LETTER TO THE EDITOR

Differential invariants and regularity

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Abstract. We explore the role of differential invariants on the regularity of spacetime. Specifically, we consider spherically symmetric, static spacetimes, which have regular curvature invariants at the origin, but a diverging higher-order differential invariant. Exact 'solutions' which meet this criterion and satisfy all energy conditions are constructed. We then examine the physical source of the divergence in the differential invariants and conclude by showing that the singularities are weak.

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Curvature invariants (in particular scalars polynomial in the Riemann tensor) are often used to probe the singular structure of spacetime. Moreover, if all curvature invariants are regular then the spacetime is often assumed to be regular. Differential invariants such as $R_{\alpha\beta\gamma\delta;\mu}R^{\alpha\beta\gamma\delta;\mu}$ have not been widely studied. In this letter we seek static spherically symmetric spacetimes which have regular curvature invariants, satisfy energy conditions but have diverging differential invariants. We then seek to understand the 'source' of the singularity in the differential invariants. Note that it is not our intent to generate a 'new solution to Einstein's equations', but rather to examine the role of differential invariants.

It is clear that curvature invariants alone are not a sufficient probe of the 'physics' of a solution. For example, Bonnor [1] has recently re-examined the Kinnersley 'photon rocket'

$$\begin{aligned} \mathrm{d}s^2 = {} & (1 - a^2r^2\sin^2\theta - 2ar\cos\theta - 2mr^{-1})\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}r \\ & -r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) - 2ar^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}u \\ & -\infty < u < \infty, \quad 0 < r < \infty, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi \end{aligned} \tag{1}$$

where $m = m(u)$ is the mass of a particle at the origin and $a = a(u)$ its acceleration along the negative direction of the polar axis. In addition to an examination of the Killing vectors, Bonnor points out that $a(u)$ does not enter any scalars polynomial in the Riemann tensor; it enters only at the level of the differential invariants. By choice of a singular acceleration $a(u)$ one could, therefore, introduce a 'singularity' into the metric which would not appear at the level of scalar polynomials.

We set out to construct an example of a spacetime with a singularity in the nth-order differential invariant $R_{;\alpha_1\cdots\alpha_n}R^{;\alpha_1\cdots\alpha_n}$ (where R is the Ricci scalar and n is up to order four) for spacetimes which have regular curvature invariants. The simplest possibility is a spherically symmetric static spacetime and so we consider the metric

$$\mathrm{d}s^2 = b(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 - a(r)\,\mathrm{d}t^2. \tag{2}$$

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The conditions for the regularity of all invariants polynomial in the Riemann tensor [2] at the origin are [3]

$$a(0) = 1 \qquad b(0) = 1 \qquad a'(0) = 0 \qquad b'(0) = 0 \; ('\equiv \mathrm{d}/\mathrm{d}r). \tag{3}$$

(These conditions are equivalent to those obtained by Schmidt [4] and by Maartens, see [5] and references therein.)

Using GRTensorII [6] to examine the structure of various differential invariants at the origin we are led to consider

$$a(r) = b(r) = 1 + r^{n+3/2} \qquad n = 1, 2, 3 \ldots. \tag{4}$$

which clearly satisfy (3). The Einstein tensor for the resultant spacetime is

$$G_r^r = \frac{r^{n-1/2}(1 - 2n - 2r^{n+3/2})}{2(1 + r^{n+3/2})^2} \equiv \frac{P(r)}{8\pi} \tag{5}$$

$$G_\theta^\theta = G_\phi^\phi = \frac{r^{n-1/2}(3 + 8n + 4n^2 - (6 + 4n)r^{n+3/2})}{8(1 + r^{n/2})^3} \equiv \frac{P_T(r)}{8\pi} \tag{6}$$

$$G_t^t = -\frac{r^{n-1/2}(5 + 2n + 2r^{n+3/2})}{2(1 + r^{n/2})^2} \equiv -\frac{\rho(r)}{8\pi}. \tag{7}$$

For an anisotropic spherically symmetric spacetime the Ricci scalar can be expressed as

$$R = 8\pi(\rho - P - 2P_T). \tag{8}$$

Clearly for $n = 1$ the invariant $R_{;\alpha}R^{;\alpha}$ will diverge as $r \to 0$ and this is a result of the finite differentiability of the density and pressures. We have verified that for metrics of the form (2) with (4) and $n = 1, 2, 3, 4$ the differential invariants up to order $(n-1)$ are regular at the origin while the nth-order differential invariant diverges there. (A general inductive proof for all n is difficult, since it would require an explicit general form for an arbitrary number of covariant derivatives of R.)

