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Heuristic analysis of the complete symmetry group and nonlocal symmetries for some nonlinear evolution equations

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SUMMARY

The complete symmetry group of a $1+1$ evolution equation has been demonstrated to be represented by the six-dimensional Lie algebra of point symmetries $sl(2, R) \oplus_s W$, where W is the three-dimensional Heisenberg–Weyl algebra. We construct a complete symmetry group of a nonlinear heat equation $u_t = F(u_x)u_{xx}$ for some smooth functions F, using the point symmetries admitted by each equation. The nonlinear heat equation is not specifiable by point symmetries alone even when the number of symmetries is 6. We report *Ansätze* which provide a route to the determination of the required nonlocal symmetry necessary to supplement the point symmetries for the complete specification of these nonlinear $1+1$ evolution equations. The nonlocal symmetry immediately realized is said to be *generic* to a class of equations as it gives a specific structure to an equation. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The concept of complete symmetry groups of a differential equation was introduced some 12 years ago by Krause [1, 2] to ordinary differential equations to describe the group in which its algebraic representation completely specified the differential equation under consideration. The differential equation in this instance was that of the Kepler problem for which Krause found it necessary to introduce nonlocal symmetries since the Lie point symmetries of the system of ordinary differential equations describing the Kepler problem were insufficient to specify the system completely. Indeed the symmetries corresponding to the angular momentum are not even included in the representation of the complete symmetry group. The reason for this is found in the concept of minimality of representation which forms an integral part of the definition of a complete

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symmetry group. Although nonlocal symmetries feature in a number of studies of the complete symmetry groups of certain problems [3–6], this has been more an accident of the development of the study of complete symmetry groups. The theoretical treatment and applications in terms of point symmetries have been well established for ordinary differential equations [7–9].

In general [8], a system of n second-order ordinary differential equations requires $2n+1$ symmetries to specify it completely. The Newtonian equations for the Kepler problem possess just the five Lie point symmetries of the algebra $A_2 \oplus A_{3,9}$ representing invariance under time translation and rescaling on the one hand and the rotational invariance of $so(3)$ on the other.‡ Krause had resorted to the use of nonlocal symmetries to remedy the deficit§ and devised an ingenious scheme for their determination. Unfortunately, nonlocal symmetries of differential equations in general have a property in common with symmetries of first-order differential equations. Although they are more numerous than the grains of sand by the sea, there is no finite algorithm for their general determination.

Until recently, the determination of complete symmetry groups has been confined to systems of ordinary differential equations [7]. In a study of the complete symmetry group of the $1+1$ heat equation and some related equations which arise in Financial Mathematics, some basic considerations in [14–16] showed that the number of Lie point symmetries required to specify the $1+1$ heat equation is 6. Moreover, the condition for the number of symmetries required to specify the heat equation is the same for all $1+1$ evolution equations [15] be they linear or nonlinear.

In this paper, we turn our attention to the complete characterization of a class of partial differential equations by its symmetry group involving nonlocal symmetries of the nonlinear heat equation of the form

$$u_t = F(u_x)u_{xx} \tag{1}$$

where

$$u_t = \frac{\partial u}{\partial t}, \quad u_x = \frac{\partial u}{\partial x} \quad \text{and} \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

In particular, we consider the specific case mentioned in [17] for which F¶ takes the form

$$F = u_x^n, \quad n \neq 0, -2 \tag{2}$$

We adopt a following notation for a symmetry $\mathbf{X}$ of a partial differential equation:

$$\mathbf{X} = \xi(\cdot)\partial x + \tau(\cdot)\partial t + \eta(\cdot)\partial u$$

where ξ, τ and η are functions of their arguments and with the second extension/prolongation given by

$$\mathbf{X}^{[2]} = \xi(\cdot)\partial x + \tau(\cdot)\partial t + \eta(\cdot)\partial u + \eta_x(\cdot)\partial u_x + \eta_t(\cdot)\partial u_t + \eta_{xx}(\cdot)\partial u_{xx}$$

‡We use the classification scheme of Mubarakzyanov [10–13].

