# VARIATIONAL $C^\infty-$SYMMETRIES AND EULER-LAGRANGE EQUATIONS

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**Abstract**

A generalization of the concept of variational symmetry, based on $\lambda-$prolongations, allows us to construct new methods of reduction for Euler-Lagrange equations. An adapted formulation of the Noether's theorem for the new class of symmetries is presented. Some examples illustrate how the method works in practice.

## 1 Introduction

Lie symmetry groups provides a powerful and systematic method for analyzing ordinary (and partial) differential equations. However, not every integration technique can be based on symmetry analysis, [6, 7], and require generalizations of the classical Lie methods.

A more general approach to the integration of ordinary differential equations is based on the concept of a nonlocal exponential symmetry, which first appeared in ([15], Exercise 2.31). This method was further developed in the work of Abraham–Shrauner and her collaborators, [1], and in the theory of solvable structures, [2, 8]. There exists a large variety of processes of reduction that can be explained and deduced by this theory. These ideas

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were further developed by the first author, [11], who replaces the non-local exponential terms by a new method of prolonging vector fields known as the $\lambda-$prolongation, leading to the notion of a $C^{\infty}-$symmetry or $\lambda-$symmetry. These methods have been extended to partial differential equations in the work of Cicgona, Gaeta and Morando, [3, 4, 5], who develop the concept of a $\mu$-symmetry.

For ordinary differential equations that can be derived from a variational principle

$$\mathcal{L}[u] = \int L(x, u^{(n)}) dx, \tag{1.1}$$

the existence of special types of symmetries (variational symmetries) *doubles* the power of Lie's method of reduction. Due to the special structure of the Euler-Lagrange equation derived from (1.1), the knowledge of a variational symmetry allows us to reduce the order by two.

We can expect that a generalization of the concept of variational symmetry, based on the new $\lambda-$prolongations, will generate new methods of reduction for Euler-Lagrange equations. In this paper we establish this generalization and introduce the concept of variational $C^{\infty}-$symmetry, also including generalized vector fields. Some important properties of these variational $C^{\infty}-$symmetries are presented. We also provides an algorithmic procedure to reduce by two the order of any Euler-Lagrange equation that admits a variational $C^{\infty}-$symmetry. This is a "partial" reduction, because, in general, a one-parameter family of solutions is lost when the reduced equation is considered.

The correspondence between variational symmetries and conservation laws for Euler-Lagrange equations is completely determined by celebrated Noether's Theorem [9, 14, 15]. Since every conservation law rises from an ordinary (generalized) variational symmetry, we can not expect to find new conservation laws associated with variational $C^{\infty}-$symmetries. In Section 4 we establish the corresponding version of Noether's theorem for the new symmetries (Theorem 3). This result allows us to reformulate the connection between the original Euler-Lagrange equation and the reduced equation. In addition, we will be able to obtain a conservation law for the one-parameter family of solutions lost in the reduction process, by relating the variational $C^{\infty}-$symmetry to an special pseudo-variational symmetry. We also include several examples to illustrate how this new method works in practice.

Throughout the paper, we will freely use the notations and results in [15]. We will restrict our attention to single variable integrals leading to ordinary differential equations. Extensions of these results to $\mu-$variational symmetries of multivariable variational problems and Euler–Lagrange partial differential equations will proceed in an analogous fashion, but we leave the details to another publication.

# 2 Variational $C^\infty-$symmetries

## 2.1 Some previous results

Let us consider a variational problem

$$\mathcal{L}[u] = \int L(x, u^{(n)})dx \tag{2.2}$$

where the Lagrangian $L(x, u^{(n)})$ is defined on $M^{(n)}$, for some open set $M$ of the space of independent and dependent variables $X \times U$. Let

$$E[L] \equiv \sum_{i=0}^{n} (-D)^i (\partial_{u_i} L) = 0 \tag{2.3}$$

be the associated Euler-Lagrange equation, where $D$ stands for the total derivative operator with respect to $x$. To simplify the notation, we will denote by $\mathcal{A}$ the space of smooth functions depending on $x, u$ and derivatives of $u$ up to some finite, but unspecified, order and we write $P[u] = P(x, u^{(m)})$ if we do not need to precise the order of derivatives that $P$ depends on.

Roughly speaking, a variational symmetry group of the functional (2.2) is a local group of transformations that leaves the variational integral $\mathcal{L}$ unchanged when $u = f(x)$ is transformed by the action of the group. The infinitesimal criterion of invariance ([15], pag. 253) characterizes the infinitesimal generators of connected groups of *variational symmetries*. They are the vector fields $\mathbf{v} = \xi(x, u)\partial_x + \eta(x, u)\partial_u$ such that

$$\mathbf{v}^{(n)}(L) + L\mathrm{D}(\xi) = 0. \tag{2.4}$$

The relation between symmetry groups and conservation laws was first determined by E. Noether. In modern language, the characteristic $Q = \mathbf{v}(u) - \mathbf{v}(x)u_x$ of a variational symmetry $\mathbf{v}$ is also the characteristic of a conservation law for the Euler-Lagrange equation, i.e. $QE[L] = D(P)$ for some $P \in \mathcal{A}$. The hypothesis that the vector field $\mathbf{v}$ generate a group of variational symmetries is overly restrictive to deduce the existence of a conservation law. This motivates a generalization of a variational symmetry: the infinitesimal *divergence symmetries* are the vector fields $\mathbf{v}$ such that

$$\mathbf{v}^{(n)}(L) + L\mathrm{D}(\xi) = D(B), \tag{2.5}$$

for some $B \in \mathcal{A}$.

It is well-known that a one-parameter symmetry group of variational symmetries for the Euler-Lagrange equations allows us to reduce the order by two. This is the Lagrangian counterpart of what is now known as Marsden–Weinstein reduction, [10].

It is also known that there exist ordinary differential equations without Lie symmetries that can be reduced or integrated by using different methods. One of them, that explains a large variety of these processes, is based on the existence of $C^\infty-$symmetries [11, 13]. This concept is based of a new way of prolonging vectors fields. For a given vector field $\mathbf{v} = \xi(x,u)\partial_x + \eta(x,u)\partial_u$ defined on $M \subset X \times U$ and for an arbitrary function $\lambda \in C^\infty(M^{(1)})$ the $\lambda-$*prolongation of order* $n$ *of* $\mathbf{v}$ is the vector field

$$\mathbf{v}^{[\lambda,(n)]} = \xi(x,u)\partial_x + \sum_{i=0}^{n} \eta^{[\lambda,(i)]}(x,u^{(i)})\partial_{u_i}, \tag{2.6}$$

defined on $M^{(n)}$, where $\eta^{[\lambda,(0)]}(x,u) = \eta(x,u)$ and, for $1 \leqslant i \leqslant n$,

$$\begin{aligned} \eta^{[\lambda,(i)]}(x,u^{(i)}) = \;& D\left(\eta^{[\lambda,(i-1)]}(x,u^{(i-1)})\right) - D(\xi(x,u))u_i \\ &+\lambda\left(\eta^{[\lambda,(i-1)]}(x,u^{(i-1)}) - \xi(x,u)u_i\right). \end{aligned} \tag{2.7}$$

Formally, the $\lambda-$prolongation of a vector field $\mathbf{v}$ can be identified as the ordinary prolongation of a nonlocal exponential vector field, ([15], Exercise 2.31)

$$\widehat{\mathbf{v}}^{(n)} = e^{\int \lambda dx}\mathbf{v}^{[\lambda,(n)]} \qquad \text{where} \qquad \widehat{\mathbf{v}} = e^{\int \lambda dx}\mathbf{v}..$$

Equivalent characterizations of the $\lambda-$prolongations can be consulted in [12] (Theorem 2). One of them, that will be used in this paper, states that an arbitrary prolongation of $\mathbf{v}$ to $M^{(n)}$

$$\mathbf{v}_n^* = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} + \sum_{i=1}^{n} \eta_i^*(x,u^{(i)})\frac{\partial}{\partial u_i}, \tag{2.8}$$

is the $\lambda-$prolongation of $\mathbf{v}$ if and only if

$$[\mathbf{v}_n^*, D] = \lambda\mathbf{v}_n^* - (D+\lambda)(\mathbf{v}_n^*(x))D. \tag{2.9}$$

For this kind of prolongations it is possible to calculate a complete system of differential invariants by invariant derivation of lower order invariants. This is the key to construct new methods of order reduction, based on the existence of $C^\infty-$symmetries, [11].

