Symmetries of Differential Equations via Cartan's Method of Equivalence

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Abstract. We formulate a method of computing invariant 1-forms and structure equations of symmetry pseudo-groups of differential equations based on Cartan's method of equivalence and the moving coframe method introduced by Fels and Olver. Our apparoach does not require a preliminary computation of infinitesimal defining systems, their analysis and integration, and uses differentiation and linear algebra operations only. Examples of its applications are given.

1. Introduction

The theory of symmetries of differential equations was created by Sophus Lie more than a hundred years ago. One of Lie's greatest contributions was the discovery of the connection between continuous transformation groups and their infinitesimal generators, which allows one to reduce complicated nonlinear invariance conditions of d.e.s under an action of a transformation group to much simple linear conditions of infinitesimal invariance – defining equations of symmetry algebra. Lie's method turned out to be a powerful tool for studying differential equations, finding their exact solutions, conservation laws, etc. [14, 1, 4, 12, 18, 19, 29, 13, 30]. In almost all cases the infinitesimal defining equations of the Lie pseudo-groups of symmetries of d.e.s can be derived algorithmically. Lie's method requires an integration of (over-determined) system of partial differential equations to find a symmetry group of d.e.s explicitly. In the last decade methods which do not use an integration but rather extract information about structure of symmetry groups directly from their infinitesimal defining systems were developed [23, 24, 27, 28]. It was shown how to obtain the dimension of the finite Lie group, and in [23, 24] it was also shown how to find the structure constants c^i_{jk} of the symmetry algebra in the finite-dimensional case. In [16, 17] the method of [23, 24] was generalized to the case of structurally transitive infinite Lie pseudo-groups. Specifically, it was shown how to obtain the Cartan structure equations of the symmetry pseudo-group for a system of d.e.s from its infinitesimal defining equations.

The theory of infinite Lie pseudo-groups was created by Élie Cartan [5] – [9]. It does not use infinitesimal methods and is based on the possibility to characterize an infinite

Lie pseudo-group on a manifold M as the set of projections of bundle transformations of a principal fiber bundle $M \times \mathcal{G} \to M$, where $\mathcal{G}$ is some Lie group, that preserve a collection of 1-forms τ^i on $M \times \mathcal{G}$. The equations that express the differentials $d\tau^i$ through the τ^i and modified Maurer - Cartan forms μ^α of the group $\mathcal{G}$,

$$d\tau^i = A^i_{\alpha j}\, \mu^\alpha \wedge \tau^j + T^i_{jk}\, \tau^j \wedge \tau^k,$$

are called Cartan structure equations; they include important information about the pseudo-group (see, particularly, [20, Theorem 11.16]).

In the present paper we apply Cartan's method of equivalence [9, 20] and the moving coframe method of [10, 11] to obtain invariant 1-forms of a symmetry pseudo-group of d.e.s. Unlike the approach of [16, 17], the method used here does not require a preliminary computation of infinitesimal defining systems and their reduction to the involutive form.

A system $\mathcal{R}_s$ of differential equations of order s in n independent variables and m dependent variables is locally considered to be the subbundle in the bundle $J^s(\mathcal{E})$ of s-jets of the bundle $\mathcal{E} = \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. A pseudo-group of symmetries $Sym(\mathcal{R}_s)$ of the system $\mathcal{R}_s$ is a subgroup of the pseudo-group of contact transformations of the bundle $J^s(\mathcal{E})$ and consists of those transformations which preserve the subbundle $\mathcal{R}_s$. So the problem of finding the group $Sym(\mathcal{R}_s)$ is a particular case of the general problem of equivalence of embedded submanifolds under an action of a pseudo-group. A powerful and convenient moving coframe method for solving this equivalence problem was developed in [10, 11].

Some simplifications are possible if we deal with the first order systems of d.e.s. By [22, Theorem 3.3.1.] a system $\mathcal{R}_s$ is equivalent to the system $\hat{\mathcal{R}}_1$ of the first order, which is the subbundle in $J^1(\hat{\mathcal{E}})$, where $\hat{\mathcal{E}} = J^{s-1}(\mathcal{E})$. The pseudo-group $Sym(\hat{\mathcal{R}}_1)$ of symmetries of the system $\hat{\mathcal{R}}_1$ is a subgroup of the pseudo-group $Cont(J^1(\hat{\mathcal{E}}))$ of contact transformations of the bundle $J^1(\hat{\mathcal{E}})$. By Bäcklund's theorem, [3], [20, Theorem 4.32], contact transformations on $J^1(\hat{\mathcal{E}})$ are prolongations of point transformations on $\hat{\mathcal{E}}$. Cartan's method of equivalence allows us to obtain invariant 1-forms which define the pseudo-group of contact transformations. Then we can find the invariant 1-forms of the pseudo-group $Sym(\hat{\mathcal{R}}_1)$. To do that, we should make the following steps. First, we restrict the invariant 1-forms of the pseudo-group $Cont(J^1(\hat{\mathcal{E}}))$ on the subbundle $\hat{\mathcal{R}}_1$ and obtain the set of linear dependent 1-forms. Next, we apply the procedure of normalization to the appearing conditions of linear dependence. Finally, we apply the operations of Cartan's equivalence method to the restrictions on $\hat{\mathcal{R}}_1$ of the structure equations of the pseudo-group $Cont(J^1(\hat{\mathcal{E}}))$.

2. Invariant 1-forms and structure equations of the pseudo-group of contact transformations

According to [22, Theorem 3.3.1.] a compatible system $\mathcal{R}_s$ of d.e.s of order s is equivalent to the system $\hat{\mathcal{R}}_1$ of order 1, which has more dependent variables. So it is possible to restrict our attention to the case of $s = 1$.

Let $\mathcal{R}_1$ be a system of partial differential equations of the first order, considered to be the subbundle in the bundle $J^1(\mathcal{E})$ of 1-jets of the bundle $\mathcal{E} \to X$ over an n-dimensional base manifold X, with q-dimensional fibers. Let $(x^1, x^2, ..., x^n)$ denote local coordinates of the base X and $(u^1, u^2, ..., u^q)$ denote local coordinates of the fibers of $\mathcal{E}$. Then local coordinates of the bundle $J^1(\mathcal{E})$ are $(x^1, ..., x^n, u^1, ..., u^q, p^1_1, ..., p^1_n, ..., p^q_1, ..., p^q_n)$, and a local section $f : X \to \mathcal{E}$ defined by the equalities $u^\alpha = f^\alpha(x)$, $\alpha \in \{1, ..., q\}$, has corresponding 1-jet $j_1(f) : X \to J^1(\mathcal{E})$, defined by the equalities $u^\alpha = f^\alpha(x)$, $p^\alpha_i = \frac{\partial f^\alpha(x)}{\partial x^i}$, $\alpha \in \{1, ..., q\}$, $i \in \{1, ..., n\}$.

A differential form τ on $J^1(\mathcal{E})$ is called a *contact form* if it is annihilated by all 1-jets: $j_1(f)^*\tau = 0$. In local coordinates every contact 1-form is a linear combination of the Cartan forms $\tau^\alpha = du^\alpha - p^\alpha_i \, dx^i$, $\alpha \in \{1, ..., q\}$ (here and below we use Einstein summation convention, so $p^\alpha_i \, dx^i = \sum\limits_{i=1}^{n} p^\alpha_i \, dx^i$ etc.)