§We must note that the use of nonlocal symmetries in the first application of this concept of a complete symmetry group should not be taken to imply that nonlocal symmetries are a necessary concomitant. That nonlocal symmetries have played an important role in the determination of the complete symmetry group in a number of instances [3–6] should not obscure the reality that point symmetries have played an important role in the theoretical development as well as certain applications [7–9] of complete symmetry groups.

¶Note that the conditions demanded for n contrast those given in [18]. This is because the symmetry group of the second equation in (2) is no different for all values of n except possibly for $n = 0, -2$.

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The relevant extended infinitesimals $\eta_x(\cdot), \eta_t(\cdot)$ and $\eta_{xx}(\cdot)$ are functions of their arguments and are as found on [19, p. 67]. We begin by giving a brief review of point symmetries admitted by (1) for a specific function F.

2. REVIEW OF THE LIE POINT SYMMETRIES OF $u_t = u_x^n u_{xx}$

It is well understood from [17] that Equation (1) has, *inter alia*, a realization of the algebra $A_{3,1}$, where

$$A^1_{3,1} = \langle \partial t, \partial u, \partial x \rangle \tag{3}$$

for an arbitrary smooth function F. Also the maximal invariance algebra admitted by (1) according to Theorem 3.4 of [17] is the four-dimensional Lie algebra:

$$A_{3,1} \oplus \langle 2t\partial t + x\partial x + u\partial u \rangle \tag{4}$$

The class of equations is further reported to admit the following Lie point symmetries for different smooth functions F:

$$F = u_x^n : \Gamma = nt\partial t - u\partial u, \quad n \neq 0, -2 \tag{5}$$

There are no further point symmetries of (1) with F taking one of the forms in (5) reported by [17, Theorem 3.4] to have been admitted by Equation (1). It is further accepted [20] that for the realization of a complete symmetry group of a 1 + 1 evolution equation one requires six symmetries, be they point, generalized or nonlocal. This therefore implies that one would require a nonlocal symmetry, possibly more, to specify (1) completely.

The classical heat equation, i.e. $F = u_x^n$, $n = 0$,

$$u_t = u_{xx} \tag{6}$$

as a linear partial differential equation, its complete symmetry group has been dealt with in [20].

For $F = u_x^n$, $n = -2$, in (5) one immediately realizes a linearizable equation,

$$u_t = u_x^{-2} u_{xx} \tag{7}$$

which admits the following Lie point symmetries[‖]

$$\begin{aligned}
\Lambda_1 &= \partial t\\
\Lambda_2 &= \partial u\\
\Lambda_3 &= 2t\partial t + u\partial u\\
\Lambda_4 &= xu\partial x - 2t\partial u\\
\Lambda_5 &= t^2\partial t - \tfrac{1}{4}(2t + u^2)x\partial x + ut\partial u\\
\Lambda_6 &= x\partial x\\
\Lambda_7 &= F_5(u, t)\partial x
\end{aligned} \tag{8}$$

‖The calculation of point symmetries has been made possible by one of the classic codes described in [21, 22].

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where F_5 a solution of the linear heat equation $x_t = x_{uu}$. We have a few important observations of Equation (11) to note. Firstly, contrary to [18, Theorem 3.4] we have an extension of the algebra, which has been reported by the theorem to have been impossible, as we witness the $\infty + 1 + 5$ Lie point symmetries admitted by Equation (11) with Λ_2, Λ_4 and Λ_5 constituting an $sl(2, R)$ algebra. Secondly, this equation does not seem to admit point symmetries constituting the $A_{3,1}$ algebra but rather the $A_{2,1}$ algebra given by Λ_1 and Λ_2. Thirdly, the point symmetries look very similar to those admitted by the heat equation (6) but with the x in place of u. This reflects a particular behaviour in the geometry of the problem. Furthermore, the case $n = 0$ constitutes the classical heat equation and there is no need to discuss its symmetry properties here as these are well known in the literature.

