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Symmetry solutions of a nonlinear elastic wave equation with third-order anharmonic corrections *

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Abstract Lie symmetry method is applied to analyze a nonlinear elastic wave equation for longitudinal deformations with third-order anharmonic corrections to the elastic energy. Symmetry algebra is found and reductions to second-order ordinary differential equations (ODEs) are obtained through invariance under different symmetries. The reduced ODEs are further analyzed to obtain several exact solutions in an explicit form. It was observed in the literature that anharmonic corrections generally lead to solutions with time-dependent singularities in finite times. Along with solutions with time-dependent singularities, we also obtain solutions which do not exhibit time-dependent singularities.

Key words group invariant solutions, Lie symmetries, nonlinear elasticity equations, partial differential equations

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1 Introduction

The mathematical modeling of most of the natural and physical processes leads to such nonlinear partial differential equations (PDEs) whose analytic solutions are hard to find. Therefore, the reduction to ordinary differential equations (ODEs) and construction of exact solutions are two important problems in the study of nonlinear PDEs. A powerful general technique for analyzing nonlinear PDEs is given by the classical Lie symmetry method, which can be efficiently employed to study problems having implicit or explicit symmetries. Thus, it provides the most widely applicable technique to find the closed-form solutions of differential equations and contains many efficient methods for solving differential equations like separation of variables, traveling wave solutions, self-similar solutions, and exponential self-similar solutions (as a particular case, cf. [1]). Since the modern treatment of the classical Lie symmetry theory by Ovsiannikov[2], the theory of symmetries of differential equations has been studied intensely and has substantially grown. A large amount of literature about the classical Lie symmetry theory and its applications and extensions is available, e.g., [2-16].

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This work is concerned with the investigation of symmetry reductions and exact solutions of a nonlinear wave equation for elastic motion with third-order anharmonic corrections to the elastic energy. The wave equation with anharmonic corrections has not received much attention in the continuum limit. However, in discrete models, some one-dimensional problems in discrete and semi-discrete lattices for cubic and quartic anharmonics have been studied[17-20]. Our aim is to apply the classical Lie symmetry method to investigate the exact solutions of equation of motion with third-order anharmonic corrections to the elastic energy in the continuum limit, which is quite different from the discrete lattice models. Approximate symmetries and prolongation technique were used to carry out symmetry analysis of some interesting cases of such nonlinear wave equations by Alfinito et al. in [21]. The equation of motion for longitudinal deformation with third-order anharmonic terms has also been analyzed in [22] by using quadratures and asymptotic series, where it was concluded that anharmonic corrections to the elastic energy are likely to lead to solutions involving time-dependent singularities at finite times. Interestingly, the symmetry solutions obtained in our work constitute some examples of solutions which do not exhibit time-dependent singularities at finite times.

Let us outline the equation of motion under investigation; the reader is referred to [22] for details. The elasticity models for large values of elastic deformations require the terms of order higher than second to be considered in the elastic energy. The nonlinear equations obtained are termed as anharmonic corrections to the elastic wave equation. The linear theory of elasticity is based upon the assumption that the strain tensor u_{ij} depends linearly on the vector u_i as

$$u_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{1.1}$$

The elastic energy for an isotropic body is then given by

$$E = \int \left(\frac{\lambda}{2}u_{ii}^2 + \mu u_{ij}^2\right) d\boldsymbol{r}, \tag{1.2}$$

where $\boldsymbol{r}$ is the position vector, and λ and μ are Lame's coefficients.