A local diffeomorphism $\Delta : J^1(\mathcal{E}) \to J^1(\mathcal{E})$, $\Delta : (x, u, p) \mapsto (\overline{x}, \overline{u}, \overline{p})$, is called a *contact transformation*, if for every contact form τ, the form $\Delta^*\tau$ is also a contact form; in other words, if $\Delta^*\tau^\alpha = d\overline{u}^\alpha - \overline{p}^\alpha_i \, d\overline{x}^i = \zeta^\alpha_\beta(x, u, p) \, \tau^\beta$ for some functions ζ^α_β on $J^1(\mathcal{E})$.

By Bäcklund's theorem, [3], [20, Theorem 4.32], in the case of $n > 1$ and $q > 1$ every contact transformation $\Delta : J^1(\mathcal{E}) \to J^1(\mathcal{E})$ is a prolongation of a point transformation $\Gamma : \mathcal{E} \to \mathcal{E}$, $\Gamma : (x, u) \mapsto (\overline{x}, \overline{u})$, where the functions $\overline{p}^\alpha_i$ are defined by the equalities

$$\frac{\partial \overline{u}^\alpha}{\partial x^j} + \frac{\partial \overline{u}^\alpha}{\partial u^\beta} \, p^\beta_j = \overline{p}^\alpha_i \left(\frac{\partial \overline{x}^i}{\partial x^j} + \frac{\partial \overline{x}^i}{\partial u^\beta} \, p^\beta_j \right). \tag{1}$$

To obtain a collection of invariant 1-forms of the pseudo-group of contact transformations on $J^1(\mathcal{E})$ we apply Cartan's equivalence method [9, 20]. For this purpose we consider the coframe $\{ (\tau^\alpha, dx^i, dp^\alpha_i) \, | \, \alpha \in \{1, ..., q\}, i \in \{1, ..., n\} \}$ on $J^1(\mathcal{E})$. A contact transformation Δ acts on this coframe in the following manner:

$$\Delta^* \begin{pmatrix} \tau^\alpha \\ dx^i \\ dp^\alpha_i \end{pmatrix} = S \begin{pmatrix} \tau^\alpha \\ dx^i \\ dp^\alpha_i \end{pmatrix},$$

where $S : J^1(\mathcal{E}) \to \mathcal{G}$ is an analytic function on $J^1(\mathcal{E})$ taking values in the Lie group $\mathcal{G}$ of non-degenerate block lower triangular matrices of the form

$$\begin{pmatrix} a^\alpha_\beta & 0 & 0 \\ C^i_\beta & b^i_j & 0 \\ F^\alpha_{i\beta} & G^\alpha_{ij} & h^{\alpha j}_{i\beta} \end{pmatrix}.$$

In accordance with Cartan's method of equivalence, we consider the lifted coframe on $J^1(\mathcal{E}) \times \mathcal{G}$

$$\begin{aligned} &\Theta^\alpha = a^\alpha_\beta \, \tau^\beta, \\ &\Xi^i = c^i_\beta \, \Theta^\beta + b^i_j \, dx^j, \\ &\Sigma^\alpha_i = f^\alpha_{i\beta} \, \Theta^\beta + g^\alpha_{ij} \, \Xi^j + h^{\alpha j}_{i\beta} \, dp^\beta_j, \end{aligned} \tag{2}$$

where for convenience we use the notations $c^i_\beta = C^i_\gamma \, A^\gamma_\beta$, $f^\alpha_{i\beta} = F^\alpha_{i\gamma} \, A^\gamma_\beta - G^\alpha_{ij} \, B^j_k \, c^k_\beta$, $g^\alpha_{ij} = G^\alpha_{ik} \, B^k_j$; (A^β_γ) is the inverse matrix of the matrix (a^α_β), (B^j_k) is the inverse matrix of the matrix (b^i_j), so $a^\alpha_\beta \, A^\beta_\gamma = \delta^\alpha_\gamma$ and $b^i_j \, B^j_k = \delta^i_k$. To find an invariant coframe we use the procedure of absorption and normalization of essential torsion coefficients [20, Chapter 10].

Taking exterior differentials of 1-forms Θ^α and substituting the differentials du^β, dx^j, dp^β_j expressed from the equations (2), we obtain

$$\begin{aligned} d\Theta^\alpha = &\Big(da^\alpha_\beta \, A^\beta_\gamma + a^\alpha_\beta \, B^j_k \, H^{\beta s}_{j\eta} \left(c^k_\gamma \, (\Sigma^\eta_s - f^\eta_{s\epsilon} \, \Theta^\epsilon - g^\eta_{sl} \, \Xi^l) - f^\eta_{s\gamma} \, \Xi^s\right)\Big) \wedge \Theta^\gamma \\ &+ a^\alpha_\beta \, B^j_k \, H^{\beta s}_{j\eta} \, \Xi^k \wedge \Sigma^\eta_s - a^\alpha_\beta \, B^j_k \, H^{\beta s}_{j\eta} \, g^\eta_{sl} \, \Xi^k \wedge \Xi^l, \end{aligned}$$

where the functions $H^{\beta k}_{j\gamma}$ are defined by the conditions $H^{\beta k}_{j\gamma} \, h^{\gamma i}_{k\alpha} = \delta^i_j \, \delta^\beta_\alpha$. The multipliers of $\Xi^k \wedge \Sigma^\eta_s$ and $\Xi^k \wedge \Xi^l$ are essential torsion coefficients. We normalize them by the following choice of the parameters of the Lie group $\mathcal{G}$:

$$h^{\alpha j}_{i\beta} = a^\alpha_\beta \, B^j_i, \tag{3}$$

$$g^\alpha_{ij} = g^\alpha_{ji}. \tag{4}$$

Then we have

$$d\Theta^\alpha = \Phi^\alpha_\beta \wedge \Theta^\beta + \Xi^k \wedge \Sigma^\alpha_k, \tag{5}$$

$$\Phi^\alpha_\beta = da^\alpha_\gamma \, A^\gamma_\beta + c^k_\gamma \, f^\alpha_{k\beta} \, \Theta^\gamma - f^\alpha_{k\beta} \, \Xi^k - c^k_\beta \, g^\alpha_{kj} \, \Xi^j + c^k_\beta \, \Sigma^\alpha_k. \tag{6}$$