## 2.2 The Concept of Variational $C^\infty$−Symmetries.

In this section, we show how the concept of variational symmetry can be generalized when $\lambda-$ prolongations are considered. This will generate new methods of reduction for Euler-Lagrange equations. The concept of infinitesimal divergence symmetry and the $\lambda-$prolongation formula inspire the following generalization of the definition of variational symmetry:

**Definition 2.1** *A vector field* $\mathbf{v} = \xi(x,u)\partial_x + \eta(x,u)\partial_u$ *is a variational* $C^\infty-$*symmetry of the functional* $\mathcal{L}[u] = \int L(x, u^{(n)})dx$ *if there exists* $B[u] \in \mathcal{A}$ *such that*

$$\mathbf{v}^{[\lambda,(n)]}(L) + L(D+\lambda)(\xi) = (D+\lambda)(B), \tag{2.10}$$

*for some* $\lambda \in C^\infty(M^{(1)})$. *We also say that* $\mathbf{v}$ *is a variational* $\lambda-$*symmetry to precise the function* $\lambda$ *for which (2.10) is satisfied.*

Let us observe that standard divergence variational symmetries correspond to variational $C^\infty-$symmetries for function $\lambda = 0$.

Two Lagrangians $L$ and $\widetilde{L}$ are equivalent if $L - \widetilde{L} = Df$ is a divergence term (see Theorem 4.7 in [15]). In particular, the associated Euler-Lagrange equations are the same. The next proposition states the coherence of definition 2.1: formula (2.10) remains invariant when equivalent Lagrangians are considered.

**Proposition 2.1** *Let* $\mathbf{v}$ *be a variational* $\lambda-$*symmetry of* $\mathcal{L}[u] = \int L(x, u^{(n)})dx$. *The vector field* $\mathbf{v}$ *is also a variational* $\lambda-$*symmetry of* $\widetilde{\mathcal{L}}[u] = \int \widetilde{L}(x, u^{(n)})dx$ *where* $\widetilde{L} = L + Df$, *for any* $f \in \mathcal{A}$.

Proof By using

$$\mathbf{v}^{[\lambda,(n)]}(L) + L(D+\lambda)(\xi) = (D+\lambda)(B), \tag{2.11}$$

we get:

$$\begin{aligned} &\mathbf{v}^{[\lambda,(n)]}(L+Df) + (L+Df)(D+\lambda)(\xi) = \\ &\quad = \mathbf{v}^{[\lambda,(n)]}(L) + \mathbf{v}^{[\lambda,(n)]}(Df) + L(D+\lambda)(\xi) + Df(D+\lambda)(\xi) \\ &\quad = (D+\lambda)(B) + Df(D+\lambda)(\xi) + \mathbf{v}^{[\lambda,(n)]}(Df). \end{aligned} \tag{2.12}$$

Formula (2.9), written as $[\mathbf{v}^{[\lambda,(n)]}, D] = \lambda \mathbf{v}^{[\lambda,(n)]} - (D+\lambda)(\mathbf{v}(x)) \cdot D$, applied to $f$ gives

$$\mathbf{v}^{[\lambda,(n)]}(Df) = (D+\lambda)(\mathbf{v}^{[\lambda,(n)]}f) - (D+\lambda)(\mathbf{v}(x)) \cdot Df. \tag{2.13}$$

Therefore (2.12) becomes

$$\begin{aligned}\mathbf{v}^{[\lambda,(n)]}(L+Df)+(L+Df)(D+\lambda)(\xi) =&\\ =(D+\lambda)(B)+(D+\lambda)(\mathbf{v}^{[\lambda,(n)]}f)&\\ =(D+\lambda)(B+\mathbf{v}^{[\lambda,(n)]}f).&\end{aligned} \tag{2.14}$$

■

The following result shows that a variational $C^\infty-$symmetry remains as a variational $C^\infty-$symmetry under a change of variables:

**Proposition 2.2** *Let* $\mathbf{v}$ *be a variational* $C^\infty-$*symmetry of* $\mathcal{L}[u]=\int L(x,u^{(n)})dx$ *and consider any change of variables*

$$\widetilde{x}=\widetilde{\mathcal{X}}(x,u), \quad \widetilde{u}=\widetilde{\mathcal{U}}(x,u). \tag{2.15}$$

*The vector field* $\mathbf{v}$ *in new variables,* $\widetilde{\mathbf{v}}$*, is a variational* $C^\infty-$*symmetry of the corresponding transformed functional* $\widetilde{\mathcal{L}}[\widetilde{u}]=\int \widetilde{L}(\widetilde{x},\widetilde{u}^{(n)})d\widetilde{x}$*.*

Proof

Let us denote by

$$x=\mathcal{X}(\widetilde{x},\widetilde{u}), \quad u=\mathcal{U}(\widetilde{x},\widetilde{u}). \tag{2.16}$$

the inverse change of coordinates. The two Lagrangians $L(x,u^{(n)})$ and $\widetilde{L}(\widetilde{x},\widetilde{u}^{(n)})$ are related, through the change of variables, by the formula

$$L(x,u^{(n)})=\frac{\widetilde{L}(\widetilde{x},\widetilde{u}^{(n)})}{D_{\widetilde{x}}\mathcal{X}(\widetilde{x},\widetilde{u})}. \tag{2.17}$$

We need the following useful property of $\lambda-$prolongations (see [11] for details):

$$\mathbf{v}^{[\lambda,(n)]}=\widetilde{\mathbf{v}}^{[\widetilde{\lambda},(n)]}, \qquad \text{for} \qquad \lambda=\frac{\widetilde{\lambda}}{D_{\widetilde{x}}\mathcal{X}(\widetilde{x},\widetilde{u})}. \tag{2.18}$$

By (2.18) and since $D_x=\dfrac{D_{\widetilde{x}}}{D_{\widetilde{x}}\mathcal{X}}$, formula (2.10), in new variables, becomes:

$$\widetilde{\mathbf{v}}^{[\widetilde{\lambda},(n)]}\left(\frac{\widetilde{L}}{D_{\widetilde{x}}\mathcal{X}}\right)+\frac{\widetilde{L}}{(D_{\widetilde{x}}\mathcal{X})^2}(D_{\widetilde{x}}+\widetilde{\lambda})(\widetilde{\mathbf{v}}(\widetilde{x}))=\frac{1}{D_{\widetilde{x}}\mathcal{X}}(D_{\widetilde{x}}+\widetilde{\lambda})(\widetilde{B}). \tag{2.19}$$

