

Structure of the coadjoint orbits of Lie groups ^{*}

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Abstract

We study the geometrical structure of the coadjoint orbits of an arbitrary complex or real Lie algebra \mathfrak{g} containing some ideal \mathfrak{n} . It is shown that any coadjoint orbit in \mathfrak{g}^* is a bundle with the affine subspace of \mathfrak{g}^* as its fibre. This fibre is an isotropic submanifold of the orbit and is defined only by the coadjoint representations of the Lie algebras \mathfrak{g} and \mathfrak{n} on the dual space \mathfrak{n}^* . The use of this fact and an application of methods of symplectic geometry give a new insight into the structure of coadjoint orbits and allow us to generalize results derived earlier in the case when \mathfrak{g} is a split extension using the Abelian ideal \mathfrak{n} (a semidirect product). As applications, a new proof of the formula for the index of Lie algebra and a necessary condition of integrality of a coadjoint orbit are obtained.

1 Introduction

A Lie algebra is a semidirect product if it is a split extension using its Abelian ideal. The structure of the coadjoint orbits of a semidirect product is well understood and known due to papers of Rawnsley [1], Baguis [2], Panyushev [3, 4, 5] and others [6, 7, 8, 9]. According to [1], the coadjoint orbits of a semidirect product are classified by the coadjoint orbits of so-called little-groups (reduced-groups) which are isotropy subgroups of some representations. In fact, the fibre bundles having these coadjoint orbits as fibres, completely characterize the coadjoint orbits of the semidirect product. Our paper is devoted to a generalization of these results of Rawnsley for arbitrary Lie algebras. While in [1] and [2] for calculations the exact multiplication formulas were used, our approach in the general case is completely different.

Let G be a connected Lie group with a normal connected subgroup N and let \mathfrak{g} and \mathfrak{n} be their Lie algebras. Since \mathfrak{n} is an ideal of \mathfrak{g} , the coadjoint action of G on \mathfrak{g}^* induces the G -action \cdot on \mathfrak{n}^* . Our considerations in the article are based on the following two facts:

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An arbitrary coadjoint orbit \mathcal{O} in \mathfrak{g}^* is a bundle with some affine subspace $\mathcal{A} \subset \mathcal{O} \subset \mathfrak{g}^*$ of dimension $\dim \mathcal{A} = \dim(G \cdot \nu) - \dim(N \cdot \nu)$ as its fibre, where $\nu = \sigma | \mathfrak{n} \in \mathfrak{n}^*$ and $\sigma \in \mathcal{A}$. The affine subspace \mathcal{A} is an isotropic submanifold of the orbit \mathcal{O} with respect to the canonical Kirillov-Kostant-Souriau symplectic structure on \mathcal{O} . (*)

The identity component of the isotropy group $N_\nu = \{n \in N : n \cdot \nu = \nu\}$ of ν acts transitively on the affine subspace $\mathcal{A} \subset \mathcal{O}$. (**)

The fact (**) is equivalent to the so-called ‘‘Stages Hypothesis’’, which is a sufficient condition for a general reduction by stages theorem and was formulated in the paper of Marsden et al. [10]. In their monograph [11] this hypothesis was verified for all split extensions \mathfrak{g} using the Lie algebra \mathfrak{n} . Reformulating ‘‘Stages Hypothesis’’ in the form (**) we found a short Lie-algebraic proof of this hypothesis for all Lie algebras \mathfrak{g} in our paper [12]. A slight modification of this proof and the using of Rawnsley’s approach [1] allow us to prove the fact (*) in this paper (Theorem 18) and to generalize results derived earlier in the case of semidirect products by Rawnsley [1]. In this direction our aim is to give, on one hand, a description of the geometrical structure of the coadjoint orbits in terms of fibre bundles having little (reduced) algebra coadjoint orbits as fibres (Proposition 17). On the other hand, we investigate in detail the structure of the isotropy subgroups with respect to the coadjoint representation of the Lie algebra \mathfrak{g} and the little (reduced) algebra (Proposition 26) and apply this to formulate necessary conditions for the integrality of the coadjoint orbit of \mathfrak{g} (Proposition 27). Proving the non-sufficiencies of this condition even in the semidirect product case, we show that the assertion [1, Corollary to Prop.2] is not correct (see Remark 30).

The index of a Lie algebra is defined as the codimension in the dual space of a coadjoint orbit of the maximal dimension. The description of the geometrical structure of the coadjoint orbits mentioned above gives us a new proof of the formula for the index of Lie algebra (Theorem 13 and Corollary 15) obtained by Panasyuk for arbitrary Lie algebras [13]. Moreover, our approach allows us to find the direct connection of this formula with the geometrical structure of the coadjoint orbits of the little (reduced) Lie algebra. Remark that the proof in [13] is based on the so-called ‘‘symplectic reduction by stages’’ scheme [11] and on calculations of the ranks of some Poisson submanifold of \mathfrak{g}^* by the construction of the dual pairs of Poisson manifolds. This formula for index is a generalization of the well-known Raïs’ formula for semidirect products [6]. As another generalization of the Raïs’ formula we can mention Panyushev’s index formula [14] for some subclass of graded Lie algebras. Remark also that the index of representations associated with stabilizers and so-called representations with good index behavior was considered by Panyushev and Yakimova in the paper [15].

Summarizing the results of this article, we mention the following point:

- the properties (*) and (**) guarantee the existence of some natural linear structure on the space of N_ν -orbits and, consequently, the interpretation

of this space as the dual space of some (reduced) Lie algebra, the interpretation of the orbits in this space as the coadjoint orbits in this dual space.