Now the exterior differentials of Ξ^i and Σ^α_i become

$$d\Xi^i = \Psi^i_k \wedge \Xi^k + \Pi^i_\gamma \wedge \Theta^\gamma, \tag{7}$$

$$d\Sigma^\alpha_i = \Phi^\alpha_\gamma \wedge \Sigma^\gamma_i - \Psi^k_i \wedge \Sigma^\alpha_k + \Lambda^\alpha_{i\beta} \wedge \Theta^\beta + \Omega^\alpha_{ij} \wedge \Xi^j, \tag{8}$$

where

$$\Psi^i_k = db^i_j \, B^j_k - c^i_\beta \, \Sigma^\beta_k, \tag{9}$$

$$\Pi^i_\gamma = dc^i_\gamma + c^i_\beta \, \Phi^\beta_\gamma - c^k_\gamma \, \Psi^i_k - c^k_\gamma \, c^i_\beta \, \Sigma^\beta_k, \tag{10}$$

$$\begin{aligned} \Lambda^\alpha_{i\beta} = &\, df^\alpha_{i\beta} + f^\alpha_{i\gamma} \, \Phi^\gamma_\beta + g^\alpha_{ij} \, \Pi^j_\beta - f^\gamma_{i\beta} \, (\Phi^\alpha_\gamma - c^k_\epsilon \, f^\alpha_{k\gamma} \, \Theta^\epsilon + f^\alpha_{k\gamma} \, \Xi^k \\ &+ c^k_\gamma \, g^\alpha_{kj} \, \Xi^j - c^k_\gamma \, \Sigma^\alpha_k) + f^\alpha_{k\beta} \, (\Psi^k_i + c^k_\gamma \, \Sigma^\gamma_i) + c^k_\beta \, f^\alpha_{k\gamma} \, \Sigma^\gamma_i, \end{aligned} \tag{11}$$

$$\begin{aligned} \Omega^\alpha_{ij} = &\, dg^\alpha_{ij} + g^\alpha_{ik} \, \Psi^k_j + g^\alpha_{jk} \, \Psi^k_i - f^\alpha_{i\beta} \, \Sigma^\beta_j - f^\alpha_{j\beta} \, \Sigma^\beta_i \\ &- g^\gamma_{ij} \, (\Phi^\alpha_\gamma - c^k_\beta \, f^\alpha_{k\gamma} \, \Theta^\beta + f^\alpha_{k\gamma} \Xi^k + c^k_\gamma \, g^\alpha_{ks} \, \Xi^s - c^k_\gamma \, \Sigma^\alpha_k). \end{aligned} \tag{12}$$

We note that the conditions (4) imply

$$\Omega^\alpha_{ij} = \Omega^\alpha_{ji}. \tag{13}$$

Thus the specifications (3) and (4) of the group parameters of the coframe (2) give the lifted coframe

$$\Theta^\alpha = a^\alpha_\beta\,(du^\beta - p^\beta_j\,dx^j), \tag{14}$$

$$\Xi^i = c^i_\beta\,\Theta^\beta + b^i_j\,dx^j, \tag{15}$$

$$\Sigma^\alpha_i = f^\alpha_{i\beta}\,\Theta^\beta + g^\alpha_{ij}\,\Xi^j + a^\alpha_\beta\,B^j_i\,dp^\beta_j \tag{16}$$

on $J^1(\mathcal{E}) \times \mathcal{H}$, where $\mathcal{H}$ is the subgroup of the group $\mathcal{G}$ defined by the equalities (3) and (4). The structure equations (7), (8) do not contain any torsion coefficients, while the structure equations (5) contain only constant torsion coefficients.

The structure equations (5), (7), (8) remain unchanged if we make the following change of the modified Maurer - Cartan forms Φ^α_β, Ψ^i_k, Π^i_γ, $\Lambda^\alpha_{i\beta}$, Ω^α_{ij} :

$$\begin{array}{lcl}
\Phi^\alpha_\beta & \mapsto & \Phi^\alpha_\beta + K^\alpha_{\gamma\epsilon}\,\Theta^\epsilon, \\
\Psi^i_k & \mapsto & \Psi^i_k + L^i_{kj}\,\Xi^j + M^i_{k\gamma}\,\Theta^\gamma, \\
\Pi^i_\gamma & \mapsto & \Pi^i_\gamma + M^i_{k\gamma}\,\Xi^k + N^i_{\gamma\epsilon}\,\Theta^\epsilon, \\
\Lambda^\alpha_{i\beta} & \mapsto & \Lambda^\alpha_{i\beta} + P^\alpha_{i\beta\gamma}\,\Theta^\gamma + Q^\alpha_{i\beta k}\,\Xi^k + K^\alpha_{\gamma\beta}\,\Sigma^\gamma_i - M^k_{i\beta}\,\Sigma^\alpha_k, \\
\Omega^\alpha_{ij} & \mapsto & \Omega^\alpha_{ij} + Q^\alpha_{i\beta j}\,\Theta^\beta + R^\alpha_{ijk}\,\Xi^k - L^k_{ij}\,\Sigma^\alpha_k,
\end{array}$$

where $K^\alpha_{\gamma\epsilon}$, L^i_{kj}, $M^i_{k\gamma}$, $N^i_{\gamma\epsilon}$, $P^\alpha_{i\beta\gamma}$, $Q^\alpha_{i\beta k}$, R^α_{ijk} are arbitrary functions on $J^1(\mathcal{E})\times\mathcal{H}$ satisfying the following symmetry conditions:

$$\begin{array}{l}
K^\alpha_{\gamma\epsilon} = K^\alpha_{\epsilon\gamma}, \quad L^i_{kj} = L^i_{jk}, \quad N^i_{\gamma\epsilon} = N^i_{\epsilon\gamma}, \\
P^\alpha_{i\beta\gamma} = P^\alpha_{i\gamma\beta}, \quad Q^\alpha_{i\beta k} = Q^\alpha_{k\beta i}, \quad R^\alpha_{ijk} = R^\alpha_{ikj} = R^\alpha_{jik}.
\end{array} \tag{17}$$

Their number

$$\begin{array}{l}
r^{(1)} = \frac{1}{2}\,q^2\,(q+1) + \frac{1}{2}\,n^2\,(n+1) + n^2\,q + \frac{1}{2}\,n\,q\,(q+1) + \frac{1}{2}\,n\,q^2\,(q+1) \\
\quad + \frac{1}{2}\,n\,q^2\,(n+1) + \frac{1}{6}\,q\,n\,(n+1)\,(n+2)
\end{array}$$

is the degree of indeterminancy [20, Definition 11.2] of the lifted coframe Θ^α, Ξ^i, Σ^α_i.

Using the conditions (13), it is not hard to compute the reduced characters [20, Definition 11.4] of this coframe: $s'_1 = s'_2 = ... = s'_q = q + n + n\,q$, $s'_{q+1} = n + n\,q$, $s'_{q+2} = n + (n-1)\,q$, $s'_{q+2} = n + (n-2)\,q$, ... , $s'_{q+n-1} = n + 2\,q$, $s'_{q+n} = n + q$, $s'_{q+n+1} = s'_{q+n+2} = ... = s'_{q+n+nq} = 0$. It is easy to verify that the Cartan test

$$r^{(1)} = s'_1 + 2\,s'_2 + 3\,s'_3 + ... + (q + n + n\,q)\,s'_{q+n+nq}$$

is satisfied, so by definition 11.7 of [20] the lifted coframe (14), (15), (16) is involutive, and by theorem 11.16 of [20], since the last non-zero reduced character s'_{q+n} is equal to $q + n$, the transformations of the invariance pseudo-group of this coframe depend on

$q+n$ functions of $q+n$ variables, as it should be. It is easy to verify directly that the transformation $\Upsilon : J^1(\mathcal{E}) \times \mathcal{H} \to J^1(\mathcal{E}) \times \mathcal{H}$ satisfies the conditions

$$\Upsilon^* \, \Theta^\alpha = \Theta^\alpha, \quad \Upsilon^* \, \Xi^i = \Xi^i, \quad \Upsilon^* \, \Sigma^\alpha_i = \Sigma^\alpha_i \tag{18}$$

if and only if it is projectable on $J^1(\mathcal{E})$ and its projection $\Delta : J^1(\mathcal{E}) \to J^1(\mathcal{E})$, $\Delta : (x,u,p) \mapsto (\overline{x}, \overline{u}, \overline{p})$, is the prolongation of the transformation $\Gamma : \mathcal{E} \to \mathcal{E}$, $\Gamma : (x,u) \mapsto (\overline{x}, \overline{u})$, such that the conditions (1) are satisfied. Thus the equalities (18) really define the pseudo-group of contact transformations on $J^1(\mathcal{E})$, when $q > 1$ and $n > 1$.

