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The equivalence problem for the Euler–Bernoulli beam equation via Cartan's method

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Abstract
We completely solve the local point equivalence problem for the Euler–Bernoulli beam equation using Cartan's method of equivalence. We obtain five equivalence classes. For each equivalence class, we establish the necessary and sufficient conditions for similarity, and derive a basis of differential invariants as well as operators of invariant differentiation.

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1. Introduction

Modern architectural prowess owes a lot to the development of vibration theories. Indeed vibration is the single most frequent cause of failure in architectural structures such as bridges and buildings. Therefore the understanding of the response of these structures to unwanted vibrations is of paramount importance in their design and durability. Scientists such as Leonardo Da Vinci and Galileo already anticipated the need of sound vibration theories. Da Vinci's drawings and multiple attempts to rationalize them are vivid testimonies to his quest for such theories. Although very advanced in his time, Da Vinci lacked mathematical tools and physical laws such as calculus and Newton's laws which postdate him. We owe to Jacob Bernoulli (1654–1705) the first consistent theory of elastic beams that uses the language of calculus and a specific constitutive law. Daniel Bernoulli (1700–1782), relying on the seminal work of his uncle Jacob, derived the differential equation governing the motion of vibrating beams. Leonard Euler (1707–1783) validated the Bernoullis theory by studying the shape of loaded thin elastic beams. It is worth noting that in the Euler–Bernoulli beam theory, rotary inertia and shear distortion are neglected. A theory including all these effects is due to Timoshenko [1]. For a recent review of the development of models for transversely vibrating thin elastic beams, we refer the reader to Han *et al* [2], and Park and Gao [3].

Our focus in this paper is on the *equivalence problem* for the unloaded one-dimensional Euler–Bernoulli equation

$$\frac{\partial^2}{\partial x^2}\left(f(x)\frac{\partial^2 u}{\partial x^2}\right) + m(x)\frac{\partial^2 u}{\partial t^2} = 0, \qquad t > 0, \quad 0 < x < L, \tag{1}$$

where $f(x) > 0$ is the flexural rigidity, $m(x) > 0$ is the lineal mass density and $u(t, x)$ is the transversal displacement at time t and position x from one end of the beam taken as origin. Equation (1) is customarily solved subject to initial and boundary conditions such as clamped ends, hinged ends and free ends boundary conditions.

Solving the equivalence problem for (1) amounts to finding necessary and sufficient conditions under which equation of the form (1) is mapped by an invertible transformation to a given equation of the same class. The equivalence problem is considered solved if we exhaust all the equivalence classes. There are typically two proven approaches to the equivalence problem for differential equations: *Lie's infinitesimal method* and *Cartan's equivalence method*. The two methods are loosely dual of each other: Lie's method uses vector fields whereas Cartan's method employs differential forms. However a major difference between the two methods is that Lie's infinitesimal method requires integration in the solution process whereas Cartan's method necessitates only differentiation. There are other subtle similarities and differences between the two methods. In order to appreciate these subtleties, the reader is referred to the books [4–6]. Arguably, Lie's method has received its fair share of popularization since the early 1980s. The same cannot be said about Cartan's method. A possible reason can be found in Weyl's review [7] of Cartan's book [8]: 'Nevertheless, I must admit that I found the book like most Cartan's papers, hard reading'. Therefore, it comes as no surprise that Cartan's students such as Ehresmann [9, 10] and Chern [11] dedicated a great deal of their work to the justification and explanation of Cartan's computations. In the process they introduced two fundamental concepts that lie at the heart of modern exposés of Cartan's method: jet spaces and G-structures. Undoubtedly, the books by Gardner [4], and Olver[6] made the case for Cartan's method even stronger by presenting the method in modern language supported by rigorous justifications of all calculations and constructions.

Our main goal in this paper is to implement Cartan's equivalence method on (1). Previous works on the equivalence problem for the Euler–Bernoulli equation include the paper by Gottlieb [12] where the author was concerned by beams equivalent to the uniform beam (i.e. m and f constant), the paper by Wafo Soh [13] dealing with Lie's approach to the equivalence problem for (1).

We have divided our exposé into four sections including this introduction. In section 2, we establish that the contact symmetry Lie algebra of (1) coincides with its point symmetry Lie algebra. Thus the contact equivalence problem for (1) is equivalent to the point equivalence problem. In section 3 that deals with the point equivalence problem for (1), the result of section 2 comes in handy since setting up the point equivalence problem in Cartan's method is generally more involved than the contact equivalence problem. In the final section, we sum up our findings.

2. Contact symmetries of the Euler–Bernoulli beam equation

In this section, we compute the infinitesimal contact symmetries of the Euler–Bernoulli beam equation. The main result is that the contact symmetry Lie algebra of the Euler–Bernoulli beam equation coincides with its point symmetry Lie algebra. Owing to this result, the contact equivalence problem for the Euler–Bernoulli beam equation is identical to the point equivalence problem.

We assume throughout this section that the reader is familiar with the theory of contact transformations and contact symmetries. For an authoritative introduction to these theories, the reader is referred to [14–16].

In order to simplify calculations we rewrite the Euler–Bernoulli equation as a system of first-order partial differential equations (PDEs).

$$\begin{cases} u^1_x = u^2, \quad u^2_x = u^3, \quad u^3_x = u^4, \quad u^1_t = u^5, \quad u^2_t = u^5_x \\ u^4_x = -2\dfrac{f'(x)}{f(x)}u^4 - \dfrac{f''(x)}{f(x)}u^3 - (g'(x))^4 u^5_t, \end{cases} \tag{2}$$

where the subscripts stand for partial differentiation, $u^1 = u$, and $m(x) = f(x)(g'(x))^4$ for later convenience. If X is a contact symmetry of the Euler–Bernoulli equation, its fourth prolongation $X^{[4]}$ rewritten in the variables t, x, u^1 to u^5 becomes a point symmetry of (2). However, a point symmetry of (2) when projected on the space with coordinates $(t, x, u^1 = u)$, is a contact symmetry of the Euler–Bernoulli equation if and only if the symmetry coefficients depend on $t, x, u^1 = u, u^2 = u_x$ and $u^5 = u_t$. This remark gives us a way to obtain the contact symmetries of the Euler–Bernoulli equation from the point symmetries of (2).