Nonlinear contributions to elasticity appear through the strain tensor and the higher-order terms in the elastic energy, which is given by

$$E = \int \left(\frac{\lambda}{2}u_{ii}^2 + \mu u_{ij}^2 + \frac{1}{3}Au_{ij}u_{jk}u_{ki} + Bu_{ij}^2u_{kk} + \frac{1}{3}Cu_{ii}^3\right) d\boldsymbol{r}, \tag{1.3}$$

where A, B and C are constants. In Eq. (1.3), only third-order terms are retained and higher-order terms are neglected. For a longitudinal displacement $u_1(x_1) = u(x)$, the energy E takes the form

$$E = \int \left(\alpha u_x^2 + \frac{1}{6}\beta u_x^3\right) \rho d\boldsymbol{r}, \tag{1.4}$$

where $\alpha = \frac{\lambda+2\mu}{\rho}$, $\beta = \frac{3(\lambda+2\mu)+2(A+3B+C)}{\rho}$, and ρ is the density. For the elastic energy given by Eq. (1.4), the equation of motion can be written as

$$\frac{\partial^2 u}{\partial t^2} = \left(\alpha + \beta\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2}. \tag{1.5}$$

This nonlinear wave equation is the continuum limit of the Fermi-Pasta-Ulam equation[21-23]. Symmetry solutions of some cases of Eq. (1.5), other than the case considered in this work, have been recently found in [24].

The Lame coefficients for different materials are usually considered to be constants. It is clear that $\beta = 3\alpha$ for $A = B = C = 0$, but in this case the explicit third-order contributions to the energy are absent. It is worth noting that for $\beta = 3\alpha$, the nonlinearities do occur in the energy through the nonlinear terms in the strain tensor. In this case, Eq. (1.5) can be written as

$$\frac{\partial^2 u}{\partial t^2} = \alpha\left(1 + 3\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2}. \tag{1.6}$$

The aim of this paper is to investigate different cases, where group-invariant exact solutions of Eq. (1.6) can be found. The classical Lie symmetry method is used for the reduction of Eq. (1.6) to second-order ODEs and thus for the construction of its exact solutions. Symmetries of Eq. (1.6) are found in Section 2. These symmetries are exploited in Section 3 to perform symmetry reductions and to obtain several exact solutions of Eq. (1.6). A number of special solutions obtained here provide solutions with no singularities in time, but they also contain solutions illustrating the presence of anticipated time-dependent singularities occurring due to anharmonic corrections.

2 The Lie symmetry algebra

The method of determining the classical symmetries of a PDE is standard, which is described in many books, e.g., [9, 15-16]. To obtain the symmetry algebra of the PDE (1.6), we take the infinitesimal generator of symmetry algebra of the form

$$X = \xi_1(x,t,u)\frac{\partial}{\partial x} + \xi_2(x,t,u)\frac{\partial}{\partial t} + \varphi_1(x,t,u)\frac{\partial}{\partial u}.$$

Using the invariance condition, i.e., applying the 2nd prolongation $X^{[2]}$ to Eq. (1.6), yields the following system of 11 determining equations:

$$\begin{aligned}
&(\xi_1)_u = 0, \quad (\xi_2)_u = 0, \quad (\phi_1)_{u,u} = 0, \quad (\xi_1)_t = 0, \quad (\xi_2)_x = 0,\\
&-(\xi_1)_{x,x} + 2(\phi_1)_{x,u} = 0, \quad -(\xi_1)_{x,x} + 2(\phi_1)_{x,u} + 3(\phi_1)_{x,x} = 0,\\
&(\phi_1)_{t,t} - \alpha(\phi_1)_{x,x} = 0, \quad -3(\xi_1)_x + 2(\xi_2)_t + (\phi_1)_u = 0,\\
&-2(\xi_1)_x + 2(\xi_2)_t + 3(\phi_1)_x = 0, \quad -(\xi_2)_{t,t} + 2(\phi_1)_{t,u} = 0.
\end{aligned}$$

Solving determining equations, we obtain the following infinitesimals:

$$\xi_1 = k_3 + k_4 x, \quad \xi_2 = k_2 + \left(k_4 - \frac{3k_6}{2}\right)t, \quad \phi_1 = k_5 + k_1 t + k_6 x + (k_4 + 3k_6)u.$$

Hence, the associated symmetry algebra of Eq. (1.6) is spanned by the vector fields

$$\begin{aligned}
&X_1 = t\frac{\partial}{\partial u}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial x},\\
&X_4 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \quad X_5 = \frac{\partial}{\partial u}, \quad X_6 = -\frac{3t}{2}\frac{\partial}{\partial t} + (3u + x)\frac{\partial}{\partial u}.
\end{aligned}$$

The commutation relations of the Lie algebra $\mathcal{G}$ are in Table 1.