Since the forms Θ^α, Ξ^i, Σ^α_i are preserved by the pseudo-group transformations, their exterior differentials are preserved also, so $\Upsilon^* d\Theta^\alpha = d\Theta^\alpha$, $\Upsilon^* d\Xi^i = d\Xi^i$, $\Upsilon^* d\Sigma^\alpha_i = d\Sigma^\alpha_i$, therefore we have

$$\begin{aligned}
&\Upsilon^*(\Phi^\alpha_\beta \wedge \Theta^\beta + \Xi^k \wedge \Sigma^\alpha_k) = (\Upsilon^* \Phi^\alpha_\beta) \wedge \Theta^\beta + \Xi^k \wedge \Sigma^\alpha_k = \Phi^\alpha_\beta \wedge \Theta^\beta + \Xi^k \wedge \Sigma^\alpha_k,\\
&\Upsilon^*(\Psi^i_k \wedge \Xi^k + \Pi^i_\gamma \wedge \Theta^\gamma) = (\Upsilon^* \Psi^i_k) \wedge \Xi^k + (\Upsilon^* \Pi^i_\gamma) \wedge \Theta^\gamma = \Psi^i_k \wedge \Xi^k + \Pi^i_\gamma \wedge \Theta^\gamma,\\
&\Upsilon^*(\Phi^\alpha_\gamma \wedge \Sigma^\gamma_i - \Psi^k_i \wedge \Sigma^\alpha_k + \Lambda^\alpha_{i\beta} \wedge \Theta^\beta + \Omega^\alpha_{ij} \wedge \Xi^j)\\
&\qquad = \Upsilon^*(\Phi^\alpha_\gamma) \wedge \Sigma^\gamma_i - (\Upsilon^* \Psi^k_i) \wedge \Sigma^\alpha_k + (\Upsilon^* \Lambda^\alpha_{i\beta}) \wedge \Theta^\beta + (\Upsilon^* \Omega^\alpha_{ij}) \wedge \Xi^j\\
&\qquad = \Phi^\alpha_\gamma \wedge \Sigma^\gamma_i - \Psi^k_i \wedge \Sigma^\alpha_k + \Lambda^\alpha_{i\beta} \wedge \Theta^\beta + \Omega^\alpha_{ij} \wedge \Xi^j,
\end{aligned}$$

and thus

$$\begin{aligned}
\Upsilon^* \Phi^\alpha_\beta &= \Phi^\alpha_\beta + K^\alpha_{\gamma\epsilon} \, \Theta^\epsilon,\\
\Upsilon^* \Psi^i_k &= \Psi^i_k + L^i_{kj} \, \Xi^j + M^i_{k\gamma} \, \Theta^\gamma,\\
\Upsilon^* \Pi^i_\gamma &= \Pi^i_\gamma + M^i_{k\gamma} \, \Xi^k + N^i_{\gamma\epsilon} \, \Theta^\epsilon,\\
\Upsilon^* \Lambda^\alpha_{i\beta} &= \Lambda^\alpha_{i\beta} + P^\alpha_{i\beta\gamma} \, \Theta^\gamma + Q^\alpha_{i\beta k} \, \Xi^k + K^\alpha_{\gamma\beta} \, \Sigma^\gamma_i - M^k_{i\beta} \, \Sigma^\alpha_k,\\
\Upsilon^* \Omega^\alpha_{ij} &= \Omega^\alpha_{ij} + Q^\alpha_{i\beta j} \, \Theta^\beta + R^\alpha_{ijk} \, \Xi^k - L^k_{ij} \, \Sigma^\alpha_k
\end{aligned} \tag{19}$$

for some functions $K^\alpha_{\gamma\epsilon}$, L^i_{kj}, $M^i_{k\gamma}$, $N^i_{\gamma\epsilon}$, $P^\alpha_{i\beta\gamma}$, $Q^\alpha_{i\beta k}$, R^α_{ijk} on $J^1(\mathcal{E}) \times \mathcal{H}$ satisfying the conditions (17).

3. Symmetries of differential equations

A suitable method for studying geometrical properties of embedded submanifolds under an action of finite-dimensional Lie groups or infinite Lie pseudo-groups was developed in [10, 11]. For its application to the problem of finding symmetries of a system of d.e.s $\mathcal{R}_1$ we restrict the lifted coframe (14), (15), (16) on $\mathcal{R}_1$. That is, we consider the set of 1-forms $\theta^\alpha = \iota^* \Theta^\alpha$, $\xi^i = \iota^* \Xi^i$, $\sigma^\alpha_i = \iota^* \Sigma^\alpha_i$, where $\iota : \mathcal{R}_1 \to J^1(\mathcal{E})$ is the embedding (for brevity we identify the map $\iota \times id : \mathcal{R}_1 \times \mathcal{H} \to J^1(\mathcal{E}) \times \mathcal{H}$ with $\iota : \mathcal{R}_1 \to J^1(\mathcal{E})$). The 1-forms θ^α, ξ^i, σ^α_i are linearly dependent, i.e., there exists a non-trivial set of functions U_α, V_i, W^i_α on $\mathcal{R}_1 \times \mathcal{H}$, such that $U_\alpha \, \theta^\alpha + V_i \, \xi^i + W^i_\alpha \, \sigma^\alpha_i \equiv 0$.

Setting these functions equal to some constants allows one to express a part of parameters a^α_β, b^i_j, c^i_β, $f^\alpha_{i\beta}$, g^α_{ij} of the group $\mathcal{H}$ as functions of coordinates of $\mathcal{R}_1$ and other group parameters. Substituting the obtained values of parameters into the modified Maurer - Cartan forms $\phi^\alpha_\beta = \iota^*\Phi^\alpha_\beta$, $\psi^i_k = \iota^*\Psi^i_k$, $\pi^i_\beta = \iota^*\Pi^i_\beta$, $\lambda^\alpha_{i\beta} = \iota^*\Lambda^\alpha_{i\beta}$, $\omega^\alpha_{ij} = \iota^*\Omega^\alpha_{ij}$ makes a part of these forms independent of all differentials of the group parameters. Since the transformation Υ^* changes the forms Φ^α_β, Ψ^i_k, Π^i_β by the rules (19), in the case when the obtained form ϕ^α_β does not depend on all differentials of the group parameters, its coefficients at σ^γ_j and ξ^j are lifted invariants of the pseudo-group, and if the obtained forms ψ^i_k or π^i_β are independent of all differentials of the group parameters, their coefficients at σ^γ_j are lifted invariants also. Normalizing these lifted invariants to be constants allows us to express a part of the group parameters as functions of coordinates on $\mathcal{R}_1$ and other group parameters. If not all group parameters are expressed, we should substitute the expressed parameters into the forms ϕ^α_β, ψ^i_k, π^i_γ, which depend on their differentials, and repeat the process. If the process is completed, but not all group parameters are expressed as functions on $\mathcal{R}_1$, we should substitute the modified Maurer - Cartan forms ϕ^α_β, ψ^i_k, π^i_γ, $\lambda^\alpha_{i\beta}$, ω^α_{ij}, which were reduced during the process of normalization, into the reduced structure equations

