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Moving coframes and symmetries of differential equations

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Abstract

This paper studies the application of the structure theory of infinite-dimensional pseudo-groups to computing symmetries of differential equations. The main tool is a combination of Cartan's method of equivalence and the moving coframe method introduced by Fels and Olver. Our approach does not require a preliminary computation of infinitesimal defining systems, their analysis and integration, and uses differentiation and linear algebra operations only. Examples of its main features are given.

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1. Introduction

The theory of symmetries of differential equations (DEs) was created by Sophus Lie more than a 100 years ago. One of Lie's greatest contributions was the discovery of the connection between continuous transformation groups and their infinitesimal generators, which allows one to reduce complicated nonlinear invariance conditions of DEs under the action of a transformation group to much simpler linear conditions of infinitesimal invariance—defining equations of symmetry algebra. Lie's method turned out to be a powerful tool for studying differential equations, finding their exact solutions, conservation laws, etc [1, 4, 13–15, 18, 19, 29, 30]. It requires integration of the (over-determined) system of defining equations to find a symmetry group admitted by DEs explicitly. In the last decade, methods which do not use integration but rather extract information about the structure of symmetry groups directly from their infinitesimal defining systems were developed by Reid and Schwarz, [23, 24, 27, 28]. It was shown how to calculate the dimension of the finite Lie group, and in [23, 24] it was also shown how to find the structure constants of the symmetry algebra in the finite-dimensional case. In [16, 17] the method of [23, 24] was generalized to the case of structurally transitive infinite Lie pseudo-groups. Specifically, it was shown how to obtain the Cartan structure equations of the symmetry pseudo-group admitted by a system of DEs from its infinitesimal defining equations.

Élie Cartan's theory of infinite Lie pseudo-groups [5–9] does not use infinitesimal methods and is based on the possibility of characterizing an infinite Lie pseudo-group on a manifold M as the set of projections of bundle transformations on a principal fibre bundle $M \times \mathcal{G} \to M$, where $\mathcal{G}$ is some Lie group, that preserve a collection of 1-forms ϑ^i on $M \times \mathcal{G}$. The equations that express the differentials $\mathrm{d}\vartheta^i$ through the ϑ^i and modified Maurer–Cartan forms μ^α of the group $\mathcal{G}$,

$$\mathrm{d}\vartheta^i = A^i_{\alpha j}\mu^\alpha \wedge \vartheta^j + T^i_{jk}\vartheta^j \wedge \vartheta^k$$

are called Cartan structure equations; they include important information about the pseudo-group (see, particularly, [20, theorem 11.16]).

In the present paper we apply Cartan's method of equivalence, [9, 12, 20], and the moving coframe method introduced by Fels and Olver, [10, 11], to computing invariant 1-forms of a symmetry pseudo-group of partial differential equations. We treat the case when either there are more than one dependent variable, or the order of DEs is greater than 1, since the pseudo-group of contact transformations acts transitively on the set of partial differential equations of the first order with a single dependent variable, [13, section 14.1]. The case of ordinary differential equations is treated in [10, section 9]; see also [20, ch 12] and references therein.

A system $\mathcal{R}_s$ of DEs of order s in n independent variables and m dependent variables is locally considered to be the sub-bundle in the bundle $J^s(\mathcal{E})$ of s-jets of the bundle $\mathcal{E} = \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. A pseudo-group of symmetries $\mathrm{Lie}(\mathcal{R}_s)$ of the system $\mathcal{R}_s$ is a subgroup of the pseudo-group $\mathrm{Lie}(J^s(\mathcal{E}))$ of contact transformations of the bundle $J^s(\mathcal{E})$ and consists of those transformations which preserve the sub-bundle $\mathcal{R}_s$. So the problem of finding the group $\mathrm{Lie}(\mathcal{R}_s)$ is a particular case of the general problem of equivalence of embedded submanifolds under the action of a pseudo-group, and the moving coframe method, [10, 11], could be used for solving this equivalence problem.

Some simplifications are possible if we deal with the first-order systems of DEs. From [22, theorem 3.3.1.] a system $\mathcal{R}_s$ is equivalent to the system $\hat{\mathcal{R}}_1$ of first order, which is the sub-bundle in $J^1(\hat{\mathcal{E}})$, where $\hat{\mathcal{E}} = J^{s-1}(\mathcal{E})$, and, from the theorem of [13, section 17.4], the symmetry groups $\mathrm{Lie}(\mathcal{R}_s)$ and $\mathrm{Lie}(\hat{\mathcal{R}}_1)$ of the systems $\mathcal{R}_s$ and $\hat{\mathcal{R}}_1$ are isomorphic. The pseudo-group $\mathrm{Lie}(\hat{\mathcal{R}}_1)$ is a subgroup of the pseudo-group $\mathrm{Lie}(J^1(\hat{\mathcal{E}}))$. From Bäcklund's theorem [3], contact transformations on $J^1(\hat{\mathcal{E}})$ are prolongations of point transformations on $\hat{\mathcal{E}}$. Cartan's method of equivalence allows us to compute invariant 1-forms which define the pseudo-group of contact transformations. Then we can find the invariant 1-forms of the pseudo-group $\mathrm{Lie}(\hat{\mathcal{R}}_1)$. To do that, we should take the following steps. First, we restrict the invariant 1-forms of the pseudo-group $\mathrm{Lie}(J^1(\hat{\mathcal{E}}))$ on the sub-bundle $\hat{\mathcal{R}}_1$ and obtain the set of linear dependent 1-forms. Next, we apply the procedure of normalization to the resulting linear dependences among the restricted 1-forms. Finally, we apply the operations of Cartan's equivalence method to the restrictions on $\hat{\mathcal{R}}_1$ of the structure equations of the pseudo-group $\mathrm{Lie}(J^1(\hat{\mathcal{E}}))$.

2. Invariant 1-forms and structure equations of the pseudo-group of contact transformations

According to [22, theorem 3.3.1.] a compatible system $\mathcal{R}_s$ of DEs of order s is equivalent to the system $\hat{\mathcal{R}}_1$ of order 1, which has more dependent variables. So it is possible to restrict our attention to the case of $s = 1$.

Let $\mathcal{R}_1$ be a system of partial differential equations of first order, considered to be the sub-bundle in the bundle $J^1(\mathcal{E})$ of 1-jets of the bundle $\mathcal{E} \to X$ over an n-dimensional base manifold X, with q-dimensional fibres. Let $(x^1, x^2, \ldots, x^n)$ denote the local coordinates of the base X and $(u^1, u^2, \ldots, u^q)$ denote the local coordinates of the fibres of $\mathcal{E}$. Then the local coordinates of the bundle $J^1(\mathcal{E})$ are $\left(x^1, \ldots, x^n, u^1, \ldots, u^q, p^1_1, \ldots, p^1_n, \ldots, p^q_1, \ldots, p^q_n\right)$, and a local section $f : X \to \mathcal{E}$, defined by the equalities $u^\alpha = f^\alpha(x)$, $\alpha \in \{1, \ldots, q\}$, has the corresponding 1-jet $j_1(f) : X \to J^1(\mathcal{E})$, defined by the equalities $u^\alpha = f^\alpha(x)$, $p^\alpha_i = \frac{\partial f^\alpha(x)}{\partial x^i}$, $\alpha \in \{1, \ldots, q\}$, $i \in \{1, \ldots, n\}$.