Table 1 Commutator table for the Lie algebra $\mathcal{G}$

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	X_5	0	0	0	$-\frac{9X_1}{2}$
X_2	$-X_5$	0	0	$-X_2$	0	$\frac{3X_2}{2}$
X_3	0	0	0	$-X_3$	0	$-X_5$
X_4	0	X_2	X_3	0	X_5	0
X_5	0	0	0	$-X_5$	0	$-3X_5$
X_6	$\frac{9X_1}{2}$	$-\frac{3X_2}{2}$	X_5	0	$3X_5$	0

3 Exact solutions

In this section, we give invariant solutions of the PDE (1.6) corresponding to different symmetries of the PDE (1.6). It is clear that for ODEs, a solvable group leads to a reduction of order which is equal to the dimension of the group, but for PDEs, one has to introduce new dependent and independent variables so that the problem does not degenerate to constant solutions. The standard method is used for this by introducing similarity variables. These are new independent variables as basic invariant functions which do not involve the dependent variables. The dependent variables are defined implicitly involving invariants which contain the original variables, see [7] for details. Hence, the transformations through similarity variables lead to a reduction in the number of variables. The cases below provide many exact similarity solutions of Eq. (1.6), as well as illustrations of reductions using similarity variables. For each case, the similarity variables of the respective symmetry are used to reduce the PDE (1.6) to a second-order ODE. These ODEs are further solved to obtain classes of exact solutions of Eq. (1.6), which are invariant under the considered symmetry.

3.1 Invariant solutions for $X = X_4 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$

Solving the characteristic system for $XI = 0$ gives $I_1 = \frac{t}{x}$ and $I_2 = \frac{u}{x}$ as the differential invariants of $X = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$. Hence, the similarity variables for X are

$$\xi(x,t) = \frac{t}{x} \quad \text{and} \quad V(\xi) = \frac{u}{x}.$$

Substitution of similarity variables in Eq. (1.6) implies that the solution of Eq. (1.6) is of the form $u = xV(\xi)$, where $V(\xi)$ satisfies the ODE

$$\frac{d^2V}{d\xi^2}\left[1 - \alpha\xi^2 + 3\alpha\xi^2\left(-V + \xi\frac{dV}{d\xi}\right)\right] = 0.$$

The equation $1 - \alpha\xi^2 + 3\alpha\xi^2\left(-V + \xi\frac{dV}{d\xi}\right) = 0$ yields $V(\xi) = \frac{1}{9\alpha\xi^2} - \frac{1}{3} + C_1\xi$. Thus, an exact solution of the PDE (1.6), invariant under the symmetry $x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$, is given by

$$u(x,t) = \frac{x^3}{9\alpha t^2} - \frac{x}{3} + C_1 t.$$

3.2 Invariant solutions for $X = X_1 + X_2 = \frac{\partial}{\partial t} + t\frac{\partial}{\partial u}$

Calculating the differential invariants of $X = \frac{\partial}{\partial t} + t\frac{\partial}{\partial u}$ gives $I_1 = x$ and $I_2 = u - \frac{t^2}{2}$. Hence, the similarity variables for X are

$$\xi(x,t) = x \quad \text{and} \quad V(\xi) = u - \frac{t^2}{2},$$

which imply that the solution of Eq. (1.6) is of the form $u = V(\xi) + \frac{t^2}{2}$, where $V(\xi)$ satisfies the ODE

