Reduction Properties of Ordinary Differential Equations of Maximal Symmetry

SIBUSISO MOYO

WITH

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ABSTRACT

- We study the ordinary differential equation

$$yy^{(n+1)} + \alpha y' y^{(n)} = 0$$

and show that this equation is always integrable for a certain value of α.

- We also note that there is a special case for a particular α for which this equation has a nonlocal symmetry which enables one to reduce it to an equation of maximal symmetry.

- Different features of the differential equation and its intrinsic connection to the $sl(2, R)$ subalgebra are illustrated including the connection to integrating factors. Here, we look at the reduction properties of these equations from an algebraic point of view.

Synopsis *videlicit* Motivation

- The original motivation comes from the Ermakov-Pinney equation [1, 2] which in its simplest form is

$$w'' + \frac{K}{w^3} = 0, \tag{1}$$

where K is a constant.

- In theoretical discussions the sign of the constant K is immaterial and in fact it is often rescaled to unity. In practical applications it would be negative to avoid 'collapse into the origin' due to its interpretation as the square of angular momentum [3, 4].

- The general form of (1), *videlicet*

$$\ddot{\rho} + \omega^2(t)\rho = \frac{1}{\rho^3} \tag{2}$$

occurs in the study of the time-dependent linear oscillator, be it the classical or the quantal problem, as the differential equation which determines the time-dependent rescaling of the space variable and the definition of 'new time'. In this context we mention the references [5, 6, 7, 8, 9, 10, 11].

- Another origin of (1) – of particular interest in this work – is as an integral of the third-order equation of maximal symmetry which in its elemental form is

$$y''' = 0. \tag{3}$$

- The integration of (3), which is a feature of the calculation of the symmetries of all linear ordinary differential equations of maximal symmetry [18], by means of an integrating factor gives a variety of results depending upon the integrating factor used. This includes the one relevant to (1).

- Some obvious integrating factors give

$$\begin{array}{rcl} 1.y''' = 0 & \longrightarrow & I_3 = y'' \\ x.y''' = 0 & & I_2 = xy'' - y' \\ \frac{1}{2}x^2.y''' = 0 & & I_1 = \frac{1}{2}x^2y'' - xy' + y \\ y''.y''' = 0 & & J = \frac{1}{2}y''^2 \quad \text{ie} \quad \frac{1}{2}I_3^2 \\ y.y''' = 0 & & y''y^3 + K = 0 \end{array} \tag{4}$$

  and the last of these is to (1) when the integral is interpreted as an equation. (The numbering of the fundamental first integrals follows the convention given in Flessas *et al*

[19, 20].)

- To illustrate the point on integrating factors we consider the equation of maximal symmetry (3) which has seven Lie point symmetries. These are

$$G_1 = \partial_y$$
$$G_2 = x\partial_y$$
$$G_3 = x^2\partial_y$$
$$G_4 = y\partial_y$$
$$G_5 = \partial_x$$
$$G_6 = x\partial_x + y\partial_y$$
$$G_7 = x^2\partial_x + 2xy\partial_y.$$

The algebra is $3A_1, \{sl(2,R) \oplus_s A_1\}$ and $3A_1$. The autonomous integrating factors for (3) are y'' and y as mentioned above. We list the symmetries and algebra when each of the integrating factors is treated as an equation and as a function.

When we multiply $y''' = 0$ by the integrating factor y'' we obtain $y''y''' = 0$. Integrating this expression gives $\frac{1}{2}y''^2 = k$, where k is a constant of integration. This may be rewritten as $y'' = k$ without loss of generality. This may be treated as the function y'' or as an equation $y'' - k = 0$ in which k is a parameter. We then have three cases for which we list the symmetries as follows:

$y'' = 0$	$y'' = k$	y''
$G_1 = \partial_y$	$G_1 = \partial_y$	$G_1 = \partial_y$
$G_2 = x\partial_y$	$G_2 = x\partial_y$	$G_2 = x\partial_y$
$G_3 = y\partial_y$	$G_3 = (\frac{1}{2}x^2k - y)\partial_y$	$G_3 = \partial_x$
$G_4 = \partial_x$	$G_4 = \partial_x + 2xk\partial_y$	$G_4 = x\partial_x + 2y$
$G_5 = x\partial_x$	$G_5 = x\partial_x + x^2k\partial_y$	
$G_6 = x^2\partial_x + xy\partial_y$	$G_6 = x^2\partial_x + (xy + \frac{1}{2}x^3k)\partial_y$	
$G_7 = y\partial_x$	$G_7 = (y - \frac{3}{2}x^2k)\partial_x - x^3k^2\partial_y$	
$G_8 = xy\partial_x + y^2\partial_y$	$G_8 = (xy - \frac{1}{2}x^3k)\partial_x +$	
	$(y^2 - \frac{1}{4}x^4k^2)\partial_y$	

- It is well known that when a symmetry is used to determine a first integral for a differential equation, the symmetry provides an integrating factor for the equation and remains as a symmetry of the first integral.

- **Definition:** We define a first integral I for an equation of maximal symmetry $E = y^{(n)} = 0$ as $I = f(y, y', y'', ..., y^{(n-1)})$ where

$$\frac{\mathrm{d}I}{\mathrm{d}x}_{|E=0} = 0 \iff \frac{\mathrm{d}f}{\mathrm{d}x}_{|E=0} = 0. \tag{5}$$

This means that if $g(x, y, y', y'', ..., y^{(n-1)})$ is an integrating factor then

$$\frac{\mathrm{d}I}{\mathrm{d}x}_{|E=0} = gE(x, y, y', ..., y^{(n)})_{|E=0} = 0. \tag{6}$$

Some Remarks

- The algebra of the symmetries listed in columns one and two is $sl(3,R) : 2A_1 \oplus_s \{sl(2,R) \oplus A_1\} \oplus 2A_1$ whereas that for the third column is $A_{4,9}^1 : A_2 \oplus_s 2A_1$.

- This is a clear indication of the distinction of the algebraic properties between first integrals, that is the function of column three, and configurational invariants, that the equations of columns one and two.

- If y is used an the integrating factor we obtain $yy''' = 0$. Integrating this equation gives

$$yy'' - \tfrac{1}{2}y'^2 = k \tag{7}$$

which can be written as

$$(y^{1/2})'' = \frac{k}{(y^{1/2})^3} \tag{8}$$

and is the simplest form of the Ermakov-Pinney equation [1, 2].

- As before we write down the point symmetries corresponding to the three cases of the differential equation $u'' = k/u^3$ where $u = y^{1/2}$. We have the following:

$$
\begin{array}{lll}
u'' = 0 & u'' = k/u^3 & u''u^3 \\
G_1 = \partial_u & G_1 = \partial_x & G_1 = \partial_x \\
G_2 = x\partial_u & G_2 = 2x\partial_x + u\partial_u & G_2 = 2x\partial_x + u\partial_u \\
G_3 = u\partial_u & G_3 = x^2\partial_x + xu\partial_u & G_3 = x^2\partial_x + xu\partial_u. \\
G_4 = \partial_x & & \\
G_5 = x\partial_x & & \\
G_6 = x^2\partial_x + xu\partial_u & & \\
G_7 = u\partial_x & & \\
G_8 = xu\partial_x + u^2\partial_u & &
\end{array}
\tag{9}
$$

- The transformation of $yy'' - \frac{1}{2}y'^2 = k$ to $u'' = k/u^3$ does not make a difference in terms of the symmetries as we just have a point transformation in this case.

- The characteristic feature of the Ermakov-Pinney equation, (1), is that it possesses the three-element algebra of Lie point symmetries $sl(2, R)$ which in itself is characteristic of all scalar ordinary differential equations of maximal symmetry.

- One of the interesting things to do will be to investigate higher-order analogues of the Ermakov-Pinney equation. The basic criterion is algebraic.