2 Coadjoint orbits and their affine subspaces defined by the ideal

2.1 Definitions and notation

Let \mathfrak{g} be a Lie algebra over the ground field \mathbb{F} , where $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , and $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ be its finite-dimensional representation. Denote by $\rho^* : \mathfrak{g} \rightarrow \text{End}(V^*)$ the dual representation of \mathfrak{g} . An element $w \in V^*$ is called *\mathfrak{g} -regular* whenever its isotropy algebra $\mathfrak{g}_w = \{\xi \in \mathfrak{g} : \rho^*(\xi)w = 0\}$ has minimal dimension. The set of all \mathfrak{g} -regular elements is open and dense in V^* . Moreover, this set is Zariski open in V^* . The non-negative integer $\dim \mathfrak{g}_w$, where $w \in V^*$ is \mathfrak{g} -regular, is called the *index* of the representation ρ and is denoted by $\text{ind}(\mathfrak{g}, V)$. The *index* $\text{ind } \mathfrak{g}$ of the Lie algebra \mathfrak{g} is the index of its coadjoint representation, or equivalently, the dimension of the isotropy algebra of a \mathfrak{g} -regular element in the dual \mathfrak{g}^* (with respect to the coadjoint representation). The set of all \mathfrak{g} -regular elements in \mathfrak{g}^* is denoted by $R(\mathfrak{g}^*)$.

For any subspace $\mathfrak{a} \subset \mathfrak{g}$ (resp. $V \subset \mathfrak{g}^*$) denote by $\mathfrak{a}^\perp \subset \mathfrak{g}^*$ (resp. V^\perp) its annihilator in \mathfrak{g}^* (resp. in \mathfrak{g}). It is clear that $(\mathfrak{a}^\perp)^\perp = \mathfrak{a}$. A subset $\mathcal{A} \subset \mathfrak{g}^*$ will be called an affine k -subspace if it is of the form $\mathcal{A} = \sigma + V$ where $\sigma \in \mathfrak{g}^*$ is an element and $V \subset \mathfrak{g}^*$ is a subspace of dimension k . The direct and semi-direct products of Lie algebras are denoted by \times and \ltimes respectively. The direct sums of spaces are denoted by $\dot{+}$. The identity component of an arbitrary Lie group H is denoted by H^0 . We will write π_j for the j -homotopy group of a manifold. Also we will often use the following well known statement on the topology of homogeneous spaces (see [16, Ch.III, §6.6] and [17, Ch.1, §3.4]):

Lemma 1. *For a connected Lie group K and its (not necessary closed) subgroup H the following holds: 1) if H is a normal subgroup of K and $\pi_1(K) = 0$ then $H = \overline{H}$ and $\pi_1(H^0) = \pi_1(K/H^0) = 0$; 2) if $H = \overline{H}$, $\pi_1(K/H) = \pi_2(K/H) = 0$ then the Lie subgroup H is connected, i.e. $|H/H^0| = 1$, and $\pi_1(K) \simeq \pi_1(H)$; 3) if $H = \overline{H}$ and $\pi_1(K) = 0$ then $\pi_1(K/H) \simeq H/H^0$. Here \overline{H} denotes the closure of H in K .*

2.2 Coadjoint orbits and their isotropy groups

Let G be a connected real or complex Lie group with a normal connected subgroup $N \subset G$ (not necessary closed). Denote by \mathfrak{g} and \mathfrak{n} the corresponding Lie algebras. Since the Lie group N is a normal subgroup of G , we have

$$\text{Ad}_n \xi - \xi \in \mathfrak{n} \quad \text{for all } n \in N, \xi \in \mathfrak{g}. \quad (1)$$

This fact is well known if the subgroup N is closed. To prove (1) in our general case it is sufficient to remark that the curve $n(\exp(t\xi)n^{-1}\exp(-t\xi))$ is the curve in N passing through the identity element.

Let $\text{Ad}^* : G \rightarrow \text{End}(\mathfrak{g}^*)$ be the coadjoint representation of the Lie group G on the dual space \mathfrak{g}^* . Since we shall consider also some subgroups of G , by Ad_g^* and ad_ξ^* we shall denote only the operators on the space \mathfrak{g}^* , by Ad_g and ad_ξ the operators on the Lie algebra \mathfrak{g} . Fix some linear functional $\sigma \in \mathfrak{g}^*$. Denote by G_σ the isotropy group of σ (with respect to the coadjoint representation of G) and by \mathfrak{g}_σ its Lie algebra. Put $N_\sigma = N \cap G_\sigma$ and $\mathfrak{n}_\sigma = \mathfrak{n} \cap \mathfrak{g}_\sigma$. The subgroup N_σ is a closed subgroup in N with the Lie algebra \mathfrak{n}_σ . By the definition,

$$\mathfrak{g}_\sigma = \{\xi \in \mathfrak{g} : \langle \sigma, [\xi, \mathfrak{g}] \rangle = 0\} \quad \text{and} \quad \mathfrak{n}_\sigma = \{y \in \mathfrak{n} : \langle \sigma, [y, \mathfrak{g}] \rangle = 0\}. \quad (2)$$

Since the subalgebra \mathfrak{n} is an ideal of \mathfrak{g} , the adjoint representations of \mathfrak{g} induce the representation ρ of \mathfrak{g} in \mathfrak{n} , the adjoint action $\text{Ad} : G \rightarrow \text{End}(\mathfrak{g})$ of G induces G -action on \mathfrak{n} : $G \times \mathfrak{n} \rightarrow \mathfrak{n}$, $(g, y) \mapsto \text{Ad}_g y$. For the dual representation ρ^* of \mathfrak{g} in \mathfrak{n}^* we have:

$$\langle \rho_\xi^* \mu, y \rangle = \langle \mu, [\xi, y] \rangle, \quad \text{where } \xi \in \mathfrak{g}, \mu \in \mathfrak{n}^*, y \in \mathfrak{n}.$$

The corresponding G -action on \mathfrak{n}^* is defined by the equation $\langle g \cdot \mu, y \rangle = \langle \mu, \text{Ad}_g y \rangle$. The restriction of this action on the subgroup $N \subset G$ is its coadjoint action. Moreover, the canonical projection $\Pi_1^{\mathfrak{g}} : \mathfrak{g}^* \rightarrow \mathfrak{n}^*$, $\beta \mapsto \beta|_{\mathfrak{n}}$ is a G -equivariant mapping with respect to these two actions of G on the spaces \mathfrak{g}^* and \mathfrak{n}^* respectively:

$$\Pi_1^{\mathfrak{g}}(\text{Ad}_g^* \beta) = g \cdot \Pi_1^{\mathfrak{g}}(\beta), \quad \text{for all } \beta \in \mathfrak{g}^*, g \in G.$$