A differential form ϑ on $J^1(\mathcal{E})$ is called a *contact form* if it is annihilated by all 1-jets: $j_1(f)^*\vartheta = 0$. In local coordinates every contact 1-form is a linear combination of the Cartan forms $\vartheta^\alpha = \mathrm{d}u^\alpha - p^\alpha_i\,\mathrm{d}x^i$, $\alpha \in \{1, \ldots, q\}$ (here and later we use the Einstein summation convention, so $p^\alpha_i\,\mathrm{d}x^i = \sum_{i=1}^n p^\alpha_i\,\mathrm{d}x^i$ etc).

A local diffeomorphism $\Delta : J^1(\mathcal{E}) \to J^1(\mathcal{E})$, $\Delta : (x, u, p) \mapsto (\bar{x}, \bar{u}, \bar{p})$, is called a *contact transformation*, if for every contact form ϑ, the form $\Delta^*\bar{\vartheta}$ is also a contact form; in other words, if $\Delta^*\bar{\vartheta}^\alpha = \mathrm{d}\bar{u}^\alpha - \bar{p}^\alpha_i\,\mathrm{d}\bar{x}^i = \zeta^\alpha_\beta(x, u, p)\vartheta^\beta$ for some functions ζ^α_β on $J^1(\mathcal{E})$.

From Bäcklund's theorem [3], in the case of $n > 1$ and $q > 1$ every contact transformation $\Delta : J^1(\mathcal{E}) \to J^1(\mathcal{E})$ is a prolongation of a point transformation $\Gamma : \mathcal{E} \to \mathcal{E}$, $\Gamma : (x, u) \mapsto (\bar{x}, \bar{u})$, where the functions $\bar{p}^\alpha_i$ are defined by the equalities

$$\frac{\partial \bar{u}^\alpha}{\partial x^j} + \frac{\partial \bar{u}^\alpha}{\partial u^\beta} p^\beta_j = \bar{p}^\alpha_i \left(\frac{\partial \bar{x}^i}{\partial x^j} + \frac{\partial \bar{x}^i}{\partial u^\beta} p^\beta_j \right). \tag{1}$$

To obtain a collection of invariant 1-forms of the pseudo-group of contact transformations on $J^1(\mathcal{E})$, we apply Cartan's equivalence method [9, 20]. For this purpose we consider the coframe $\left\{\left(\vartheta^\alpha, \mathrm{d}x^i, \mathrm{d}p^\alpha_i\right) \,\middle|\, \alpha \in \{1, \ldots, q\}, i \in \{1, \ldots, n\}\right\}$ on $J^1(\mathcal{E})$. A contact transformation Δ acts on this coframe in the following manner:

$$\Delta^* \begin{pmatrix} \bar{\vartheta}^\alpha \\ \mathrm{d}\bar{x}^i \\ \mathrm{d}\bar{p}^\alpha_i \end{pmatrix} = S \begin{pmatrix} \vartheta^\alpha \\ \mathrm{d}x^i \\ \mathrm{d}p^\alpha_i \end{pmatrix}$$

where $S : J^1(\mathcal{E}) \to \mathcal{G}$ is an analytic function on $J^1(\mathcal{E})$, taking values in the Lie group $\mathcal{G}$ of non-degenerate block lower triangular matrices of the form

$$\begin{pmatrix} a^\alpha_\beta & 0 & 0 \\ C^i_\beta & b^i_j & 0 \\ F^\alpha_{i\beta} & G^\alpha_{ij} & h^{\alpha j}_{i\beta} \end{pmatrix}.$$

In accordance with Cartan's method of equivalence, we consider the lifted coframe on $J^1(\mathcal{E}) \times \mathcal{G}$

$$\Theta^\alpha = a^\alpha_\beta \vartheta^\beta \qquad \Xi^i = c^i_\beta \Theta^\beta + b^i_j\,\mathrm{d}x^j \qquad \Sigma^\alpha_i = f^\alpha_{i\beta}\Theta^\beta + g^\alpha_{ij}\,\Xi^j + h^{\alpha j}_{i\beta}\,\mathrm{d}p^\beta_j \tag{2}$$

where for convenience we use the notations $c^i_\beta = C^i_\gamma A^\gamma_\beta$, $f^\alpha_{i\beta} = F^\alpha_{i\gamma} A^\gamma_\beta - G^\alpha_{ij} B^j_k c^k_\beta$, $g^\alpha_{ij} = G^\alpha_{ik} B^k_j$; $\left(A^\beta_\gamma\right)$ is the inverse of the the matrix $\left(a^\alpha_\beta\right)$, $\left(B^j_k\right)$ is the inverse of the matrix $\left(b^i_j\right)$, so $a^\alpha_\beta A^\beta_\gamma = \delta^\alpha_\gamma$ and $b^i_j B^j_k = \delta^i_k$. To find an invariant coframe we use the procedure of absorption and normalization of essential torsion coefficients [20, ch 10].

Taking exterior differentials of 1-forms Θ^α and substituting the differentials $\mathrm{d}u^\beta$, $\mathrm{d}x^j$, $\mathrm{d}p^\beta_j$ expressed from equations (2), we obtain

$$\begin{aligned} \mathrm{d}\Theta^\alpha = {} & \left(\mathrm{d}a^\alpha_\beta A^\beta_\gamma + a^\alpha_\beta B^j_k H^{\beta s}_{j\eta} \left(c^k_\gamma \left(\Sigma^\eta_s - f^\eta_{s\epsilon}\Theta^\epsilon - g^\eta_{sl}\,\Xi^l\right) - f^\eta_{s\gamma}\,\Xi^s\right)\right) \wedge \Theta^\gamma \\ & + a^\alpha_\beta B^j_k H^{\beta s}_{j\eta}\,\Xi^k \wedge \Sigma^\eta_s - a^\alpha_\beta B^j_k H^{\beta s}_{j\eta} g^\eta_{sl}\,\Xi^k \wedge \Xi^l \end{aligned}$$

where the functions $H^{\beta k}_{j\gamma}$ are defined by the conditions $H^{\beta k}_{j\gamma} h^{\gamma i}_{k\alpha} = \delta^i_j \delta^\beta_\alpha$. The multipliers of $\Xi^k \wedge \Sigma^\eta_s$ and $\Xi^k \wedge \Xi^l$ are essential torsion coefficients. We normalize them by the following choice of the parameters of the Lie group $\mathcal{G}$:

$$h^{\alpha j}_{i\beta} = a^\alpha_\beta B^j_i \tag{3}$$
$$g^\alpha_{ij} = g^\alpha_{ji} \tag{4}$$

Then we have

$$\mathrm{d}\Theta^\alpha = \Phi^\alpha_\beta \wedge \Theta^\beta + \Xi^k \wedge \Sigma^\alpha_k \tag{5}$$
$$\Phi^\alpha_\beta = \mathrm{d}a^\alpha_\gamma A^\gamma_\beta + c^k_\gamma f^\alpha_{k\beta} \Theta^\gamma - f^\alpha_{k\beta} \Xi^k - c^k_\beta g^\alpha_{kj} \Xi^j + c^k_\beta \Sigma^\alpha_k. \tag{6}$$