- In investigations of the Emden-Fowler equation [21, 22, 23, 24] the existence of a Lie point symmetry for certain indices* is intimately connected [25, 26, 27] with the solution of the fourth-order equation

$$2yy'''' + 5y'y''' + \alpha y''' = 0, \tag{10}$$

  where α is a parameter which occurs in both the Emden-Fowler equation and in the symmetry [26, 27, 28, 29, 30, 31].

- The determination of the first integral associated with the symmetry is not possible in closed-form for nonzero α. When $\alpha = 0$, the Lie symmetry becomes a Noether symmetry and the associated integral follows directly from an application of Noether's theorem [32]. Equation (10) possesses just two Lie point symmetries.

- However, in the case that $\alpha = 0$, *ie*, when the equation

*For instance when the Emden-Fowler equation of index two given by $y'' = f(x)y^2$ gives rise to (10). See [25] for a detailed treatment of this equation.

has the form

$$2yy'''' + 5y'y''' = 0, \tag{11}$$

there are the three Lie point symmetries

$$\Gamma_1 = \partial_x, \quad \Gamma_2 = x\partial_x \quad \text{and} \quad \Gamma_3 = y\partial_y. \tag{12}$$

Reduction by Γ_1 leads to a third-order equation which also has three Lie point symmetries. Two of these are the descendants of Γ_2 and Γ_3 as one would expect since Γ_1 is the normal subgroup in both cases. The third symmetry of the reduced equation,

$$\Lambda_4 = 2u^2\partial_u + uv\partial_v \tag{13}$$

(the variables of the reduced equation are $u = y$ and $v = y'$) is a hidden symmetry of Type II [33, 34, 35] and has its origin in the nonlocal symmetry of the fourth-order equation,

$$\Gamma_4 = 3\left(\int y\mathrm{dx}\right)\partial_x + 2y^2\partial_y. \tag{14}$$

- When the symmetry (13) is used to reduce the third-order equation to a second-order equation, the resulting equation is of maximal symmetry and so is linear when expressed in the correct coordinates [25]. Without this hidden symmetry the reduction by the three symmetries given in (12) leads to an Abel's equation of the second kind.

- Equation (11) was used by Euler *et al* [36] as an example in their study of the integrability properties of equations of the form

$$y^{(n+1)} = h\left(y, y^{(n)}\right) y' \tag{15}$$

and they showed that the equation could be reduced to $\mathrm{d}^4\mathrm{Y}/\mathrm{d}\mathrm{X}^4 = 0$, that is the fourth-order equation of maximal symmetry, by means of a complex sequence of nonlocal transformations which by most curious happenstance included the very Emden-Fowler equation from which it arose.

- In this work we draw together various features to which we have alluded above to make a coherent study. We commence with first integrals which possess three Lie point symmetries with the algebra $sl(2, R)$ and have a structure resembling that of (1). The associated differential equation is of the form of (15) with an explicit form of the function h, that is the imposition of the algebraic constraint provides a precise definition of the associated differential equation. Equation (11) does not fit into this structure. Equation (11) is a particular case of the two-parameter family of differential equations,

$$yy^{(n+1)} + \alpha y' y^{(n)} = 0, \tag{16}$$

and we make a study of the point symmetries for general values of the parameter α using the parameter-testing facility of Program LIE [37, 38].

- We find that there is a value of the parameter, α, for which (16) is always integrable. We see that (11) and its useful nonlocal symmetry is peculiar.

Higher-order analogues of the Ermakov-Pinney equation

– As we indicated above, there are several approaches which may be taken. Here we assume that the integral has the same structure as (1) and possesses the Lie algebra $sl(2,R)$ of point symmetries.