For $F = u_x^n,\ n \neq 0, -2$, in (5) one immediately realizes the not-so-simple equation,

$$u_t = u_x^n u_{xx}, \quad n \neq 0, -2 \tag{9}$$

which is invariant under the action of the following Lie point symmetries:

$$\begin{aligned} \Gamma_1 &= \partial t \\ \Gamma_2 &= \partial u \\ \Gamma_3 &= \partial x \\ \Gamma_4 &= (n+2)t\partial t + x\partial x \\ \Gamma_5 &= nt\partial t - u\partial u \end{aligned} \tag{10}$$

We note here that Equation (9) admits a finite number of Lie point symmetries but this does not reflect any strange behaviour associated with the geometry as in the previous case. The homogeneity and solution symmetries are lacking. This behaviour is very much in line with the stipulations of [18, Theorem 3.4] in terms the presence of $A_{3,1}$ realizations given by Γ_1, Γ_2 and Γ_3. Also Γ_5 has already been reported in (5), however we have an extra nongeneric point symmetry Γ_4.

3. COMPLETE SYMMETRY GROUP OF $u_t = u_x^n u_{xx}$

In this section, we turn our attention to the construction of a complete symmetry group of the class of equations:

$$u_t = u_x^n u_{xx} \tag{11}$$

The complete symmetry group of (11) for $n = 0$, i.e. the classical heat equation, is well understood and was treated in [15]. In this section, we construct the complete symmetry group of (11) for $n = -2$ and $n \neq -2$. We begin with the former case.

3.1. Specification of $u_t = u_x^n u_{xx}$ for $n = -2$

To construct a complete symmetry group of (7), we start with a general second-order evolution equation *videlicet*:

$$u_{xx} = f(x, t, u, u_x, u_t) \tag{12}$$

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where f is a function of its arguments to be specified. For Equation (12) to be invariant under Λ_1 and Λ_2 it must be of the form

$$u_{xx} = f(x, u_x, u_t) \tag{13}$$

The second extension of Λ_6 is given by

$$\Lambda_6^{[2]} = x\partial x - u_x \partial u_x - 2u_{xx}\partial u_{xx} \tag{14}$$

The application of (14) to (13) produces the following associated Lagrange's system upon taking (13) into account:

$$\frac{\mathrm{d}f}{-2f} = \frac{\mathrm{d}x}{x} = \frac{\mathrm{d}u_x}{-u_x} = \frac{\mathrm{d}u_t}{0} \tag{15}$$

One may carefully select the characteristics so that (13) takes the form

$$u_{xx} = u_x^2 h(u_t, xu_x) \tag{16}$$

The application of the second extension of Λ_3, given by

$$\Lambda_3^{[3]} = u\partial u + 2t\partial t + u_x\partial u_x - u_t\partial u_t + u_{xx}\partial u_{xx} \tag{17}$$

to (16) produces the associated Lagrange's system

$$\frac{\mathrm{d}h}{-h} = \frac{\mathrm{d}v}{v} = \frac{\mathrm{d}u_t}{-u_t} \tag{18}$$

upon letting $v = xu_x$. One's choice of characteristics may reduce (16) to

$$u_{xx} = u_t u_x^2 g(w) \tag{19}$$

where $w = xu_t u_x$. The application of the second extension of Λ_4, i.e.

$$\Lambda_4^{[2]} = xu\partial x - 2t\partial u - uu_x\partial u_x - 2u_t\partial u_t - 2uu_{xx}\partial u_{xx} \tag{20}$$

to (19) produces, after some simplification,

$$g + wg' = 0 \tag{21}$$

upon the integration of which (19) becomes

$$u_{xx} = \frac{u_x}{x}\kappa \tag{22}$$

where κ is a constant of integration. It is clear that we are by no means obtaining the intended equation as we note that the term u_t has disappeared completely in Equation (22) and there is no way to recover it from the application of Λ_5. This surely is not an desired outcome since we started with a general second-order *time-evolution* equation, applied point symmetries admitted by an *time-evolution* equation hoping to achieve an *time-evolution* equation. In fact what we obtain is politely expressed as absurd! Why?