Now, a vector

$$X = \xi^1(t, x, u^j)\frac{\partial}{\partial t} + \xi^2(t, x, u^j)\frac{\partial}{\partial x} + \eta^i(t, x, u^j)\frac{\partial}{\partial u^i} \tag{3}$$

is a point symmetry of (2) if

$$\begin{cases} X^{[1]}\left(u^1_x - u^2\right) = 0, \quad X^{[1]}\left(u^2_x - u^3\right) = 0, \quad X^{[1]}\left(u^3_x - u^4\right) = 0, \\ X^{[1]}\left(u^1_t - u^5\right) = 0, \quad X^{[1]}\left(u^2_t - u^5_x\right) = 0 \\ X^{[1]}\left(u^4_x + 2\dfrac{f'(x)}{f(x)}u^4 + \dfrac{f''(x)}{f(x)}u^3 + (g'(x))^4 u^5_t\right) = 0, \end{cases} \tag{4}$$

whenever (2) is satisfied. In (4), $X^{[1]}$ is the first prolongation of X defined by

$$X^{[1]} = X + \eta^i_{,t}\frac{\partial}{\partial u^i_t} + \eta^i_{,x}\frac{\partial}{\partial u^i_x}, \tag{5}$$

$$\eta^i_{,t} = D_t(\eta^i) - u^i_t D_t(\xi^1) - u^i_x D_t(\xi^2), \tag{6}$$

$$\eta^i_{,x} = D_x(\eta^i) - u^i_t D_x(\xi^1) - u^i_x D_x(\xi^2), \tag{7}$$

$$D_t = \frac{\partial}{\partial t} + u^i_t\frac{\partial}{\partial u^i}, \qquad D_x = \frac{\partial}{\partial x} + u^i_x\frac{\partial}{\partial u^i}\cdot \tag{8}$$

Expanding the system (4) yields an over-determined system of linear PDEs. After lengthy albeit simple calculations and simplifications, we obtain

$$\xi^1 = 2c_1 t + c_3, \qquad \xi^2 = c_1\frac{g(x)}{g'(x)} + \frac{c_2}{g(x)}, \qquad \eta^1 = a(x)u^1 + b(t, x) \tag{9}$$

$$\eta^2 = D_x(\eta^1) - u^2 D_x(\xi^2), \eta^3 = D_x^2(\eta^1) - u^2 D_x^2(\xi^2) - 2u^3 D_x(\xi^2), \tag{10}$$

$$\eta^4 = D_x^3(\eta^1) - u^2 D_x^3(\xi^2) - 3u^3 D_x^2(\xi^2) - 3u^4 D_x(\xi^2), \tag{11}$$

$$\eta^5 = D_t(\eta^1) - u^5 D_t(\xi^1), \tag{12}$$

where the functions $a(x), b(t, x), f(x)$ and $g(x)$ are constrained by the following equations:

$$\partial_x^4 b + 2\frac{f'}{f}\partial_x^3 + \frac{f''}{f}\partial_x^2 b + g'^4\partial_t^2 b = 0, \tag{13}$$

$$a^{(4)} + 2\frac{f'}{f}a^{(3)} + \frac{f''}{f}a'' = 0, \tag{14}$$

$$\begin{aligned}2a' + c_1\left(\frac{f'}{f} + \frac{gf''}{g'f} - \frac{gf'^2}{g'f^2} - \frac{gg''f'}{g'^2f} + 3\frac{g''}{g} - 6\frac{gg''^2}{g'^3} + 3\frac{gg^{(3)}}{g'^2}\right)\\ + c_2\left(\frac{f''}{fg'} - \frac{f'^2}{f^2g'} - \frac{f'g''}{fg'^2} + 6\frac{g''^2}{g'^3} + 3\frac{g^{(3)}}{g'^2}\right) = 0,\end{aligned} \tag{15}$$

$$\begin{aligned}6\frac{f'}{f}a'' + 2\frac{f''}{f}a' + 4a^{(3)} + c_1\left(\frac{f''g''}{fg'} - 6\frac{f'g''^2}{fg'^2} + 2\frac{f''gg''^2}{fg'^3} + 12\frac{f'gg''^3}{fg'^4} + 12\frac{g''^3}{g'^3}\right.\\ - 24\frac{gg''^4}{g'^5} + 4\frac{f'g^{(3)}}{fg'} + \frac{f''gg^{(3)}}{fg'^2} - 12\frac{f'gg^{(3)}}{fg'^3} + 14\frac{g''g^{(3)}}{g'^2} + 36\frac{gg''^2g^{(3)}}{g'^4}\\ \left. + 6\frac{g(g^{(3)})^2}{g'^3} + 2\frac{f'gg^{(4)}}{fg'^2} + 3\frac{g^{(4)}}{g'} - 8\frac{gg''g^{(4)}}{g'^3} + \frac{gg^{(5)}}{g'^2}\right)\\ + c_2\left(12\frac{f'g''^3}{fg'^4} - 2\frac{f''g''^2}{fg'^3} + 24\frac{g''^4}{g'^5} + \frac{f''g^{(3)}}{fg'^2} - 12\frac{f'g''g^{(3)}}{fg'^3}\right.\\ \left. + 36\frac{g''^2g^{(3)}}{g'^4} - 6\frac{(g^{(3)})^2}{g'^3} + 2\frac{f'g^{(4)}}{g'^3} + 8\frac{g''g^{(4)}}{g'^3} + \frac{g^{(5)}}{g'^2}\right) = 0,\end{aligned} \tag{16}$$

$$\begin{aligned}6a'' + 6\frac{f'}{f}a' + c_1\left(2\frac{f''}{f} - \frac{f'f''g}{f^2g'^2} + 6\frac{f'g''}{fg'} - 2\frac{f''gg''}{fg'^2} + 12\frac{f'gg''^2}{fg'^3} - 12\frac{g''^2}{g'^2}\right.\\ \left. + 24\frac{gg''^3}{g'^4} + \frac{f^{(3)}g}{fg'} + 6\frac{f'gg^{(3)}}{fg'^2} + 8\frac{g^{(3)}}{g'} - 24\frac{gg''g^{(3)}}{g'^3} + 4\frac{gg^{(4)}}{g'^2}\right)\end{aligned} \tag{17}$$

$$\begin{aligned}+ c_2\left(-\frac{ff''}{f^2g'} - 2\frac{f''g''}{fg'^2} - 12\frac{f'g''^2}{fg'^3} + 24\frac{g''^3}{g'^4}\right.\\ \left. + \frac{f^{(3)}}{fg'} + 6\frac{f'g^{(3)}}{fg'^2} + 24\frac{g''g^{(3)}}{g'^3} + 4\frac{g^{(4)}}{g'^2}\right) = 0.\end{aligned} \tag{18}$$

The projection of the symmetry vector X on the space with coordinates $(t, x, u^1 = u)$ yields

$$\tilde{X} = (2c_1t + c_3)\frac{\partial}{\partial t} + \left(c_1\frac{g(x)}{g'(x)} + \frac{c_2}{g(x)}\right)\frac{\partial}{\partial x} + (a(x)u^1 + b(t, x))\frac{\partial}{\partial u}\cdot \tag{19}$$

The coefficients of $\tilde{X}$ are independent of u_t and u_x. Thus any contact symmetry of the Euler–Bernoulli equation is a point symmetry. The converse is trivial.

3. Point equivalence problem for the Euler–Bernoulli beams equations

In this section we consider the local equivalence problem for (1) under the action of the pseudo-group of point transformations. Two equations are said to be equivalent if there exists a point transformation which maps the equations to each other. We apply Élie Cartan's structure theory of Lie pseudo-groups (see [17, 22]), to obtain necessary and sufficient conditions under which equivalence mappings can be found. This theory describes a Lie pseudo-group in terms of a set of invariant differential 1-forms called *Maurer–Cartan forms*. Expressions of exterior differentials of Maurer–Cartan forms in terms of the forms themselves yield *Cartan structure equations* for the pseudo-group. The Maurer–Cartan forms contain all information about the pseudo-group, in particular, they give basic invariants and operators of invariant differentiation and allow one to solve equivalence problems for submanifolds under the action of the pseudo-group.