– Suppose that there exists an integral of the form

$$I = y^{(n)}y^{\alpha}, \tag{17}$$

when α is a parameter to be determined, with the associated $(n+1)$th-order equation

$$yy^{(n+1)} + \alpha y' y^{(n)} = 0 \tag{18}$$

such that the integral, I, has the $sl(2,R)$ symmetries appropriate [18] to the nth-order equation of maximal symmetry, $y^{(n)} = 0$, *ie*

$$\begin{aligned} \Gamma_1 &= \partial_x \\ \Gamma_2 &= x\partial_x + \tfrac{1}{2}(n-1)y\partial_y \\ \Gamma_3 &= x^2\partial_x + (n-1)xy\partial_y. \end{aligned} \tag{19}$$

– The first symmetry, Γ_1, is implied by the autonomy of (17). Since the Lie Bracket, $[\Gamma_1, \Gamma_3]_{LB} = 2\Gamma_1$, we need use only Γ_1 and Γ_3. The $(n+1)$th extension of Γ_3 is

$$\Gamma_3^{[n+1]} = x^2\partial_x + (n-1)xy\partial_y + \left[(n-1)y + (n-3)xy'\right]\partial_{y'}$$
$$+\left[2(n-2)y' + (n-5)xy''\right]\partial_{y''} + \left[3(n-3)y'' + (n-7)xy'''\right]\partial_{y'''} +$$
$$+\left[n-(2n+1)\right]xy^{(n)}\partial_{y^{(n)}} - \left[(n+1)y^{(n)} + (n+3)xy^{(n+1)}\right]\partial_{y^{(n+1)}}.$$

Since both integral and equation are autonomous, (20) may be split into an x-free part and an x-dependent part. The former does not contain any operators of relevance to the integral and so gives zero automatically. In the case of the latter we obtain

$$-(n+1)xy^{(n)}y^{\alpha} + (n-1)\alpha xyy^{(n)}y^{\alpha-1} \qquad (21)$$

which is zero provided the parameter α takes the value $(n+1)/(n-1)$. Evidently $n \neq 1$, *ie* there does not exist a first-order Ermakov-Pinney equation.

– We recall that the Ermakov-Pinney equation has the form

$$w'' = \frac{K}{w^3} \qquad (22)$$

in the notation adopted above whereas its primitive form is

$$yy'' - \tfrac{1}{2}y'^2 = \tfrac{1}{2}K \tag{23}$$

as the direct integral of $y''' = 0$ in association with the integrating factor y. The transformation is $y = \frac{1}{2}w^2$. Does a similar property persist at the higher order?

– Consider the general equation (18) constrained to possess the representation of the $sl(2, R)$ subalgebra given in (19)[†],

$$yy^{(n+1)} + \frac{n+1}{n-1}y'y^{(n)} = 0, \tag{24}$$

which is the derivative of the Ermakov-Pinney equation

$$y^{(n)} + \frac{K}{y^{(n+1)/(n-1)}} = 0. \tag{25}$$

– For the purposes of this treatment equation (25) defines the general Ermakov-Pinney equation for $n > 1$. The property holds for $n = 2$. For $n = 3$ the Ermakov-Pinney equation is

$$y''' + \frac{K}{y^2} = 0 \tag{26}$$

[†]Equation (24) is not the most general nth order ordinary differential equation invariant under $sl(2, R)$. See [39] for a detailed treatment of this question.

and the corresponding fourth-order equation is

$$yy'''' + 2y'y''' = 0. \tag{27}$$

– To see if this equation has a simple form we set $y = w^\alpha$, where α is a parameter to be determined. The fourth-order equation becomes, after a modicum of simplification,

$$w^3w'''' + 2(3\alpha - 1)w^2w'w''' + 3(\alpha - 1)w^2w''^2 + 12(\alpha - 1)^2ww'^2w'' + 3(\alpha - 1)^2(\alpha - 2)w'^4 = 0. \tag{28}$$

It is evident that the original form of the equation is the simplest available under this class of transformations. In the case of $n = 2$ the ability to reduce the nonlinear equation obtained by differentiation of the second-order Ermakov-Pinney equation to the third-order equation of maximal symmetry was accidental and not an intrinsic property of Ermakov-Pinney equations.