Now the exterior differentials of Ξ^i and Σ^α_i become

$$\mathrm{d}\Xi^i = \Psi^i_k \wedge \Xi^k + \Pi^i_\gamma \wedge \Theta^\gamma \tag{7}$$
$$\mathrm{d}\Sigma^\alpha_i = \Phi^\alpha_\gamma \wedge \Sigma^\gamma_i - \Psi^k_i \wedge \Sigma^\alpha_k + \Lambda^\alpha_{i\beta} \wedge \Theta^\beta + \Omega^\alpha_{ij} \wedge \Xi^j \tag{8}$$

where

$$\Psi^i_k = \mathrm{d}b^i_j B^j_k - c^i_\beta \Sigma^\beta_k \tag{9}$$
$$\Pi^i_\gamma = \mathrm{d}c^i_\gamma + c^i_\beta \Phi^\beta_\gamma - c^k_\gamma \Psi^i_k - c^k_\gamma c^i_\beta \Sigma^\beta_k \tag{10}$$
$$\begin{aligned}\Lambda^\alpha_{i\beta} = {} & \mathrm{d}f^\alpha_{i\beta} + f^\alpha_{i\gamma} \Phi^\gamma_\beta + g^\alpha_{ij} \Pi^j_\beta - f^\gamma_{i\beta} \left(\Phi^\alpha_\gamma - c^k_\epsilon f^\alpha_{k\gamma} \Theta^\epsilon + f^\alpha_{k\gamma} \Xi^k + c^k_\gamma g^\alpha_{kj} \Xi^j - c^k_\gamma \Sigma^\alpha_k\right) \\ & + f^\alpha_{k\beta} \left(\Psi^k_i + c^k_\gamma \Sigma^\gamma_i\right) + c^k_\beta f^\alpha_{k\gamma} \Sigma^\gamma_i \end{aligned} \tag{11}$$
$$\begin{aligned}\Omega^\alpha_{ij} = {} & \mathrm{d}g^\alpha_{ij} + g^\alpha_{ik} \Psi^k_j + g^\alpha_{jk} \Psi^k_i - f^\alpha_{i\beta} \Sigma^\beta_j - f^\alpha_{j\beta} \Sigma^\beta_i - g^\gamma_{ij} \left(\Phi^\alpha_\gamma - c^k_\beta f^\alpha_{k\gamma} \Theta^\beta \right. \\ & \left. + f^\alpha_{k\gamma} \Xi^k + c^k_\gamma g^\alpha_{ks} \Xi^s - c^k_\gamma \Sigma^\alpha_k\right). \end{aligned} \tag{12}$$

We note that the conditions (4) imply

$$\Omega^\alpha_{ij} = \Omega^\alpha_{ji}. \tag{13}$$

Thus the specifications (3) and (4) of the group parameters of the coframe (2) give the lifted coframe

$$\Theta^\alpha = a^\alpha_\beta \left(\mathrm{d}u^\beta - p^\beta_j \, \mathrm{d}x^j\right) \tag{14}$$
$$\Xi^i = c^i_\beta \Theta^\beta + b^i_j \, \mathrm{d}x^j \tag{15}$$
$$\Sigma^\alpha_i = f^\alpha_{i\beta} \Theta^\beta + g^\alpha_{ij} \Xi^j + a^\alpha_\beta B^j_i \, \mathrm{d}p^\beta_j \tag{16}$$

on $J^1(\mathcal{E}) \times \mathcal{H}$, where $\mathcal{H}$ is the subgroup of the group $\mathcal{G}$ defined by the equalities (3) and (4). The structure equations (7) and (8) do not contain any torsion coefficient, while the structure equations (5) contain only constant torsion coefficients.

The structure equations (5), (7) and (8) remain unchanged if we make the following change of the modified Maurer–Cartan forms Φ^α_β, Ψ^i_k, Π^i_γ, $\Lambda^\alpha_{i\beta}$, Ω^α_{ij}:

$$\begin{aligned} \Phi^\alpha_\beta &\mapsto \Phi^\alpha_\beta + K^\alpha_{\beta\gamma} \Theta^\gamma \\ \Psi^i_k &\mapsto \Psi^i_k + L^i_{kj} \Xi^j + M^i_{k\gamma} \Theta^\gamma \\ \Pi^i_\gamma &\mapsto \Pi^i_\gamma + M^i_{k\gamma} \Xi^k + N^i_{\gamma\epsilon} \Theta^\epsilon \\ \Lambda^\alpha_{i\beta} &\mapsto \Lambda^\alpha_{i\beta} + P^\alpha_{i\beta\gamma} \Theta^\gamma + Q^\alpha_{i\beta k} \Xi^k + K^\alpha_{\gamma\beta} \Sigma^\gamma_i - M^k_{i\beta} \Sigma^\alpha_k \\ \Omega^\alpha_{ij} &\mapsto \Omega^\alpha_{ij} + Q^\alpha_{i\beta j} \Theta^\beta + R^\alpha_{ijk} \Xi^k - L^k_{ij} \Sigma^\alpha_k \end{aligned}$$

where $K^\alpha_{\gamma\epsilon}$, L^i_{kj}, $M^i_{k\gamma}$, $N^i_{\gamma\epsilon}$, $P^\alpha_{i\beta\gamma}$, $Q^\alpha_{i\beta k}$, R^α_{ijk} are arbitrary functions on $J^1(\mathcal{E}) \times \mathcal{H}$ satisfying the following symmetry conditions:

$$\begin{aligned} &K^\alpha_{\gamma\epsilon} = K^\alpha_{\epsilon\gamma} \qquad L^i_{kj} = L^i_{jk} \qquad N^i_{\gamma\epsilon} = N^i_{\epsilon\gamma} \\ &P^\alpha_{i\beta\gamma} = P^\alpha_{i\gamma\beta} \qquad Q^\alpha_{i\beta k} = Q^\alpha_{k\beta i} \qquad R^\alpha_{ijk} = R^\alpha_{ikj} = R^\alpha_{jik}. \end{aligned} \tag{17}$$

Their number

$$\begin{aligned} r^{(1)} = \tfrac{1}{2}q^2(q+1) + \tfrac{1}{2}n^2(n+1) + n^2q + \tfrac{1}{2}nq(q+1) + \tfrac{1}{2}nq^2(q+1) \\ + \tfrac{1}{2}nq^2(n+1) + \tfrac{1}{6}qn(n+1)(n+2) \end{aligned}$$

is the degree of indeterminancy, [20, definition 11.2], of the lifted coframe Θ^α, Ξ^i, Σ^α_i.

Using conditions (13), it is not hard to compute the reduced characters, [20, definition 11.4], of this coframe: $s'_1 = s'_2 = \cdots = s'_q = q + n + nq$, $s'_{q+1} = n + nq$, $s'_{q+2} = n + (n-1)q$, $s'_{q+2} = n + (n-2)q$, $\ldots$, $s'_{q+n-1} = n + 2q$, $s'_{q+n} = n + q$, $s'_{q+n+1} = s'_{q+n+2} = \cdots = s'_{q+n+nq} = 0$. It is easy to verify that the Cartan test

$$r^{(1)} = s'_1 + 2s'_2 + 3s'_3 + \cdots + (q+n+nq)s'_{q+n+nq}$$

is satisfied, so by definition 11.7 of [20] the lifted coframe (14), (15), (16) is involutive, and by theorem 11.16 of [20], since the last non-zero reduced character s'_{q+n} is equal to $q+n$, the transformations of the invariance pseudo-group of this coframe depend on $q+n$ functions of $q+n$ variables, as it should be. It is easy to directly verify that the transformation $\Upsilon\colon J^1(\mathcal{E}) \times \mathcal{H} \to J^1(\mathcal{E}) \times \mathcal{H}$ satisfies the conditions

$$\Upsilon^*\bar{\Theta}^\alpha = \Theta^\alpha \qquad \Upsilon^*\bar{\Xi}^i = \Xi^i \qquad \Upsilon^*\bar{\Sigma}^\alpha_i = \Sigma^\alpha_i \tag{18}$$

if and only if it is projectable on $J^1(\mathcal{E})$ and its projection $\Delta\colon J^1(\mathcal{E}) \to J^1(\mathcal{E})$, $\Delta\colon (x, u, p) \mapsto (\bar{x}, \bar{u}, \bar{p})$, is the prolongation of the transformation $\Gamma\colon \mathcal{E} \to \mathcal{E}$, $\Gamma\colon (x, u) \mapsto (\bar{x}, \bar{u})$, such that conditions (1) are satisfied. Thus the equalities (18) really define the pseudo-group of contact transformations on $J^1(\mathcal{E})$, when $q > 1$ and $n > 1$.

