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Lie symmetry analysis for classification on pattern in excitable media

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Abstract

The dynamics of wave front in two dimensional excitable media is described by the derivative Burgers' equation. In this paper, we will carry out Lie symmetry analysis to the equation for constructing particular solutions associated with chemical patterns. The form of the variable coefficient in the reduced equation by using symmetries classifies the invariant solutions into three cases and the solutions include arc, circle, knee, spiral and double scroll patterns.

1. Introduction

Pattern formation by chemical reaction in excitable media has been investigated by theoretical or experimental studies [1–11]. Analytical treatment to the phenomena is mainly performed by a probe into the reaction–diffusion equations which are evolution equations to the concentrations of the activator and the inhibitor. Yamada and Nozaki [11] have investigated isolated wave fronts in two dimension and assume a width of a boundary layer between the rest and excited region of the activator are enough thin. For this situation, they have derived a derivative Burgers' equation (DB equation) as an equation governing the dynamics of the boundary layer (i.e. wave front) by using a singular perturbation method. As a result of their study, we can treat various chemical patterns in one frame, i.e. the DB equation. This equation can be linearized to the heat equation by the Cole–Hopf transformation. Yamada and Nozaki have focused on group invariant solutions under the Lie symmetry of the heat equation. Their result implies that various patterns are classified into three types by the Lie symmetries.

Lie symmetry analysis is one of the most powerful methods to get particular solutions of differential equations [12]. It is based on the study of their invariance with respect to one-parameter Lie group of point transformations whose infinitesimal generators are represented as vector fields. Once the Lie groups that leave the differential equations invariant are known, we can construct an exact solution called a group invariant solution which is invariant under the transformation.

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The aim of this paper is to present a neat formalism to complete classification of group invariant solutions of the DB equation. Lie symmetry analysis provides a list of symmetries and reduced equations described by invariant variables only. Analytical expressions of the solutions (i.e. shapes of patterns) may not be related to symmetry properties directly. The simplest way to notice a relation between the solutions and shapes of patterns is to classify the solutions by the reduced DB equation and to plot their figures. The results include that the variable coefficient in the reduced equation is a key to perform classification, and show some figures of the solutions. And furthermore, the boundary value problem for patterns with an edge are discussed.

2. Lie symmetry analysis to the DB equation

By using the multiplescale technique, the reaction–diffusion equations are reduced into the DB equation which is an equation of the tangential velocity along with the boundary layer between the chemical activities [11]. The DB equation is written as

$$u_{xt} = (uu_x + u_{xx})_x, \tag{1}$$

where $s = x/(\sqrt{\epsilon}K)$, t and $\alpha = \sqrt{\epsilon}dKu(x,t)$ are the arclength, the time and the tangential velocity, respectively; d is a diffusion constant of the activator, $1/K$ a characteristic length which has relation to parameters representing pattern and ϵ a small parameter in multiplescale expansion.

An infinitesimal generator of Lie symmetry is given in the following form:

$$\mathbf{V} = \tau\partial_t + \xi\partial_x + \phi\partial_u. \tag{2}$$

Eq. (1) admits the Lie symmetry whose coefficients are

$$\tau = c_1t^2 + c_2t + c_3,$$
$$\xi = \frac{1}{2}\tau_t x + \tau\sqrt{\tau}\bar{\xi}_t,$$
$$\phi = -\frac{1}{2}\tau_t u - \frac{1}{2}\tau_{tt}x - (\tau\sqrt{\tau}\bar{\xi})_t,$$

where c_i $(i = 1,2,3)$ are arbitrary constants and $\bar{\xi}$ is an arbitrary function of the time t. The symmetry (2) is derived through the standard procedure.

The group invariant solution is a solution of a transformed differential equation by similarity variables, which is obtained by solving the Lie equation associated with the symmetry (2)

$$\frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}u}{\phi}. \tag{3}$$

Solving Eq. (3) yields the similarity variables

$$y = \frac{x}{\sqrt{\tau}} - \bar{\xi}(t), \qquad u = -\frac{\tau_t y + U(y) + \tau_t\bar{\xi} + 2\tau\bar{\xi}_t}{2\sqrt{\tau}}. \tag{4}$$

Under the transformation (4), we can reduce Eq. (1) to an ordinary differential equation of the form

$$2U''' - (U')^2 - UU'' + D = 0, \tag{5}$$

where prime denotes differentiation with respect to y and $D = c_2^2 - 4c_1c_3$. Eq. (5) can be integrated twice and be linearized by the Cole–Hopf transformation $u = -4V'/V$, we get

$$-4V'' + F(y)V = 0, \quad \text{where } F(y) = \frac{D}{4}y^2 + ay + b, \tag{6}$$

where a and b are integration constants. Since the function $F(y)$ are in three cases i.e. constant, linear and quadrature. In the next section, we enumerate all group invariant solutions in the cases.