The answer lies in the consideration of the geometry of the equation. Symmetries (8) of (7) reflect the *rotation* and *reflection* in space. The reflection behaviour is amenable to continuous

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transformations only but not to continuous point transformations. This in turn translates into the fact that this equation is not connected to the heat equation via a continuous transformation but rather a discrete transformation,** *videlicet*,

$$t = -T, \quad x = U, \quad u = X, \quad T \neq 0, \ \ U \neq 0 \tag{23}$$

Transformation (23) clearly is a continuous but not a point transformation and one immediately observes a reflection symmetry and consequently (7) cannot be specified using point symmetries only and so we require nonlocal symmetries to remove the deficit and we do this in the next section.††

3.2. Specification of $u_t = u_x^n u_{xx}$ for $n \neq -2$

As before, we commence with a general second-order time-evolution equation:

$$u_{xx} = f(x, t, u, u_x, u_t) \tag{24}$$

where f is a function to be specified. For the intended equation to be invariant under Γ_1, Γ_2 and Γ_3 of (10), it must be of the form

$$u_{xx} = f(u_x, u_t) \tag{25}$$

The second extension of Γ_5 is given by

$$\Gamma_5^{[2]} = nt\partial t - u\partial u - u_x\partial u_x - (n+1)u_t\partial u_t - u_{xx}\partial u_{xx} \tag{26}$$

and its application to (26) produces

$$u_{xx} = u_x \frac{\partial f}{\partial u_x} + (n+1)u_t \frac{\partial f}{\partial u_t} \tag{27}$$

With the associated Lagrange's system given by

$$\frac{\mathrm{d}f}{f} = \frac{\mathrm{d}u_x}{u_x} = \frac{\mathrm{d}u_t}{-(n+1)u_t}$$

the characteristics may be chosen such that Equation (24) takes the form

$$u_{xx} = u_x h\left(\frac{u_x^{n+1}}{u_t}\right) \tag{28}$$

The second extension of the remaining point symmetry is

$$(n+2)t\partial t + x\partial x - u_x\partial u_x - (n+2)u_t\partial u_t - 2u_{xx} \tag{29}$$

Its application to (28) yields after some simplification

$$h = -wh' \tag{30}$$

**One needs to take heed when constructing similarity solutions of (1) with $F = u_x^n$, $n = -2$. using point symmetries to avoid disaster!

††This feature is in fact important to illustrate the fact that the maximal number of point symmetries possessed by a given equation is not adequate to imply its complete specification with point symmetries only as claimed in [14].

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where we have set $w = u_x^{n+1}/u_t$. The solution of this equation is child's play and it produces

$$h = \frac{\imath}{w}$$

and consequently (28) takes the form

$$u_t = \imath u_{xx} u_x^n, \quad n \neq -2 \tag{31}$$

where $\imath$ is a constant to be specified. To specify the constant, one requires a nonlocal symmetry as we have exhausted the available point symmetries of (10). Note that unlike the previous case, Equation (33) is in line with the equation we are trying to specify. The not-so-pleasant journey to search for nonlocal symmetries is inevitable in both these circumstances and for that reason we devote the entire Section 4 to this exercise.

4. ALGORITHMIC APPROACH FOR FINDING NONLOCAL SYMMETRIES

In this section, we return to the question of finding nonlocal symmetries of (1). We begin with the case for which

$$F = u_x^n, \quad n = -2 \tag{32}$$

where we have the $\infty + 1 + 5$ Lie point symmetries (8), and later the case where $n \neq -2$. From the previous section, it has become apparent that the two point symmetries Λ_4 and Λ_5 are not specifying Equation (1). Hence one requires two more symmetries, which must be nonlocal, to remove the deficit so as to satisfy the requirements of the conjecture in [15]. To accomplish this, we commence with the general second-order evolution equation:

$$u_{xx} = f(x, t, u, u_x, u_t) \tag{33}$$

where f is a function yet to be determined. We proceed to apply the symmetries of the algebra $A_{2,1}$, where

$$A_{2,1}^1 = \langle \partial t, \partial u \rangle \tag{34}$$

to Equation (33) and consider both cases for n.