Indeed, for any $y \in \mathfrak{n}$

$$\langle \Pi_1^{\mathfrak{g}}(\text{Ad}_g^* \beta), y \rangle = \langle \text{Ad}_g^* \beta, y \rangle = \langle \beta, \text{Ad}_g y \rangle = \langle \Pi_1^{\mathfrak{g}}(\beta), \text{Ad}_g y \rangle = \langle g \cdot \Pi_1^{\mathfrak{g}}(\beta), y \rangle. \quad (3)$$

On the other hand, the canonical homomorphism $\pi : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{n}$ induces the canonical linear embedding $\pi^* : (\mathfrak{g}/\mathfrak{n})^* \rightarrow \mathfrak{g}^*$. The following lemma is known. We will prove it for completeness and also because the proof will be used to give a more general result.

Lemma 2. *The canonical linear embedding $\pi^* : (\mathfrak{g}/\mathfrak{n})^* \rightarrow \mathfrak{g}^*$ maps each coadjoint orbit $\mathcal{O}_{\mathfrak{b}}$ of the quotient Lie algebra $\mathfrak{b} = \mathfrak{g}/\mathfrak{n}$ onto some coadjoint orbit $\mathcal{O}_{\mathfrak{g}}$ of \mathfrak{g} . This map defines a one-to-one correspondence between the set of all coadjoint orbits in $(\mathfrak{g}/\mathfrak{n})^*$ and the set of all coadjoint orbits in \mathfrak{g}^* belonging to the annihilator $\mathfrak{n}^\perp \subset \mathfrak{g}^*$. Moreover, the restriction $\pi^* : \mathcal{O}_{\mathfrak{b}} \rightarrow \mathcal{O}_{\mathfrak{g}}$ of the map π^* is a symplectic map, i.e. $(\pi^*|_{\mathcal{O}_{\mathfrak{b}}})^*(\omega_{\mathfrak{g}}) = \omega_{\mathfrak{b}}$, where $\omega_{\mathfrak{g}}$ and $\omega_{\mathfrak{b}}$ are the canonical Kirillov-Kostant-Souriau symplectic 2-forms on the coadjoint orbits $\mathcal{O}_{\mathfrak{g}} \subset \mathfrak{g}^*$ and $\mathcal{O}_{\mathfrak{b}} \subset \mathfrak{b}^*$ respectively.*

Proof. Since \mathfrak{n} is an ideal of \mathfrak{g} , there exists a unique homomorphism φ of the Lie group G into the group of all automorphisms of the Lie algebra $\mathfrak{b} = \mathfrak{g}/\mathfrak{n}$ such that $\varphi(g) \circ \pi = \pi \circ \text{Ad}_g$, $g \in G$. The connected Lie group $\text{Ad}(G)$ is

the group of inner automorphisms of \mathfrak{g} . Since each inner derivation of \mathfrak{b} is induced by some inner derivation of \mathfrak{g} , the image $B = \varphi(G)$ of G is the Lie group of inner automorphisms of the Lie algebra \mathfrak{b} . Taking into account that $\text{Ad}_g^* \circ \pi^* = \pi^* \circ (\varphi(g))^*$, $\pi^*(\mathfrak{b}^*) = \mathfrak{n}^\perp$ and $\text{Ad}^*(G)(\mathfrak{n}^\perp) = \mathfrak{n}^\perp$, we complete the proof of the first assertion.

Choose some element $\beta' \in \mathcal{O}_{\mathfrak{b}}$ and put $\beta = \pi^*(\beta')$. To prove the second assertion, remark that the map π^* is linear. Then $d(\pi^*)(\beta') = \pi^*$ and for any $\xi, \eta \in \mathfrak{g}$

$$\begin{aligned} \langle d(\pi^*)(\beta')(\widetilde{\text{ad}}_{\pi(\xi)}^* \beta'), \eta \rangle &= \langle \widetilde{\text{ad}}_{\pi(\xi)}^* \beta', \pi(\eta) \rangle = \langle \beta', [\pi(\xi), \pi(\eta)]_{\mathfrak{b}} \rangle \\ &= \langle \beta', \pi([\xi, \eta]) \rangle = \langle \text{ad}_\xi^* \beta, \eta \rangle. \end{aligned}$$

Thus $((\pi^*)^* \omega_{\mathfrak{g}})(\beta')(\widetilde{\text{ad}}_{\pi(\xi)}^* \beta', \widetilde{\text{ad}}_{\pi(\eta)}^* \beta') = \omega_{\mathfrak{g}}(\beta)(\text{ad}_\xi^* \beta, \text{ad}_\eta^* \beta)$ and by definition of $\omega_{\mathfrak{g}}$

$$\omega_{\mathfrak{g}}(\beta)(\text{ad}_\xi^* \beta, \text{ad}_\eta^* \beta) \stackrel{\text{def}}{=} \langle \beta, [\xi, \eta] \rangle = \langle \beta', [\pi(\xi), \pi(\eta)]_{\mathfrak{b}} \rangle \stackrel{\text{def}}{=} \omega_{\mathfrak{b}}(\beta')(\widetilde{\text{ad}}_{\pi(\xi)}^* \beta', \widetilde{\text{ad}}_{\pi(\eta)}^* \beta').$$

i.e. $(\pi^*|_{\mathcal{O}_{\mathfrak{b}}})^*(\omega_{\mathfrak{g}}) = \omega_{\mathfrak{b}}$. \square

Remark 3. As follows from Lemma 2 the set of all coadjoint orbits of the Lie algebra \mathfrak{g} contains the coadjoint orbits of all its quotient algebras.