Are these ‘solutions’ physically reasonable? Since *any* static spherically symmetric metric can be interpreted as an anisotropic matter distribution, it is important to check energy conditions as a first step. For the general case (4) the weak, strong, and dominant energy conditions are satisfied for $0 \le r < \infty$. The density (ρ), radial and tangential pressures (P and P_T) all vanish at the origin $r = 0$ and as $r \to \infty$ (this allows the spacetime to be interpreted as an anisotropic matter distribution which forms a ‘thick shell’). Moreover, it can be verified that not only are all invariants polynomial in the Riemann tensor for this spacetime regular at the origin, they *vanish* there. We also find

$$\frac{r}{2m(r)} = \frac{1 + r^{n+3/2}}{r^{n+3/2}} \tag{9}$$

where $m(r)$ is the effective gravitational mass ($\equiv (g_{\theta\theta})^{3/2}R_{\theta\phi}{}^{\theta\phi}/2$), so that $r > 2m(r)$ for $0 \le r < \infty$.

Consider, for example, the specific case $n = 1$. The thick shell can, but need not, be joined to a Schwarzschild exterior at (and only at) $r = (3/2)^{2/5} \equiv r_0$ where $P = P_T = 0$. Note that P and $P_T < 0$ for $r > r_0$. Evaluation of $R_{;\alpha}R^{;\alpha}$ gives

$$\frac{(7 - 254r^{5/2} + 146r^5 + 32r^{15/2})^2}{64(1 + r^{5/2})^9 r} \tag{10}$$

which diverges as $r \to 0$ like $1/r$. Note also that the gradients of ρ, P and P_T all diverge like $1/r^{1/2}$ as $r \to 0$ and that $\rho \sim 7P/3$ and $P \sim 4P_T/5$ as $r \to 0$.

The generalized Tolman (Oppenheimer–Volkoff) equation is

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{(\rho + P)(m(r) + 4\pi Pr^3)}{r(r - 2m(r))} - \frac{2(P - P_T)}{r}. \tag{11}$$

Recently Baumgarte and Rendall [7] have re-examined this fundamental equation and generalized and simplified previous arguments. In particular, they have shown that with isotropic pressure ($P = P_T$) if $\rho > 0$ and $\rho \in C^0$ with $P(0) = P_0$ where $0 < P_0 < \infty$, then $r > 2m(r)$ and $P \in C^1$ for $P > 0$. (There is then *no* isotropic pressure counterpart to the spacetimes considered here.) In the anisotropic pressure case they prove a similar theorem (with $P(0) = P_T(0)$) but require $P_T \in C^1$ in a closed neighbourhood of $r = 0$. For the case $n = 1$ P_T is *not* C^1 at $r = 0$. This then can be considered as the 'source' of the singularity in the differential invariants. Solutions with a divergence in a higher-order ($n > 1$) differential invariant will have a density and pressures which *do* meet the differentiability requirements of Baumgarte and Rendall and which satisfy $r > 2m(r)$.

Finally, we examine the nature of a singularity as regards its 'strength'. The radial null geodesics of the metric (2) reduce to

$$t^* = \frac{1}{a(r)} \tag{12}$$

$$r^* = \frac{\epsilon}{(a(r)b(r))^{1/2}} \tag{13}$$

where $* \equiv \mathrm{d}/\mathrm{d}\lambda$ for affine λ and $\epsilon = \pm 1$. We have chosen the future orientation $t^* > 0$. The equations can be considered solved by quadrature. From these equations it follows that

$$R_{\alpha\beta} l^\alpha l^\beta = \frac{(a(r)b(r))'}{(a(r)b(r))^2 r} \tag{14}$$

and so the 'singularity' at the origin is not strong for any value of n [8].

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