4.1. Nonlocal symmetry for $F = u_x^n,\ n = -2$

In this case, we use the realization of (34) to remove t and u from f to obtain

$$u_{xx} = f(x, u_x, u_t) \tag{35}$$

At this juncture, we must specify the desired general structure of the remaining function and then work backwards to find the nonlocal symmetry which would specify that structure.[‡‡] In this case, we desire that the function f takes the form

$$u_{xx} = u_t h(x, u_x) \tag{36}$$

[‡‡]This is what makes the nonlocal symmetry immediately realized *generic* to a particular class of the equations having a similar underlying structure.

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and we impose this structure and conduct a search for this miraculous nonlocal symmetry. This structure, (36), would have been obtained by setting $f=u_t h(x,u_x)$ which can only be specified by the choice of characteristics obtained from the associated Lagrange's system:

$$\frac{\mathrm{d}f}{f}=\frac{\mathrm{d}u_t}{u_t}=\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}u_x}{0} \tag{37}$$

The associated Lagrange's system above results from the application of the second extension of our as yet *unknown* nonlocal symmetry, say Λ_8, given by

$$\Lambda_8^{[2]}=\tau(\cdot)\partial t+\eta(\cdot)\partial u+(0)\partial x+\eta_t\partial u_t+(0)\partial u_x+u_{xx}\partial u_{xx} \tag{38}$$

where τ and η are functions of unspecified arguments, to Equation (35). Note that from the associated Lagrange's system above we deduce that $\xi(\cdot)=0$. The extended infinitesimals of (38) insinuate the system of partial differential determining equations [23]:

$$\begin{aligned}\eta_x &: \frac{\partial\eta}{\partial x}-\frac{\partial\tau}{\partial x}u_t=0\\ \eta_t &: \frac{\partial\eta}{\partial t}-\frac{\partial\tau}{\partial t}u_t=u_t\\ \eta_{xx} &: \frac{\partial^2\eta}{\partial x^2}-2\frac{\partial\tau}{\partial x}u_{tx}-\frac{\partial^2\tau}{\partial x^2}u_t=u_{xx}\end{aligned} \tag{39}$$

in which differentiation with respect to t or x is total, with the assumption that η is free of u. The first and third equations give

$$\frac{\partial\tau}{\partial x}=-\frac{u_{xx}}{u_{tx}} \tag{40}$$

Differentiating the first equation with respect to t and the second equation with respect to x, taking (40) into account and after some manipulation we obtain

$$\frac{\partial\tau}{\partial t}=-1-\frac{u_{xx}u_{tt}}{u_{tx}} \tag{41}$$

With (41) the second equation of (39) produces

$$\frac{\partial\eta}{\partial t}=\frac{u_{xx}u_{tt}u_t}{u_{tx}} \tag{42}$$

Also Equation (40) with the first equation of (39) yields

$$\frac{\partial\eta}{\partial x}=-\frac{u_{xx}u_t}{u_{tx}} \tag{43}$$

With (40)–(43) one immediately gathers the required coefficients

$$\begin{aligned}\tau(\cdot)&=-\left(\int\frac{u_{xx}}{u_{tx}}\,\mathrm{d}x+\int\frac{u_{xx}u_{tt}}{u_{tx}}\,\mathrm{d}t+t\right)\\ \eta(\cdot)&=\int\frac{u_{xx}u_{tt}u_t}{u_{tx}}\,\mathrm{d}t-\int\frac{u_{xx}u_t}{u_{tx}}\,\mathrm{d}x\end{aligned} \tag{44}$$

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and the required nonlocal symmetry is given by

$$\Lambda_8 = \tau(\cdot)\partial t + \eta(\cdot)\partial u \tag{45}$$

with τ and η defined by (44). The search for a nonlocal symmetry[§§] of Equation (1) has been made possible by what we term the *Choice*[¶¶] algorithm. Furthermore, the nonlocal symmetry is said to be generic to a class of equations having a structure similar to (36). Accordingly, upon application of a nonlocal symmetry (45), Equation (35) takes the form (36).