For the element $\sigma \in \mathfrak{m}^*$, consider its restriction $\nu = \sigma|_{\mathfrak{n}} \in \mathfrak{n}^*$. Denote by G_ν and N_ν the isotropy groups of the element ν with respect to the ρ^* -action, by \mathfrak{g}_ν and \mathfrak{n}_ν the corresponding Lie algebras. It is clear that $\mathfrak{n}_\nu = \mathfrak{n} \cap \mathfrak{g}_\nu$ and the subgroup $N_\nu = N \cap G_\nu$ is a normal subgroup of G_ν . Remark here, that N_ν is also the usual isotropy group for coadjoint representation of the Lie group N on the dual space \mathfrak{n}^* .

Since $[\mathfrak{g}, \mathfrak{n}] \subset \mathfrak{n}$, by the definition,

$$\mathfrak{g}_\nu = \{\xi \in \mathfrak{g} : \rho_\xi^* \nu = 0\} = \{\xi \in \mathfrak{g} : \langle \nu, [\xi, \mathfrak{n}] \rangle = 0\} = \{\xi \in \mathfrak{g} : \langle \sigma, [\xi, \mathfrak{n}] \rangle = 0\}, \quad (4)$$

$$\mathfrak{n}_\nu = \{y \in \mathfrak{n} : \langle \nu, [y, \mathfrak{n}] \rangle = 0\} = \{y \in \mathfrak{n} : \langle \sigma, [y, \mathfrak{n}] \rangle = 0\}, \quad (5)$$

and

$$G_\nu = \{g \in G : g \cdot \nu = \nu\} = \{g \in G : \text{Ad}_g^* \sigma|_{\mathfrak{n}} = \sigma|_{\mathfrak{n}} = \nu\}. \quad (6)$$

Note that

$$\text{Ad}(G_\nu)(\mathfrak{n}_\nu) = \mathfrak{n}_\nu \quad (7)$$

because $\text{Ad}(G_\nu)(\mathfrak{g}_\nu) = \mathfrak{g}_\nu$ and $\text{Ad}(G)(\mathfrak{n}) = \mathfrak{n}$ (by definition (5) $\langle \nu, [\mathfrak{n}_\nu, \mathfrak{n}] \rangle = 0$). Also by the identity $\text{Ad}^*(G)(\mathfrak{n}^\perp) = \mathfrak{n}^\perp$,

$$G_\nu = \{g \in G : \text{Ad}_g^*(\mathcal{A}_\nu) = \mathcal{A}_\nu\}, \text{ where} \quad (8)$$

$$\mathcal{A}_\nu \stackrel{\text{def}}{=} \sigma + \mathfrak{n}^\perp = \{\alpha \in \mathfrak{g}^* : \alpha|_{\mathfrak{n}} = \nu\}. \quad (9)$$

Let $\mathcal{O}^\sigma(G) = \{\text{Ad}_g^* \sigma, g \in G\}$ be the coadjoint orbit of the Lie group G in \mathfrak{g}^* through the point σ and let $\mathcal{O}^\nu(G) = G \cdot \nu$ be the corresponding G -orbit in \mathfrak{n}^* . Consider also the orbit $\mathcal{O}^\sigma(G_\nu) \subset \mathcal{O}^\sigma(G)$ in \mathfrak{g}^* of the Lie group G_ν .

Lemma 4. *The restriction $p_1 = \Pi_1^{\mathfrak{g}}|_{\mathcal{O}^\sigma(G)}$ of the natural projection $\Pi_1^{\mathfrak{g}} : \mathfrak{g}^* \rightarrow \mathfrak{n}^*$ is a G -equivariant submersion of the coadjoint orbit $\mathcal{O}^\sigma(G)$ onto the orbit $\mathcal{O}^\nu(G)$. This map $\Pi_1^{\mathfrak{g}} : \mathcal{O}^\sigma(G) \rightarrow \mathcal{O}^\nu(G)$ is a bundle with the total space $\mathcal{O}^\sigma(G)$, the base $\mathcal{O}^\nu(G)$ and the fibre $\mathcal{O}^\sigma(G_\nu)$.*

To prove the lemma it is sufficient to remark that $\mathcal{O}^\sigma(G) \simeq G/G_\sigma$, $\mathcal{O}^\nu(G) \simeq G/G_\nu$ and $\mathcal{O}^\sigma(G_\nu) \simeq G_\nu/G_\sigma$.

The coadjoint orbit $\mathcal{O}^\sigma(G) \subset \mathfrak{g}^*$ is a symplectic manifold with the symplectic Kirillov-Kostant-Souriau 2-form ω :

$$\omega(\sigma)(\text{ad}_\xi^* \sigma, \text{ad}_\eta^* \sigma) \stackrel{\text{def}}{=} \langle \sigma, [\xi, \eta] \rangle, \quad \text{where } \xi, \eta \in \mathfrak{g}. \quad (10)$$

Here the tangent space $T_\sigma \mathcal{O}^\sigma(G)$ is identified, as usual, with the subspace $\text{ad}_\mathfrak{g}^* \sigma$ of \mathfrak{g}^* . We will say that a submanifold $M \subset \mathcal{O}^\sigma(G)$ is an *isotropic submanifold* of the orbit $\mathcal{O}^\sigma(G)$ if for each point $\alpha \in M$ the tangent space $T_\alpha M$ is an isotropic subspace of $T_\alpha \mathcal{O}^\sigma(G)$ with respect to the form ω , i.e. $\omega(\alpha)(T_\alpha M, T_\alpha M) = 0$.