3. Classification for group invariant solutions and related patterns

We present a complete classification of solutions of the linearized equation (6) and typical patterns in the following figures.

3.1. The case $F(y) = b$

In this case, we find three types of solutions according to the sign of b:

$$\text{the case } b > 0, \quad V = C_1 \exp(\sqrt{b}y/2) + C_2 \exp(-\sqrt{b}y/2), \tag{7}$$

$$\text{the case } b = 0, \quad V = C_1 y + C_2, \tag{8}$$

$$\text{the case } b < 0, \quad V = C_1 \cos(\sqrt{|b|}y/2) + C_2 \sin(\sqrt{|b|}y/2), \tag{9}$$

where C_1 and C_2 are arbitrary constants. In Figs. 1–4, we illustrate shapes of patterns associated with the solution for $b = 4$. The solution (7) becomes $\cosh(y)$ and $\sinh(y)$ in $C_1 = C_2 = 1/2$ and $C_1 = -C_2 = 1/2$, respectively. The function τ are chosen as 1 and $(t+1)^2$. Especially, an evolution of pattern are shown at the time $t = 0$, 1, 2 in Fig. 2. In Figs. 5 and 6, we choose the function $V(y)$ as y with $\tau = 1$ and $(t+1)^2$, respectively. In Figs. 7 and 8, we use only a part of $\cos(y)$ and let values of parameters, $C_1 = 1$ and $b = -4$. Fig. 8 shows shapes at the time $t = 1$, 2 and 3.

3.2. The case $F(y) = ay + b$ $(a \neq 0)$

In this case, we get

$$V = C_1 Ai\left\{\left(\frac{a}{4}\right)^{1/3}\left(y + \frac{b}{a}\right)\right\} + C_2 Bi\left\{\left(\frac{a}{4}\right)^{1/3}\left(y + \frac{b}{a}\right)\right\}, \tag{10}$$

where Ai and Bi are the Airy's functions and C_1 and C_2 are arbitrary constants. Let the parameters, $a = 4$ and $b = 0$. Figs. 9–12 show patterns in all combination of two Airy's function in V and $\tau = 1$, $(t+1)^2$.

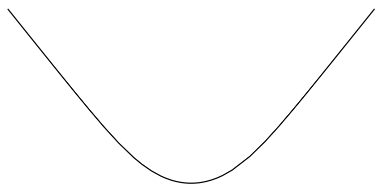

Fig. 1. The case $F(y) = b$ ($D = 0$, $a = 0$, $b > 0$): $V = \cosh(y)$, $b = 4$, $\tau = 5$.

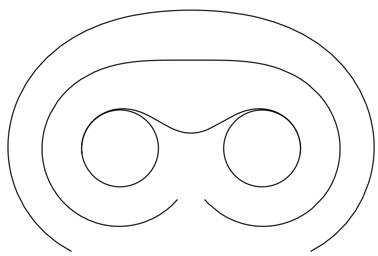

Fig. 2. The case $F(y) = b$ ($D = 0$, $a = 0$, $b > 0$): $V = \cosh(y)$, $b = 4$, $\tau = (t+1)^2$.

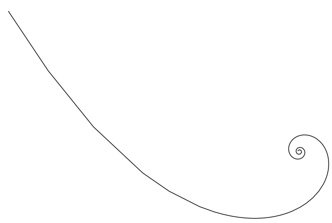

Fig. 3. The case $F(y) = b$ ($D = 0$, $a = 0$, $b > 0$): $V = \sinh(y)$, $b = 4$, $\tau = 1$.

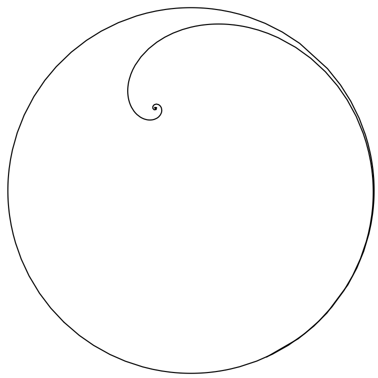

Fig. 4. The case $F(y) = b$ ($D = 0$, $a = 0$, $b > 0$): $V = \sinh(y)$, $b = 4$, $\tau = (t+1)^2$.

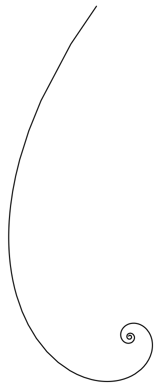

Fig. 5. The case $F(y) = b$ ($D = 0$, $a = 0$, $b = 0$): $V = y$, $\tau = 1$.