Since the forms Θ^α, Ξ^i, Σ^α_i are preserved by the pseudo-group transformations, their exterior differentials are also preserved, so $\Upsilon^*\,\mathrm{d}\bar{\Theta}^\alpha = \mathrm{d}\Theta^\alpha$, $\Upsilon^*\,\mathrm{d}\bar{\Xi}^i = \mathrm{d}\Xi^i$, $\Upsilon^*\,\mathrm{d}\bar{\Sigma}^\alpha_i = \mathrm{d}\Sigma^\alpha_i$; therefore we have

$$\begin{aligned} &\Upsilon^*\left(\bar{\Phi}^\alpha_\beta \wedge \bar{\Theta}^\beta + \bar{\Xi}^k \wedge \bar{\Sigma}^\alpha_k\right) = \left(\Upsilon^*\bar{\Phi}^\alpha_\beta\right) \wedge \Theta^\beta + \Xi^k \wedge \Sigma^\alpha_k = \Phi^\alpha_\beta \wedge \Theta^\beta + \Xi^k \wedge \Sigma^\alpha_k \\ &\Upsilon^*\left(\bar{\Psi}^i_k \wedge \bar{\Xi}^k + \bar{\Pi}^i_\gamma \wedge \bar{\Theta}^\gamma\right) = \left(\Upsilon^*\bar{\Psi}^i_k\right) \wedge \Xi^k + \left(\Upsilon^*\bar{\Pi}^i_\gamma\right) \wedge \Theta^\gamma = \Psi^i_k \wedge \Xi^k + \Pi^i_\gamma \wedge \Theta^\gamma \\ &\Upsilon^*\left(\bar{\Phi}^\alpha_\gamma \wedge \bar{\Sigma}^\gamma_i - \bar{\Psi}^k_i \wedge \bar{\Sigma}^\alpha_k + \bar{\Lambda}^\alpha_{i\beta} \wedge \bar{\Theta}^\beta + \bar{\Omega}^\alpha_{ij} \wedge \bar{\Xi}^j\right) \\ &\qquad = \left(\Upsilon^*\bar{\Phi}^\alpha_\gamma\right) \wedge \Sigma^\gamma_i - \left(\Upsilon^*\bar{\Psi}^k_i\right) \wedge \Sigma^\alpha_k + \left(\Upsilon^*\bar{\Lambda}^\alpha_{i\beta}\right) \wedge \Theta^\beta + \left(\Upsilon^*\bar{\Omega}^\alpha_{ij}\right) \wedge \Xi^j \\ &\qquad = \Phi^\alpha_\gamma \wedge \Sigma^\gamma_i - \Psi^k_i \wedge \Sigma^\alpha_k + \Lambda^\alpha_{i\beta} \wedge \Theta^\beta + \Omega^\alpha_{ij} \wedge \Xi^j \end{aligned}$$

and thus

$$\begin{aligned} \Upsilon^*\bar{\Phi}^\alpha_\beta &= \Phi^\alpha_\beta + K^\alpha_{\beta\gamma}\Theta^\gamma \\ \Upsilon^*\bar{\Psi}^i_k &= \Psi^i_k + L^i_{kj}\,\Xi^j + M^i_{k\gamma}\,\Theta^\gamma \\ \Upsilon^*\bar{\Pi}^i_\gamma &= \Pi^i_\gamma + M^i_{k\gamma}\,\Xi^k + N^i_{\gamma\epsilon}\Theta^\epsilon \\ \Upsilon^*\bar{\Lambda}^\alpha_{i\beta} &= \Lambda^\alpha_{i\beta} + P^\alpha_{i\beta\gamma}\Theta^\gamma + Q^\alpha_{i\beta k}\,\Xi^k + K^\alpha_{\gamma\beta}\Sigma^\gamma_i - M^k_{i\beta}\Sigma^\alpha_k \\ \Upsilon^*\bar{\Omega}^\alpha_{ij} &= \Omega^\alpha_{ij} + Q^\alpha_{i\beta j}\Theta^\beta + R^\alpha_{ijk}\,\Xi^k - L^k_{ij}\Sigma^\alpha_k \end{aligned} \tag{19}$$

for some functions $K^\alpha_{\gamma\epsilon}$, L^i_{kj}, $M^i_{k\gamma}$, $N^i_{\gamma\epsilon}$, $P^\alpha_{i\beta\gamma}$, $Q^\alpha_{i\beta k}$, R^α_{ijk} on $J^1(\mathcal{E}) \times \mathcal{H}$ satisfying conditions (17).

3. Symmetries of differential equations

For application of the moving coframe method, [10, 11], to the problem of finding symmetries of a system of DEs $\mathcal{R}_1$, we restrict the lifted coframe (14), (15), (16) on $\mathcal{R}_1$. That is, we consider the set of 1-forms $\theta^\alpha = \iota^*\Theta^\alpha$, $\xi^i = \iota^*\Xi^i$, $\sigma^\alpha_i = \iota^*\Sigma^\alpha_i$, where $\iota : \mathcal{R}_1 \to J^1(\mathcal{E})$ is the embedding (for brevity we identify the map $\iota \times \mathrm{id} : \mathcal{R}_1 \times \mathcal{H} \to J^1(\mathcal{E}) \times \mathcal{H}$ with $\iota: \mathcal{R}_1 \to J^1(\mathcal{E})$). The 1-forms θ^α, ξ^i, σ^α_i are linearly dependent, i.e. there exists a non-trivial set of functions U_α, V_i, W^i_α on $\mathcal{R}_1 \times \mathcal{H}$, such that $U_\alpha\theta^\alpha + V_i\xi^i + W^i_\alpha\sigma^\alpha_i \equiv 0$.

Setting these functions equal to some constants allows one to express a part of the parameters a^α_β, b^i_j, c^i_β, $f^\alpha_{i\beta}$, g^α_{ij} of the group $\mathcal{H}$ as functions of coordinates of $\mathcal{R}_1$ and other group parameters. Substitution of the obtained values of the parameters into the modified Maurer–Cartan forms $\phi^\alpha_\beta = \iota^*\Phi^\alpha_\beta$, $\psi^i_k = \iota^*\Psi^i_k$, $\pi^i_\beta = \iota^*\Pi^i_\beta$, $\lambda^\alpha_{i\beta} = \iota^*\Lambda^\alpha_{i\beta}$, $\omega^\alpha_{ij} = \iota^*\Omega^\alpha_{ij}$ makes a part of these forms, or their linear combinations, independent of all differentials of the group parameters. Since the transformation Υ^* changes the forms Φ^α_β, Ψ^i_k, Π^i_β by the rules (19), in the case when the obtained form ϕ^α_β does not depend on all differentials of the group parameters, its coefficients at σ^γ_j and ξ^j are lifted invariants of the pseudo-group, and if the obtained forms ψ^i_k or π^i_β are independent of all differentials of the group parameters, their coefficients at σ^γ_j are also lifted invariants. Normalizing these lifted invariants to be constants allows us to express a part of the group parameters as functions of coordinates on $\mathcal{R}_1$ and other group parameters. If not all group parameters are expressed, we should substitute the expressed parameters into the forms ϕ^α_β, ψ^i_k, π^i_γ, which depend on their differentials, and repeat the process. If the process is completed, but not all group parameters are expressed as functions on $\mathcal{R}_1$, we should substitute the modified Maurer–Cartan forms ϕ^α_β, ψ^i_k, π^i_γ, $\lambda^\alpha_{i\beta}$, ω^α_{ij}, which were reduced during the process of normalization, into the reduced structure equations

$$\mathrm{d}\theta^\alpha = \phi^\alpha_\beta \wedge \theta^\beta + \xi^k \wedge \sigma^\alpha_k$$
$$\mathrm{d}\xi^i = \psi^i_k \wedge \xi^k + \pi^i_\gamma \wedge \theta^\gamma$$
$$\mathrm{d}\sigma^\alpha_i = \phi^\alpha_\gamma \wedge \sigma^\gamma_i - \psi^k_i \wedge \sigma^\alpha_k + \lambda^\alpha_{i\beta} \wedge \theta^\beta + \omega^\alpha_{ij} \wedge \xi^j.$$