Let us consider two orbits $\mathcal{O}^\sigma(N)$ and $\mathcal{O}^\sigma(G_\nu)$ (submanifolds of $\mathcal{O}^\sigma(G)$) through the point σ . It follows immediately from definition (10) that the tangent space $T_\sigma \mathcal{O}^\sigma(G_\nu) = \text{ad}_{\mathfrak{g}_\nu}^* \sigma$ is an orthogonal complement to the tangent space $T_\sigma \mathcal{O}^\sigma(N) = \text{ad}_\mathfrak{n}^* \sigma$ in $T_\sigma \mathcal{O}^\sigma(G)$ with respect to the symplectic form ω :

$$\omega(\sigma)(\text{ad}_\xi^* \sigma, \text{ad}_\mathfrak{n}^* \sigma) \stackrel{\text{def}}{=} \langle \sigma, [\xi, \mathfrak{n}] \rangle = 0 \iff \xi \in \mathfrak{g}_\nu. \quad (11)$$

Since the form ω is non-degenerate and the isotropy algebra \mathfrak{g}_σ is a subalgebra of \mathfrak{g}_ν (see definition (2)), we have

$$\dim \mathfrak{g} - \dim \mathfrak{g}_\sigma = (\dim \mathfrak{n} - \dim \mathfrak{n}_\sigma) + (\dim \mathfrak{g}_\nu - \dim \mathfrak{g}_\sigma). \quad (12)$$

This identity can be easily rewritten in the following form

$$\dim \mathfrak{n}_\sigma = \dim \mathfrak{n} - (\dim \mathfrak{g} - \dim \mathfrak{g}_\nu), \quad (13)$$

i.e. the dimension of the Lie algebra $\dim \mathfrak{n}_\sigma$ depends on its restriction $\nu = \sigma|_{\mathfrak{n}}$ alone. Moreover, by the commutation relation $[\mathfrak{g}, \mathfrak{n}] \subset \mathfrak{n}$, the algebra \mathfrak{n}_σ also depends only on this restriction ν :

$$\mathfrak{n}_\sigma \stackrel{\text{def}}{=} \{y \in \mathfrak{n} : \langle \sigma, [y, \mathfrak{g}] \rangle = 0\} = \{y \in \mathfrak{n} : \langle \nu, [y, \mathfrak{g}] \rangle = 0\}. \quad (14)$$

This Lie algebra and the corresponding connected Lie subgroup of N_ν will be denoted by $\mathfrak{n}_{\nu\nu}$ and $N_{\nu\nu}^0$ respectively. In other words, for each element $\alpha \in \mathfrak{g}^*$ such that $\alpha|_{\mathfrak{n}} = \sigma|_{\mathfrak{n}}$:

$$\mathfrak{n}_\alpha = \mathfrak{n}_\sigma = \mathfrak{n}_{\nu\nu} \quad \text{and} \quad N_\alpha^0 = N_\sigma^0 = N_{\nu\nu}^0. \quad (15)$$

In particular, $N_{\nu\nu}^0$ is a closed subgroup of the Lie groups N and N_ν . Moreover, this subgroup is the connected component of the closed subgroup $N_{\nu\nu}$ of $N_\nu \subset N$, where

$$N_{\nu\nu} = \{n \in N : \text{Ad}_n^*(\alpha) = \alpha \text{ for all } \alpha \in \mathcal{A}_\nu\} = \bigcap_{\alpha \in \mathcal{A}_\nu} N_\alpha. \quad (16)$$

However, we can rewrite identity (13) in the following form:

$$\dim \mathfrak{n}_\nu - \dim \mathfrak{n}_\sigma = \dim \mathfrak{g} - (\dim \mathfrak{n} + \dim \mathfrak{g}_\nu - \dim \mathfrak{n}_\nu). \quad (17)$$

The right-hand side of this identity is the codimension of the subspace $\mathfrak{n} + \mathfrak{g}_\nu$ in \mathfrak{g} because by definition $\mathfrak{n}_\nu = \mathfrak{g}_\nu \cap \mathfrak{n}$. The left-hand side of (17) is the dimension of the subspace $\text{ad}_{\mathfrak{n}_\nu}^* \sigma \subset \mathfrak{g}^*$. But $\text{ad}_{\mathfrak{n}_\nu}^* \sigma$ is a subspace of $(\mathfrak{n} + \mathfrak{g}_\nu)^\perp$ because $\text{ad}_{\mathfrak{n}_\nu}^* \sigma(\mathfrak{n}) = 0$ by definition (5) and $\text{ad}_{\mathfrak{n}_\nu}^* \sigma(\mathfrak{g}_\nu) = 0$ by (4). Therefore from (17) it follows that

$$\dim(\mathfrak{n}_\nu/\mathfrak{n}_\sigma) = \dim(\mathfrak{n} + \mathfrak{g}_\nu)^\perp \quad \text{and, consequently,} \quad \text{ad}_{\mathfrak{n}_\nu}^* \sigma = (\mathfrak{n} + \mathfrak{g}_\nu)^\perp. \quad (18)$$

Remark 5. The subspace $\text{ad}_{\mathfrak{n}_\nu}^* \sigma \subset \mathfrak{g}^*$ is the tangent space to the orbit $\mathcal{O}^\sigma(N_\nu) = \mathcal{O}^\sigma(G_\nu) \cap \mathcal{O}^\sigma(N)$ of the Lie group N_ν through the point $\sigma \in \mathfrak{g}^*$ and, as we shown above, this space is the null space of the restrictions $\omega|_{T_\sigma \mathcal{O}^\sigma(G_\nu)}$ and $\omega|_{T_\sigma \mathcal{O}^\sigma(N)}$.