The structure of Euler *et al*

A second approach to the investigation of equations of the structure of the Ermakov-Pinney equation is to begin from the structure treated by Euler *et al* [36]. The model equation which they treated had the general form

$$y^{(n+1)} = h\left(y, y^{(n)}\right) y'. \tag{29}$$

We impose an $sl(2,R)$ algebraic structure on this equation. We take the structure to be

$$\begin{aligned} \Gamma_1 &= \partial_x \\ \Gamma_2 &= x\partial_x + my\partial_y \\ \Gamma_3 &= x^2\partial_x + 2mxy\partial_y, \end{aligned} \tag{30}$$

where the parameter m is at our disposal.

The structure assumed for (29) makes the possession of Γ_1 automatic. The action of the $(n+1)$th extension of Γ_2 leads to

$$my\frac{\partial h}{\partial y} + (m-n)y^{(n)}\frac{\partial h}{\partial y^{(n)}} = -nh \tag{31}$$

$$\frac{\mathrm{d}y}{my} = \frac{\mathrm{d}y^{(\mathrm{n})}}{(m-n)y^{(n)}} = \frac{\mathrm{d}h}{-nh} \tag{32}$$

and from the associated Lagrange's system, (32), we find that the characteristics of the first-order linear partial differential equation, (31), are

$$u = hy^{n/m} \quad \text{and} \quad v = \frac{y^{(n)}}{y^{(m-n)/m}} \tag{33}$$

so that the form of (29) invariant under the actions of Γ_1 and Γ_2 is

$$y^{n/m}y^{(n+1)} = g\left(\frac{y^{(n)}}{y^{(m-n)/m}}\right)y'. \tag{34}$$

We now turn to Γ_3. The $(n+1)$th extension is

$$\begin{aligned}\Gamma_3^{[n+1]} = {} & x^2\partial_x + 2mxy\partial_y + 2\left[my + (m-1)xy'\right)\partial_{y'} \\ & +2\left[(2m-1)y' + (m-2)xy''\right]\partial_{y''} + \ldots + \\ & 2\left\{\left[nm - \tfrac{1}{2}n(n-1)\right]y^{(n-1)} + (m-n)xy^{(n)}\right\}\partial_{y^{(n)}} \\ & +2\left\{\left[(n+1)m - \tfrac{1}{2}n(n+1)\right]y^{(n)} + (m-n-1)xy^{(n+1)}\right\}\partial_{y^{(n+1)}} \quad (3\end{aligned}$$

When this is applied to (34), there is no need to consider the part which has x as coefficient since the actions of Γ_1 and Γ_2 have already done that. The effective part of the operator gives

$$\begin{aligned} & y^{n/m}\left[(n+1)m - \tfrac{1}{2}n(n+1)\right]y^{(n)} \\ & = gmy + \frac{g'}{y^{(m-n)/m}}\left[nm - \tfrac{1}{2}n(n-1)\right]y^{(n-1)}y'. \end{aligned} \tag{36}$$

However, $y^{(n-1)}$ is not permitted. The coefficient of $y^{(n-1)}$ must be zero, *ie* $m = \frac{1}{2}(n-1)$. With this restriction on the value of m we find from (36) that

$$g = -\frac{n+1}{n-1} y^{\displaystyle\frac{n+1}{n-1}} y^{(n)}$$

and the equation is specifically (24). Thus the $sl(2,R)$ equation in combination with the constraint of Euler *et al* is unique at all orders.

Conclusion

- We have studied the differential equation $yy^{(n+1)} + \alpha y' y^{(n)} = 0$ and shown that this equation is always integrable for a certain value of α.
- $\alpha = 0$ is a special case for which (10) (a special case of the above equation) has a nonlocal symmetry which enables one to reduce it to an equation of maximal symmetry.
- Different features of the differential equation and its intrinsic connection to the $sl(2,R)$ subalgebra are illustrated including the connection to integrating factors.

- We have shown how the integrating factors, for example, of the third ordinary differential equation give different symmetry properties depending on which integrating factor is used.

- It is important to mention that if y is an integrating factor of $y^{(n)} = 0$, then the integral obtained by using this integrating factor always has the $sl(2, R)$ subalgebra.

*

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References

[1] Ermakov, V.: Second-order differential equations. Conditions of complete integrability *Universita Izvestia Kiev Series III*, **9**, 1-25 (trans Harin AO), 1880.