$$
\begin{aligned}
&d\theta^\alpha = \phi^\alpha_\beta \wedge \theta^\beta + \xi^k \wedge \sigma^\alpha_k,\\
&d\xi^i = \psi^i_k \wedge \xi^k + \pi^i_\gamma \wedge \theta^\gamma,\\
&d\sigma^\alpha_i = \phi^\alpha_\gamma \wedge \sigma^\gamma_i - \psi^k_i \wedge \sigma^\alpha_k + \lambda^\alpha_{i\beta} \wedge \theta^\beta + \omega^\alpha_{ij} \wedge \xi^j.
\end{aligned}
$$

If the essential torsion coefficients dependent on the group parameters appear, then we should normalize them to constants and find some new part of the group parameters, which, being substituted into the reduced modified Maurer - Cartan forms, allows us to repeat the procedure of normalization. There are two possible results of this process. The first one, when the reduced lifted coframe appears to be involutive, outputs the desired set of invariant 1-forms which characterize the pseudo-group $Sym(\mathcal{R}_1)$. In the second one, when the coframe is not involutive, we should apply the procedure of prolongation [20, Chapter 12].

3.1. Example 1: Burgers' equation

For an application of the above method to finding invariant 1-forms of the symmetry group of the Burgers' equation

$$u_t = u_{xx} + u\, u_x,$$

we take the equivalent system of the first order

$$u_x = v, \qquad v_x = u_t - u\, v.$$

Denoting $x = x^1$, $t = x^2$, $v = u^1$, $u = u^2$, $v_x = p^1_1$, $v_t = p^1_2$, $u_x = p^2_1$, $u_t = p^2_2$, we consider this system as a subbundle of the bundle $J^1(\mathcal{E})$, $\mathcal{E} = \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, with local coordinates $\{x^1, x^2, u^1, u^2, p^1_1, p^1_2, p^2_1, p^2_2\}$, where the embedding ι is defined by the equalities

$$p^1_1 = p^2_2 - u^1\, u^2, \qquad p^2_1 = u^1.$$

The forms $\theta^\alpha = \iota^*\,\Theta^\alpha$, $\alpha \in \{1, 2\}$, $\xi^i = \iota^*\,\Xi^i$, $i \in \{1, 2\}$, are linearly independent, whereas the forms $\sigma^\alpha_i = \iota^*\,\Sigma^\alpha_i$ are linearly dependent. The group parameters a^α_β, b^i_j must satisfy the conditions $det\left(a^\alpha_\beta\right) \neq 0$, $det\left(b^i_j\right) \neq 0$. Moreover, without loss of generality, we can consider that $a^1_1 \neq 0$, $a^2_2 \neq 0$, $b^1_1 \neq 0$, $b^2_2 \neq 0$. Computing the linear dependence conditions of forms σ^α_i by means of MAPLE, we obtain sequentially the group parameters a^2_1, b^2_1, b^2_2, g^2_{12}, g^2_{11}, g^1_{11}, f^2_{12}, f^2_{11}, g^2_{22}, f^2_{22}, f^2_{211} as the functions of other group parameters and the local coordinates $\{x^1, x^2, u^1, u^2, p^1_2, p^2_2\}$ of $\mathcal{R}_1$. Particularly,

$$a^2_1 = 0, \quad b^2_1 = 0, \quad b^2_2 = \frac{b^1_1 a^2_2}{a^1_1}, \quad g^2_{12} = -\frac{\left(-p^2_2 b^1_2 + u^1 u^2 b^1_2 + p^1_2 b^1_1\right) a^1_1}{(b^1_1)^3},$$

$$g^2_{11} = -\frac{a^2_2\,(p^2_2 - u^1 u^2)}{(b^1_1)^2}, \quad g^1_{11} = \frac{(u^1)^2 a^1_1 - a^1_2 p^2_2 - a^1_1 p^1_2 - u^1 (u^2)^2 a^1_1 + p^2_2 a^1_1 u^2 + u^1 u^2 a^1_2}{(b^1_1)^2},$$

$$f^2_{12} = \frac{(b^1_1)^2 a^1_2 + p^1_2 (a^1_1)^2 c^2_2 b^1_1 + u^1 u^2 (a^1_1)^2 b^1_2 c^2_2 - u^1 u^2 a^2_2 c^1_2 b^1_1 a^1_1 - p^2_2 (a^1_1)^2 b^1_2 c^2_2 + p^2_2 a^2_2 c^1_2 b^1_1 a^1_1}{(b^1_1)^3 a^1_1},$$

$$f^2_{11} = -\frac{u^1 u^2 (c^1_1 a^2_2 b^1_1 a^1_1 - (a^1_1)^2 b^1_2 c^2_1) - p^2_2 c^1_1 a^2_2 b^1_1 a^1_1 + p^2_2 (a^1_1)^2 b^1_2 c^2_1 - p^1_2 (a^1_1)^2 c^2_1 b^1_1 + a^2_2 (b^1_1)^2}{(b^1_1)^3 a^1_1},$$

while the expressions for g^2_{22}, f^2_{22} and f^2_{21} are too big to write them out in full here.

The linear dependence between the forms σ^α_i is $\sigma^1_1 = \sigma^2_2$, $\sigma^2_1 = 0$.

The analysis of the modified Maurer - Cartan forms ϕ^α_β, ψ^i_k, π^i_γ at the obtained values of the group parameters gives the following normalizations:

$$\phi^2_1 \equiv c^2_1\,\sigma^2_2 + \frac{a^2_2}{b^1_1 a^1_1}\,\xi^1 \pmod{\theta^1,\ \theta^2,\ \xi^2,\ \sigma^1_2} \quad \Rightarrow \quad c^2_1 = 0, \quad b^1_1 = \frac{a^2_2}{a^1_1};$$

$$\psi^2_2 - 2\,\psi^1_1 = (2\,c^1_1 - c^2_2)\,\sigma^1_2 \quad \Rightarrow \quad c^2_2 = 2\,c^1_1;$$

$$\psi^1_1 + \phi^1_1 - \phi^2_2 \equiv -2\,c^1_1 \sigma^2_2 \pmod{\theta^1,\ \theta^2,\ \xi^1,\ \xi^2,\ \sigma^1_2} \quad \Rightarrow \quad c^1_1 = 0;$$

$$\phi^2_1 \equiv -\left(f^1_{11} + \frac{(a^1_2 a^2_2 - a^1_1 a^2_2 u^2 + b^1_2 (a^1_1)^2}{a^2_2}\right) \xi^2 \pmod{\theta^1,\ \theta^2,\ \xi^1,\ \sigma^1_2,\ \sigma^2_2}$$

$$\Rightarrow \quad f^1_{11} = -\frac{a^1_2 a^2_2 - a^1_1 a^2_2 u^2 + b^1_2 (a^1_1)^2}{a^2_2}.$$