Our interest now centers on the two orbits in \mathfrak{g}^* (through the element σ) mentioned above: $\mathcal{O}^\sigma(G_\nu)$ and $\mathcal{O}^\sigma(N_\nu)$. First of all, we will show that $\mathcal{O}^\sigma(N_\nu^0) = \sigma + (\mathfrak{n} + \mathfrak{g}_\nu)^\perp$, i.e. this orbit is an affine subspace of \mathfrak{g}^* . To this end, we consider the kernel $\mathfrak{n}_\nu^\natural \subset \mathfrak{n}_\nu$ of the restriction $\nu|_{\mathfrak{n}_\nu}$, i.e. $\mathfrak{n}_\nu^\natural = \ker \nu \cap \mathfrak{n}_\nu$. Remark that $\mathfrak{n}_\nu^\natural = \mathfrak{n}_\nu$ or $\dim(\mathfrak{n}_\nu/\mathfrak{n}_\nu^\natural) = 1$. By (4)

$$[\mathfrak{g}_\nu, \mathfrak{n}] \subset \ker \nu \quad \text{and} \quad [\mathfrak{g}_\nu, \mathfrak{n}_\nu] \subset (\mathfrak{g}_\nu \cap \mathfrak{n}) \cap \ker \nu = \mathfrak{n}_\nu^\natural, \quad (19)$$

so that the subspace $\mathfrak{n}_\nu^\natural$ is an ideal in \mathfrak{g}_ν . Moreover, since $h \cdot \nu = \nu$, $\text{Ad}_h(\mathfrak{n}_\nu) = \mathfrak{n}_\nu$ for all $h \in G_\nu$ (see (7)) and, by the definition, $\langle h \cdot \nu, y \rangle = \langle \nu, \text{Ad}_h y \rangle$ for $y \in \mathfrak{n}$, we have

$$\text{Ad}_h(\mathfrak{n}_\nu^\natural) = \mathfrak{n}_\nu^\natural \quad \text{for all } h \in G_\nu. \quad (20)$$

Let N_ν^{fin} be the subgroup of N_ν generated by all elements $n \in N_\nu$ such that the power $(\text{Ad}_n)^m \in \text{Ad}(N_\nu^0)$ for some integer $m \in \mathbb{Z}$. This group is a closed Lie subgroup of N_ν because it contains the identity component N_ν^0 of N_ν . We claim that

$$\text{Ad}_n \xi - \xi \in \mathfrak{n}_\nu^\natural \quad \text{for all } \xi \in \mathfrak{g}_\nu \text{ and } n \in N_\nu^{\text{fin}} \supset N_\nu^0. \quad (21)$$

Relations (21) were established in [11, §5.2] in the case when the Lie group N_ν is connected. We will prove (21) modifying the method used in [11]. To this end consider the representation $n \mapsto \text{Ad}_n|_{\mathfrak{g}_\nu}$ of the Lie group $N_\nu \subset G_\nu$. This representation induces the trivial representation of the identity component $N_\nu^0 \subset N_\nu$ in the quotient algebra $\mathfrak{g}_\nu/\mathfrak{n}_\nu^\natural$ because $[\mathfrak{n}_\nu, \mathfrak{g}_\nu] \subset \mathfrak{n}_\nu^\natural$ (the corresponding homomorphism of Lie algebras is trivial). Thus relations (21) hold for all $n \in N_\nu^0$, i.e. $\langle \nu, \text{Ad}_n \xi - \xi \rangle = 0$ for all such n .

Since N_ν is a normal (not necessary closed) subgroup of G_ν , we have $\text{Ad}_n \xi - \xi \in \mathfrak{n}_\nu$ for all $n \in N_\nu$ and $\xi \in \mathfrak{g}_\nu$ (see (1)). Now to prove (21) we will show that the mapping

$$\chi_\xi : N_\nu \rightarrow \mathbb{F}, \quad \chi_\xi(n) = \langle \nu, \text{Ad}_n \xi - \xi \rangle, \quad \xi \in \mathfrak{g}_\nu,$$

is a homomorphism of the group N_ν into the additive group \mathbb{F} . Indeed, for $n_1, n_2 \in N_\nu$,

$$\begin{aligned}\langle \nu, \text{Ad}_{n_1 n_2} \xi - \xi \rangle &= \langle \nu, \text{Ad}_{n_1}(\text{Ad}_{n_2} \xi - \xi) + (\text{Ad}_{n_1} \xi - \xi) \rangle \\ &= \langle \nu, (\text{Ad}_{n_2} \xi - \xi) + (\text{Ad}_{n_1} \xi - \xi) \rangle,\end{aligned}$$

because $n_1 \cdot \nu = \nu$. Now, if $(\text{Ad}_n)^m \in \text{Ad}(N_\nu^0)$ then

$$m\chi_\xi(n) = \chi_\xi(n^m) = \langle \nu, (\text{Ad}_n)^m \xi - \xi \rangle = 0.$$

The proof of (21) is completed.

For the element $\sigma \in \mathfrak{g}^*$ denote by τ its restriction $\sigma|_{\mathfrak{g}_\nu}$. Using the pair of covectors $\nu \in \mathfrak{n}^*$ and $\tau \in \mathfrak{g}_\nu^*$ define the affine subspace $\mathcal{A}_{\nu\tau} \subset \mathcal{A}_\nu \subset \mathfrak{g}^*$ as follows:

$$\mathcal{A}_{\nu\tau} = \{\alpha \in \mathfrak{g}^* : \alpha|_{\mathfrak{n}} = \nu, \alpha|_{\mathfrak{g}_\nu} = \tau\} = \sigma + (\mathfrak{n} + \mathfrak{g}_\nu)^\perp. \quad (22)$$

It is clear that

$$\begin{aligned}\dim \mathcal{A}_{\nu\tau} &= \text{codim}(\mathfrak{n} + \mathfrak{g}_\nu) \\ &= \dim \mathfrak{g} - (\dim \mathfrak{n} + \dim \mathfrak{g}_\nu - \dim \mathfrak{n}_\nu) \\ &= \dim(G/G_\nu) - \dim(N/N_\nu) \\ &= \dim(G/N) - \dim(G_\nu/N_\nu).\end{aligned} \quad (23)$$