By derivation, the first term of (2.19) becomes

$$\frac{D_{\widetilde{x}}\mathcal{X}\widetilde{\mathbf{v}}^{[\widetilde{\lambda},(n)]}(\widetilde{L}) - \widetilde{L}\widetilde{\mathbf{v}}^{[\widetilde{\lambda},(n)]}(D_{\widetilde{x}}\mathcal{X})}{(D_{\widetilde{x}}\mathcal{X})^2} \tag{2.20}$$

and formula (2.9) applied to $\mathcal{X}$ gives

$$\widetilde{\mathbf{v}}^{[\widetilde{\lambda},(n)]}(D_{\widetilde{x}}(\mathcal{X})) = (D_{\widetilde{x}} + \widetilde{\lambda})(\mathcal{X}) - (D_{\widetilde{x}} + \widetilde{\lambda})(\widetilde{\mathbf{v}}(\widetilde{x}))D_{\widetilde{x}}\mathcal{X}. \tag{2.21}$$

By replacing (2.20) and (2.21) into (2.19) and simplifying, we finally get

$$\widetilde{\mathbf{v}}^{[\widetilde{\lambda},(n)]}(\widetilde{L}) + \widetilde{L}(D_{\widetilde{x}} + \widetilde{\lambda})(\widetilde{\mathbf{v}}(\widetilde{x})) = (D_{\widetilde{x}} + \widetilde{\lambda})(\widetilde{B}). \tag{2.22}$$

This proves the result.

■

Near any point where the vector field $\mathbf{v} \neq 0$, we can introduce a particular change of variables (2.15) such that $\mathbf{v}$ takes the canonical form $\widetilde{\mathbf{v}} = \partial_{\widetilde{u}}$, formula (2.22) becomes

$$\widetilde{\mathbf{v}}^{[\widetilde{\lambda},(n)]}(\widetilde{L}) = (D_{\widetilde{x}} + \widetilde{\lambda})(\widetilde{B}). \tag{2.23}$$

Suppose that $\widetilde{B} = -\partial_{\widetilde{u}}(A)$ for some function $A$. Then the Lagrangian

$$\widehat{L} \equiv \widetilde{L}(\widetilde{x}, \widetilde{u}^{(n)}) + D_{\widetilde{x}}(A) \tag{2.24}$$

and $\widetilde{L}$ have the same Euler-Lagrange expression, $E_{\widetilde{u}}[\widehat{L}]$. Now formula (2.9) becomes $[\widetilde{\mathbf{v}}^{[\widetilde{\lambda},(n)]}, D_{\widetilde{x}}] = \widetilde{\lambda}\widetilde{\mathbf{v}}^{[\widetilde{\lambda},(n)]}$, which, when applied to $A$, provides:

$$\widetilde{\mathbf{v}}^{[\widetilde{\lambda},(n)]}(D_{\widetilde{x}}(A)) = (D_{\widetilde{x}} + \widetilde{\lambda})(\widetilde{\mathbf{v}}^{[\widetilde{\lambda},(n)]}(A)). \tag{2.25}$$

By (2.23) and (2.25), we get

$$\widetilde{\mathbf{v}}^{[\widetilde{\lambda},(n)]}(\widehat{L}) = 0. \tag{2.26}$$

Let $w = w(\widetilde{x}, \widetilde{u}, \widetilde{u}_1)$ be a first order invariant for $\widetilde{\mathbf{v}}^{[\widetilde{\lambda},(1)]}$, that is

$$\partial_{\widetilde{u}}(w) + \widetilde{\lambda}\partial_{\widetilde{u}_1}(w) = 0. \tag{2.27}$$

A very important property of $\lambda-$prolongations is that a complete system of invariants of the $n$-th order $\lambda-$prolongation can be constructed by successive derivations of lower order invariants, [12]. In this case, by successive derivations of $w$ with respect to $\widetilde{x}$ we obtain a system of coordinates $\{\widetilde{x}, \widetilde{u}, w, \cdots, w_{n-1}\}$, such that $\widetilde{\mathbf{v}}^{[\widetilde{\lambda},(n)]} = \partial_{\widetilde{u}}$. Let us also denote by

$\widehat{L}(\widetilde{x}, w^{(n-1)})$ the Lagrangian (2.24) in the $(\widetilde{x}, \widetilde{u}, w^{(n-1)})$ variables. By (2.26), $\widehat{L}$ does not depend on $\widetilde{u}$.

By means of the transformation $\{\widetilde{x} = \widetilde{x}, w = w(\widetilde{x}, \widetilde{u}, \widetilde{u}_1)\}$ we get the following relation between $E_{\widetilde{u}}[\widehat{L}]$ and the Euler-Lagrange equation of $\widehat{L}(\widetilde{x}, w^{(n-1)})$ (see Exercise 5.49 in [15]):

$$\begin{aligned} E_{\widetilde{u}}[\widehat{L}] &= D_w^*(E_w[\widehat{L}]) - D_{\widetilde{x}}^*(D_{\widetilde{x}} w E_w[\widehat{L}]) \\ &= D_w^*(E_w[\widehat{L}]) \\ &= \partial_{\widetilde{u}}(w) E_w[\widehat{L}] - D_{\widetilde{x}}(\partial_{\widetilde{u}_{\widetilde{x}}}(w) E_w[\widehat{L}]) \end{aligned} \tag{2.28}$$

Therefore, by (2.27):

$$E_{\widetilde{u}}[\widehat{L}] = (D_{\widetilde{x}} + \widetilde{\lambda})[-\partial_{\widetilde{u}_{\widetilde{x}}}(w) E_w[\widehat{L}]]. \tag{2.29}$$

Let $w = H(\widetilde{x}, C_1, \cdots, C_{2n-2})$ be the general solution of the reduced Euler-Lagrange equation $E_w[\widehat{L}] = 0$. When $w$ is written in terms of $\{\widetilde{x}, \widetilde{u}, \widetilde{u}_1\}$, we have a first order ordinary differential equation for $\widetilde{u}$:

$$w(\widetilde{x}, \widetilde{u}, \widetilde{u}_1) = H(\widetilde{x}, C_1, \cdots, C_{2n-2}), \tag{2.30}$$

whose general solution $\widetilde{u} = G(\widetilde{x}, C_1, \cdots, C_{2n-1})$ yields a $(2n-1)-$parameter family of solutions

$$\widetilde{u}(x, u) = G(\widetilde{x}(x, u), C_1, \cdots, C_{2n-1}) \tag{2.31}$$

to the original Euler-Lagrange equation $E_u[L] = 0$.