If the essential torsion coefficients dependent on the group parameters appear, then we should normalize them to constants and find some new part of the group parameters, which, on being substituted into the reduced modified Maurer–Cartan forms, allows us to repeat the procedure of normalization. There are two possible results of this process. The first result, when the reduced lifted coframe appears to be involutive, outputs the desired set of invariant 1-forms which characterize the pseudo-group Lie($\mathcal{R}_1$). In the second result, when the coframe is not involutive, we should apply the procedure of prolongation [20, ch 12].

3.1. Example 1: Burgers' equation

For the application of the above method to finding invariant 1-forms of the symmetry group admitted by Burgers' equation

$$u_t = u_{xx} + uu_x$$

we take the equivalent system of first order

$$u_x = v \qquad v_x = u_t - uv.$$

Denoting $x = x^1$, $t = x^2$, $v = u^1$, $u = u^2$, $v_x = p^1_1$, $v_t = p^1_2$, $u_x = p^2_1$, $u_t = p^2_2$, we consider this system as a sub-bundle of the bundle $J^1(\mathcal{E})$, $\mathcal{E} = \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, with local coordinates $\left\{x^1, x^2, u^1, u^2, p^1_1, p^1_2, p^2_1, p^2_2\right\}$, where the embedding ι is defined by the equalities

$$p^1_1 = p^2_2 - u^1u^2 \qquad p^2_1 = u^1.$$

The forms $\theta^\alpha = \iota^*\Theta^\alpha$, $\alpha \in \{1,2\}$, $\xi^i = \iota^*\Xi^i$, $i \in \{1,2\}$, are linearly independent, whereas the forms $\sigma_i^\alpha = \iota^*\Sigma_i^\alpha$ are linearly dependent. The group parameters a_β^α, b_j^i must satisfy the conditions $\det\big(a_\beta^\alpha\big) \neq 0$, $\det\big(b_j^i\big) \neq 0$. Moreover, without loss of generality, we can consider that $a_1^1 \neq 0$, $a_2^2 \neq 0$, $b_1^1 \neq 0$, $b_2^2 \neq 0$. Computing the linear dependence conditions of forms σ_i^α by means of MAPLE, we find sequentially the group parameters a_1^2, b_1^2, b_2^2, g_{12}^2, g_{11}^2, g_{11}^1, f_{12}^2, f_{11}^2, g_{22}^2, f_{22}^2, f_{211}^2 as functions of other group parameters and the local coordinates $\{x^1, x^2, u^1, u^2, p_2^1, p_2^2\}$ of $\mathcal{R}_1$. Particularly,

$$a_1^2 = 0 \qquad b_1^2 = 0 \qquad b_2^2 = \frac{b_1^1 a_2^2}{a_1^1} \qquad g_{12}^2 = -\frac{\left(-p_2^2 b_2^1 + u^1u^2 b_2^1 + p_2^1 b_1^1\right) a_1^1}{\left(b_1^1\right)^3}$$

$$g_{11}^2 = -\frac{a_2^2\left(p_2^2 - u^1u^2\right)}{\left(b_1^1\right)^2} \qquad g_{11}^1 = \frac{(u^1)^2 a_1^1 - a_2^1 p_2^2 - a_1^1 p_2^1 - u^1(u^2)^2 a_1^1 + p_2^2 a_1^1 u^2 + u^1u^2 a_2^1}{\left(b_1^1\right)^2}$$

$$f_{12}^2 = \frac{\left(b_1^1\right)^2 a_2^1 + p_2^1\left(a_1^1\right)^2 c_2^2 b_1^1 + u^1u^2\left(a_1^1\right)^2 b_2^1 c_2^2 - u^1u^2 a_2^2 c_2^1 b_1^1 a_1^1 - p_2^2\left(a_1^1\right)^2 b_2^1 c_2^2 + p_2^2 a_2^2 c_2^1 b_1^1 a_1^1}{\left(b_1^1\right)^3 a_1^1}$$

$$f_{11}^2 = -\frac{u^1u^2\left(c_1^1 a_2^2 b_1^1 a_1^1 - \left(a_1^1\right)^2 b_2^1 c_1^2\right) - p_2^2 c_1^1 a_2^2 b_1^1 a_1^1 + p_2^2\left(a_1^1\right)^2 b_2^1 c_1^2 - p_2^1\left(a_1^1\right)^2 c_1^2 b_1^1 + a_2^2\left(b_1^1\right)^2}{\left(b_1^1\right)^3 a_1^1}$$

while the expressions for g_{22}^2, f_{22}^2 and f_{21}^2 are too long to be written out in full here.

The linear dependences between the forms σ_i^α are $\sigma_1^1 = \sigma_2^2$ and $\sigma_1^2 = 0$.

The analysis of the modified Maurer—Cartan forms ϕ_β^α, ψ_k^i, π_γ^i at the obtained values of the group parameters gives the following normalizations:

$$\phi_1^2 \equiv c_1^2\sigma_2^2 + \frac{a_2^2}{b_1^1 a_1^1}\xi^1 \left(\operatorname{mod}\theta^1, \theta^2, \xi^2, \sigma_2^1\right) \Rightarrow c_1^2 = 0 \qquad b_1^1 = \frac{a_2^2}{a_1^1}$$

$$\psi_2^2 - 2\psi_1^1 = \left(2c_1^1 - c_2^2\right)\sigma_2^1 \Rightarrow c_2^2 = 2c_1^1$$

$$\psi_1^1 + \phi_1^1 - \phi_2^2 \equiv -2c_1^1\sigma_2^2 \left(\operatorname{mod}\theta^1, \theta^2, \xi^1, \xi^2, \sigma_2^1\right) \Rightarrow c_1^1 = 0$$

$$\phi_1^2 \equiv -\left(f_{11}^1 + \frac{a_2^1 a_2^2 - a_1^1 a_2^2 u^2 + b_2^1\left(a_1^1\right)^2}{a_2^2}\right)\xi^2 \left(\operatorname{mod}\theta^1, \theta^2, \xi^1, \sigma_2^1, \sigma_2^2\right)$$

$$\Rightarrow f_{11}^1 = -\frac{a_2^1 a_2^2 - a_1^1 a_2^2 u^2 + b_2^1\left(a_1^1\right)^2}{a_2^2}.$$