Now the analysis of the structure equations gives step by step the following essential torsion coefficients and the corresponding normalizations:

$$d\theta^1 = -c^1_2\,\theta^2 \wedge \sigma^2_2 + \ldots \quad \Rightarrow \quad c^1_2 = 0;$$

$$d\theta^1 = \left((a^2_2)^3 f^1_{12} - (a^1_2)^2 a^2_2 + a^1_1 a^1_2 a^2_2 u^2 - (a^1_1)^2 a^1_2 b^1_2\right) \theta^2 \wedge \xi^1 + \left(f^1_{22} + \frac{a^1_2}{a^2_2} f^1_{21}\right) \theta^2 \wedge \xi^2 + ...$$

$$\Rightarrow \quad f^1_{12} = \frac{a^1_2(a^1_2 a^2_2 - a^1_1 a^1_2 a^2_2 u^2 + (a^1_1)^2 a^1_2 b^1_2)}{(a^2_2)^3}, \quad f^1_{22} = -\frac{a^1_2}{a^2_2} f^1_{21};$$

$$d\xi^2 = \frac{2\,(2\,a^1_2 a^2_2 - a^1_1 a^2_2 u^2 + b^1_2 (a^1_1)^2)}{(a^2_2)^2} \xi^1 \wedge \xi^2 + ... \quad \Rightarrow \quad a^1_2 = \frac{a^1_1\,(a^2_2 u^2 - b^1_2 a^1_1)}{2\,a^2_2};$$

$$d\xi^1 = \left(f^2_{21} + \frac{(a^1_1)^2\,(4\,(a^2_2)^2 u^1 - 2\,a^2_2 b^1_2 a^1_1 u^2 + (a^2_2)^2 (u^2)^2 + (b^1_2)^2 (a^1_1)^2)}{(a^2_2)^4}\right) \xi^1 \wedge \xi^2 + ...$$

$$\Rightarrow \quad f^2_{21} = -\frac{(a^1_1)^2\,(4\,(a^2_2)^2 u^1 - 2\,a^2_2 b^1_2 a^1_1 u^2 + (a^2_2)^2 (u^2)^2 + (b^1_2)^2 (a^1_1)^2)}{(a^2_2)^4};$$

$$d\sigma^1_2 = -\frac{(a^1_1)^2\,(b^1_2 a^1_1 - a^2_2 u^2)}{(a^2_2)^4} \theta^1 \wedge \theta^2 + ... \quad \Rightarrow \quad b^1_2 = \frac{a^2_2 u^2}{a^1_1};$$

$$d\sigma^2_2 = \frac{(a^1_1)^3\,(p^2_2 - u^1 u^2)}{(a^2_2)^3} \theta^2 \wedge \xi^1 + ... \quad \Rightarrow \quad a^2_2 = \frac{a^1_1}{(p^2_2 - u^1 u^2)^{1/3}};$$

$$d\theta^2 = \frac{1}{3\,a^1_1\,(p^2_2 - u^1 u^2)^{2/3}} \theta^2 \wedge \sigma^2_2 + ... \quad \Rightarrow \quad a^1_1 = \frac{1}{(p^2_2 - u^1 u^2)^{2/3}};$$

$$d\theta^1 = -\left(\frac{2\,g^1_{12}}{3} + \frac{2\,u^1}{(p^2_2 - u^1 u^2)^{2/3}}\right) \theta^1 \wedge \xi_2 + ... \quad \Rightarrow \quad g^1_{12} = -\frac{3\,u^1}{(p^2_2 - u^1 u^2)^{2/3}};$$

$$d\sigma^2_2 = \left(-g^1_{22} + \frac{2\,(-2\,(p^2_2)^2 + 7\,u^1 u^2 p^2_2 - 5\,(u^1 u^2)^2 + 2\,(u^1)^3 - 3\,u^1\,p^1_2)}{(p^2_2 - u^1 u^2)^2}\right) \xi^1 \wedge \xi_2 + ...$$

$$\Rightarrow \quad g^1_{22} = \frac{2\,(-2\,(p^2_2)^2 + 7\,u^1 u^2 p^2_2 - 5\,(u^1 u^2)^2 + 2\,(u^1)^3 - 3\,u^1\,p^1_2)}{(p^2_2 - u^1 u^2)^2}.$$

Thus all the group parameters are expressed as the functions of the local coordinates $\{x^1, x^2, u^1, u^2, p^1_2, p^2_2\}$ of the equation $\mathcal{R}_1$. The result of all normalizations is the invariant coframe

$$\theta^1 = \frac{du^1 - (p^2_2 - u^1 u^2)\,dx^1 - p^1_2\,dx^2}{(p^2_2 - u^1 u^2)^{2/3}},$$

$$\theta^2 = \frac{du^2 - u^1\,dx^1 - p^2_2\,dx^2}{(p^2_2 - u^1 u^2)^{1/3}},$$

$$\xi^1 = (p^2_2 - u^1 u^2)^{1/3}\,(dx^1 + u^2\,dx^2),$$

$$\xi^2 = (p^2_2 - u^1 u^2)^{2/3}\,dx^2,$$

$$\sigma^1_2 = \frac{dp^1_2 - u^2 dp^2_2 + ((u^2)^2 - 2\,u^1)\,du^1 + u^1 u^2\,du_2}{(p^2_2 - u^1 u^2)^{4/3}}$$
$$+ \frac{u^1 (p^2_2 - u^1 u^2)\,dx^1 + (4\,(u^1)^3 - 7\,(u^1 u^2)^2 + 11\,u^1 u^2 p^2_2 - 4\,u^1 p^1_2 - 4\,(p^2_2)^2)\,dx_2}{(p^2_2 - u^1 u^2)^{4/3}},$$

$$\sigma^2_2 = \frac{dp^2_2 - u^2 du^1 - u^1 du^2 - (p^1_2 + u^1 (u^2)^2 - (u^1)^2 - u^2 p^2_2)\,dx^1}{p^2_2 - u^1 u^2}$$

$$+\frac{(4\,(u^1)^2u^2+(u^2)^2p_2^2-u^1(u^2)^3-u^2p-3\,u^1p_2^2)\,dx^2}{p_2^2-u^1u^2}.$$