$$\alpha\left(1 + 3\frac{dV}{d\xi}\right)\frac{d^2V}{d\xi^2} - 1 = 0.$$

Its first integral, via the substitution $\frac{dV}{d\xi} = F(\xi)$, yields $\alpha\left(\frac{dV}{d\xi} + \frac{3}{2}\left(\frac{dV}{d\xi}\right)^2\right) - \xi = C_1$. Solving this ODE leads to exact solutions of the PDE (1.6), which do not contain singularities in time. The solutions are

$$u(x,t) = \frac{t^2}{2} - \frac{x}{3} + \frac{(\alpha + 6x + 6C_1)^{3/2}}{27\sqrt{\alpha}} + C_2, \tag{3.1}$$

$$u(x,t) = \frac{t^2}{2} - \frac{x}{3} - \frac{(\alpha + 6x + 6C_1)^{3/2}}{27\sqrt{\alpha}} + C_2. \tag{3.2}$$

3.3 Invariant solutions for $X = X_6 = -\frac{3t}{2}\frac{\partial}{\partial t} + (3u + x)\frac{\partial}{\partial u}$

The differential invariants of $X = -\frac{3t}{2}\frac{\partial}{\partial t}+(3u+x)\frac{\partial}{\partial u}$ are given by $I_1 = x$ and $I_2 = ut^2 + \frac{xt^2}{3}$. Thus, the similarity variables for X are

$$\xi(x,t) = x \quad \text{and} \quad V(\xi) = ut^2 + \frac{xt^2}{3},$$

and the solution of Eq. (1.6) is of the form $u = \frac{V(\xi)}{t^2} - \frac{x}{3}$, where $V(\xi)$ satisfies the ODE

$$V - \frac{\alpha}{2}\frac{dV}{d\xi}\frac{d^2V}{d\xi^2} = 0.$$

The substitution $\frac{dV}{d\xi} = F(V)$ provides its first integral. It follows that $V(\xi)$ satisfies $\frac{dV}{d\xi} = \frac{(3V^2+C_1)^{1/3}}{\alpha^{1/3}}$, and is thus determined by

$$\xi - \frac{\alpha^{1/3}V(\xi)\,{}_2F_1\left(\frac{1}{2}, \frac{1}{3}, \frac{3}{2}; -3\frac{V(\xi)^2}{C_1}\right)}{C_1^{1/3}} + C_2 = 0 \quad (C_1 \neq 0).$$

Here, ${}_2F_1$ denotes the Gauss hypergeometric function ${}_2F_1(a, b, c; x)$ defined as

$${}_2F_1(a,b,c;x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)b(b+1)(b+2)\cdots(b+n-1)}{n!c(c+1)(c+2)\cdots(c+n-1)},$$

whose properties can be found, for example, in [25-26]. For $C_1 = 0$, the solution is given by

$$u(x,t) = \frac{(x+C)^3}{9\alpha t^2} - \frac{x}{3}.$$

3.4 Invariant solutions for $X = X_1 + X_2 + X_3 = \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + t\frac{\partial}{\partial u}$

The differential invariants of $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + t\frac{\partial}{\partial u}$ are found to be $I_1 = t - x$ and $I_2 = u + \frac{x}{2}(x - 2t)$. Thus, the similarity variables are

$$\xi(x,t) = t - x \quad \text{and} \quad V(\xi) = u + \frac{x}{2}(x - 2t),$$

and the solution of Eq. (1.6) is of the form $u = V(\xi) - \frac{x}{2}(x-2t)$, where $V(\xi)$ satisfies the ODE

$$\alpha + 3\alpha\xi + 3\alpha\frac{dV}{d\xi}\left(-1 + \frac{d^2V}{d\xi^2}\right) - (-1 + \alpha + 3\alpha\xi)\frac{d^2V}{d\xi^2} = 0.$$