[2] Pinney, E.: The nonlinear differential equation $y''(x) + p(x)y + cy^{-3} = 0$, *Proceedings of the American Mathematical Society*, **1**, 681, 1950.

[3] Whittaker, E.T.: *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Dover, New York, 1944.

[4] Camiz, P., Geradi, A., Marchioro, C., Presutti, E. and Scacciatelli, E.: Exact solution of a time-dependent quan-

tal harmonic oscillator with a singular perturbation, *Journal of Mathematical Physics*, **12**, 2040-2043, 1971.

[5] Lewis, H. Ralph Jr.: Classical and quantum systems with time-dependent harmonic oscillator-type Hamiltonians, *Physical Review Letters*, **18**, 510-512, 1967.

[6] Lewis, H. Ralph Jr.: Motion of a time-dependent harmonic oscillator and of a charged particle in a time-dependent axially symmetric electromagnetic field, *Physical Review*, **172**, 1313-1315, 1968.

[7] Lewis, H. Ralph Jr.: Class of exact invariants for classical and quantum time-dependent harmonic oscillators, *kept Journal of Mathematical Physics*, **9**, 1976-1986, 1968.

[8] Lewis, H. Ralph Jr. and Riesenfeld, W.B.: An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field, *Journal of Mathematical Physics*, **10**, 1458-1473, 1969.

[9] Leach, P.G.L.: On a Direct method for the Determination of an Exact Invariant for the Time-Dependent Harmonic

Oscillator, *Journal of the Australian Mathematical Society, Series B*, **20**, 97-105, 1977.

[10] Leach, P.G.L.: Invariants and Wave-Functions for some Time-Dependent Harmonic Oscillator-Type Hamiltonians, *Journal of Mathematical Physics*, **18**, 1902-1907, 1977.

[11] Leach, P.G.L.: Berry's phase and wave functions for time-dependent Hamiltonian systems, *Journal of Physics A: Mathematical and General*, **23**, 2695-2699, 1990.

[12] Athorne, C., Rogers, C., Ramgulam, U. and Osbaldestin, A.: On linearization of the Ermakov system, *Physics Letters A*, **143**, 207-212, 1990.

[13] Govinder, K.S., Athorne, C. and Leach, P.G.L.: The algebraic structure of generalized Ermakov systems in three dimensions *Journal of Physics A: Mathematical and General* **26** 4035-4046, 1993.

[14] Ray, J.R.: Nonlinear superposition law for generalized Ermakov systems, *Physics Letters A*, **78**, 4-6, 1980.

[15] Ray, J.R. and Reid, J.L.: Noether's theorem and Ermakov systems for nonlinear equations of motion, Il Nuovo Cimento, **59A**, 134-140, 1980.

[16] Ray, J.R. and Reid, J.L.: Ermakov systems, velocity dependent potentials and nonlinear superposition, *Journal of Mathematical Physics*, **22**, 91-95, 1981.

[17] Sarlet, W.: Further generalizations of Ray-Reid systems, *Physics Letters*, **82A**, 161-164, 1981.

[18] Mahomed, F.M. and Leach P.G.L.: Symmetry Lie algebras of nth order ordinary differential equations, *Journal of Mathematical Analysis and Applications*, **151**, 80-107, 1990.

[19] Flessas, G.P., Govinder, K.S. and Leach, P.G.L.:Remarks on the symmetry Lie algebras of first integrals of scalar third order ordinary differential equations with maximal symmetry, *Bulletin of the Greek Mathematical Society*, **36**, 63-79, 1994.

[20] Flessas, G.P., Govinder, K.S. and Leach, P.G.L.: Characterisation of the algebraic properties of first integrals

of scalar ordinary differential equations of maximal symmetry, *Journal of Mathematical Analysis and Applications*, **212**, 349-374, 1997.

[21] Lane, I.J. Homer.: On the theoretical temperature of the Sun under the hypothesis of a gaseous mass maintaining its volume by its internal heat and depending on the laws of gases known to terrestrial experiment, *American Journal of the Sciences and Arts*, **4**, 57-74, 1869-70.