In this way, we have managed to construct a reduced Lagrangian, of order $n-1$, whose corresponding Euler-Lagrange equation (of order $2n-2$) provides a $2n-1-$parameter family of solutions to the original Euler-Lagrange equations. In other words, we have proved that variational $C^\infty-$symmetries generate new reduction procedures for Euler-Lagrange equations, as spelled out in the following theorem:

**Theorem 1 Reduction of order**

*Let $\mathcal{L}[u] = \int L(x, u^{(n)}) dx$ be an $n-$th order variational problem with Euler-Lagrange equation $E_u[L] = 0$, of order $2n$. Let $\mathbf{v}$ be a variational $\lambda-$symmetry, where $\lambda \in \mathcal{C}^\infty(M^{(1)})$. Then there exists a variational problem $\widehat{\mathcal{L}}[w] = \int \widehat{L}(\widetilde{x}, w^{(n-1)}) d\widetilde{x}$ of order $n-1$, with Euler-Lagrange equation $E_w[\widehat{L}] = 0$ of order $2n-2$, such that a $(2n-1)-$parameter family of solutions of $E_u[L] = 0$ can be found by solving a first order equation from the solutions of the Euler-Lagrange reduced equation $E_w[\widehat{L}] = 0$.*

## 3 Generalized variational $C^\infty-$symmetries

A significant generalization of the notion of symmetry group is obtained by allowing the components $\xi$ and $\eta$ of an infinitesimal generator to depend also on derivatives of $u$. A *generalized vector field* will be a formal expression of the form $\mathbf{v} = \xi[u]\partial_x + \eta[u]\partial_u$ in which $\xi$ and $\eta$ depends on $x, u$ and derivatives of $u$ with respect to $x$ up to some finite (but unspecified) order. Formally, the prolongation of generalized vector fields is obtained in the same manner as for ordinary vector fields. In a similar way, we can consider $\lambda-$prolongations of generalized vector fields, for functions $\lambda$ depending on $x, u$ and derivatives of $u$ (see [13] for details). Based on this generalizations, the concepts of generalized symmetry and generalized $\lambda-$symmetry are straightforward. In particular, we can also define generalized variational $C^\infty-$symmetries as follows:

**Definition 3.1** *A generalized vector field* $\mathbf{v} = \xi[u]\partial_x + \eta[u]\partial_u$ *is a generalized variational* $C^\infty-$*symmetry of the functional* $\mathcal{L}[u] = \int L(x, u^{(n)})dx$ *if there exists* $B[u] \in \mathcal{A}$ *such that*

$$\mathbf{v}^{[\lambda,(n)]}(L) + L(D+\lambda)(\xi) = (D+\lambda)(B), \tag{3.32}$$

*for some* $\lambda \in \mathcal{A}$.

To simplify the terminology, in what follows we will not specify the term *generalized* if it is clear from the context which type of variational $C^\infty-$symmetry is considered.

Any vector field $\mathbf{v} = \xi[u]\partial_x + \eta[u]\partial_u$ has an associated evolutionary representative, $\mathbf{v}_Q = Q\partial_u$ where $Q = \eta[u] - \xi[u]u_1$ is the characteristic of $\mathbf{v}$. We have the following alternative expression for the $\lambda-$prolongation of the vector field, [13]:

$$\mathbf{v}^{[\lambda,(n)]} = \mathbf{v}_Q^{[\lambda,(n)]} + \xi[u]D, \qquad \text{where } \mathbf{v}_Q^{[\lambda,(n)]} = \sum_{i=0}^{n}(D+\lambda)^i(Q)\partial_{u_i}. \tag{3.33}$$

The vector field $\mathbf{v}$ and its evolutionary form $\mathbf{v}_Q$ determine essentially the same variational $C^\infty-$symmetry:

**Proposition 3.1** *A vector field* $\mathbf{v}$ *is a variational* $\lambda-$*symmetry of (2.2) if and only if* $\mathbf{v}_Q$ *is a variational* $\lambda-$*symmetry.*

PROOF By (3.33), the identity (3.32) is satisfied if and only if:

$$\mathbf{v}_Q^{[\lambda,(n)]}(L) + \xi D(L) + L(D+\lambda)(\xi) = (D+\lambda)(B). \tag{3.34}$$

This expression can be written as follows:

$$\mathbf{v}_Q^{[\lambda,(n)]}(L) = (D+\lambda)(B - \xi L), \tag{3.35}$$

which proves the result.

■

The term *strict variational symmetry* is used to distingue standard variational symmetries from divergence variational symmetries. Similarly, we will say that $\mathbf{v}$ is a strict variational $C^\infty-$symmetry when the second member of (2.10) is identically null. The following result proves, in particular, that there exists strict variational $C^\infty-$symmetry associated to any given variational $C^\infty-$symmetry:

**Proposition 3.2** *Let* $\mathbf{v}$ *be a variational* $\lambda-$*symmetry of* $\mathcal{L}[u] = \int L(x, u^{(n)})dx$ *and* $f \in \mathcal{A}$. *The vector field* $\mathbf{v}^{[\lambda,(n)]} + fD$ *is also a variational* $C^\infty-$*symmetry. As a consequence,* $\mathbf{v}^{[\lambda,(n)]} - \frac{B}{L}D$ *is a strict variational* $C^\infty-$*symmetry.*

PROOF

A simple calculation gives

$$\begin{aligned}(\mathbf{v}^{[\lambda,(n)]} + fD)(L) &+ L(D+\lambda)(\xi + f) \\ &= \mathbf{v}^{[\lambda,(n)]}(L) + L(D+\lambda)(\xi) + fD(L) + D(fL) + L\lambda f.\end{aligned} \tag{3.36}$$

By (3.32), the right member of (3.36) becomes $(D+\lambda)(B) + D(fL) + L\lambda f$, for some $B \in \mathcal{A}$ and thus

$$\begin{aligned}(\mathbf{v}^{[\lambda,(n)]} + fD)(L) + L(D+\lambda)(\xi + f)& \\ &= (D+\lambda)B + (D+\lambda)(fL) \\ &= (D+\lambda)(B + fL).\end{aligned} \tag{3.37}$$

This proves that $\mathbf{v}^{[\lambda,(n)]} + fD$ is a variational $C^\infty-$symmetry. For $f = -\frac{B}{L}$, the second member of (3.37) is identically null, which proves that the modified vector field $\mathbf{v}^{[\lambda,(n)]} - \frac{B}{L}D$ is a strict variational $C^\infty-$symmetry.

■

The following proposition will also be useful.

**Proposition 3.3** *Let* $\mathbf{v}$ *be a variational* $\lambda-$*symmetry of* $\mathcal{L}[u]=\int L(x,u^{(n)})dx$ *and let* $f\in\mathcal{A}$ *be an arbitrary non null function. Then* $f\mathbf{v}$ *is a variational* $\widetilde{\lambda}-$*symmetry of* $\mathcal{L}[u]=\int L(x,u^{(n)})dx$ *for* $\widetilde{\lambda}=\lambda-\frac{D(f)}{f}$.

Proof

The proof is based on the following property of $\lambda-$prolongations ([11], Lemma 5.1):

$$f\mathbf{v}^{[\lambda,(n)]}=(f\mathbf{v})^{[\widetilde{\lambda},(n)]} \text{ for } \widetilde{\lambda}=\lambda-\frac{D(f)}{f}. \tag{3.38}$$

By multiplying both members of (2.10) by $f$, replacing $\lambda$ by $\widetilde{\lambda}+\frac{D(f)}{f}$ and by (3.38), we get:

$$(f\mathbf{v})^{[\widetilde{\lambda},(n)]}(L)+fL(D+\widetilde{\lambda})\xi+fL\frac{D(f)}{f}\xi=f(D+\widetilde{\lambda})B+f\frac{D(f)}{f}B. \tag{3.39}$$

Successive simplifications of expression (3.39):

$$(f\mathbf{v})^{[\widetilde{\lambda},(n)]}(L)+fL(D+\widetilde{\lambda})\xi+LD(f)\xi=f(D+\widetilde{\lambda})B+D(f)B$$
$$(f\mathbf{v})^{[\widetilde{\lambda},(n)]}(L)+L(fD(\xi)+f\widetilde{\lambda}\xi+D(f)\xi)=fD(B)+f\widetilde{\lambda}B+D(f)B$$

lead to the desired result:

$$(f\mathbf{v})^{[\widetilde{\lambda},(n)]}(L)+L(D+\widetilde{\lambda})(f\xi)=(D+\widetilde{\lambda})(fB). \tag{3.40}$$

■

# 4 Noether's theorem and variational $C^{\infty}-$symmetries

Theorem 1 provides a method to reduce by two the order of a given Euler-Lagrange equation. This is a "partial" reduction, meaning that, in general, a one-parameter family of solutions can not be derived from the solutions of the corresponding reduced equation. It all has to do with relation (2.29). Solutions of the reduced equation annihilates the expression in the brackets of second member of (2.29). However there could be solutions of the Euler-Lagrange equation for which this expression is neither null nor constant. In other words, in general, that expression is not a first integral of the Euler-Lagrange equation.