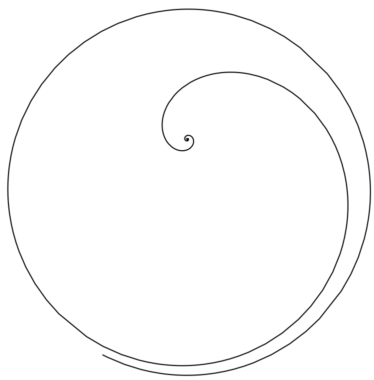

Fig. 6. The case $F(y) = b$ ($D = 0$, $a = 0$, $b = 0$): $V = y$, $\tau = (t+1)^2$.

3.3. The case $F(y) = \frac{D}{4}y^2 + ay + b$ ($D \neq 0$)

In this case, the solution of Eq. (6) are given as,
the case $D > 0$,

$$V = \exp\left\{-\frac{(4a + Dy)y}{8\sqrt{D}}\right\}\left\{C_1 H_n\left(\frac{2a + Dy}{2D^{3/4}}\right) + C_{21}F_1\left(-\frac{n}{2}, \frac{1}{2}, \frac{(2a + Dy)^2}{4D\sqrt{D}}\right)\right\}, \tag{11}$$

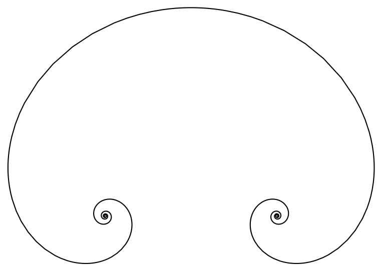

Fig. 7. The case $F(y) = b$ ($D = 0$, $a = 0$, $b < 0$): $V = \cos(y)$, $b = -4$, $\tau = 1$.

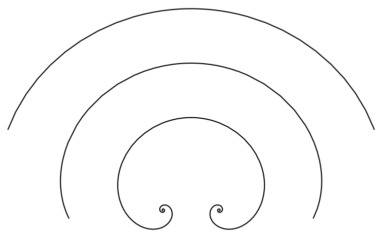

Fig. 8. The case $F(y) = b$ ($D = 0$, $a = 0$, $b < 0$): $V = \cos(y)$, $b = -4$, $\tau = (t + 1)^2$.

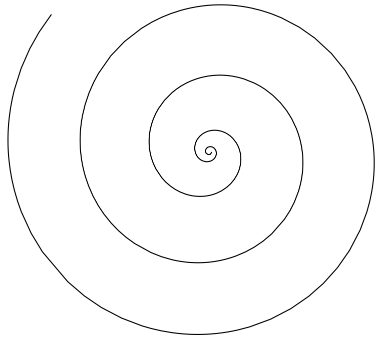

Fig. 9. The case $F(y) = ay + b$ ($D = 0$, $a \neq 0$): $V = Ai(y)$, $a = 4$, $b = 0$, $\tau = 1$.

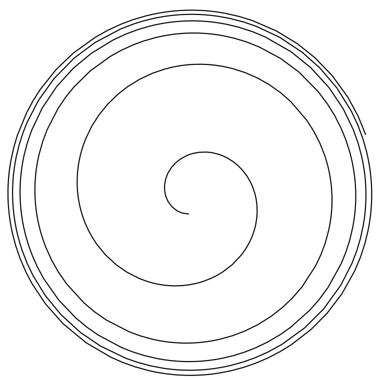

Fig. 10. The case $F(y) = ay + b$ ($D = 0$, $a \neq 0$): $V = Ai(y)$, $a = 4$, $b = 0$, $\tau = (t + 1)^2$.

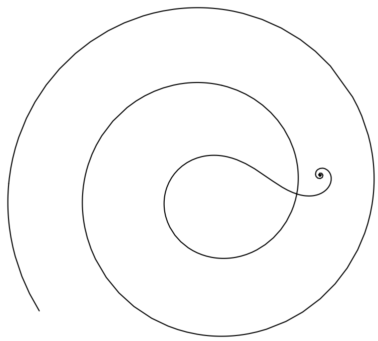

Fig. 11. The case $F(y) = ay + b$ ($D = 0$, $a \neq 0$): $V = Bi(y)$, $a = 4$, $b = 0$, $\tau = 1$.