Solving this ODE results in the following complex solutions of the PDE (1.6):

$$\begin{aligned} u(x,t) =& -\frac{x}{2}(x-2t) + \frac{(t-x)^2}{2} + \frac{t-x}{3} - \frac{t-x}{3\alpha} \\ &+ \frac{\mathrm{i}}{27\alpha^2}\left[2\alpha - 1 + 6\alpha(t-x) - \alpha^2(1+9C_1)\right]^{3/2} + C_2, \\ u(x,t) =& -\frac{x}{2}(x-2t) + \frac{(t-x)^2}{2} + \frac{t-x}{3} - \frac{t-x}{3\alpha} \\ &- \frac{\mathrm{i}}{27\alpha^2}\left[2\alpha - 1 + 6\alpha(t-x) - \alpha^2(1+9C_1)\right]^{3/2} + C_2. \end{aligned}$$

As an application, the functions $v(x,t) = \mathrm{Re}(u(x,t))$ and $w(x,t) = \mathrm{Im}(u(x,t))$ provide real-valued solutions to the coupled system of PDEs

$$\frac{\partial^2 v}{\partial t^2} = \alpha\left(\frac{\partial^2 v}{\partial x^2} + 3\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} - 3\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right), \quad \frac{\partial^2 w}{\partial t^2} = \alpha\left(\frac{\partial^2 w}{\partial x^2} + 3\frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x^2} + 3\frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial x^2}\right).$$

3.5 Invariant solutions for $X = X_1 + X_2 + X_3 + X_5 = \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + (t+1)\frac{\partial}{\partial u}$

The similarity variables for $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + (t+1)\frac{\partial}{\partial u}$ are

$$\xi(x,t) = t - x \quad \text{and} \quad V(\xi) = u + \frac{x}{2}(x - 2t - 2).$$

Thus, the solution of Eq. (1.6) is of the form $u = V(\xi) - \frac{x}{2}(x-2t-2)$, where $V(\xi)$ satisfies the ODE

$$-\alpha\left[4 + 3\xi + 3\frac{dV}{d\xi}\left(-1 + \frac{d^2V}{d\xi^2}\right)\right] + [-1 + \alpha(4+3\xi)]\frac{d^2V}{d\xi^2} = 0.$$

Its solution gives the following complex solutions of the PDE (1.6):

$$\begin{aligned} u(x,t) = &-\frac{1}{216\alpha^2}\{3 - 96\alpha^2 + 768\alpha^4 + 72\alpha t - 288\alpha^2 t - 108\alpha^2 t^2 - 72\alpha x + 72\alpha^2 x + 54\alpha^2 C_1 \\ &+ 864\alpha^4 C_1 + 243\alpha^4 C_1^2 + 8\mathrm{i}\left[8\alpha - 1 + 6\alpha(t-x) - \alpha^2(16+9C_1)\right]^{3/2} - 216\alpha^2 C_2\}, \\ u(x,t) = &-\frac{1}{216\alpha^2}\{3 - 96\alpha^2 + 768\alpha^4 + 72\alpha t - 288\alpha^2 t - 108\alpha^2 t^2 - 72\alpha x + 72\alpha^2 x + 54\alpha^2 C_1 \\ &+ 864\alpha^4 C_1 + 243\alpha^4 C_1^2 - 8\mathrm{i}\left[8\alpha - 1 + 6\alpha(t-x) - \alpha^2(16+9C_1)\right]^{3/2} - 216\alpha^2 C_2\}. \end{aligned}$$

Thus, the functions $v(x,t) = \mathrm{Re}(u(x,t))$ and $w(x,t) = \mathrm{Im}(u(x,t))$ provide real-valued solutions to the coupled system of PDEs

$$\frac{\partial^2 v}{\partial t^2} = \alpha\left(\frac{\partial^2 v}{\partial x^2} + 3\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} - 3\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right), \quad \frac{\partial^2 w}{\partial t^2} = \alpha\left(\frac{\partial^2 w}{\partial x^2} + 3\frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x^2} + 3\frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial x^2}\right).$$

3.6 Invariant solutions for $X = 3X_4 - X_6 = 3x\frac{\partial}{\partial x} + \frac{9t}{2}\frac{\partial}{\partial t} - x\frac{\partial}{\partial u}$