In this section we investigate the form of the well-known Noether's theorem for when $\lambda-$prolongations are considered. The version presented here is inspired on the proof of Noether's theorem used in [15]. From this standpoint, the essence of this theorem is reduced to the formula

$$\mathbf{v}^{(n)}(L) = Q \cdot E[L] + D(A), \text{ for some } A \in \mathcal{A}, \tag{4.41}$$

which is based on techniques of integration by parts. First of all, we prove a formula that is similar to (4.41), but adapted to $\lambda-$prolongations.

**Lemma 4.1** *Suppose $Q, \lambda \in \mathcal{A}$. There exists $A \in \mathcal{A}$ such that*

$$\mathbf{v}_Q^{[\lambda,(n)]}(L) = Q \cdot E[L] + (D+\lambda)(A). \tag{4.42}$$

Proof

Let us prove that for any $F, G$ and for each $i \in \mathrm{N}$, there exists $A_i \in \mathcal{A}$ such that

$$(D+\lambda)^i(F)G = F(-D)^i(G) + (D+\lambda)(A_i). \tag{4.43}$$

For $i = 0$ and $i = 1$, formula (4.43) is satisfied for $A_0 = 0$ and $A_1 = FG$, respectively. Let us assume that (4.43) is true for $i-1$. Then there exists a function $\widetilde{A}_{i-1}$ such that

$$(D+\lambda)^{i-1}((D+\lambda)(F))G = (D+\lambda)(F)(-D)^{i-1}(G) + (D+\lambda)(\widetilde{A}_{i-1}).$$

It is clear that

$$\begin{array}{rcl} (D+\lambda)^i(F)G & = & -FD((-D)^{i-1}(G)) + (D+\lambda)(\widetilde{A}_1) + (D+\lambda)(\widetilde{A}_{i-1}) \\ & = & F(-D)^i(G) + (D+\lambda)(A_i), \end{array}$$

where $\widetilde{A}_1 = F(-D)^{i-1}(G)$ and $A_i = \widetilde{A}_{i-1} + \widetilde{A}_1$. Then we can write

$$\begin{array}{rcl} \mathbf{v}_Q^{[\lambda,(n)]}(L) & = & \sum_{i=0}^n (D+\lambda)^i(Q)\partial_{u_i}(L) \\ & = & \sum_{i=0}^n Q(-D)^i(\partial_{u_i}(L)) + (D+\lambda)(A) \\ & = & QE[L] + (D+\lambda)(A). \end{array} \tag{4.44}$$

■

Next we present a result which is formally similar to Noether's theorem, for variational $\lambda-$symmetries.

**Theorem 2** *Let $\mathbf{v}$ be a variational $\lambda-$symmetry of the variational problem $\mathcal{L}[u] = \int L(x, u^{(n)})dx$ and $Q$ the corresponding characteristic of $\mathbf{v}$. Then there exists $P[u] \in \mathcal{A}$ such that*

$$QE[L] = (D+\lambda)(P). \tag{4.45}$$

PROOF

By (3.35) and (4.42) we deduce

$$0 = QE[L] + (D+\lambda)(A - B + \xi L). \tag{4.46}$$

Therefore (4.45) is satisfied for $P = -A + B - \xi L$.

■

Every variational symmetry of a variational problem is necessarily a symmetry of the corresponding Euler-Lagrange equation. This can be proved by means of an important commutation formula ([15], pag. 332):

$$E[\mathbf{v}_Q^{(n)}(L)] = \mathbf{v}_Q^{(2n)}(E[L]) + D_Q^*(E[L]), \tag{4.47}$$

where $L, Q \in \mathcal{A}$, and $D_Q^*$ denotes the Frechet derivative operator.

Our next goal is to prove that variational $C^\infty-$symmetries are *conditional* $C^\infty-$symmetries of the Euler-Lagrange equation. That means that variational $C^\infty-$symmetries satisfy the invariance criterion $\mathbf{v}^{[\lambda,(2n)]}(E[L]) = 0$ only for a particular class of solutions of $E[L] = 0$.

With this aim, let us investigate the form of formula (4.47) when $\lambda-$prolongations are considered. First we introduce some notations and technical formulas that will be used to prove subsequent results. The following operators are similar to the Frechet derivative operator and its adjoint, when $D$ is replaced by $D + \lambda$ :

$$(D+\lambda)_P(Q) = \mathbf{v}_Q^{[\lambda,(n)]}(P) = \sum_{i=0}^{n} (\partial_{u_i} P)(D+\lambda)^i(Q), \tag{4.48}$$

$$(D+\lambda)_Q^*(P) = \sum_{i=0}^{n} (-(D+\lambda))^i(\partial_{u_i}(Q) \cdot P). \tag{4.49}$$

Let us observe that $(D+\lambda)_R^*(1)$ corresponds to the Euler-Lagrange operator when $D$ is replaced by $D + \lambda$, so we will write

$$E^\lambda[Q] = (D+\lambda)_Q^*(1).$$

In Lemma 4.2 formula (4.47) for $\lambda$-prolongations will be stated. The proof we present uses the following relations, that can be checked without too much difficulty:

$$
\begin{aligned}
E^{\lambda}[P+Q] &= E^{\lambda}[P]+E^{\lambda}[Q] && (4.50)\\
E^{\lambda}[PQ] &= (D+\lambda)^{*}_{P}(Q)+(D+\lambda)^{*}_{Q}(P) && (4.51)\\
E^{\lambda}[(D+\lambda)P] &= (D+\lambda)^{*}_{\lambda}(P) && (4.52)\\
(D+\lambda)_{E[L]}(Q) &= (D+\lambda)^{*}_{E[L]}(Q). && (4.53)
\end{aligned}
$$

**Lemma 4.2** *Suppose* $L, Q$ *and* $\lambda \in \mathcal{A}$. *Then*

$$
E^{\lambda}[\mathbf{v}_Q^{[\lambda,(n)]}(L)] = \mathbf{v}_Q^{[\lambda,(2n)]}(E[L]) + (D+\lambda)^{*}_{Q}(E[L]) + (D+\lambda)^{*}_{\lambda}(A) \qquad (4.54)
$$

*for* $A$ *given in Lemma 4.1.*

PROOF By (4.50)-(4.53) and (4.42):

$$
\begin{aligned}
E^{\lambda}[\mathbf{v}_Q^{[\lambda,(n)]}(L)] &= E^{\lambda}[QE[L]] + E^{\lambda}[(D+\lambda)A]\\
&= (D+\lambda)^{*}_{E[L]}(Q) + (D+\lambda)^{*}_{Q}(E[L]) + (D+\lambda)^{*}_{\lambda}(A)\\
&= \mathbf{v}_Q^{[\lambda,(2n)]}(E[L]) + (D+\lambda)^{*}_{Q}(E[L]) + (D+\lambda)^{*}_{\lambda}(A).
\end{aligned}
$$