We claim that this affine subspace $\mathcal{A}_{\nu\tau} \subset \mathfrak{g}^*$ is invariant with respect to the action of the Lie group $N_\nu^{\text{fin}} \subset N_\nu$ (containing the identity component N_ν^0 of N_ν). Indeed, let $\alpha \in \mathcal{A}_{\nu\tau}$ and $n \in N_\nu^{\text{fin}}$. Since $N_\nu^{\text{fin}} \subset N_\nu = N \cap G_\nu$ and $\mathcal{A}_{\nu\tau} \subset \mathcal{A}_\nu$, by (8) $\text{Ad}_n^* \alpha|_{\mathfrak{n}} = \nu$. To prove that $\text{Ad}_n^* \alpha|_{\mathfrak{g}_\nu} = \tau$ remark that by (21) for all vectors $\xi \in \mathfrak{g}_\nu$ we have

$$\langle \text{Ad}_n^* \alpha - \alpha, \xi \rangle = \langle \alpha, \text{Ad}_n \xi - \xi \rangle \in \langle \alpha, \mathfrak{n}_\nu^\natural \rangle = \langle \nu, \mathfrak{n}_\nu^\natural \rangle = 0,$$

i.e.

$$\text{Ad}_n^*(\mathcal{A}_{\nu\tau}) \subset \mathcal{A}_{\nu\tau} \quad \text{for all } n \in N_\nu^{\text{fin}} \supset N_\nu^0. \quad (24)$$

As we have shown above, the Lie algebra \mathfrak{n}_σ is defined by the restriction $\sigma|_{\mathfrak{n}} = \nu$ alone, therefore $\mathfrak{n}_\sigma = \mathfrak{n}_\alpha$ (see (15)). By definition $N_\alpha \subset N_\nu$ and, consequently, $N_\alpha^0 \subset N_\nu^0$. Taking into account (18), we obtain that the $\text{Ad}^*(N_\nu^0)$ -orbit in the space $\mathcal{A}_{\nu\tau}$ through the element α (isomorphic to the quotient space $N_\nu^0/(N_\alpha \cap N_\nu^0)$) is an open subset of $\mathcal{A}_{\nu\tau}$:

$$\dim N_\nu^0/(N_\alpha \cap N_\nu^0) = \dim(\mathfrak{n}_\nu/\mathfrak{n}_\alpha) = \dim(\mathfrak{n}_\nu/\mathfrak{n}_\sigma) = \dim(\mathfrak{n} + \mathfrak{g}_\nu)^\perp = \dim \mathcal{A}_{\nu\tau}.$$

Since the space $\mathcal{A}_{\nu\tau}$ is connected, this orbit is the whole space $\mathcal{A}_{\nu\tau}$, i.e. $\text{Ad}^*(N_\nu^0)$ acts transitively on $\mathcal{A}_{\nu\tau}$. Since the affine space $\mathcal{A}_{\nu\tau}$ is contractible, by Lemma 1 the isotropy group $N_\alpha \cap N_\nu^0$ is connected, that is, it is equal to N_α^0 (the identity component of $N_\alpha \subset N_\nu$). Similarly, the group N_ν^{fin} acts transitively on $\mathcal{A}_{\nu\tau}$ and, consequently,

$$N_\nu^{\text{fin}}/N_\nu^0 \simeq (N_\nu^{\text{fin}} \cap N_\alpha)/N_\alpha^0.$$

Also by Lemma 1,

$$\pi_1(N_\nu^0) \simeq \pi_1(N_\alpha^0)$$

because $\pi_1(\mathcal{A}_{\nu\tau}) = \pi_2(\mathcal{A}_{\nu\tau}) = 0$. Thus

$$\mathcal{A}_{\nu\tau} \simeq N_\nu^0/N_\sigma^0 = N_\nu^0/N_{\nu\nu}^0 \quad \text{and} \quad \mathcal{A}_{\nu\tau} \simeq N_\nu^{\text{fin}}/(N_\sigma \cap N_\nu^{\text{fin}}). \quad (25)$$

Since the action of N_ν^0 on $\mathcal{A}_{\nu\tau}$ is transitive and the isotropy group $N_\alpha^0 = N_{\nu\nu}^0$ is the same for all points $\alpha \in \mathcal{A}_{\nu\tau}$, the group $N_{\nu\nu}^0$ is a normal subgroup of N_ν^0 .

Consider now the isotropy group G_ν . The algebra \mathfrak{g}_ν is its tangent Lie algebra. For the element $\tau = \sigma|_{\mathfrak{g}_\nu}$ denote by $G_{\nu\tau}$ the isotropy group of $\tau \in \mathfrak{g}_\nu^*$ with respect to the natural co-adjoint action of G_ν on \mathfrak{g}_ν^* , which we denote by $\widehat{\text{Ad}}^*$. Let $\mathcal{O}^\tau(G_\nu) \subset \mathfrak{g}_\nu^*$ be the corresponding $\widehat{\text{Ad}}^*$ -orbit of G_ν passing through the point τ (the union of disjoint coadjoint orbits in \mathfrak{g}_ν^*). Then $\mathcal{O}^\tau(G_\nu) \simeq G_\nu/G_{\nu\tau}$. Taking into account that the $\widehat{\text{Ad}}^*$ -action of G_ν on \mathfrak{g}_ν is determined by the Ad-action of G on \mathfrak{g} , we obtain that the natural projection

$$\Pi_2^g : \mathfrak{g}^* \rightarrow \mathfrak{g}_\nu^*, \quad \beta \mapsto \beta|_{\mathfrak{g}_\nu}, \quad (26)$$

is a G_ν -equivariant map with respect to the coadjoint actions Ad^* and $\widehat{\text{Ad}}^*$ of G_ν . Hence

$$\mathcal{O}^\tau(G_\nu) \stackrel{\text{def}}{=} \{\widehat{\text{Ad}}_g^* \tau, g \in G_\nu\} = \Pi_2^g(\mathcal{O}^\sigma(G_\nu)) = \{(\text{Ad}_g^* \sigma)|_{\mathfrak{g}_\nu}, g \in G_\nu\} \quad (27)$$

and

$$G_{\nu\tau} = \{g \in G : \text{Ad}_g^* \sigma|_{\mathfrak{n}} = \sigma|_{\mathfrak{n}} = \nu, \text{Ad}_g^* \sigma|_{\mathfrak{g}_\nu} = \sigma|_{\mathfrak{g}_\nu} = \tau\}. \quad (28)$$