[22] Emden, R.: *Gaskugeln, Anwedungen der mechanishen Warmentheorie auf Kosmologie und meteorologische Probleme*, BG Teubner, Leipzig, 1987.

[23] Fowler, R.H.: The form near infinity of real, continuous solutions of a certain differential equation of the second order *Quarterly Journal of Mathematics* , **45**, 289-350, 1914.

[24] Fowler, R. H.: Further studies of Emden's and similar differential equations, *Quarterly Journal of Mathematics (Oxford)*, **2**, 259-288, 1931.

[25] Leach, P.G.L. and Govinder, K.S.: Hidden symmetries and integration of the generalized Emden-Fowler equation of index two, *Proceedings 14th IMACS World Congress on Computational and Applied Mathematics*, Ames, W.F. (Eds.), Georgia Institute of Technology, Atlanta, 300-303, 1994.

[26] Leach, P.G.L.: An exact invariant for a class of time-dependent anharmonic oscillators with cubic anharmonicity, *Journal of Mathematical Physics*, **22**, 465-470, 1981.

[27] Leach, P.G.L.: First integrals for the modified Emden equation $\ddot{q} + \alpha(t)\dot{q} + q^n = 0$, *Journal of Mathematical Physics*, **26**, 2510-2514, 1985.

[28] Leach, P.G.L. and Maharaj, S.D.: A first integral for a class of time-dependent anharmonic oscillators with multiple anharmonicities, *Journal of Mathematical Physics*, **33**, 2023-2030, 1992.

[29] Leach, P.G.L., Maartens, R. and Maharaj, S.D.:Self-similar solutions of the generalized Emden-Fowler equa-

tion, *International Journal of Nonlinear Mechanics*, **27**, 575-582, 1992.

[30] Maharaj, S.D., Leach, P.G.L. and Maartens, R.: Shear-free spherically symmetric solutions with conformal symmetry *General Relativity and Gravitation*, **23**, 261-267, 1991.

[31] Mellin, C. M., Mahomed, F.M. and Leach, P.G.L.: Solution of generalized Emden-Fowler equations with two symmetries, *International Journal of Nonlinear Mechanics*, **29**, 529-538, 1994.

[32] Noether, E.: Invariante Variationsprobleme, *Königlich Gesellschaft der Wissenschaften Göttingen Nachrichten Mathematik-physik Klasse*, **2**, 235-267, 1918.

[33] Abraham, S.B. and Guo, A.: Hidden symmetries associated with the projective group of nonlinear first order ordinary differential equations, *Journal of Physics A: Mathematical and General*, **25**, 5597-5608, 1992.

[34] Abraham, S.B.: Hidden symmetries and linearization of the modified Painlevé-Ince equation, *Journal of Mathe-

matical Physics, **34**, 4809-4816, 1993.

[35] Abraham, S.B. and Leach, P.G.L.: Hidden symmetries of nonlinear ordinary differential equations, *Exploiting symmetry in Applied and Numerical Analysis*, Allgower,E., Georg, K. and Miranda, R. (Eds.), Lectures in Applied Mathematics, **29** , AMS, Providence, 1-10, 1993.

[36] Euler, N. and Leach, P.G.L.: First integrals and reduction of a class of nonlinear higher order ordinary differential equations, *Journal of Mathematical Analysis and Applications*, **287**, 337-347, 2003.

[37] Head, A.K.: `LIE`, a PC program for Lie analysis of differential equations, *kept computer Physics Communications*, **77**, 241-248, 1993.

[38] Sherring, J., Head, A.K. and Prince, G.E.: Dimsym and LIE: symmetry determining packages, *Mathematical and Computer Modelling*, **25**, 153-164, 1997.

[39] Leach, P.G.L. and Govinder, K.S.: On the uniqueness of the Schwarzian and linearisation by nonlocal contact trans-

formation, *Journal of Mathematical Analysis and Application*, **235**, 84-107, 1999.