Now the analysis of the structure equations gives, step by step, the following essential torsion coefficients and the corresponding normalizations:

$$\mathrm{d}\theta^1 = -c_2^1\theta^2 \wedge \sigma_2^2 + \cdots \Rightarrow c_2^1 = 0$$

$$\mathrm{d}\theta^1 = \left(\left(a_2^2\right)^3 f_{12}^1 - \left(a_2^1\right)^2 a_2^2 + a_1^1 a_2^1 a_2^2 u^2 - \left(a_1^1\right)^2 a_2^1 b_2^1\right)\theta^2 \wedge \xi^1 + \left(f_{22}^1 + \frac{a_2^1}{a_2^2} f_{21}^1\right)\theta^2 \wedge \xi^2 + \cdots$$

$$\Rightarrow f_{12}^1 = \frac{a_2^1\left(a_2^1 a_2^2 - a_1^1 a_2^1 a_2^2 u^2 + \left(a_1^1\right)^2 a_2^1 b_2^1\right)}{\left(a_2^2\right)^3} \qquad f_{22}^1 = -\frac{a_2^1}{a_2^2} f_{21}^1$$

$$\mathrm{d}\xi^2 = \frac{2\left(2a_2^1 a_2^2 - a_1^1 a_2^2 u^2 + b_2^1\left(a_1^1\right)^2\right)}{\left(a_2^2\right)^2}\xi^1 \wedge \xi^2 + \cdots \Rightarrow a_2^1 = \frac{a_1^1\left(a_2^2 u^2 - b_2^1 a_1^1\right)}{2a_2^2}$$

$$\mathrm{d}\xi^1 = \left(f^2_{21} + \frac{\left(a^1_1\right)^2 \left(4\left(a^2_2\right)^2 u^1 - 2a^2_2 b^1_2 a^1_1 u^2 + \left(a^2_2\right)^2 (u^2)^2 + \left(b^1_2\right)^2 \left(a^1_1\right)^2\right)}{\left(a^2_2\right)^4} \right) \xi^1 \wedge \xi^2 + \cdots$$

$$\Rightarrow f^2_{21} = -\frac{\left(a^1_1\right)^2 \left(4\left(a^2_2\right)^2 u^1 - 2a^2_2 b^1_2 a^1_1 u^2 + \left(a^2_2\right)^2 (u^2)^2 + \left(b^1_2\right)^2 \left(a^1_1\right)^2\right)}{\left(a^2_2\right)^4}$$

$$\mathrm{d}\sigma^1_2 = -\frac{\left(a^1_1\right)^2 \left(b^1_2 a^1_1 - a^2_2 u^2\right)}{\left(a^2_2\right)^4} \theta^1 \wedge \theta^2 + \cdots \Rightarrow b^1_2 = \frac{a^2_2 u^2}{a^1_1}$$

$$\mathrm{d}\sigma^2_2 = \frac{\left(a^1_1\right)^3 \left(p^2_2 - u^1 u^2\right)}{\left(a^2_2\right)^3} \theta^2 \wedge \xi^1 + \cdots \Rightarrow a^2_2 = \frac{a^1_1}{\left(p^2_2 - u^1 u^2\right)^{1/3}}$$

$$\mathrm{d}\theta^2 = \frac{1}{3a^1_1 \left(p^2_2 - u^1 u^2\right)^{2/3}} \theta^2 \wedge \sigma^2_2 + \cdots \Rightarrow a^1_1 = \frac{1}{\left(p^2_2 - u^1 u^2\right)^{2/3}}$$

$$\mathrm{d}\theta^1 = -\left(\frac{2g^1_{12}}{3} + \frac{2u^1}{\left(p^2_2 - u^1 u^2\right)^{2/3}} \right) \theta^1 \wedge \xi_2 + \cdots \Rightarrow g^1_{12} = -\frac{3u^1}{\left(p^2_2 - u^1 u^2\right)^{2/3}}$$

$$\mathrm{d}\sigma^2_2 = \left(-g^1_{22} + \frac{2\left(-2\left(p^2_2\right)^2 + 7u^1 u^2 p^2_2 - 5(u^1 u^2)^2 + 2(u^1)^3 - 3u^1 p^1_2\right)}{\left(p^2_2 - u^1 u^2\right)^2} \right) \xi^1 \wedge \xi_2 + \cdots$$

$$\Rightarrow g^1_{22} = \frac{2\left(-2\left(p^2_2\right)^2 + 7u^1 u^2 p^2_2 - 5(u^1 u^2)^2 + 2(u^1)^3 - 3u^1 p^1_2\right)}{\left(p^2_2 - u^1 u^2\right)^2}.$$

Thus all the group parameters are expressed as functions of the local coordinates $\left\{x^1, x^2, u^1, u^2, p^1_2, p^2_2\right\}$ of the equation $\mathcal{R}_1$. The result of all normalizations is the invariant coframe

$$\theta^1 = \frac{\mathrm{d}u^1 - \left(p^2_2 - u^1 u^2\right) \mathrm{d}x^1 - p^1_2 \, \mathrm{d}x^2}{\left(p^2_2 - u^1 u^2\right)^{2/3}}$$

$$\theta^2 = \frac{\mathrm{d}u^2 - u^1 \, \mathrm{d}x^1 - p^2_2 \, \mathrm{d}x^2}{\left(p^2_2 - u^1 u^2\right)^{1/3}}$$

$$\xi^1 = \left(p^2_2 - u^1 u^2\right)^{1/3} (\mathrm{d}x^1 + u^2 \, \mathrm{d}x^2)$$

$$\xi^2 = \left(p^2_2 - u^1 u^2\right)^{2/3} \, \mathrm{d}x^2$$

$$\sigma^1_2 = \frac{\mathrm{d}p^1_2 - u^2 \, \mathrm{d}p^2_2 + ((u^2)^2 - 2u^1) \, \mathrm{d}u^1 + u^1 u^2 \, \mathrm{d}u_2}{\left(p^2_2 - u^1 u^2\right)^{4/3}} + \frac{u^1 \left(p^2_2 - u^1 u^2\right) \mathrm{d}x^1 + \left(4(u^1)^3 - 7(u^1 u^2)^2 + 11 u^1 u^2 p^2_2 - 4u^1 p^1_2 - 4\left(p^2_2\right)^2\right) \mathrm{d}x_2}{\left(p^2_2 - u^1 u^2\right)^{4/3}}$$

$$\sigma^2_2 = \frac{\mathrm{d}p^2_2 - u^2 \, \mathrm{d}u^1 - u^1 \, \mathrm{d}u^2 - \left(p^1_2 + u^1 (u^2)^2 - (u^1)^2 - u^2 p^2_2\right) \mathrm{d}x^1}{p^2_2 - u^1 u^2} + \frac{\left(4(u^1)^2 u^2 + (u^2)^2 p^2_2 - u^1 (u^2)^3 - u^2 p - 3u^1 p^2_2\right) \mathrm{d}x^2}{p^2_2 - u^1 u^2}.$$