4.2. *Nonlocal symmetry for* $F = u_x^n,\ n \neq -2$

In this case we apply the third realization of (34) to obtain

$$u_{xx} = f(u, u_x, u_t) \tag{46}$$

and requires that (46) takes the form

$$u_{xx} = u_t h(u, u_x) \tag{47}$$

This can be obtained by a setting $f = u_t h(u, u_x)$ from the characteristics f/u_t, u and u_x. These characteristics are obtained from the associated Lagrange's system:

$$\frac{\mathrm{d}f}{f} = \frac{\mathrm{d}u_t}{u_t} = \frac{\mathrm{d}u}{0} = \frac{\mathrm{d}u_x}{0}$$

The above system is the result of the application of an as yet unknown nonlocal symmetry, Γ_6 say. The system tells us that the nonlocal symmetry can only be of the form

$$\Gamma_6 = \zeta(\cdot)\partial t + \xi(\cdot)\partial x \tag{48}$$

where ζ and ξ are to be specified. The extended infinitesimals produce the following system of symmetry determining equations:

$$\begin{aligned}
\eta_x &: \frac{\partial \xi}{\partial x} u_x - \frac{\partial \tau}{\partial x} u_t = 0 \\
\eta_t &: \frac{\partial \tau}{\partial t} u_t + \frac{\partial \xi}{\partial t} u_t = -u_t \\
\eta_{xx} &: \frac{\partial^2 \xi}{\partial x^2} u_x + \frac{\partial^2 \tau}{\partial x^2} + 2\frac{\partial^2 \xi}{\partial x} u_{xx} + 2\frac{\partial^2 \tau}{\partial x^2} u_{xt} - = -u_{xx}
\end{aligned} \tag{49}$$

[§§]One can also use the methods treated in [24–28] to find nonlocal symmetries. However, for our purposes this is a more direct method for finding nonlocal symmetries.

[¶¶]Choice, since the nonlocal symmetry realized after the application of the algorithm depends on the choices made by the user. It must be noted that the *choice* algorithm produces a nonunique nonlocal symmetry due to different possible choices of assumptions.

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which upon solving as we did previously produces the required coefficients of ∂t and ∂x to be

$$\begin{aligned}\zeta &= -\left[\int \frac{u_{xx}}{u_{tx}}\,\mathrm{d}x + \int \frac{u_{xx}u_{tt}}{u_{tx}}\,\mathrm{d}t + t\right]\\ \xi &= \int \frac{u_{xx}u_{tt}u_t}{u_{tx}}\,\mathrm{d}t - \int \frac{u_{xx}u_t}{u_{tx}}\,\mathrm{d}x\end{aligned} \tag{50}$$

and consequently (48), which has been obtained by the choice algorithm. All that is left now is to specify the equations completely.

5. COMPLETE SPECIFICATION OF THE EQUATIONS

5.1. *Complete specification of* $u_t = u_x^n u_{xx}$ *for* $n = -2$

To specify completely the equation, one applies the remaining useful symmetries to (36). The second extension of Λ_6 is given by

$$\Lambda_6^{[2]} = x\partial x - u_x\partial u_x - 2u_{xx}\partial u_{xx} \tag{51}$$

and its application to (36) produces the associated Lagrange's system:

$$\frac{\mathrm{d}h}{-2h} = \frac{\mathrm{d}x}{x} = \frac{\mathrm{d}u_x}{-u_x} \tag{52}$$