For convenience of computations we let $h(x) = g'(x)$ in (2). From the calculations of section 2, we infer that two equations from the class (1) are equivalent with respect to point transformations whenever their corresponding systems (2) are equivalent with respect to the pseudo-group $\mathrm{Cont}(J^1(\pi))$ of point transformations on the bundle $J^1(\pi)$ of the first-order jets for the bundle $\pi : \mathbb{R}^2 \times \mathbb{R}^5 \to \mathbb{R}^2, \pi : (t, x, u^1, \ldots u^5) \mapsto (t, x)$.

As is shown in [23], the following differential 1-forms,

$$\begin{aligned}
\Theta^\alpha &= a^\alpha_\beta (\mathrm{d}u^\beta - u^\beta_{x^j}\,\mathrm{d}x^j),\\
\Xi^i &= b^i_j\,\mathrm{d}x^j + c^i_\beta \Theta^\beta,\\
\Sigma^\alpha_i &= a^\alpha_\beta B^j_i\,\mathrm{d}u^\beta_{x^j} + q^\alpha_{i\beta}\Theta^\beta + r^\alpha_{ij}\Xi^j,
\end{aligned}$$

are Maurer–Cartan forms of $\mathrm{Cont}(J^1(\pi))$. They are defined on $J^1(\pi) \times \mathcal{H}$, where

$$\mathcal{H} = \left\{\left(a^\alpha_\beta, b^i_j, c^i_\beta, q^\alpha_{i\beta}, r^\alpha_{ij}\right) | \alpha, \beta \in \{1, \ldots, 5\}, i, j \in \{1, 2\}, \det\left(a^\alpha_\beta\right)\cdot\det\left(b^i_j\right) \neq 0, r^\alpha_{ij} = r^\alpha_{ji}\right\},$$

$\left(B^i_j\right)$ is the inverse matrix for $\left(b^i_j\right)$, and we have renamed the independent variables as $t = x^1, x = x^2$. They satisfy the structure equations

$$\begin{aligned}
\mathrm{d}\Theta^\alpha &= \Phi^\alpha_\beta \wedge \Theta^\beta + \Xi^k \wedge \Sigma^\alpha_k,\\
\mathrm{d}\Xi^i &= \Psi^i_k \wedge \Xi^k + \Pi^i_\gamma \wedge \Theta^\gamma,\\
\mathrm{d}\Sigma^\alpha_i &= \Phi^\alpha_\gamma \wedge \Sigma^\gamma_i - \Psi^k_i \wedge \Sigma^\alpha_k + \Lambda^\alpha_{i\beta} \wedge \Theta^\beta + \Omega^\alpha_{ij} \wedge \Xi^j,
\end{aligned}$$

where the forms $\Phi^\alpha_\beta, \Psi^i_j, \Pi^i_\beta, \Lambda^\alpha_{i\beta}$ and Ω^α_{ij} depend on differentials of the coordinates of $\mathcal{H}$.

System (2) defines a submanifold $\mathcal{R} \subset J^1(\pi)$. The Maurer–Cartan forms for its symmetry pseudo-group $\mathrm{Cont}(\mathcal{R})$ can be found from restrictions $\theta^\alpha = \iota^*\Theta^\alpha$, $\xi^i = \iota^*\Xi^i$ and $\sigma^\alpha_i = \iota^*\Sigma^\alpha_i$, where $\iota = \iota_0 \times \mathrm{id} : \mathcal{R} \times \mathcal{H} \to J^1(\pi) \times \mathcal{H}$ with $\iota_0 : \mathcal{R} \to J^1(\pi)$ defined by (2). In order to compute the Maurer–Cartan forms for the symmetry pseudo-group, we implement Cartan's equivalence method. Firstly, the forms $\theta^\alpha, \xi^i, \sigma^\alpha_i$ are linearly dependent, i.e. there exists a non-trivial set of functions $U_\alpha, V_i, W^i_\alpha$ on $\mathcal{R} \times \mathcal{H}$ such that $U_\alpha\theta^\alpha + V_i\xi^i + W^i_\alpha\sigma^\alpha_i \equiv 0$. Setting these functions equal to some appropriate constants allows one to express a part of the coordinates of $\mathcal{H}$ as functions of the other coordinates of $\mathcal{R} \times \mathcal{H}$. Secondly, we substitute the obtained values into the forms $\phi^\alpha_\beta = \iota^*\Phi^\alpha_\beta$ and $\psi^i_k = \iota^*\Psi^i_k$ and find their linear combinations which are semi-basic with respect to the projection $\mathcal{R} \times \mathcal{H} \to \mathcal{R}$. The coefficients of semi-basic forms ϕ^α_β at σ^γ_j, ξ^j, and the coefficients of semi-basic forms ψ^i_j at σ^γ_j are lifted invariants of $\mathrm{Cont}(\mathcal{R})$. We set them equal to appropriate constants and get expressions for the next part of the coordinates of $\mathcal{H}$ as functions of the other coordinates of $\mathcal{R} \times \mathcal{H}$. Thirdly, we analyze the reduced structure equations

$$\begin{aligned}
\mathrm{d}\theta^\alpha &= \phi^\alpha_\beta \wedge \theta^\beta + \xi^k \wedge \sigma^\alpha_k,\\
\mathrm{d}\xi^i &= \psi^i_k \wedge \xi^k + \pi^i_\gamma \wedge \theta^\gamma,\\
\mathrm{d}\sigma^\alpha_i &= \phi^\alpha_\gamma \wedge \sigma^\gamma_i - \psi^k_i \wedge \sigma^\alpha_k + \lambda^\alpha_{i\beta} \wedge \theta^\beta + \omega^\alpha_{ij} \wedge \xi^j.
\end{aligned}$$

If their coefficients contain coordinates of $\mathcal{H}$, we normalize them and repeat the process. Applying, if necessary, the procedure of prolongation, [6, chapter 12], we finally obtain Maurer–Cartan forms of $\mathrm{Cont}(\mathcal{R})$ together with its structure equations, differential invariants and invariant differentiations. The differential invariants parametrize classifying manifolds associated with system $\mathcal{R}$. Cartan's solution to the equivalence problem states that two systems are (locally) equivalent if and only if their classifying manifolds (locally) overlap.

For system (2) the first step yields linear dependences

$$\sigma^1_1 = 0, \qquad \sigma^1_2 = 0, \qquad \sigma^2_2 = 0, \qquad \sigma^3_2 = 0, \qquad \sigma^2_1 = \sigma^5_2, \qquad \sigma^4_2 = \sigma^5_1$$

after the following normalizations of the matrices (a^α_β) and (b^i_j),

$$\left(a^\alpha_\beta\right) = \begin{pmatrix} a^1_1 & 0 & 0 & 0 & 0 \\ a^2_1 & a^2_2 & 0 & 0 & 0 \\ a^3_1 & a^3_2 & a^3_3 & 0 & 0 \\ a^4_1 & a^4_2 & a^4_3 & \frac{(a^5_5)^2}{a^2_2 h^4} & 0 \\ a^5_1 & a^5_2 & 0 & 0 & a^5_5 \end{pmatrix}, \qquad \left(b^i_j\right) = b^1_1 \cdot \begin{pmatrix} 1 & 0 \\ \frac{a^4_5}{a^5_5} + \frac{a^5_2}{a^2_2} & \frac{a^5_5}{a^2_2} \end{pmatrix}$$

and normalizations of the parameters r^1_{ij} with $i, j \in \{1, 2\}$, r^α_{i2} with $\alpha \in \{2, \ldots, 5\}$, $i \in \{1, 2\}$, $q^\alpha_{2\beta}$ with $\alpha \in \{1, \ldots, 3\}$, $\beta \in \{1, \ldots, 5\}$, $r^5_{i\beta}$ with $i \in \{1, 2\}$, $\beta \in \{1, \ldots, 5\}$, which are too long to be written explicitly.