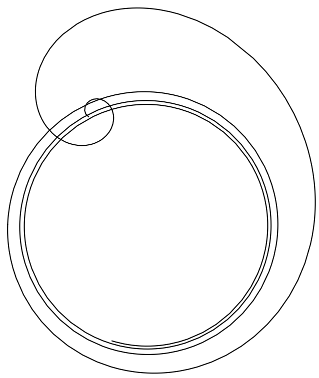

Fig. 12. The case $F(y) = ay + b$ ($D = 0$, $a \neq 0$): $V = Bi(y)$, $a = 4$, $b = 0$, $\tau = (t + 1)^2$.

the case $D < 0$,

$$V = \exp\left\{\frac{\mathrm{i}(4a - Dy)y}{8\sqrt{D}}\right\}\left\{C_1 H_m\left(\frac{(-1)^{1/4}(-2a + Dy)}{2D^{3/4}}\right) + C_{21}F_1\left(-\frac{m}{2}, \frac{1}{2}, \frac{\mathrm{i}(2a + Dy)^2}{4D\sqrt{D}}\right)\right\}, \tag{12}$$

where $H_n(z)$, ${}_1F_1(a, b, c)$ are the Hermitian polynomial, the hypergeometric function of confluent type and

$$n \equiv \frac{a^2 - (b + \sqrt{D})D}{2\sqrt{D}D} \quad \text{and} \quad m \equiv \frac{a^2 + bD + \mathrm{i}D\sqrt{D}}{2\sqrt{D}D}\mathrm{i}.$$

We concentrate only the Hermitian polynomial term in the solution (11). For $D = 1$ and $a = 0$, the values of -1, -3 and -5 of b correspond to the zeroth, first and second order Hermitian polynomials, i.e. H_0, H_1 and H_2, respectively. In Figs. 13–15, patterns with $\tau = (t^2 + 1)/2$ are shown, and one of the zeroth order at various times $t = 1$, 2 and 3 are depicted.

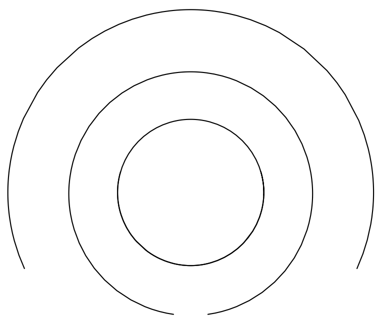

Fig. 13. The case $F(y) = Dy^2/2 + ay + b$ ($D \neq 0$): $V = \exp(-y^2/8)H_0(y/2)$, $D = 1$, $a = 0$, $b = -1$, $\tau = (t^2 + 1)/2$.

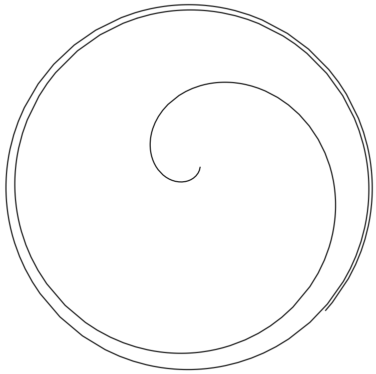

Fig. 14. The case $F(y) = Dy^2/2 + ay + b$ $(D \neq 0)$: $V = \exp(-y^2/8)H_1(y/2)$, $D = 1$, $a = 0$, $b = -3$, $\tau = (t^2 + 1)/2$.

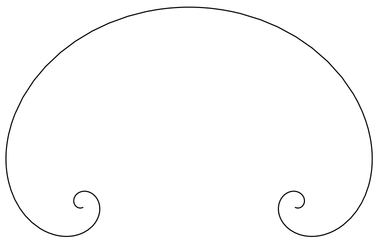

Fig. 15. The case $F(y) = Dy^2/2 + ay + b$ $(D \neq 0)$: $V = \exp(-y^2/8)H_2(y/2)$, $D = 1$, $a = 0$, $b = -5$, $\tau = (t^2 + 1)/2$.

Remark. For patterns having an edge (for example, spiral), we should have to discuss a boundary value problem at the edge. As a boundary condition, we consider the situation that the edge remains at the edge. Then the patterns do not admit translational symmetry for the edge. Since the invariance y are rewritten as

$$y = \frac{x - \sqrt{\tau}\bar{\xi}}{\sqrt{\tau}},$$

the function $\bar{\xi}(t)$ should vanish.

4. Summary

We present a complete description of group invariant solutions under the Lie symmetry of the DB equation. Requiring invariance under the symmetry yields reduced ordinary differential equations with one variable coefficient whose form can takes in three cases. So, the invariant solutions are classified into three cases associated with the form of the coefficient and all solutions are able to be listed, especially, the solution described by the hypergeometric function of confluent type is a new result. We show some figures of typical patterns admitted by the DB equations. On the patterns, double scroll is obtained as a newly result. In some figures, we show specific shapes associated with the solutions. Since the patterns cannot be classified by the form of the function $F(y)$, we come to a conclusion that the classification of the invariant solution dose not give one of their patterns. In remark, we further give a short discussion on a boundary value problem to patterns with an edge. If patterns with an edge are invariant under the symmetry, the arbitrariness $\bar{\xi}(t)$ vanishes.

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