Its structure equations are

$$\begin{aligned}
d\theta^1 &= I\,\theta^1\wedge\xi^1+\tfrac{2}{3}\,\theta^1\wedge\sigma_2^2+\xi^1\wedge\sigma_2^2+\xi^2\wedge\sigma_2^1,\\
d\theta^2 &= -\theta^1\wedge\xi^1+\tfrac{1}{2}\,I\,\theta^2\wedge\xi^1+\tfrac{1}{3}\,\theta^2\wedge\sigma_2^2+\xi^2\wedge\sigma_2^2,\\
d\xi^1 &= \theta^2\wedge\xi^2-\tfrac{1}{3}\,\xi^1\wedge\sigma_2^2,\\
d\xi^2 &= I\,\xi^1\wedge\xi^2-\tfrac{2}{3}\,\xi^2\wedge\sigma_2^2,\\
d\sigma_2^1 &= -\theta^1\wedge\xi^1-6\,I\,\theta^1\wedge\xi^2-\tfrac{3}{2}\,I\,\theta^2\wedge\xi^1-\theta^2\wedge\sigma_2^2-15\,I\,\xi^1\wedge\xi^2\\
&\quad -2\,I\,\xi^1\wedge\sigma_2^1+7\,\xi^2\wedge\sigma_2^2+\tfrac{4}{3}\,\sigma_2^1\wedge\sigma_2^2,\\
d\sigma_2^2 &= -3\,\theta^1\wedge\xi^2+\theta^2\wedge\xi^1-\tfrac{3}{2}\,I\,\theta^2\wedge\xi^2+\xi^1\wedge\sigma_2^1-\tfrac{3}{2}\,I\,\xi^1\wedge\sigma_2^2,
\end{aligned}$$

where the only invariant I is of form

$$I=\frac{2\,(p_2^1+u^1(u^2)^2-(u^1)^2-u^2p_2^2)}{3\,(p_2^2-u^1u^2)^{4/3}}.$$

Taking its exterior differential, we have

$$dI=-\tfrac{2}{3}\,\theta^2-2\,I^2\,\xi^1+2\,\xi^2+\tfrac{2}{3}\,\sigma_2^1-\tfrac{4}{3}\,I\,\sigma_2^2,$$

so all differential invariants of the group are functionally expressed as functions of I, the rank of the coframe [20, Proposition 8.18] is equal to 1, and its symmetry group is 5-dimensional [20, Theorem 8.22] (as it should be; for the full details of finding infinitesimal generators of this group by Lie's method see, e.g., [30, Chapter 3, § 5].)

3.2. Example 2: One-dimensional equations of gas dynamics in Lagrange coordinates

One-dimensional dynamics of polytropic gas in Lagrange coordinates is described by the system of d.e.s [26]

$$\begin{aligned}
&\rho_t+\rho^2u_m=0,\\
&u_t+p_m=0,\\
&p_t+\gamma\,\rho\,p\,u_m=0.
\end{aligned}\tag{20}$$

Denoting $\rho=u^1$, $u=u^2$, $p=u^3$, $t=x^1$, $m=x^2$ and using the above method, we obtain the invariant coframe of the symmetry group of the system (20)

$$\theta^1=\frac{1}{u^1}\,\left(du^1+(u^1)^2p_2^2\,dx^1-p_2^1\,dx^2\right),$$

$$\theta^2=\sqrt{\frac{u^1}{\gamma u^3}}\,\left(du^2+p_2^3\,dx^1-p_2^2\,dx^2\right),$$

$$\theta^3=\frac{1}{\gamma u^3}\,\left(du^3+\gamma\,u^1\,u^3\,p_2^2\,dx^1-p_2^3\,dx^2\right),$$

$$
\begin{aligned}
\xi^1 &= \sqrt{\frac{u^1}{\gamma u^3}}\, dx^2, \qquad (21)\\
\xi^2 &= u^1\, p^2_2\, dx^1,\\
\sigma^1_2 &= \frac{1}{u^1\, p^2_2}\sqrt{\frac{\gamma u^3}{u^1}}\,\left(dp^1_2 - \frac{p^1_2}{u^1}\, du^1 - \frac{(\gamma-1)\,(u^1)^3(p^2_2)^2 u^3 - (p^1_2)^2(u^3)^2 - (p^3_2)^2(u^1)^2}{2\, u^1\,(u^3)^2}\, dx^2\right),\\
\sigma^2_2 &= \frac{1}{u^1\, p^2_2}\,\left(dp^2_2 + \frac{\gamma-1}{2}\,(u^1)^2(p^2_2)^2\, dx^1 + p^1_2\, p^2_2\, dx^2\right),\\
\sigma^3_2 &= \frac{1}{p^2_2\,\sqrt{\gamma\, u^1\, u^3}}\,\left(dp^3_2 + \gamma\, u^1\, p^2_2\, p^3_2\, dx^1 - \frac{\gamma-1}{2}\, u^1\,(p^2_2)^2\, dx^2\right)
\end{aligned}
$$

(since from considering the physical meaning we have $u^1 = \rho > 0$ and $u^3 = p > 0$, therefore there is no need to worry about the signs of the expressions under the square roots).

The structure equations of this coframe are

$$
\begin{aligned}
d\theta^1 &= \theta^1\wedge\xi^2 + \xi^1\wedge\sigma^1_2 - \xi^2\wedge\sigma^2_2,\\
d\theta^2 &= \tfrac{1}{2}\,\theta^1\wedge\theta^2 + \tfrac{\gamma}{2}\,\theta^2\wedge\theta^3 + I_1\,\theta^2\wedge\xi^1 + \tfrac{\gamma-1}{2}\,\theta^2\wedge\xi^2 + \xi^1\wedge\sigma^2_2 - \xi^2\wedge\sigma^3_2,\\
d\theta^3 &= \theta^1\wedge\xi^2 + I_2\,\theta^3\wedge\xi^1 + \xi^1\wedge\sigma^3_2 - \xi^2\wedge\sigma^2_2,\\
d\xi^1 &= \tfrac{1}{2}\theta^1\wedge\xi^1 - \tfrac{\gamma}{2}\theta^3\wedge\xi^1 - \xi^1\wedge\sigma^2_2,\\
d\xi^2 &= \theta^1\wedge\xi^2 - \xi^2\wedge\sigma^2_2,\\
d\sigma^1_2 &= \tfrac{1}{2}\,\gamma\,(\gamma-1)\,\theta^1\wedge\xi^1 - \tfrac{1}{2}\,\theta^1\wedge\sigma^1_2 - \tfrac{1}{2}\,\left(2\, I^2_2 - \gamma^2 + \gamma\right)\,\theta^3\wedge\xi^1\\
&\quad + \tfrac{\gamma}{2}\,\theta^3\wedge\sigma^1_2 + I_1\,\xi^1\wedge\sigma^1_2 + \gamma\,(\gamma-1)\,\xi^1\wedge\sigma^2_2 - \gamma\, I_2\,\xi^1\wedge\sigma^3_2 + \sigma^1_2\wedge\sigma^2_2,\\
d\sigma^2_2 &= \tfrac{\gamma-1}{2}\,\theta^1\wedge\xi^2 - \xi^1\wedge\sigma^1_2 - \tfrac{\gamma-1}{2}\,\xi^2\wedge\sigma^2_2,\\
d\sigma^3_2 &= -\tfrac{\gamma-1}{2}\,\theta^1\wedge\xi^1 + I_2\,\theta^1\wedge\xi^2 - \tfrac{1}{2}\,\theta^1\wedge\sigma^3_2 - \tfrac{\gamma}{2}\,\theta^3\wedge\sigma^3_2 + (\gamma-1)\,\xi^1\wedge\sigma^2_2\\
&\quad - I_1\,\xi^1\wedge\sigma^3_2 - I_2\,\xi^2\wedge\sigma^2_2 - \sigma^2_2\wedge\sigma^3_2.
\end{aligned}
$$

The invariants I_1 and I_2 are defined by the equalities

$$
I_1 = \sqrt{\frac{\gamma\, u^1}{u^3}}\,\frac{p^3_2\, u^1 - p^1_2\, u^3}{2\,(u^1)^2\, p^2_2}, \qquad I_2 = \sqrt{\frac{\gamma}{u^1\, u^3}}\,\frac{p^3_2}{p^2_2}.
$$

Their exterior differentials are

$$
\begin{aligned}
dI_1 &= -\frac{I_1}{2}\,\theta^1 + \frac{\gamma}{2}\,(I_1 - I_2)\,\theta^3 + \frac{1}{2}\,\sigma^1_2 - I_1\,\sigma^2_2 + \frac{\gamma}{2}\,\sigma^3_2,\\
dI_2 &= -\frac{I_2}{2}\,\theta^1 + \gamma\,\left(I_1 - \frac{I_2}{2}\right)\,\theta^2 + \left(\frac{\gamma\,(\gamma-1)}{2} - I_1 I_2\right)\,\xi^1 - I_2\,\sigma^1_2 + \gamma\,\sigma^3_2,
\end{aligned}
$$

so all differential invariants of the symmetry group depend functionally on I_1 and I_2. Thus the coframe (21) has the rank 2, and the symmetry group of the system (20) is 6-dimensional. In [2, Chapter 3] the explicit form of the infinitesimal generators of this group is given.