After the normalizations of the second step, the only free coordinates on $\mathcal{H}$ are a^1_1, a^2_2 and r^α_{11} with $\alpha \in \{2, \ldots, 5\}$, while all the other coordinates are expressed in terms of these ones, the functions $f(x)$ and $h(x)$, and their derivatives (we have omitted the explicit expressions since they are too long).

At the third step, we obtain the reduced structure equations

$$\begin{aligned} \mathrm{d}\theta^1 &= \eta_1 \wedge \theta^1 + \xi^1 \wedge \theta^5 + \xi^2 \wedge \theta^2, \\ \mathrm{d}\theta^2 &= \eta_2 \wedge \theta^2 + \xi^1 \wedge \sigma^5_2 + \xi^2 \wedge \theta^3, \\ \mathrm{d}\theta^3 &= (2\eta_2 - \eta_1) \wedge \theta^3 + \xi^1 \wedge \sigma^3_1 + \xi^2 \wedge \theta^4 + \tfrac{1}{8}\left(a^2_2\right)^2\left(a^1_1\right)^{-2} K_1 \xi^2 \wedge \theta^2, \\ \mathrm{d}\theta^4 &= (3\eta_2 - 2\eta_1) \wedge \theta^4 + \xi^1 \wedge \sigma^4_1 - \xi^2 \wedge \sigma^5_1 + \tfrac{3}{8}\left(a^2_2\right)^3\left(a^1_1\right)^{-3} h^{-1}(hK_1' - 2K_1 h')\xi^2 \wedge \theta^2 \\ &\quad + \tfrac{3}{8}\left(a^2_2\right)^2\left(a^1_1\right)^{-2} K_1 \xi^2 \wedge \theta^3, \end{aligned} \tag{20}$$

where

$$K_1 = 10\frac{h''}{h} - 15\left(\frac{h'}{h}\right)^2 + 4\frac{f''}{f} - 3\left(\frac{f'}{f}\right)^2,$$

and the differential forms η_1, η_2 depend on $\mathrm{d}a^1_1$ and $\mathrm{d}a^2_2$. To proceed with normalizations, we have to impose some restrictions on the functions $f(x)$ and $h(x)$. As a result of these restrictions, the following cases arise.

Case 1. When $K_1 \not\equiv 0$, $K_1 h^{-2} \not\equiv \text{const}$, we set the ratio of the coefficients at $\xi^2 \wedge \theta^2$ and $\xi^2 \wedge \theta^3$ in (20) to 1 and obtain

$$a^1_1 = a^2_2 \cdot \left(\frac{K_1'}{K_1} - 2\frac{h'}{h}\right).$$

Case 2. If $K_1 h^{-2} \equiv \kappa$, where κ is a non-zero constant, then we equate the coefficient of $\xi^2 \wedge \theta^3$ in (20) to $3\kappa/8$. This yields

$$a^1_1 = h a^2_2.$$

If $K_1 \equiv 0$, then we have the following structure equations:

$$\begin{aligned} \mathrm{d}\theta^1 &= \eta_1 \wedge \theta^1 + \xi^1 \wedge \theta^5 + \xi^2 \wedge \theta^2, \\ \mathrm{d}\theta^2 &= \eta_2 \wedge \theta^2 + \xi^1 \wedge \sigma^5_2 + \xi^2 \wedge \theta^3, \\ \mathrm{d}\theta^3 &= (2\eta_2 - \eta_1) \wedge \theta^3 + \xi^1 \wedge \sigma^3_1 + \xi^2 \wedge \theta^4, \end{aligned}$$

$$\begin{aligned}
d\theta^4 &= (3\eta_2 - 2\eta_1) \wedge \theta^4 + \xi^1 \wedge \sigma_1^4 - \xi^2 \wedge \sigma_1^5,\\
d\theta^5 &= -(\eta_1 + 2\eta_2) \wedge \theta^5 - \tfrac{1}{100}\left(a_2^2\right)^4\left(a_1^1\right)^{-4} K_2 \xi^1 \wedge \theta^1 + \xi^2 \wedge \sigma_2^5 + \xi^1 \wedge \sigma_1^5,\\
d\xi^1 &= 2(\eta_1 - \eta_2) \wedge \xi^1,\\
d\xi^2 &= (\eta_1 - \eta_2) \wedge \xi^2,\\
d\sigma_1^3 &= (4\eta_2 - 3\eta_1) \wedge \sigma_1^3 + \eta_3 \wedge \xi^1 + \xi^2 \wedge \sigma_1^4,\\
d\sigma_1^4 &= (5\eta_2 - 4\eta_1) \wedge \sigma_1^4 + \eta_6 \wedge \xi^1 - \eta_5 \wedge \xi^2,\\
d\sigma_1^5 &= (4\eta_2 - 3\eta_1) \wedge \sigma_1^5 + \eta_5 \wedge \xi^1 + \eta_4 \wedge \xi^2\\
&\quad + \tfrac{1}{100}\left(a_2^2\right)^5 h^{-1}\left(a_1^1\right)^{-5}\left(hK_2' - 4K_2h'\right)\xi^2 \wedge \theta^1\\
&\quad + \tfrac{1}{100}\left(a_2^2\right)^4\left(a_1^1\right)^{-4} K_2 \xi^2 \wedge \theta^2,\\
d\sigma_2^5 &= (3\eta_2 - 2\eta_1) \wedge \sigma_2^5 + \eta_4 \wedge \xi^1 - \sigma_1^3 \wedge \xi^2
\end{aligned} \tag{21}$$

where

$$K_2 = 10\frac{f^{(4)}}{f} - 10\frac{f'f'''}{f^2} - 11\left(\frac{f''}{f}\right)^2 + 24\frac{(f')^2 f''}{f^3} - 9\left(\frac{f'}{f}\right)^4.$$

Then new cases arise.

Case 3. When $K_1 \equiv 0$, $K_2 \not\equiv 0$ and $K_2 h^{-4} \not\equiv \text{const}$, we let the ratio of the coefficients at $\xi^2 \wedge \theta^1$ and $\xi^2 \wedge \theta^2$ in (21) equal to 1 and obtain

$$a_1^1 = a_2^2 \cdot \left(\frac{K_2'}{K_2} - 4\frac{h'}{h}\right).$$

Case 4. When $K_1 \equiv 0$ and $K_2 h^{-4} \equiv \kappa$, where κ is a non-zero constant, we assign the coefficient at $\xi^2 \wedge \theta^2$ in (21) to $\kappa/100$. Then we have

$$a_1^1 = h a_2^2.$$

Finally, we have

Case 5. $K_1 \equiv 0$ and $K_2 \equiv 0$.