The similarity variables for $X = 3x\frac{\partial}{\partial x} + \frac{9t}{2}\frac{\partial}{\partial t} - x\frac{\partial}{\partial u}$ are

$$\xi(x,t) = \frac{t}{x^{3/2}} \quad \text{and} \quad V(\xi) = u + \frac{x}{3}.$$

Thus, the solution of Eq. (1.6) is of the form $u = V(\xi) - \frac{x}{3}$ with $V(\xi)$ determined from

$$8\frac{d^2V}{d\xi^2} + 135\alpha\xi^2\left(\frac{dV}{d\xi}\right)^2 + 81\alpha\xi^3\frac{dV}{d\xi}\frac{d^2V}{d\xi^2} = 0.$$

Its first integral, via the substitution $\frac{dV}{d\xi} = F(\xi)$, yields $2\left(\frac{dV}{d\xi}\right)^{4/5} + 9\left(\frac{dV}{d\xi}\right)^{9/5}\alpha x^3 - C = 0$. It follows that, for $C = 0$, an X-invariant exact solution of the PDE (1.6) is given by

$$u(x,t) = -\frac{x}{3} + \frac{x^3}{9\alpha t^2} + C_1.$$

3.7 Invariant solutions for $X = X_2 + X_4 = x\frac{\partial}{\partial x} + (t+1)\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$

Calculating the differential invariants of $X = x\frac{\partial}{\partial x} + (t+1)\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$ gives $I_1 = \frac{1+t}{x}$ and $I_2 = \frac{u}{x}$. Hence, the similarity variables for X are

$$\xi(x,t) = \frac{1+t}{x} \quad \text{and} \quad V(\xi) = \frac{u}{x}.$$

Thus, the solution of Eq. (1.6) is of the form $u = xV(\xi)$ with $V(\xi)$ satisfying the ODE

$$\frac{d^2V}{d\xi^2}\left[1 - \alpha\xi^2 + 3\alpha\xi^2\left(-V + \xi\frac{dV}{d\xi}\right)\right] = 0.$$

Solving this equation gives an X-invariant exact solution

$$u(x,t) = \frac{x^3 - 3\alpha(1+t)^2\left[x - 3(1+t)C_1\right]}{9\alpha(1+t)^2} \tag{3.3}$$

of the PDE (1.6) with providing an example of a solution that does not exhibit time-dependent singularities.

3.8 Invariant solutions for $X = X_3 + X_4 = (x+1)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$

The similarity variables for $X = (x+1)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$ are

$$\xi(x,t) = \frac{t}{1+x} \quad \text{and} \quad V(\xi) = \frac{u}{1+x},$$

and $V(\xi)$ is determined from

$$\frac{d^2V}{d\xi^2}\left[1 - \alpha\xi^2 + 3\alpha\xi^2\left(-V + \xi\frac{dV}{d\xi}\right)\right] = 0.$$

Hence,

$$u(x,t) = \frac{(1+x)^3 + 3\alpha t^2\left(-1 - x + 3tC_1\right)}{9\alpha t^2}$$

is an exact solution of the PDE (1.6), which is invariant under $(x+1)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$.

3.9 Invariant solutions for $X = X_2 + X_3 + X_4 = (x+1)\frac{\partial}{\partial x} + (t+1)\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$

The similarity variables for $X = (x+1)\frac{\partial}{\partial x} + (t+1)\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$ are

$$\xi(x,t) = \frac{1+t}{1+x} \quad \text{and} \quad V(\xi) = \frac{u}{1+x}.$$

Substituting them in Eq. (1.6) reduces the question to finding the solution $V(\xi)$ of the ODE

$$\frac{d^2V}{d\xi^2}\left[1 - \alpha\xi^2 + 3\alpha\xi^2\left(-V + \xi\frac{dV}{d\xi}\right)\right] = 0.$$

It follows that

$$u(x,t) = \frac{(1+x)^3 + 3\alpha(1+t)^2\left[-1 - x + 3(1+t)C_1\right]}{9\alpha(1+t)^2} \tag{3.4}$$

is an exact solution of the PDE (1.6) with no time-dependent singularities.