Its structure equations are

$$\mathrm{d}\theta^1 = I\theta^1 \wedge \xi^1 + \tfrac{2}{3}\theta^1 \wedge \sigma_2^2 + \xi^1 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^1$$
$$\mathrm{d}\theta^2 = -\theta^1 \wedge \xi^1 + \tfrac{1}{2}I\theta^2 \wedge \xi^1 + \tfrac{1}{3}\theta^2 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^2$$
$$\mathrm{d}\xi^1 = \theta^2 \wedge \xi^2 - \tfrac{1}{3}\xi^1 \wedge \sigma_2^2$$
$$\mathrm{d}\xi^2 = I\xi^1 \wedge \xi^2 - \tfrac{2}{3}\xi^2 \wedge \sigma_2^2$$
$$\mathrm{d}\sigma_2^1 = -\theta^1 \wedge \xi^1 - 6I\theta^1 \wedge \xi^2 - \tfrac{3}{2}I\theta^2 \wedge \xi^1 - \theta^2 \wedge \sigma_2^2 - 15I\xi^1 \wedge \xi^2 - 2I\xi^1 \wedge \sigma_2^1 + 7\xi^2 \wedge \sigma_2^2 + \tfrac{4}{3}\sigma_2^1 \wedge \sigma_2^2$$
$$\mathrm{d}\sigma_2^2 = -3\theta^1 \wedge \xi^2 + \theta^2 \wedge \xi^1 - \tfrac{3}{2}I\theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma_2^1 - \tfrac{3}{2}I\xi^1 \wedge \sigma_2^2$$

where the only invariant I has the form

$$I = \frac{2\big(p_2^1 + u^1(u^2)^2 - (u^1)^2 - u^2 p_2^2\big)}{3\big(p_2^2 - u^1u^2\big)^{4/3}}.$$

Taking its exterior differential, we obtain

$$\mathrm{d}I = -\tfrac{2}{3}\theta^2 - 2I^2\xi^1 + 2\xi^2 + \tfrac{2}{3}\sigma_2^1 - \tfrac{4}{3}I\sigma_2^2$$

so all derived invariants of the group are functionally expressed as functions of I. Therefore the rank of the coframe, [20, proposition 8.18], is equal to 1, and, by theorem 8.22 of [20], its symmetry group is five-dimensional (as it should be; for full details of finding infinitesimal generators of this group by Lie's method see, e.g., [30, ch 3, section 5].)

3.2. *Example 2: one-dimensional equations of gas dynamics in Lagrange coordinates*

One-dimensional dynamics of polytropic gas in Lagrange coordinates is described [26], by the system of DEs

$$\rho_t + \rho^2 u_m = 0 \qquad u_t + p_m = 0 \qquad p_t + \gamma\rho p u_m = 0. \tag{20}$$

Denoting $\rho = u^1$, $u = u^2$, $p = u^3$, $t = x^1$, $m = x^2$ and using the above method, we obtain the invariant coframe of the symmetry group of the system (20)

$$\theta^1 = \frac{1}{u^1}\left(\mathrm{d}u^1 + (u^1)^2 p_2^2\,\mathrm{d}x^1 - p_2^1\,\mathrm{d}x^2\right)$$
$$\theta^2 = \sqrt{\frac{u^1}{\gamma u^3}}\left(\mathrm{d}u^2 + p_2^3\,\mathrm{d}x^1 - p_2^2\,\mathrm{d}x^2\right)$$
$$\theta^3 = \frac{1}{\gamma u^3}\left(\mathrm{d}u^3 + \gamma u^1 u^3 p_2^2\,\mathrm{d}x^1 - p_2^3\,\mathrm{d}x^2\right)$$
$$\xi^1 = \sqrt{\frac{u^1}{\gamma u^3}}\,\mathrm{d}x^2$$
$$\xi^2 = u^1 p_2^2\,\mathrm{d}x^1 \tag{21}$$
$$\sigma_2^1 = \frac{1}{u^1 p_2^2}\sqrt{\frac{\gamma u^3}{u^1}}\left(\mathrm{d}p_2^1 - \frac{p_2^1}{u^1}\,\mathrm{d}u^1 - \frac{(\gamma-1)(u^1)^3\left(p_2^2\right)^2 u^3 - \left(p_2^1\right)^2(u^3)^2 - \left(p_2^3\right)^2(u^1)^2}{2u^1(u^3)^2}\,\mathrm{d}x^2\right)$$
$$\sigma_2^2 = \frac{1}{u^1 p_2^2}\left(\mathrm{d}p_2^2 + \frac{\gamma-1}{2}(u^1)^2\left(p_2^2\right)^2\,\mathrm{d}x^1 + p_2^1 p_2^2\,\mathrm{d}x^2\right)$$
$$\sigma_2^3 = \frac{1}{p_2^2\sqrt{\gamma u^1 u^3}}\left(\mathrm{d}p_2^3 + \gamma u^1 p_2^2 p_2^3\,\mathrm{d}x^1 - \frac{\gamma-1}{2}u^1\left(p_2^2\right)^2\mathrm{d}x^2\right)$$

(since from the physical meaning we have $u^1 = \rho > 0$ and $u^3 = p > 0$; therefore there is no need to worry about the signs of the expressions under the square roots).

The structure equations of this coframe are

$$\begin{aligned}
d\theta^1 &= \theta^1 \wedge \xi^2 + \xi^1 \wedge \sigma_2^1 - \xi^2 \wedge \sigma_2^2 \\
d\theta^2 &= \tfrac{1}{2}\theta^1 \wedge \theta^2 + \tfrac{\gamma}{2}\theta^2 \wedge \theta^3 + I_1\theta^2 \wedge \xi^1 + \tfrac{\gamma-1}{2}\theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma_2^2 - \xi^2 \wedge \sigma_2^3 \\
d\theta^3 &= \theta^1 \wedge \xi^2 + I_2\theta^3 \wedge \xi^1 + \xi^1 \wedge \sigma_2^3 - \xi^2 \wedge \sigma_2^2 \\
d\xi^1 &= \tfrac{1}{2}\theta^1 \wedge \xi^1 - \tfrac{\gamma}{2}\theta^3 \wedge \xi^1 - \xi^1 \wedge \sigma_2^2 \\
d\xi^2 &= \theta^1 \wedge \xi^2 - \xi^2 \wedge \sigma_2^2 \\
d\sigma_2^1 &= \tfrac{1}{2}\gamma(\gamma-1)\theta^1 \wedge \xi^1 - \tfrac{1}{2}\theta^1 \wedge \sigma_2^1 - \tfrac{1}{2}\left(2I_2^2 - \gamma^2 + \gamma\right)\theta^3 \wedge \xi^1 + \tfrac{\gamma}{2}\theta^3 \wedge \sigma_2^1 \\
&\quad + I_1\xi^1 \wedge \sigma_2^1 + \gamma(\gamma-1)\xi^1 \wedge \sigma_2^2 - \gamma I_2\xi^1 \wedge \sigma_2^3 + \sigma_2^1 \wedge \sigma_2^2 \\
d\sigma_2^2 &= \tfrac{\gamma-1}{2}\theta^1 \wedge \xi^2 - \xi^1 \wedge \sigma_2^1 - \tfrac{\gamma-1}{2}\xi^2 \wedge \sigma_2^2 \\
d\sigma_2^3 &= -\tfrac{\gamma-1}{2}\theta^1 \wedge \xi^1 + I_2\theta^1 \wedge \xi^2 - \tfrac{1}{2}\theta^1 \wedge \sigma_2^3 - \tfrac{\gamma}{2}\theta^3 \wedge \sigma_2^3 + (\gamma-1)\xi^1 \wedge \sigma_2^2 \\
&\quad - I_1\xi^1 \wedge \sigma_2^3 - I_2\xi^2 \wedge \sigma_2^2 - \sigma_2^2 \wedge \sigma_2^3.
\end{aligned}$$

The invariants I_1 and I_2 are defined by the equalities

$$I_1 = \sqrt{\frac{\gamma u^1}{u^3}}\,\frac{p_2^3 u^1 - p_2^1 u^3}{2(u^1)^2 p_2^2} \qquad I_2 = \sqrt{\frac{\gamma}{u^1 u^3}}\,\frac{p_2^3}{p_2^2}.$$

Their exterior differentials are

$$\begin{aligned}
dI_1 &= -\frac{I_1}{2}\theta^1 + \frac{\gamma}{2}(I_1 - I_2)\theta^3 + \frac{1}{2}\sigma_2^1 - I_1\sigma_2^2 + \frac{\gamma}{2}\sigma_2^3 \\
dI_2 &= -\frac{I_2}{2}\theta^1 + \gamma\left(I_1 - \frac{I_2}{2}\right)\theta^2 + \left(\frac{\gamma(\gamma-1)}{2} - I_1 I_2\right)\xi^1 - I_2\sigma_2^1 + \gamma\sigma_2^3
\end{aligned}$$

so all derived invariants of the symmetry group depend functionally on I_1 and I_2. Thus the coframe (21) has rank 2, and the symmetry group of the system (20) is six-dimensional. In [2, ch 3] the explicit form of the infinitesimal generators of this group is given.