so that after a suitable choice of characteristics the function h takes the form

$$h = u_x^2 g(xu_x) \tag{53}$$

and consequently the equation takes the form

$$u_{xx} = u_t u_x^2 g(xu_x) \tag{54}$$

A similar application of $\Lambda_3^{[2]}$ to (54) reduces (54) to

$$u_{xx} = \gamma u_t u_x^2 \tag{55}$$

where γ is a constant of integration. At this point, it seems we have run out of useful point symmetries as the application of any of the remaining symmetries does not produce the equation. We therefore have to make use of our algorithm to find a nonlocal symmetry to specify the arbitrary constant. We require that $\lambda = 1$. There are several ways to achieve this, one of which would be to demand that the coefficients of ∂u, ∂u_x, ∂u_t and ∂u_{xx} be 0, u_x, 0 and $-2u_{xx}$, respectively. Using the algorithm described above and after a certain amount of effort, we obtain

$$\begin{aligned}\tau(\cdot) &= 2\left[\int\left(\frac{u_xu_{xx}}{u_{tx}-u_tu_{xx}}\right)\mathrm{d}x - \int\left(\frac{u_xu_{xx}}{u_{tx}-u_tu_{xx}}\right)\mathrm{d}t\right] + t\\ \xi(\cdot) &= 2\int\left(\frac{u_xu_{xx}}{u_{tx}-u_tu_{xx}}\right)\mathrm{d}x + \int\left(\frac{u_t}{u_x}\left(1 + 2\frac{u_xu_{xx}}{u_{tx}-u_tu_{xx}}\right)\right)\mathrm{d}t + t\end{aligned} \tag{56}$$

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The second nonlocal symmetry is given by

$$\Lambda_9 = \tau\partial t + \xi\partial x \tag{57}$$

with τ and ξ given by (56) and consequently Equation (33) has been completely specified.

5.2. *Complete specification of $u_t = u_x^n u_{xx}$ for $n \neq -2$*

The complete specification (33) for the case $n \neq -2$ can be achieved in two routes. One may apply the nonlocal symmetry (57) to Equation (31), since the nonlocal symmetry serves to specify a constant to equal one, to achieve the desired Equation (9). Alternatively one may apply the remaining symmetries[‖‖] of (10) to (47) as follows. The application of Γ_2 to Equation (47) produces

$$u_{xx} = u_t g(u_x) \tag{58}$$

The application of the second extension of Γ_5 produces

$$ng = -u_x g' \tag{59}$$

so that (58) becomes

$$u_{xx} = \imath u_t u_x^{-n} \tag{60}$$

It is interesting to note that the remaining symmetry gives no useful information when applied to specify the constant! This is a nice example to show that a complete symmetry group of a given partial differential equation is not *unique* once nonlocal symmetries have been summoned to specify a given equation. Hence one turns back to a nonlocal symmetry (57) to specify the constant and consequently Equation (9) follows immediately.

The other unused symmetries of (8) and (10) are still necessary for inclusion in the representation of complete symmetry group of (33) due to meeting the requirement of the closure property. Consequently, we represent them as

$$\langle \Lambda_i, i = 1, 9\rangle_{\text{csg}} \tag{61}$$

for the case $n = -2$ and

$$\langle \Gamma_j, j = 1, 6\rangle_{\text{csg}} \tag{62}$$

for the case $n \neq -2$.

6. CONCLUSION

We started by showing the fact that, although the equation under consideration may posses a sufficient number of point symmetries in terms of [15], some of those point symmetries may not be useful in specification of the equation as observed in Section 3. We further (Section 4) provided a choice algorithm which allows the determination of the required nonlocal symmetries necessary to supplement the point symmetries. The nonlocal symmetry realized is *generic* to the structure

‖‖Point symmetries remaining after finding a nonlocal symmetry.

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imposed upon the equation. As a result all equations having the structure (36) and (47) admit the nonlocal symmetries, Λ_8 and Γ_6, respectively. Further, these nonlocal symmetries need not be unique. However, the choice algorithm provides a much more convenient way to find nonlocal symmetries.

We have also seen that once nonlocal symmetries come into the picture one loses the apparent uniqueness property of complete symmetry group for a given equation which was a nice feature for partial differential equations. What has transpired in this paper is that, with the help of the choice algorithm, one can, (if one so choose), completely specify the given equation using nonlocal symmetries only.

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