In case 1, after a prolongation, we have the following structure equations:

$$\begin{aligned}
d\theta^1 &= \eta_1 \wedge \theta^1 + \xi^1 \wedge \theta^5 + \xi^2 \wedge \theta^2,\\
d\theta^2 &= \left(\eta_1 - \tfrac{1}{2}\left(1 + 2\mathbb{D}(L_1)L_1^{-1}\right)\xi^2\right) \wedge \theta^2 + \xi^1 \wedge \sigma_2^5 + \xi^2 \wedge \theta^3,\\
d\theta^3 &= \left(\eta_1 - \left(1 + 2\mathbb{D}(L_1)L_1^{-1}\right)\xi^2\right)\theta^3 + \xi^1 \wedge \sigma_1^3 + \xi^2 \wedge \left(\theta^4 + \tfrac{1}{8}L_1^{-2}\theta^2\right),\\
d\theta^4 &= \left(\eta_1 - \tfrac{3}{2}\left(1 + 2\mathbb{D}(L_1)L_1^{-1}\right)\xi^2\right)\theta^4 + \xi^1 \wedge \sigma_1^4 - \xi^2 \wedge \left(\sigma_1^5 - \tfrac{3}{8}L_1^{-2}(\theta^2 + \theta^3)\right),\\
d\theta^5 &= \left(\eta_1 - \left(1 + 2\mathbb{D}(L_1)L_1^{-1}\right)\xi^2\right) \wedge \theta^5 + \xi^1 \wedge \sigma_1^5 + \xi^2 \wedge \sigma_2^5\\
&\quad + \tfrac{1}{400}\left(60L_1\mathbb{D}(L_1) + 90L_1^2 - 4L_2 - 9\right)L_1^{-4}\xi^1 \wedge \theta^1,\\
d\xi^1 &= -\left(1 + 2\mathbb{D}(L_1)L_1^{-1}\right)\xi^1 \wedge \xi^2,\\
d\xi^2 &= 0,\\
d\sigma_1^3 &= \left(\eta_1 - 2\left(1 + 2\mathbb{D}(L_1)L_1^{-1}\right)\xi^2\right) \wedge \sigma_1^3 + \eta_2 \wedge \xi^1 + \xi^2 \wedge \left(\sigma_1^4 + \tfrac{1}{8}L_1^{-2}\sigma_2^5\right),\\
d\sigma_1^4 &= \left(\eta_1 - \tfrac{5}{2}\left(1 + 2\mathbb{D}(L_1)L_1^{-1}\right)\xi^2\right) \wedge \sigma_1^4 + \eta_5 \wedge \xi^1 + \xi^2 \wedge \left(\eta_4 + \tfrac{3}{8}L_1^{-2}(\sigma_1^3 + \sigma_2^5)\right),\\
d\sigma_1^5 &= \left(\eta_1 - 2\left(1 + 2\mathbb{D}(L_1)L_1^{-1}\right)\xi^2\right) \wedge \sigma_1^5 + \eta_4 \wedge \xi^1 + \eta_3 \wedge \xi^2 + \tfrac{1}{200}\left(2\mathbb{D}(L_2) + 4L_2\right.\\
&\quad \left. - 30L_1\mathbb{D}^2(L_1) - 30(\mathbb{D}(L_1))^2 - 150L_1\mathbb{D}(L_1) - 90L_1^2 + 9\right)L_1^{-4}\theta^1 \wedge \xi^2\\
&\quad + e\tfrac{1}{400}\left(60L_1\mathbb{D}(L_1) + 90L_1^2 - 4L_2 - 9\right)L_1^{-4}\theta^2 \wedge \xi^2,
\end{aligned}$$

$$
\begin{aligned}
\mathrm{d}\sigma_2^5 &= \left(\eta_1 - \tfrac{3}{2}\left(1 + 2\mathbb{D}(L_1)L_1^{-1}\right)\xi^2\right) \wedge \sigma_2^5 + \eta_3 \wedge \xi^1 - \sigma_1^3 \wedge \xi^2,\\
\mathrm{d}\eta_1 &= 0,\\
\mathrm{d}\eta_2 &= \left(\eta_1 - 3\left(1 + 2\mathbb{D}(L_1)L_1^{-1}\right)\xi^2\right) \wedge \eta_2 + \pi_1 \wedge \xi^1 - \left(\eta_5 + \tfrac{1}{8}L_1^{-2}\eta_3\right) \wedge \xi^2,\\
\mathrm{d}\eta_3 &= \left(\eta_1 - \tfrac{5}{2}\left(1 + 2\mathbb{D}(L_1)L_1^{-1}\right)\xi^2\right) \wedge \eta_3 + \pi_2 \wedge \xi^1 - \eta_2 \wedge \xi^2,\\
\mathrm{d}\eta_4 &= \left(\eta_1 - 3\left(1 + 2\mathbb{D}(L_1)L_1^{-1}\right)\xi^2\right) \wedge \eta_4 + \pi_3 \wedge \xi^1 + \pi_2 \wedge \xi^2 - \tfrac{1}{200}\left(2\mathbb{D}(L_2) + 4L_2\right.\\
&\qquad \left. - 30L_1\mathbb{D}^2(L_1) - 30(\mathbb{D}(L_1))^2 - 150L_1\mathbb{D}(L_1) - 90L_1^2 + 9\right) L_1^{-4}\theta^5 \wedge \xi^2\\
&\qquad + \tfrac{1}{400}\left(60L_1\mathbb{D}(L_1) + 90L_1^2 - 4L_2 - 9\right) L_1^{-4}\xi^2 \wedge \sigma_2^5,\\
\mathrm{d}\eta_5 &= \left(\eta_1 - \tfrac{7}{2}\left(1 + 2\mathbb{D}(L_1)L_1^{-1}\right)\xi^2\right) \wedge \eta_5 + \pi_4 \wedge \xi^1 - \left(\pi_3 + \tfrac{3}{8}L_1^{-2}(\eta_2 + \eta_3)\right) \wedge \xi^2,
\end{aligned}
$$

where we denote

$$
L_1 = \frac{K_1'}{|K_1|^{3/2}} - 2\frac{h'}{h|K_1|^{1/2}}, \qquad L_2 = \frac{K_2}{K_1^2}, \tag{22}
$$

and the invariant differentiation

$$
\mathbb{D} = \frac{1}{L_1|K_1|^{1/2}}\frac{\partial}{\partial x} \tag{23}
$$

is defined by the identity $\mathrm{d}R = R_x\mathrm{d}x = \mathbb{D}(R)\xi^2$ for an arbitrary function $R = R(x)$.