Remark 1 Though it is difficult to find exact solutions of the PDE (1.6), in many cases one can succeed in reducing it to a first-order ODE. The systematic procedure for such reductions is illustrated in the following example.

Example We reduce the PDE (1.6) using the symmetry $X = X_3 + X_6 = \frac{\partial}{\partial x} - \frac{3t}{2}\frac{\partial}{\partial t} + (3u+x)\frac{\partial}{\partial u}$. The similarity variables are

$$\xi(x,t) = \mathrm{e}^{\frac{3x}{2}}t \quad \text{and} \quad V(\xi) = \frac{1}{9}\mathrm{e}^{-3x}(1 + 9u + 3x),$$

which reduce the PDE (1.6) to the ODE

$$\xi\left[-8 + 81\alpha\xi^2\left(2V + \xi\frac{dV}{d\xi}\right)\right]\frac{d^2V}{d\xi^2} + 81\alpha\xi\left(2V + \xi\frac{dV}{d\xi}\right)\left(4V + 5\xi\frac{dV}{d\xi}\right) = 0. \tag{3.5}$$

This equation is difficult to solve so we further reduce through its symmetries. The symmetry algebra of Eq. (3.5) is found to be one-dimensional and generated by $X = \xi\frac{\partial}{\partial \xi} - 2V\frac{\partial}{\partial V}$. Since the differential invariants of X do not lead to much simplification, we find the representation of

Eq. (3.5) in the canonical coordinates, $w = \xi^2 V$ and $t = \ln \xi$, corresponding to the symmetry X as follows:

$$-8\frac{d^2w}{dt^2} + \frac{dw}{dt}\left(40 + 81\alpha\frac{d^2w}{dt^2}\right) - 48w = 0. \tag{3.6}$$

The equation (3.6) clearly admits the symmetry $\tilde{X} = \frac{\partial}{\partial t}$, whose similarity variables $r = w$ and $y(r) = \frac{dw}{dt}$ reduce it to the first-order ODE

$$\frac{dy}{dr} = \frac{48r - 40y}{81\alpha y^2 - 8y}.$$

4 Discussions of solutions and initial boundary value problems (IBVPs)

Most of the solutions obtained above exhibit time-dependent singularities, reinforcing what was pointed out in [22]. However, solutions given by Eqs. (3.1)–(3.4) provide examples of solutions that do not involve time-dependent singularities at finite times.

Consider the elastic movements and deformations governed by the PDE (1.6) with the initial displacement $u_0(x)$ and the initial velocity $v_0(x)$. Depending on the physical situation, different types of initial and boundary conditions can be applied to solutions presented in the previous section. Below, we provide instances of IBVPs where functions, derived from solutions in Eqs. (3.2)–(3.4), serve as solutions of IBVPs with the Dirichlet type boundary conditions. This is done by applying initial and boundary conditions of the form

$$u(x,0) = u_0(x), \tag{4.1}$$

$$\frac{\partial u}{\partial t}(x,0) = v_0(x), \tag{4.2}$$

$$u(0,t) = g(t), \tag{4.3}$$

$$u(L,t) = f(t). \tag{4.4}$$

4.1 Finite domains

Consider the PDE (1.6) in a finite domain $0 < x < L$ and $t > 0$, together with the condition (4.4). The solution given by Eq. (3.3) can be written in the form

$$u(x,t) = \frac{x^3 - L^3 + 3\alpha(3f(t) + L - x)(1+t)^2}{9\alpha(1+t)^2} \tag{4.5}$$

for IBVPs involving the condition (4.4). For example, for $f(t) = \frac{L^3}{9\alpha(1+t)^2} - \frac{L}{3}$, $u(x,t) = \frac{x^3}{9\alpha(1+t)^2} - \frac{x}{3}$ gives a solution for IBVP-1,