3.3. Example 3: Liouville's equation

For finding invariant 1-forms and structure equations of the symmetry pseudo-group of Liouville's equation

$$u_{tx} = e^u$$

we take the equivalent system of first order

$$u_t = v \qquad v_x = e^u.$$

Using the notations $u = u^1$, $v = u^2$, $t = x^1$, $x = x^2$ and applying the above procedure of absorption and normalization, we have $\sigma_1^1 = 0$, $\sigma_2^2 = 0$, while θ^1, θ^2, ξ^1, ξ^2, σ_2^1 and σ_1^2 constitute the lifted coframe

$$\begin{aligned}
\theta^1 &= du^1 - u^2\,dx^1 - p_2^1\,dx^2 \\
\theta^2 &= a_2^2\left(du^2 - p_1^2\,dx^1 - \exp(u^1)\,dx^2\right) \\
\xi^1 &= \left(a_2^2\right)^{-1} dx^1
\end{aligned} \tag{22}$$

$$
\begin{aligned}
\xi^2 &= a_2^2 \exp(u^1)\,\mathrm{d}x^2 \\
\sigma_2^1 &= \left(a_2^2\right)^{-1} \exp(-u^1)\,\mathrm{d}p_2^1 - \left(a_2^2\right)^{-1} \mathrm{d}x^1 + a_2^2 g_{22}^1 \exp(u^1)\,\mathrm{d}x^2 \\
\sigma_1^2 &= \left(a_2^2\right)^2 \left(\mathrm{d}p_1^2 - u^2\,\mathrm{d}u^2 + \left(\left(a_2^2\right)^{-3} g_{11}^2 + u^2 p_1^2\right)\mathrm{d}x^1\right).
\end{aligned}
$$

The exterior differentials of these forms are

$$
\begin{aligned}
\mathrm{d}\theta^1 &= -\theta^2 \wedge \xi^1 + \xi^1 \wedge \sigma_2^1 \\
\mathrm{d}\theta^2 &= \chi_1 \wedge \theta^2 - \theta^1 \wedge \xi^2 + \xi^1 \wedge \sigma_1^2 \\
\mathrm{d}\xi^1 &= -\chi_1 \wedge \xi^1 \\
\mathrm{d}\xi^2 &= \chi_1 \wedge \xi^2 + \theta^1 \wedge \xi^2 \\
\mathrm{d}\sigma_2^1 &= \chi_2 \wedge \xi^2 - \chi_1 \wedge \sigma_2^1 - \theta^1 \wedge \left(\sigma_2^1 + \xi^1\right) \\
\mathrm{d}\sigma_1^2 &= \chi_3 \wedge \xi^1 + 2\chi_1 \wedge \sigma_1^2
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
\chi_1 &= \left(a_2^2\right)^{-1} \mathrm{d}a_2^2 + a_2^2 u^2 \xi^1 \\
\chi_2 &= \mathrm{d}g_{22}^1 + 2g_{22}^1(\chi_1 + \theta^1) + \left(a_2^2\right)^{-1} \exp(-u^1) p_2^1 \left(\xi^1 - \sigma_2^1\right) + w_1 \xi^2 \\
\chi_3 &= \mathrm{d}g_{11}^2 - 3g_{11}^2 \chi_1 + \left(a_2^2\right)^2 \left(p_1^2 + (u^2)^2\right)(\theta^2 + \xi^2) + 3a_2^2 u^2 \sigma_1^2 + w_2 \xi^1
\end{aligned}
\tag{24}
$$

w_1 and w_2 are free parameters. The structure equations (23) do not contain any torsion coefficient depending on the group parameters. The coframe (22) is not involutive, because its degree of indeterminancy $r^{(1)}$ is 2, whereas the reduced characters are $s_1' = 3$, $s_2' = \cdots = s_6' = 0$, so Cartan's test is not satisfied. Therefore we should use the procedure of prolongation [20, ch 12]. For this purpose we unite both coframes (22) and (24) into the new base coframe, whereas w_1 and w_2 turn into the new group parameters. Finding exterior differentials of χ_1, χ_2 and χ_3, we have

$$
\begin{aligned}
\mathrm{d}\chi_1 &= \theta^2 \wedge \xi^1 - \xi^1 \wedge \xi^2 \\
\mathrm{d}\chi_2 &= \nu_1 \wedge \xi^2 - 2\theta^1 \wedge \chi_1 - 2\chi_1 \wedge \chi_2 \\
\mathrm{d}\chi_3 &= \nu_2 \wedge \xi^1 + 2(\theta^2 + \xi^2) \wedge \sigma_1^2 + 3\chi_1 \wedge \chi_2
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
\nu_1 = {} & \mathrm{d}w_1 + 3w_1\left(\theta^1 + \chi_1\right) + \left(\left(a_2^2\right)^{-1} \exp(-2u^1)\left(p_2^1\right)^2 - g_{22}^1\right)\left(\xi^1 + \sigma_2^1\right) \\
& - \left(a_2^2\right)^{-1} \exp(-u^1) p_2^1 \chi_2 \\
\nu_2 = {} & \mathrm{d}w_2 + 4w_2\chi_2 + 2\left(\left(a_2^2\right)^3 (u^2)^3 - g_{11}^2\right)(\theta^2 + \xi^2) + 2\left(a_2^2\right)^2\left((u^2)^2 - 2p_1^2\right)\sigma_1^2 + 3a_2^2 u^2 \chi_3.
\end{aligned}
$$

The structure equations (25) admit the change

$$
\nu_1 \mapsto \nu_1 + z_1 \xi^2 \qquad \nu_2 \mapsto \nu_2 + z_2 \xi^1
$$

with the free parameters z_1 and z_2. So the degree of indeterminancy of the coframe (22), (24) is $r^{(1)} = 2$ again, while the reduced characters now are $s_1' = 2$, $s_2' = \cdots = s_9' = 0$. Cartan's test is therefore satisfied, and the coframe (22), (24) is involutive. Since the last non-zero reduced character is $s_1' = 2$, the symmetry pseudo-group transformations depend on two arbitrary functions of one variable. This agrees with the result found by Lie [15, Bd. 3, S. 469–478]. In [16, 17] the structure equations of this pseudo-group are obtained using a different method; see also [25].

4. Conclusion

We have demonstrated the possibility of applying the combination of Cartan's equivalence method and the moving coframe method to computing invariant 1-forms and structure equations of symmetry pseudo-groups of DEs. The approach used here does not require finding infinitesimal defining systems, analysis of their involutivity and integration and includes only differentiation and linear algebra operations. So it is algorithmic *in principle*, although the labyrinth of the corresponding computations is very intricate. Further work should be done to reduce the complexity of computations by means of using invariant 1-forms which define contact transformations on bundles of higher order jets, see [21].

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