■

**Lemma 4.3** *Let* $\mathbf{v}$ *be a variational* $\lambda-$*symmetry of the variational problem* $\mathcal{L}[u] = \int L(x, u^{(n)})dx$. *Let* $Q$ *be the corresponding characteristic, and let* $P[u]$ *be given by Theorem 2. Then*

$$
\mathbf{v}_Q^{[\lambda,(2n)]}(E[L]) = -(D+\lambda)^{*}_{Q}(E[L]) + (D+\lambda)^{*}_{\lambda}(P). \qquad (4.55)
$$

PROOF By formula (3.35), the left hand side of (4.54) becomes

$$
\begin{aligned}
E^{\lambda}[\mathbf{v}_Q^{[\lambda,(n)]}(L)] &= E^{\lambda}[(D+\lambda)(B-\xi L)]\\
&= (D+\lambda)^{*}_{\lambda}(B-\xi L)\\
&= (D+\lambda)^{*}_{\lambda}(P) + (D+\lambda)^{*}_{\lambda}(A)
\end{aligned} \qquad (4.56)
$$

and by (4.54) we get the result.

■

Let us observe that evaluating (4.55) when $E[L]=0$ gives

$$
\mathbf{v}_Q^{[\lambda,(2n)]}(E[L])_{|E[L]=0} = (D+\lambda)^{*}_{\lambda}(P)_{|E[L]=0} \qquad (4.57)
$$

and the second member, in general, is not null. Therefore, variational $C^\infty-$symmetries are not, in general, $C^\infty-$symmetries of the Euler-Lagrange equation. However, $\mathbf{v}_Q^{[\lambda,(2n)]}(E[L]) = 0$ on solutions to the combined system $E[L] = 0, P = 0$. This explains, from another point of view, the "partial" reduction for the Euler-Lagrange equation stated by Theorem 1. Next result connects function $P$ of Theorem 2 with the expression in brackets of formula (2.29) and, consequently, with the reduced equation of Theorem 1.

**Theorem 3** *Let* $\mathbf{v}$ *be a variational* $\lambda-$*symmetry of the variational problem* $\mathcal{L}[u] = \int L(x, u^{(n)})dx$, *and* $P[u]$ *given by Theorem 2. Then* $\mathbf{v}$ *is a* $\lambda-$*symmetry of the equation* $P[u] = 0$. *The reduced equation of* $P[u] = 0$ *through this* $\lambda-$*symmetry is (up to multipliers) the reduced equation of the Euler-Lagrange equation corresponding to* $\mathbf{v}$, *according to Theorem 1.*

Proof Let us retain the notation of Theorem 1. We can assume that $\mathbf{v}$ is a proper $C^\infty-$symmetry, i.e., not equivalent to a standard variational symmetry. In terms of $(\widetilde{x}, \widetilde{u}, w^{2n-1})$, (4.45) becomes

$$E_{\widetilde{u}}[\widehat{L}] = (D_{\widetilde{x}} + \widetilde{\lambda})(\widetilde{P}), \tag{4.58}$$

where $\widetilde{P}$ stands for $P$ in the new variables. By (2.29),

$$(D_{\widetilde{x}} + \widetilde{\lambda})(\widetilde{P} + \partial_{\widetilde{u}_{\widetilde{x}}}(w)E_w[\widehat{L}]) = 0. \tag{4.59}$$

We set $\widetilde{P} + \partial_{\widetilde{u}_{\widetilde{x}}}(w)E_w[\widehat{L}] = H(\widetilde{x}, \widetilde{u}, w^{(2n-1)})$ and (4.59) can be written:

$$\begin{array}{rcl} 0 & = & D_{\widetilde{x}}(H(\widetilde{x}, \widetilde{u}, w^{(2n-1)})) + \widetilde{\lambda}(\widetilde{x}, \widetilde{u}, w)H(\widetilde{x}, \widetilde{u}, w^{(2n-1)}) \\ & = & \dfrac{\partial H}{\partial \widetilde{x}}(\widetilde{x}, \widetilde{u}, w^{(2n-1)}) + \dfrac{\partial H}{\partial \widetilde{u}}(\widetilde{x}, \widetilde{u}, w^{(2n-1)}) \cdot D_{\widetilde{x}}\widetilde{u} + \cdots \\ & & + \dfrac{\partial H}{\partial w_{2n-1}}(\widetilde{x}, \widetilde{u}, w^{(2n-1)}) \cdot w_{2n} + \widetilde{\lambda}(\widetilde{x}, \widetilde{u}, w)H(\widetilde{x}, \widetilde{u}, w^{(2n-1)}). \end{array} \tag{4.60}$$

The only term where $w_{2n}$ appears is $\dfrac{\partial H}{\partial w_{2n-1}}(\widetilde{x}, \widetilde{u}, w^{(2n-1)}) \cdot w_{2n}$, and so its coefficient must vanish: $\dfrac{\partial H}{\partial w_{2n-1}}(\widetilde{x}, \widetilde{u}, w^{(2n-1)})$. Therefore, $H$ does not depend on $w_{2n-1}$, and (4.60) becomes

$$\begin{array}{rcl} 0 & = & \dfrac{\partial H}{\partial \widetilde{x}}(\widetilde{x}, \widetilde{u}, w^{(2n-2)}) + \dfrac{\partial H}{\partial \widetilde{u}}(\widetilde{x}, \widetilde{u}, w^{(2n-2)}) \cdot D_{\widetilde{x}}\widetilde{u} + \cdots \\ & & + \dfrac{\partial H}{\partial w_{2n-2}}(\widetilde{x}, \widetilde{u}, w^{(2n-2)}) \cdot w_{2n-1} + \widetilde{\lambda}(\widetilde{x}, \widetilde{u}, w)H(\widetilde{x}, \widetilde{u}, w^{(2n-2)}). \end{array} \tag{4.61}$$

The variable $w_{2n-1}$ only appears in $\dfrac{\partial H}{\partial w_{2n-2}}(\widetilde{x}, \widetilde{u}, w^{(2n-2)}) \cdot w_{2n-1}$, and, as above, we deduce that $H$ does not depend on $w_{2n-1}$. By continuing this process, we obtain

$$0 = \frac{\partial H}{\partial \widetilde{x}}(\widetilde{x}, \widetilde{u}) + \frac{\partial H}{\partial \widetilde{u}}(\widetilde{x}, \widetilde{u}) \cdot D_{\widetilde{x}}\widetilde{u} + \widetilde{\lambda}(\widetilde{x}, \widetilde{u}, w) H(\widetilde{x}, \widetilde{u}), \tag{4.62}$$

that is: $0 = D_{\widetilde{x}}(H) + \widetilde{\lambda} \cdot H$. *If* function $H$ is not null, $\widetilde{\lambda} - \frac{D_{\widetilde{x}}(1/H)}{1/H} = 0$ and Proposition 3.3 proves that $\dfrac{1}{H}\widetilde{\mathbf{v}}$ is a (standard) variational symmetry, which we excluded at the start of the proof.