The structure equations in case 2 read

$$
\begin{aligned}
\mathrm{d}\theta^1 &= \eta_1 \wedge \theta^1 + \xi^1 \wedge \theta^5 + \xi^2 \wedge \theta^2,\\
\mathrm{d}\theta^2 &= \eta_1 \wedge \theta^2 + \xi^1 \wedge \sigma_2^5 + \xi^2 \wedge \theta^3,\\
\mathrm{d}\theta^3 &= \eta_1 \wedge \theta^3 + \xi^1 \wedge \sigma_1^3 + \xi^2 \wedge \left(\theta^4 + \tfrac{1}{8}\kappa\theta^2\right),\\
\mathrm{d}\theta^4 &= \eta_1 \wedge \theta^4 + \xi^1 \wedge \sigma_1^4 - \xi^2 \wedge \left(\sigma_1^5 - \tfrac{3}{8}\kappa\theta^3\right),\\
\mathrm{d}\theta^5 &= \eta_1 \wedge \theta^5 + \xi^1 \wedge \left(\sigma_1^5 - \tfrac{1}{400}(4M + 9\kappa^2)\theta^1\right) + \xi^2 \wedge \sigma_2^5,\\
\mathrm{d}\xi^1 &= 0,\\
\mathrm{d}\xi^2 &= 0,\\
\mathrm{d}\sigma_1^3 &= \eta_1 \wedge \sigma_1^3 + \eta_2 \wedge \xi^1 + \xi^2 \wedge \left(\sigma_1^4 + \tfrac{1}{8}\kappa\sigma_2^5\right),\\
\mathrm{d}\sigma_1^4 &= \eta_1 \wedge \sigma_1^4 + \eta_5 \wedge \xi^1 + \xi^2 \wedge \left(\eta_4 + \tfrac{3}{8}\kappa\sigma_1^3\right),\\
\mathrm{d}\sigma_1^5 &= \eta_1 \wedge \sigma_1^5 + \eta_4 \wedge \xi^1 - \xi^2 \wedge \left(\eta_3 - \tfrac{1}{100}\mathbb{D}(M)\theta^1 - \tfrac{1}{400}(4M + 9\kappa^2)\theta^2\right),\\
\mathrm{d}\sigma_2^5 &= \eta_1 \wedge \sigma_2^5 + \eta_3 \wedge \xi^1 - \sigma_1^3 \wedge \xi^2,\\
\mathrm{d}\eta_1 &= 0,\\
\mathrm{d}\eta_2 &= \eta_1 \wedge \eta_2 + \pi_1 \wedge \xi^1 - \left(\eta_5 + \tfrac{1}{8}\kappa\eta_3\right) \wedge \xi^2,\\
\mathrm{d}\eta_3 &= \eta_1 \wedge \eta_3 + \pi_2 \wedge \xi^1 - \eta_2 \wedge \xi^2,\\
\mathrm{d}\eta_4 &= \eta_1 \wedge \eta_4 + \pi_3 \wedge \xi^1 + \left(\pi_2 + \tfrac{1}{100}\mathbb{D}(M)\theta^5 + \tfrac{1}{400}\left(4M + 9\kappa^2\right)\sigma_2^5\right) \wedge \xi^2,\\
\mathrm{d}\eta_5 &= \eta_1 \wedge \eta_5 + \pi_4 \wedge \xi^1 - \left(\pi_3 + \tfrac{3}{8}\kappa\eta_2\right) \wedge \xi^2,
\end{aligned}
$$

where we denote

$$
M = K_2h^{-4}, \tag{24}
$$

while now

$$
\mathbb{D} = h^{-1}\frac{\partial}{\partial x}. \tag{25}
$$

In case 3 we get the following structure equations:

$$
\begin{aligned}
&d\theta^1 = \eta_1 \wedge \theta^1 + \xi^1 \wedge \theta^5 + \xi^2 \wedge \theta^2,\\
&d\theta^2 = \left(\eta_1 - \tfrac{1}{4}(1 + 4\mathbb{D}(N)N^{-1})\xi^2\right) \wedge \theta^2 + \xi^1 \wedge \sigma_2^5 + \xi^2 \wedge \theta^3,\\
&d\theta^3 = \left(\eta_1 - \tfrac{1}{2}(1 + 4\mathbb{D}(N)N^{-1})\xi^2\right) \wedge \theta^3 + \xi^1 \wedge \sigma_1^3 + \xi^2 \wedge \theta^4,\\
&d\theta^4 = \left(\eta_1 - \tfrac{3}{4}(1 + 4\mathbb{D}(N)N^{-1})\xi^2\right) \wedge \theta^4 + \xi^1 \wedge \sigma_1^4 - \xi^2 \wedge \sigma_1^5,\\
&d\theta^5 = \left(\eta_1 - \tfrac{1}{2}(1 + 4\mathbb{D}(N)N^{-1})\xi^2\right) \wedge \theta^5 + \xi^1 \wedge \left(\sigma_1^5 - \tfrac{1}{100}N^{-4}\theta^1\right) + \xi^2 \wedge \sigma_2^5,\\
&d\xi^1 = -\tfrac{1}{2}(1 + 4\mathbb{D}(N)N^{-1})\xi^1 \wedge \xi^2,\\
&d\xi^2 = 0,\\
&d\sigma_1^3 = (\eta_1 - (1 + 4\mathbb{D}(N)N^{-1})\xi^2) \wedge \sigma_1^3 + \eta_2 \wedge \xi^1 + \xi^2 \wedge \sigma_1^4,\\
&d\sigma_1^4 = \left(\eta_1 - \tfrac{5}{4}\,(1 + 4\mathbb{D}(N)N)\,\xi^2\right) \wedge \sigma_1^4 + \eta_5 \wedge \xi^1 - \eta_4 \wedge \xi^2,\\
&d\sigma_1^5 = (\eta_1 - (1 + 4\mathbb{D}(N)N^{-1})\xi^2) \wedge \sigma_1^5 + \eta_4 \wedge \xi^1 + \left(\eta_3 - \tfrac{1}{100}N^{-4}(\theta^1 + \theta^2)\right) \wedge \xi^2,\\
&d\sigma_2^5 = \left(\eta_1 + \tfrac{3}{4}\,(1 + 4\mathbb{D}(N)N)\,\xi^2\right) \wedge \sigma_2^5 + \eta_3 \wedge \xi^1 - \sigma_1^3 \wedge \xi^2,\\
&d\eta_1 = 0,\\
&d\eta_2 = \left(\eta_1 - \tfrac{3}{2}(1 + 4\mathbb{D}(N)N^{-1})\xi^2\right) \wedge \eta_2 + \pi_1 \wedge \xi^1 - \eta_5 \wedge \xi^2,\\
&d\eta_3 = \left(\eta_1 - \tfrac{5}{4}(1 + 4\mathbb{D}(N)N^{-1})\xi^2\right) \wedge \eta_3 + \pi_2 \wedge \xi^1 - \eta_2 \wedge \xi^2,\\
&d\eta_4 = \left(\eta_1 - \tfrac{3}{2}(1 + 4\mathbb{D}(N)N^{-1})\xi^2\right) \wedge \eta_4 + \pi_3 \wedge \xi^1 + \left(\pi_2 + \tfrac{1}{100}N^{-4}\left(\theta^5 + \sigma_2^5\right)\right) \wedge \xi^2,\\
&d\eta_5 = \left(\eta_1 - \tfrac{7}{4}(1 + 4\mathbb{D}(N)N^{-1})\xi^2\right) \wedge \eta_5 + \pi_4 \wedge \xi^1 - \pi_3 \wedge \xi^2,
\end{aligned}
$$