$$\text{IBVP-1}\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \alpha\left(1 + 3\dfrac{\partial u}{\partial x}\right)\dfrac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \\ u(x,0) = u_0(x), \quad \dfrac{\partial u}{\partial t}(x,0) = v_0(x), \\ u(0,t) = 0, \quad u(L,t) = \dfrac{L^3}{9\alpha(1+t)^2} - \dfrac{L}{3}, \end{cases}$$

where $u_0(x) = \frac{x^3}{9\alpha} - \frac{x}{3}$ and $v_0(x) = -\frac{2x^3}{9\alpha}$. Similarly, the conditions (4.1)–(4.3) can be applied to obtain corresponding solutions.

4.2 Infinite domains

Consider the PDE (1.6) in a semi-infinite domain $0 < x < \infty$ and $t > 0$, together with the condition (4.2). In this case, the solution (3.4) can be written in the form

$$u\left(x,t\right)=\frac{(1+x)^3+(1+t)^2\{-3\alpha-3\alpha x+(1+t)[2+6x+6x^2+2x^3+9\alpha v_0\left(x\right)]\}}{9\alpha(1+t)^2}$$

for IBVPs involving the condition (4.2). As an example, for $v_0(x) = -\frac{2(1+x)^3}{9\alpha}$, $u(x,t) = \frac{(1+x)^3}{9\alpha(1+t)^2} - \frac{1+x}{3}$ provides a solution of IBVP-2,

$$\text{IBVP-2}\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=\alpha\left(1+3\dfrac{\partial u}{\partial x}\right)\dfrac{\partial^2 u}{\partial x^2}, \quad 0<x<\infty, \quad t>0,\\ u(x,0)=u_0(x), \quad \dfrac{\partial u}{\partial t}(x,0)=-\dfrac{2(1+x)^3}{9\alpha},\\ u(0,t)=f(t),\end{cases}$$

where $u_0(x) = \frac{(1+x)^3}{9\alpha} - \frac{1+x}{3}$ and $f(t) = \frac{1}{9\alpha(1+t)^2} - \frac{1}{3}$.

Consider the PDE (1.6) in an infinite domain $-\infty < x < \infty$ and $t > 0$, together with the condition (4.1). In this case, the solution (3.3) can be written in the form

$$u\left(x,t\right)=\frac{3\alpha tx(1+t)^2-tx^3(3+3t+t^2)+9\alpha(1+t)^3u_0\left(x\right)}{9\alpha(1+t)^2}$$

for IBVPs involving the condition (4.1). As an example, for $u_0(x)=\frac{x^3}{9\alpha}-\frac{x}{3}$, $u(x,t)=\frac{x^3}{9\alpha(1+t)^2}-\frac{x}{3}$ provides a solution of IBVP-3,

$$\text{IBVP-3}\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=\alpha\left(1+3\dfrac{\partial u}{\partial x}\right)\dfrac{\partial^2 u}{\partial x^2}, \quad -\infty<x<\infty, \quad t>0,\\ u(x,0)=\dfrac{x^3}{9\alpha}-\dfrac{x}{3}, \quad \dfrac{\partial u}{\partial t}(x,0)=v_0(x),\end{cases}$$

where $v_0(x) = -\frac{2x^3}{9\alpha}$.

The solutions of IBVPs presented above illustrate physical natures of solutions (3.1)–(3.4) in the sense that these do not involve time-dependent singularities at finite times. Yet, in the case of spatial boundaries placed at infinity like IBVP-2 and IBVP-3, the solutions may exhibit unphysical character in the sense that the solutions become boundless at spatial boundaries at infinity. However, in some cases, one can derive completely physical solutions from the solutions in Section 3 which neither have singular behaviors in time nor involve boundless movement at spatial boundaries at infinity. For example, in a situation where x varies along the line $x = \sqrt{3\alpha}(1+t)$, for the large x, the solutions $u(x,t)$ given by Eq. (3.3) also stay bounded at spatial boundaries at infinity; in fact, $u(x,t) \to 0$ as $x \to \pm\infty$.

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