Thus, $H \equiv 0$ and

$$\widetilde{P} = -\partial_{\widetilde{u}_{\widetilde{x}}}(w) E_w[\widehat{L}]. \tag{4.63}$$

Since $E_w[\widehat{L}]$ does not depend on $\widetilde{u}$, we have $\mathbf{v}^{[\lambda,(2n-1)]}(P) = 0$ when $P = 0$, which proves that $\mathbf{v}$ is $\lambda-$symmetry of $P = 0$. In particular, $\widetilde{P}$ does not depend on $\widetilde{u}$ and $\widetilde{P} = 0$ is the reduced equation corresponding to $\mathbf{v}$. Formula (4.63) proves the second part of the theorem.

■

# 5 Partial conservation laws

In this section we focus our attention on the solutions of the Euler-Lagrange equation that do not arise from the reduced equation of Theorem 1. Such solutions satisfy $P \neq 0$ but $(D + \lambda)(P) = 0$. The following result states that $\frac{1}{P}\mathbf{v}$ is a pseudo-variational symmetry of the problem, which is defined as a generalized vector field $\mathbf{v} = \xi(x, u^{(k)})\partial_x + \eta(x, u^{(k)})\partial_u$ that satisfies

$$\mathbf{v}^{(n)}(L) + LD(\xi) = D(B), \tag{5.64}$$

for some $B[u] \in \mathcal{A}$, **only** on solutions of the Euler-Lagrange equations ([15], Exercise 5.38).

**Theorem 4** *Let $\mathcal{L}[u] = \int L(x, u^{(n)})dx$ be an $n-$th order variational problem with Euler-Lagrange equation $E_u[L] = 0$, of order $2n$. Let $\mathbf{v}$ be a variational $\lambda-$symmetry and let $P$ be as in Theorem 2. The generalized vector field $Y = \frac{1}{P}\mathbf{v}$ is a pseudo-variational symmetry of the problem.*

Proof According to Definition 2.10, there exists $B[u] \in \mathcal{A}$ such that

$$\mathbf{v}^{[\lambda,(n)]}(L) + L(D + \lambda)(\xi) = (D + \lambda)(B), \tag{5.65}$$

for some $\lambda \in C^\infty(M^{(1)})$. According to Proposition 3.3, the vector field $\frac{1}{P}\mathbf{v}$ is a variational $C^\infty-$symmetry for $\widetilde{\lambda} = \lambda + \frac{D(P)}{P}$. In particular,

$$\frac{1}{P}\mathbf{v}^{[\lambda,(n)]} = \left(\frac{1}{P}\mathbf{v}\right)^{[\widetilde{\lambda},(n)]}. \tag{5.66}$$

Now we multiply both members of (5.65) by $\frac{1}{P}$ :

$$\frac{1}{P}\mathbf{v}^{[\lambda,(n)]}(L) + \frac{1}{P}L(D+\lambda)\xi = \frac{1}{P}(D+\lambda)B. \tag{5.67}$$

Let us evaluate (5.67) on the solutions $u$ of the Euler-Lagrange equations such that $P[u] \neq 0$. Since $(D+\lambda)(P)[u] = 0$, the following relation holds, for any $A \in \mathcal{A}$ :

$$\frac{1}{P}(D+\lambda)(A)[u] = D\left(\frac{A}{P}\right)[u]. \tag{5.68}$$

If $(D+\lambda)(P)[u] = 0$, we also have

$$\frac{1}{P}\mathbf{v}^{[\lambda,(n)]}[u] = \left(\frac{1}{P}\mathbf{v}\right)^{[\widetilde{\lambda},(n)]}[u] = \left(\frac{1}{P}\mathbf{v}\right)^{(n)}[u]. \tag{5.69}$$

Therefore, (5.67) evaluated on $u$ becomes

$$\left(\frac{1}{P}\mathbf{v}\right)^{(n)}(L)[u] + LD\left(\frac{\xi}{P}\right)[u] \quad = \quad D\left(\frac{B}{P}\right)[u]\ . \tag{5.70}$$

This proves the theorem.

■

To every pseudo-variational symmetry of a normal variational problem there corresponds a conservation law and, moreover, there is always a true variational symmetry giving rise to the same law ([15], Exercise 5.38).

The corresponding conservation law associated to the pseudo-variational symmetry of the previous theorem can be constructed as follows. Let $\widetilde{Q}$ be the characteristic of vector field $\frac{1}{P}\mathbf{v}$ and $\left(\frac{1}{P}\mathbf{v}\right)_{\widetilde{Q}}$ the corresponding evolutionary form. The next relation always holds ([15], pag. 273):

$$\left(\frac{1}{P}\mathbf{v}\right)_{\widetilde{Q}}^{(n)}(L) = \widetilde{Q}\cdot E[L] + D(A), \tag{5.71}$$

for some function $A$. By other hand, in terms of $\left(\frac{1}{P}\mathbf{v}\right)_{\widetilde{Q}}$, formula (5.67) reads

$$\left(\frac{1}{P}\mathbf{v}\right)^{(n)}_{\widetilde{Q}}(L)[u] \quad = \quad D\left(\frac{B}{P} - L\xi\right)[u] \ . \tag{5.72}$$

From equation (5.71) evaluated when $E[L] = 0$ and equation (5.72) we finally get

$$D\left(A - \frac{B}{P} - L\xi\right) = 0 \qquad \text{when} \qquad E[L] = 0. \tag{5.73}$$

In consequence, $A - \frac{B}{P} - L\xi$ is a conservation law associated to the pseudo-variational symmetry $\frac{1}{P}\mathbf{v}$.

# 6 Some examples

## 6.1 First order Lagrangian

1. **Strict variational $C^\infty$-symmetries:**

   The vector field $\mathbf{v} = \partial_u$ is a strict variational $C^\infty-$symmetry, for $\lambda = u$, of the variational problem associated to the Lagrangian

   $$L(x, u, u_x) = x^3 + \left(u_x - \frac{u^2}{2}\right)^2. \tag{6.74}$$

   Indeed,

   $$\mathbf{v}^{[\lambda,(2)]}(L) = (\partial_u + u\partial_{u_x} + (u^2 + u_x)\partial_{u_{xx}})(L) = 0. \tag{6.75}$$

   The corresponding Euler-Lagrange equation is

   $$E[L] \equiv u^3 - 2u_{xx} = 0. \tag{6.76}$$

   The order reduction associated to the variational $C^\infty-$symmetry $\mathbf{v}$ can be constructed by using Theorem 1 or Theorem 3. Next we use both theorems to compare the two different methods.

   - Theorem 1: in coordinates $\{x, u\}$ the vector field $\mathbf{v}$ adopts the canonical form $\partial_u$ and the original Lagrangian (6.74) is an invariant for $\mathbf{v}^{[\lambda,(2)]}$ by itself. We consider a first order invariant for $\mathbf{v}^{[\lambda,(2)]}$ :

     $$w = u_x - \frac{u^2}{2}. \tag{6.77}$$

In coordinates $\{x, u, w\}$ the Lagrangian $L$ becomes $x^3 + w^2$. The associated Euler-Lagrange equation for the reduced Lagrangian $\widehat{L}(x, w) = x^3 + w^2$ is given by $2w = 0$. The general solution of this (0-th order) ordinary differential equation is $w = 0$. By (6.77) we obtain the first order differential equation

$$u_x - \frac{u^2}{2} = 0. \tag{6.78}$$

By solving this equation we recover a one-parameter family of solutions of (6.76) given by $u(x) = -\dfrac{2}{x + C}$ for $C \in \mathrm{R}$.