3.3. Example 3: Liouville's equation

For finding invariant 1-forms and structure equations of the symmetry pseudo-group of Liouville's equation

$$u_{tx} = \mathrm{e}^u$$

we take the equivalent system of the first order

$$u_t = v, \qquad v_x = \mathrm{e}^u.$$

Using the notations $u = u^1$, $v = u^2$, $t = x^1$, $x = x^2$ and applying the above procedure of absorption and normalization, we have $\sigma^1_1 = 0$, $\sigma^2_2 = 0$, while θ^1, θ^2, ξ^1, ξ^2, σ^1_2 and σ^2_1 constitute the lifted coframe

$$\begin{aligned}
\theta^1 &= du^1 - u^2 dx^1 - p^1_2 dx^2, \\
\theta^2 &= a^2_2 \left(du^2 - p^2_1 dx^1 - \mathrm{e}^{u^1} dx^2 \right), \\
\xi^1 &= (a^2_2)^{-1} dx^1, \\
\xi^2 &= a^2_2 \mathrm{e}^{u^1} \mathrm{d}x^2, \\
\sigma^1_2 &= (a^2_2)^{-1} \mathrm{e}^{-u^1} dp^1_2 - (a^2_2)^{-1} dx^1 + a^2_2 g^1_{22} \mathrm{e}^{u^1} dx^2, \\
\sigma^2_1 &= (a^2_2)^2 \left(dp^2_1 - u^2 dx^1 + ((a^2_2)^{-1} g^2_{11} + u^2 p^2_1) dx^1 \right).
\end{aligned} \tag{22}$$

The exterior differentials of these forms are

$$\begin{aligned}
d\theta^1 &= -\theta^2 \wedge \xi^1 + \xi^1 \wedge \sigma^1_2, \\
d\theta^2 &= \chi_1 \wedge \theta^2 - \theta^1 \wedge \xi^2 + \xi^1 \wedge \sigma^2_1, \\
d\xi^1 &= -\chi_1 \wedge \xi^1, \\
d\xi^2 &= \chi_1 \wedge \xi^2 + \theta^1 \wedge \xi^2, \\
d\sigma^1_2 &= \chi_2 \wedge \xi^2 - \chi_1 \wedge \sigma^1_2 - \theta^1 \wedge (\sigma^1_2 + \xi^1), \\
d\sigma^2_1 &= \chi_3 \wedge \xi^1 + 2\, \chi_1 \wedge \sigma^2_1,
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
\chi_1 &= (a^2_2)^{-1} da^2_2 + a^2_2 u^2 \xi^1, \\
\chi_2 &= dg^1_{22} + 2\, g^1_{22} (\chi_1 + \theta^1) + (a^2_2)^{-1} \mathrm{e}^{-u^1} p^1_2 (\xi^1 - \sigma^1_2) + w_1 \xi^2, \\
\chi_3 &= dg^2_{11} - 3\, g^2_{11} \chi_1 + (a^2_2)^2 (p^2_1 + (u^2)^2) (\theta^2 + \xi^2) + 3\, a^2_2 u^2 \sigma^2_1 + w_2 \xi^1,
\end{aligned} \tag{24}$$

w_1 and w_2 are free parameters. The structure equations (23) do not contain any torsion coefficient depending on the group parameters. The coframe (22) is not involutive, because its degree of indeterminancy $r^{(1)}$ is 2, whereas the reduced characters are $s'_1 = 3$, $s'_2 = ... = s'_6 = 0$, so Cartan's test is not satisfied. Therefore we should use the procedure of prolongation [20, Chapter 12]. For this purpose we unite both coframes (22) and (24) into the new base coframe, whereas w_1 and w_2 turn into the new group parameters. Finding exterior differentials of χ_1, χ_2 and χ_3, we have

$$\begin{aligned} d\chi_1 &= \theta^2 \wedge \xi^1 - \xi^1 \wedge \xi^2, \\ d\chi_2 &= \nu_1 \wedge \xi^2 - 2\,\theta^1 \wedge \chi_1 - 2\,\chi_1 \wedge \chi_2, \\ d\chi_3 &= \nu_2 \wedge \xi^1 + 2\,(\theta^2 + \xi^2) \wedge \sigma^2_1 + 3\,\chi_1 \wedge \chi_2, \end{aligned} \tag{25}$$

where

$$\begin{aligned} \nu_1 &= dw_1 + 3\,w_1(\theta^1 + \chi_1) + \left((a^2_2)^{-1}\mathrm{e}^{-2u^1}(p^1_2)^2 - g^1_{22}\right)\,(\xi^1 + \sigma^1_2) - (a^2_2)^{-1}\mathrm{e}^{-u^1}p^1_2\chi_2, \\ \nu_2 &= dw_2 + 4\,w_2\chi_2 + 2\,\left((a^2_2)^3(u^2)^3 - g^2_{11}\right)\,(\theta^2 + \xi^2) + 2\,(a^2_2)^2\left((u^2)^2 - 2\,p^2_1\right)\,\sigma^2_1 \\ &\quad +3\,a^2_2 u^2\chi_3. \end{aligned}$$

The structure equations (25) admit the change

$$\nu_1 \mapsto \nu_1 + z_1\xi^2, \qquad \nu_2 \mapsto \nu_2 + z_2\xi^1$$

for the free parameters z_1 and z_2. So the degree of indeterminancy of the coframe (22), (24) is $r^{(1)} = 2$ again, while the reduced characters now are $s'_1 = 2$, $s'_2 = ... = s'_9 = 0$. Cartan's test is therefore satisfied, and the coframe (22), (24) is involutive. Since the last non-zero reduced character is $s'_1 = 2$, the symmetry pseudo-group transformations depend on two arbitrary functions of one variable. This agrees with the result found by Liouville [15]. In [16, 17] the structure equations of this pseudo-group are derived using a different method; see also [25].

4. Conclusion

The approach to computation of symmetry groups used here does not require obtaining infinitesimal defining systems, analysis of their involutivity and integration, and includes only differentiation and linear algebra operations. So it is algorithmic *in principle*, although the labyrinth of corresponding computations is very intricate. In the future it seems that it will be possible to reduce the complexity of computations by means of using the canonical contact forms [21] on bundles of higher order jets.

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