with

$$N = \frac{K_2'}{|K_2|^{5/4}} - 4\frac{h'}{h|K_2|^{1/4}} \tag{26}$$

and

$$\mathbb{D} = \frac{1}{N|K_2|^{1/4}}\frac{\partial}{\partial x}. \tag{27}$$

In case 4 the structure equations are

$$
\begin{aligned}
&d\theta^1 = \eta_1 \wedge \theta^1 + \xi^1 \wedge \theta^5 + \xi^2 \wedge \theta^2,\\
&d\theta^2 = \eta_1 \wedge \theta^2 + \xi^1 \wedge \sigma_2^5 + \xi^2 \wedge \theta^3,\\
&d\theta^3 = \eta_1 \wedge \theta^3 + \xi^1 \wedge \sigma_1^3 + \xi^2 \wedge \theta^4,\\
&d\theta^4 = \eta_1 \wedge \theta^4 + \xi^1 \wedge \sigma_1^4 - \xi^2 \wedge \sigma_1^5,\\
&d\theta^5 = \eta_1 \wedge \theta^5 + \xi^1 \wedge \left(\sigma_1^5 - \kappa\theta^1\right) + \xi^2 \wedge \sigma_2^5,\\
&d\xi^1 = 0,\\
&d\xi^2 = 0,\\
&d\sigma_1^3 = \eta_1 \wedge \sigma_1^3 + \eta_2 \wedge \xi^1 + \xi^2 \wedge \sigma_1^4,\\
&d\sigma_1^4 = \eta_1 \wedge \sigma_1^4 + \eta_5 \wedge \xi^1 - \eta_4 \wedge \xi^2,\\
&d\sigma_1^5 = \eta_1 \wedge \sigma_1^5 + \eta_4 \wedge \xi^1 + (\eta_3 - \kappa\theta^2) \wedge \xi^2,\\
&d\sigma_2^5 = \eta_1 \wedge \sigma_2^5 + \eta_3 \wedge \xi^1 - \sigma_1^3 \wedge \xi^2,\\
&d\eta_1 = 0,\\
&d\eta_2 = \eta_1 \wedge \eta_2 + \pi_1 \wedge \xi^1 - \eta_5 \wedge \xi^2,\\
&d\eta_3 = \eta_1 \wedge \eta_3 + \pi_2 \wedge \xi^1 - \eta_2 \wedge \xi^2,
\end{aligned}
$$

$$\mathrm{d}\eta_4 = \eta_1 \wedge \eta_4 + \pi_3 \wedge \xi^1 + \left(\pi_2 + \kappa\sigma_2^5\right) \wedge \xi^2,$$
$$\mathrm{d}\eta_5 = \eta_1 \wedge \eta_5 + \pi_4 \wedge \xi^1 - \pi_3 \wedge \xi^2.$$

In case 5 we have the following structure equations:

$$\begin{aligned}
\mathrm{d}\theta^1 &= \eta_1 \wedge \theta^1 + \xi^1 \wedge \theta^5 + \xi^2 \wedge \theta^2,\\
\mathrm{d}\theta^2 &= \eta_2 \wedge \theta^2 + \xi^1 \wedge \sigma_2^5 + \xi^2 \wedge \theta^3,\\
\mathrm{d}\theta^3 &= (2\eta_2 - \eta_1) \wedge \theta^3 + \xi^1 \wedge \sigma_1^3 + \xi^2 \wedge \theta^4,\\
\mathrm{d}\theta^4 &= (3\eta_2 - 2\eta_1) \wedge \theta^4 + \xi^1 \wedge \sigma_1^4 - \xi^2 \wedge \sigma_1^5,\\
\mathrm{d}\theta^5 &= (2\eta_2 - \eta_1) \wedge \theta^5 + \xi^1 \wedge \sigma_1^5 + \xi^2 \wedge \sigma_2^5,\\
\mathrm{d}\xi^1 &= 2(\eta_1 - \eta_2) \wedge \xi^1,\\
\mathrm{d}\xi^2 &= (\eta_1 - \eta_2) \wedge \xi^2,\\
\mathrm{d}\sigma_1^3 &= (4\eta_2 - 3\eta_1) \wedge \sigma_1^3 + \eta_3 \wedge \xi^1 + \xi^2 \wedge \sigma_1^4,\\
\mathrm{d}\sigma_1^4 &= \eta_6 \wedge \xi^1 - \eta_5 \wedge \xi^2 + (5\eta_2 - 4\eta_1) \wedge \sigma_1^4,\\
\mathrm{d}\sigma_1^5 &= \eta_4 \wedge \xi^2 + \eta_5 \wedge \xi^1 + (4\eta_2 - 3\eta_1) \wedge \sigma_1^5,\\
\mathrm{d}\sigma_2^5 &= \eta_4 \wedge \xi^1 + (3\eta_2 - 2\eta_1) \wedge \sigma_2^5 - \sigma_1^3 \wedge \xi^2,\\
\mathrm{d}\eta_1 &= 0,\\
\mathrm{d}\eta_2 &= 0,\\
\mathrm{d}\eta_3 &= (6\eta_2 - 5\eta_1) \wedge \eta_3 + \pi_1 \wedge \xi^1 - \eta_6 \wedge \xi^2,\\
\mathrm{d}\eta_4 &= (5\eta_2 - 4\eta_1) \wedge \eta_4 + \pi_2 \wedge \xi^1 - \eta_3 \wedge \xi^2,\\
\mathrm{d}\eta_5 &= (6\eta_2 - 5\eta_1) \wedge \eta_5 + \pi_3 \wedge \xi^1 + \pi_2 \wedge \xi^2,\\
\mathrm{d}\eta_6 &= (7\eta_2 - 6\eta_1) \wedge \eta_6 + \pi_4 \wedge \xi^1 - \pi_3 \wedge \xi^2.
\end{aligned}$$

These computations together with results of Cartan's equivalence method (see, e.g. [6, theorem 15.12]) yield the following.

Theorem. *The class of systems (2) is divided into five subclasses $\mathcal{A}_1$ to $\mathcal{A}_5$ invariant under an action of the pseudo-group of point transformations:*

$\mathcal{A}_1$ consists of all systems (2) such that $K_1 h^{-2} \not\equiv \mathrm{const}$;
$\mathcal{A}_2$ consists of all systems (2) such that $K_1 h^{-2} \equiv \kappa$, where κ is a non-zero constant;
$\mathcal{A}_3$ consists of all systems (2) such that $K_1 \equiv 0$ and $K_2 h^{-4} \not\equiv \mathrm{const}$;
$\mathcal{A}_4$ consists of all systems (2) such that $K_1 \equiv 0$, $K_2 h^{-4} \equiv \kappa$, where κ is a non-zero constant;
$\mathcal{A}_5$ consists of all systems (2) such that $K_1 \equiv 0$ and $K_2 \equiv 0$.

The basic differential invariants for systems from the subclass $\mathcal{A}_1$ are the functions L_1 and L_2 defined by (22), the invariant differentiation $\mathbb{D}$ is defined by (23). Two systems from $\mathcal{A}_1$ are equivalent w.r.t. the pseudo-group of point transformations whenever they have the same functional dependences among the invariants L_1, L_2, $\mathbb{D}(L_1)$ and $\mathbb{D}(L_2)$.

The basic differential invariant for systems from $\mathcal{A}_2$ is the function M defined by (24), and the invariant differentiation $\mathbb{D}$ is defined by (25). Two systems from $\mathcal{A}_2$ are equivalent if and only if they have the same value of κ and the same functional dependence among the invariants M and $\mathbb{D}(M)$.

The basic differential invariant for systems from $\mathcal{A}_3$ is the function N defined by (26), and the invariant differentiation $\mathbb{D}$ is defined by (27). Two systems from $\mathcal{A}_3$ are equivalent if and only if they have the same functional dependence among the invariants N and $\mathbb{D}(N)$.

Two systems from $\mathcal{A}_4$ are equivalent if and only if they have the same value of κ.

Every system from $\mathcal{A}_5$ is equivalent to the system (2) with $f \equiv 1$ and $h \equiv 1$.