- Theorem 3: it can be checked that $P = u^2 - 2u_x$ satisfies $QE[L] = (D + \lambda)(P)$. As promised by theorem 3, the vector field $\mathbf{v} = \partial_u$ is also a $C^\infty$−symmetry, for $\lambda = u$, of the equation $P = 0$ :

$$\mathbf{v}^{[\lambda,(2)]}(P) = (\partial_u + u\partial_{u_x} + (u^2 + u_x)\partial_{u_{xx}})(u^2 - 2u_x) = 0. \tag{6.79}$$

To reduce the order of the equation $P = 0$ by means of the $C^\infty$−symmetry $\mathbf{v}$, we consider the set $\{x, w\}$, for $w$ as in (6.77), that constitutes a complete system of invariants of $\mathbf{v}^{[\lambda,(1)]}$. In terms of $\{x, w\}$ the equation $P = 0$ becomes $2w = 0$, that is equivalent to the reduction obtained by the previous method.

2. **Divergence variational $C^\infty$-symmetries:**

The vector field $\mathbf{v} = \partial_u$ is a divergence variational $C^\infty$−symmetry, for $\lambda = u$, of the variational problem associated to the Lagrangian

$$L(x, u, u_x) = x\, {u_x}^2 + \frac{u^3\, (4 + 3\, x\, u)}{12}, \tag{6.80}$$

because

$$\mathbf{v}^{[\lambda,(2)]}(L) = u\, \left(u + x\, u^2 + 2\, x\, u_x\right) = (D + u)(x\, u^2). \tag{6.81}$$

Therefore, (2.10) is satisfied for $B = x\, u^2$. The corresponding Euler-Lagrange equation is

$$E[L] \equiv u^2 + x\, u^3 - 2\, (u_x + x\, u_{xx}) = 0. \tag{6.82}$$

In coordinates $\{x, u\}$ the vector field $\mathbf{v}$ adopts the canonical form $\partial_u$ but in this case, in view of (6.81), the original Lagrangian (6.80) is *not* an invariant of $\mathbf{v}^{[\lambda,(2)]}$. According to Theorem 1, we choose some function $A$ such that $B = -\partial_u(A)$, for example, $A = \frac{-1}{3}u^3\,x$. In coordinates $\{x, u, w, w_x\}$, for $w$ given by (6.77), the corresponding $\mathbf{v}^{[\lambda,(2)]}-$invariant Lagrangian (2.24) becomes $x\,w^2$. The associated Euler-Lagrange equation for the reduced Lagrangian $\widehat{L}(x, w) = x\,w^2$ is given by $2\,x\,w = 0$. From the general solution of this 0-th order ordinary differential equation ($w = 0$) and through the first order differential equation (6.78), we obtain the one-parameter family of solutions of (6.82) given by $u(x) = -\frac{2}{x+C}$ for $C \in \mathrm{R}$.

To apply Theorem 3, $P = x\,u^2 - 2\,x\,u_x$ satisfies $QE[L] = (D + \lambda)(P)$, and the reduced equation of $P = 0$, by means of the $\lambda-$symmetry $\mathbf{v}$, is $2\,x\,w = 0$, which is equivalent to the reduction obtained by the previous method.

## 6.2 Second order Lagrangian

The vector field $\mathbf{v} = \partial_u$ is a variational $C^\infty-$symmetry, for $\lambda = u$, of the variational problem associated to the Lagrangian

$$L(x, u, u_x, u_{xx}) = x\left(u_x - \frac{u^2}{2}\right) + \frac{1}{u_{xx} - u\,u_x} \tag{6.83}$$

because

$$\mathbf{v}^{[\lambda,(2)]}(L) = (\partial_u + u\partial_{u_x})(L) = 0 \tag{6.84}$$

The corresponding Euler-Lagrange equation is a fourth order differential equation:

$$\begin{aligned} E[L] \equiv -1 + \frac{2\,u\,\left({u_x}^2 + u\,u_{xx} - u_{xxx}\right)}{\left(u\,u_x - u_{xx}\right)^3} - \frac{6\left({u_x}^2 + u\,u_{xx} - u_{xxx}\right)^2}{\left(-\left(u\,u_x\right) + u_{xx}\right)^4} \\ + \frac{2\,\left(3\,u_x\,u_{xx} + u\,u_{xxx} - u_{xxxx}\right)}{\left(u\,u_x - u_{xx}\right)^3} - x\,u = 0. \end{aligned} \tag{6.85}$$

- Theorem 1: A complete system of invariants of $\mathbf{v}^{[\lambda,(4)]}$ is given by $\{x, w, w_x, w_{xx}, w_{xxx}\}$, for $w$ as in (6.77). In coordinates $\{x, u, w, w_x\}$, (6.83) becomes $\widehat{L}(x, w, w_x) = x\,w + \frac{1}{w_x}$. The Euler-Lagrange equation that corresponds to the reduced Lagrangian $\widehat{L}$, in coordinates $\{x, w, w_x, w_{xx}\}$, becomes

$$\frac{2w_{xx}}{w_x^3} - x = 0. \tag{6.86}$$

The general solution of this second order equation is given by

$$w = \pm\sqrt{2}\,\mathrm{arctan}\left(\frac{x}{\sqrt{C_1 - x^2}}\right) + C_2, \quad C_1, C_2 \in \mathrm{R}. \tag{6.87}$$

By setting $w = u_x - \frac{u^2}{2}$ through the first order equation

$$u_x - \frac{u^2}{2} = \pm\sqrt{2}\,\mathrm{arctan}\left(\frac{x}{\sqrt{C_1 - x^2}}\right) + C_2, \tag{6.88}$$

we get a three-parameter family of solutions of $E[L] = 0$.

- We can also use Theorem 3 to effect the reduction, but the complexity of (6.85) to determine an expression $P$ that satisfies $Q\,E[L] = (D + \lambda)(P)$ suggests to use the previous method. Anyway, the reader can check that $P$ is given

$$P(x, u, u_x, u_{xx}, u_{xxx}) = \frac{2\,\left({u_x}^2 + u\,u_{xx} - u_{xxx}\right)}{\left(u\,u_x - u_{xx}\right)^3} - x. \tag{6.89}$$

  We use the $\lambda-$symmetry $\mathbf{v}$ to reduce the equation $P = 0$. In terms of the invariants $\{x, w, w_x, w_{xx}\}$ as above, the equation $P = 0$ becomes (6.86). By Theorem 3, this is also the reduced equation for the Euler-Lagrange equation (6.85).

# 7 Conclusions

The new technique of $\lambda-$prolongations and some conditions of invariance allowed us to introduce the concept of $C^\infty-$symmetry and to derive new methods of reduction for ordinary differential equations [11]. In this paper we prove that a convenient generalization of the concept of variational symmetries for Euler-Lagrange equations, based on a similar technique, also provides new algorithms of reduction for this type of equations (Theorem 1).

This generalization corresponds to the concept of variational $C^\infty-$symmetry, and some important properties have been presented. We have also provided the general algorithm to reduce by two the order of a given Euler-Lagrange equation admitting a $C^\infty-$symmetry. In general, a one-parameter family of solutions can not be derived from the solutions of the corresponding reduced equation. For this kind of solutions we have proved the existence of a conditional conservation law, associated to a pseudo-variational symmetry of the problem (Theorem 4).

The method of reduction can also be interpreted in terms of the formulation of the Noether's theorem when $\lambda-$prolongations are considered (Theorem 2). This result clarifies the relation between the original Euler-Lagrange equations and the reduced equation (Theorem 3). Finally, some examples have been included to illustrate the main results and the methods presented in this work.

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