2}.$$

Then we have

$$\mu = \frac{v_1}{K s}\big(dv_0 - ((v_0^2 - u)\,v_1 - u_z - v_0\,u_y)\,dx - v_1\,dy - (v_0\,v_1 - u_y)\,dz\big).$$

This form, in accordance with Remark 1, is a WE form of the covering.

4.3 Boyer–Finley Equation

The Boyer–Finley equation

$$u_{xy} = (e^u)_{zz} \tag{9}$$

is obtained in [2] in studying real Euclidean self-dual Einstein spaces with one rotational Killing vector. The covering equations

$$v_x = u_x + e^v\,v_z, \qquad v_y = e^{u-v}(v_z - u_z) \tag{10}$$

for (9) are found in [46]. Equations (10) define an infinite-dimensional covering over (9) with the fibre variables $v_0 = v$, $v_k = \partial^k v/\partial z^k$, $k \in \mathbb{N}$, extended total derivatives

$$\widetilde{D}_x = \bar{D}_x + \sum_{j=0}^{\infty} \widetilde{D}_z^j(u_x + e^{v_0}\,v_1)\frac{\partial}{\partial v_j},$$

$$\widetilde{D}_y = \bar{D}_y + \sum_{j=0}^{\infty} \widetilde{D}_z^j \left(e^{u-v_0}(v_1 - u_z)\right) \frac{\partial}{\partial v_j},$$

$$\widetilde{D}_z = \bar{D}_z + \sum_{j=0}^{\infty} v_{j+1} \frac{\partial}{\partial v_j},$$

and WE forms

$$\widetilde{\vartheta}_0 = dv_0 - (u_x + e^{v_0} v_1)\,dx - e^{u-v_0}(v_1 - u_z)\,dy - v_1\,dz,$$
$$\widetilde{\vartheta}_k = \widetilde{D}_z^k(\widetilde{\vartheta}_0), \quad k \in \mathbb{N}.$$

The structure equations for the symmetry pseudo-group $\mathrm{Cont}(\mathcal{E})$ of (9) have the form

$$\begin{aligned}
&d\xi^1 = \omega_1 \wedge \xi^1, \qquad d\xi^2 = \omega_2 \wedge \xi^2, \qquad d\xi^3 = \omega_3 \wedge \xi^3,\\
&d\omega_1 = \xi^1 \wedge \omega_4, \qquad d\omega_2 = \xi^2 \wedge \omega_5, \qquad d\omega_3 = 0,\\
&d\omega_4 = -\omega_1 \wedge \omega_4 + \omega_6 \wedge \xi^1, \qquad d\omega_5 = -\omega_2 \wedge \omega_5 + \omega_7 \wedge \xi^2,\\
&d\omega_6 = \alpha_1 \wedge \xi^1 - 2\,\omega_1 \wedge \omega_6, \qquad d\omega_7 = \alpha_2 \wedge \xi^2 - 2\,\omega_2 \wedge \omega_7,\\
&d\omega_8 = 0, \qquad d\omega_9 = -2\,\omega_3 \wedge \omega_9,
\end{aligned}$$

where $\omega_1, \ldots, \omega_9$ are invariant combinations of θ_0, θ_1, θ_2, ξ^1, ξ^2, ξ^3, σ_{11}, σ_{13}, σ_{22}, σ_{23}, and σ_{33}. We have

$$\xi^1 = \frac{u_{xz}}{u_z}\,dx, \qquad \xi^2 = \frac{u_z^3\,e^u}{u_{xz}}\,dy, \qquad \xi^3 = u_z\,dz,$$

$$\omega_1 = \frac{du_{xz}}{u_{xz}} - \frac{du_z}{u_z} - u_x\,dx, \qquad \omega_2 = du + 3\frac{du_z}{u_z} - \frac{du_{xz}}{u_{xz}} - u_y\,dy, \qquad \omega_3 = \frac{du_z}{u_z}.$$

For any local section $h : \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}$ of the bundle π, $h : (x, y, z) \mapsto (x, y, z, u(x, y, z))$, the forms $h^*\omega_j$ are *horizontal components* of the forms ω_j. If the function u from h is a solution of (9), then $h^*\omega_j$ and $h^*\xi^i$ are invariant w.r.t. $\mathrm{Cont}(\mathcal{E})$. Since

$$h^*\omega_3 = \frac{u_{xz}dx + u_{yz}dy + u_{zz}dz}{u_z} = h^*\xi^1 + \frac{u_{xz}u_{yz}}{u_z^4 e^u} h^*\xi^2 + \frac{u_{zz}}{u_z^2} h^*\xi^3,$$

the functions

$$I_1 = \frac{u_{zz}}{u_z^2}, \qquad I_2 = \frac{u_{xz}u_{yz}}{u_z^4 e^u}$$

are invariants of $\mathrm{Cont}(\mathcal{E})$. Invariant derivatives $\mathbb{D}_i$, $i \in \{1, 2, 3\}$, of $\mathrm{Cont}(\mathcal{E})$ are defined by the identity $j_\infty(h)^* dH = j_\infty(h)^*(\mathbb{D}_i(H))\, h^*\xi^i$ for an arbitrary function H on $\mathcal{E}^\infty$, that is,

$$\mathbb{D}_1 = \frac{u_z}{u_{xz}} \bar{D}_x, \qquad \mathbb{D}_2 = \frac{u_{xz}}{u_z^3\,e^u} \bar{D}_y, \qquad \mathbb{D}_3 = \frac{1}{u_z} \bar{D}_z.$$

It is easy to show that every differential invariant of $\mathrm{Cont}(\mathcal{E})$ is a function of $\mathbb{D}_1^{i_1} \circ \mathbb{D}_2^{i_2} \circ \mathbb{D}_3^{i_3}(I_k)$, $k \in \{1, 2\}$, but we need not this result in the sequel.

Now we consider the invariant combination

$$\mu = \omega_1 - \omega_3 - J\,\xi^1 + (J-1)\xi^2 - J\,\xi^3$$
$$= \frac{du_{xz}}{u_{xz}} - 2\frac{du_z}{u_z} - \left(u_x + J\frac{u_{xz}}{u_z}\right)dx + (J-1)\frac{u_z^3\,e^u}{u_{xz}}\,dy - J\,u_z\,dz,$$

where $J = \mathbb{D}_1(I_1)$. The change of variables

$$v_0 = \ln u_{xz} - 2\,\ln u_z, \qquad v_1 = \frac{u_{xzz}}{u_{xz}} - 2\frac{u_{zz}}{u_z} = J\,u_z,$$

gives

$$\mu = dv_0 - (u_x + e^{v_0}\,v_1)\,dx - e^{u-v_0}(v_1 - u_z)\,dy - v_1\,dz.$$

This is the WE form $\widetilde{\vartheta}_0$ of the covering.

5 Conclusion

We have shown that coverings of some nonlinear DEs can be derived from Maurer–Cartan forms of their symmetry pseudo-groups. This interesting and intriguing connection between local and nonlocal aspects of geometry of DEs will require further study. We hope that more examples of the discussed phenomenon will allow to clarify the relation and its underlying theory. Also, we believe that these results will shed additional light on the complicated methods for constructing coverings of DEs with more than two independent variables.

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