4. An illustrative example

In [12], Gottlieb obtained seven classes of non-uniform Euler–Bernoulli beams that are isospectral with the unit beam (i.e. $f = 1, m = 1$). The method he employed to derive these beams consists in transforming the Euler–Bernoulli equation into a canonical equation using Baricilon's transformation [24]. Then, he obtained conditions under which the canonical equation is a unit beam. These conditions form a system of nonlinear PDEs. The solutions of this system yield instances of non-uniform beams that can be mapped to the unit beam. Gottlieb [12] found some explicit solutions of this system and used them to derive his models. All his models fall in the class $\mathcal{A}_5$ of the above theorem. Consider from class (1) of [12] the non-uniform beam with the following physical characteristics:

$$m(x) = 3(1+x)^{-1/2}, \qquad f(x) = \tfrac{1}{27}(1+x)^{3/2}. \tag{28}$$

Simple computations show that

$$h(x) = \sqrt{3}(1+x)^{-1/2} \tag{29}$$

and

$$K_1 = 0, \qquad K_2 = 0. \tag{30}$$

Therefore, according to our theorem, we may map (1) with f and m given by (28) to the unit beam. Indeed the change of coordinates [12]

$$v = u, \qquad z = 3((1+x)^{1/2} - 1), \quad t = t \tag{31}$$

does the job.

Now consider the following boundary-value problem for the beam with physical characteristics provided by (28)

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{27}(1+x)^{3/2}\frac{\partial^2 u}{\partial x^2}\right) + 3(1+x)^{-1/2}\frac{\partial^2 u}{\partial t^2} = 0, \qquad 0 \leqslant x \leqslant 3 \tag{32}$$

$$u(0,x) = x^3(3-x)^3, \qquad u_t(0,x) = 0, \tag{33}$$

$$u(t,0) = 0, \qquad u_{xx}(t,0) = 0, \quad u(t,3) = 0, \quad u_{xx}(t,3) = 0. \tag{34}$$

The boundary conditions (34) correspond to hinged ends. It can be verified that the boundary conditions (33) and (34) are compatible. Under the change of variables (31), the boundary-value problem (32)–(34) becomes

$$v_{zzzz} + v_{tt} = 0, \qquad 0 \leqslant z \leqslant 3, \tag{35}$$

$$v(0,z) = \frac{z^3(3-z)^3(6+z)^3(9+z)^3}{9^{12}}, \qquad v_t(0,z) = 0 \tag{36}$$

$$v(t,0) = 0, \qquad v_{zz}(t,0) = 0, \qquad v(t,3) = 0, \qquad v_{zz}(t,3) = 0. \tag{37}$$

We employ separation of variables to solve (35)–(37). That is, we look for a solution in the form

$$v(t,z) = T(t)V(z), \tag{38}$$

where $T(t)$ and $V(z)$ are functions to be determined. The insertion of the ansatz (38) into (35) yields

$$-\frac{\ddot{T}}{T} = \omega^4 = \frac{V''''}{V}, \tag{39}$$

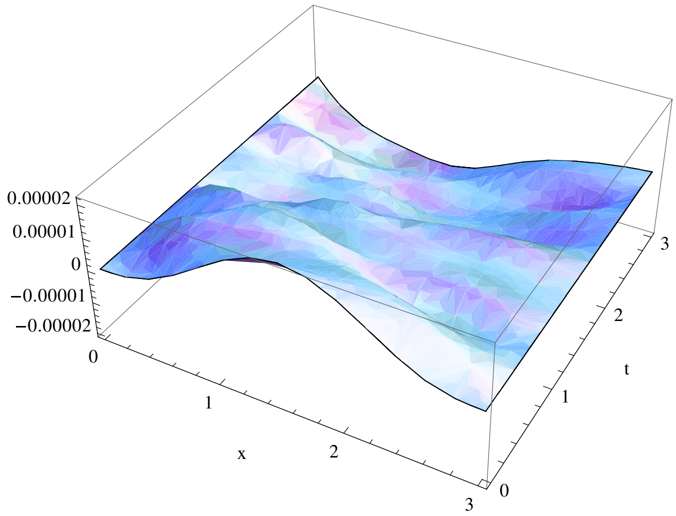

Figure 1. Solution of the boundary-value problem (32)–(34).

where the dot and prime stand for differentiations with respect to t and z respectively, and ω is a constant. Taking into account (36), the first equation of (39) yields

$$T(t) = k\cos(\omega^2 t), \tag{40}$$

where k is a constant. The function $V(z)$ satisfies the boundary-value problem

$$V'''' - \omega^4 V = 0, \qquad 0 \leqslant z \leqslant 3, \tag{41}$$

$$V(0) = 0, \qquad V''(0) = 0, \qquad V(3) = 0, \qquad V''(3) = 0. \tag{42}$$

The general solution of (41) is

$$V(z) = c_1 \sin(\omega z) + c_2 \cos(\omega z) + c_3 \sinh(\omega z) + c_4 \cosh(\omega z), \tag{43}$$

where c_1 to c_4 are constants. Imposing the boundary conditions (42) leads to the following restrictions:

$$c_2 = c_3 = c_4 = 0, \qquad \omega = \frac{k\pi}{3}, \qquad k \in \mathbb{Z} - \{0\}. \tag{44}$$

We end up with a family of solutions

$$v_n = a_n \cos\left(\frac{n^2\pi^2}{9} t\right) \sin\left(\frac{n\pi}{3} z\right), n \in \mathbb{N}, \tag{45}$$

where the $\{a_n\}$'s are constants. Since (35) is linear and homogeneous, the superposition of the solutions (45), i.e.

$$v = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n^2\pi^2}{9} t\right) \sin\left(\frac{n\pi}{3} z\right), \tag{46}$$

is again a solution of (35) and it satisfies the boundary conditions (37). We now impose the initial conditions (36) to obtain

$$a_n = \frac{2}{3 \times 9^{12}} \int_0^3 z^3 (3-z)^3 (6+z)^3 (9+z)^3 \sin\left(\frac{n\pi}{3} z\right) \mathrm{d}z. \tag{47}$$

The explicit expression of (47) is not presented since it is too large. Finally the solution of the boundary-value problem (32)–(34) is given by

$$u = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n^2\pi^2}{9}t\right)\sin(n\pi((1+x)^{1/2}-1)). \tag{48}$$

The solution (46) is plotted in figure 1.

5. Conclusion

We have completely solved the (local) point equivalence problem for the Euler–Bernoulli equation. We first show using Lie's infinitesimal method that the point equivalence problem for the Euler–Bernoulli equation is the same as the contact equivalence problem. The latter is easier to set up in the context of Cartan's equivalence method. After going through Cartan's equivalence algorithm, we found five inequivalent classes. For each class we compute a basis of differential invariants and the operators of invariant differentiations. Also, we provide for each class the necessary and sufficient conditions for equivalence.

It can be readily verified that the necessary and sufficient conditions under which (1) is reducible to the constant coefficient one is equivalent to those obtained by Wafo Soh [13] using Lie's infinitesimal method. However it remains to ascertain whether the equivalence classes obtained via Lie's infinitesimal method and Cartan's equivalence method are isomorphic. We note that, in Lie's approach, equivalence is established via smaller subalgebras (3D and 4D subalgebras) of the symmetry Lie algebra whereas Cartan's approach behooves much larger subalgebras (11D, 15D, 16D and 17D subalgebras) of the symmetry Lie algebra.

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