Since by definition, $\text{Ad}^*(G_\nu)(\mathfrak{n} + \mathfrak{g}_\nu)^\perp = (\mathfrak{n} + \mathfrak{g}_\nu)^\perp$, we have

$$G_{\nu\tau} = \{g \in G : \text{Ad}_g^*(\mathcal{A}_{\nu\tau}) = \mathcal{A}_{\nu\tau}\}. \quad (29)$$

Therefore by (24) the group $G_{\nu\tau}$ contains the identity component N_ν^0 of N_ν and, moreover, the subgroup $N_\nu^{\text{fin}} \subset N_\nu$. The Lie algebra $\mathfrak{g}_{\nu\tau}$ of $G_{\nu\tau}$ contains the Lie algebra \mathfrak{n}_ν . Remark also that by definition $G_\sigma \subset G_{\nu\tau}$ and $\mathfrak{g}_\sigma \subset \mathfrak{g}_{\nu\tau}$. Since $N_\nu^0 \subset G_{\nu\tau}^0 \subset G_{\nu\tau}$, the groups $\text{Ad}^*(G_{\nu\tau}^0)$ and $\text{Ad}^*(G_{\nu\tau})$ act transitively on the affine space $\mathcal{A}_{\nu\tau}$, that is

$$G_{\nu\tau}^0/(G_\sigma \cap G_{\nu\tau}^0) \simeq G_{\nu\tau}/G_\sigma \simeq N_\nu^0/N_\sigma^0 \simeq \mathcal{A}_{\nu\tau} \quad (30)$$

and, consequently,

$$G_{\nu\tau} = N_\nu^0 \cdot G_\sigma = G_\sigma \cdot N_\nu^0 \quad \text{and} \quad \mathfrak{g}_{\nu\tau} = \mathfrak{n}_\nu + \mathfrak{g}_\sigma. \quad (31)$$

In particular,

$$\dim \mathfrak{g}_{\nu\tau} - \dim \mathfrak{g}_\sigma = \dim \mathfrak{n}_\nu - \dim \mathfrak{n}_\sigma. \quad (32)$$

Moreover, applying Lemma 1 to the spaces in (30) we obtain that

$$G_\sigma \cap G_{\nu\tau}^0 = G_\sigma^0, \quad \pi_1(G_{\nu\tau}^0) = \pi_1(G_\sigma^0) \quad \text{and} \quad G_{\nu\tau}/G_{\nu\tau}^0 \simeq G_\sigma/G_\sigma^0. \quad (33)$$

Also $G_{\nu\tau}^0 = N_\nu^0 \cdot G_\sigma^0 = G_\sigma^0 \cdot N_\nu^0$. But the group N_ν is a normal subgroup in G_ν and, consequently, the group N_ν^0 is a normal subgroup in $G_{\nu\tau}^0 \subset G_\nu$. Similarly, by the definition the group $N_\sigma^0 = N_{\nu\nu}^0$ is a normal subgroup of G_σ . Since this group is also a normal subgroup in N_ν^0 , by (31) the group $N_{\nu\nu}^0$ is a normal subgroup in $G_{\nu\tau}$.

The group $N_\nu^0 \subset G_{\nu\tau'}$ is the same group for all $\tau' \in \mathfrak{g}_\nu$. The sum $\mathcal{A}_\nu = \bigcup_{\tau' \in \mathcal{A}_\nu | \mathfrak{g}_\nu} \mathcal{A}_{\nu\tau'}$ is the union of the orbits of the group N_ν^0 , the parallel affine subspaces of \mathcal{A}_ν with the associated vector space $(\mathfrak{n} + \mathfrak{g}_\nu)^\perp$.

Remark that $\mathcal{O}^\nu(G)$ is a disjoint union of coadjoint orbits (isomorphic to $\mathcal{O}^\nu(N) \simeq N/N_\nu$) in the dual space \mathfrak{n}^* and the group G acts transitively on the set of these orbits. Moreover, by equation (23) the dimension of $\mathcal{A}_{\nu\tau}$ is equal to the codimension of the coadjoint orbit $\mathcal{O}^\nu(N) \subset \mathfrak{n}^*$ in the G -orbit $\mathcal{O}^\nu(G) \subset \mathfrak{n}^*$. The affine space $\mathcal{A}_{\nu\tau}$ as the orbit $\mathcal{O}^\sigma(N_\nu^0) \subset \mathcal{O}^\sigma(G)$ is an isotropic submanifold of the coadjoint orbit $\mathcal{O}^\sigma(G)$ (see relations (11) and Remark 5). We have proved

Proposition 6. *The affine space $\mathcal{A}_{\nu\tau}$ (22) is an isotropic submanifold of the coadjoint orbit $\mathcal{O}^\sigma(G) \subset \mathfrak{g}^*$ containing the point σ and $\dim \mathcal{A}_{\nu\tau} = \dim \mathcal{O}^\nu(G) - \dim \mathcal{O}^\nu(N)$. The Lie subgroups $\text{Ad}^*(N_\nu^0)$, $\text{Ad}^*(N_\nu^{\text{fin}})$, $\text{Ad}^*(G_{\nu\tau}^0)$ and $\text{Ad}^*(G_{\nu\tau})$ of $\text{Ad}^*(G)$ preserve the affine subspace $\mathcal{A}_{\nu\tau} \subset \mathfrak{g}^*$. The actions of these groups on $\mathcal{A}_{\nu\tau}$ are transitive. Moreover, the orbits of the action of $\text{Ad}^*(N_\nu^0)$ on the affine subspace $\mathcal{A}_\nu \subset \mathfrak{g}^*$ are the parallel affine subspaces with the associated vector space $(\mathfrak{n} + \mathfrak{g}_\nu)^\perp$. The group $N_{\nu\nu}^0$ is a normal subgroup of the Lie groups $G_{\nu\tau}$, N_ν^0 and topologically $N_\nu^0/N_{\nu\nu}^0 \simeq (\mathfrak{n} + \mathfrak{g}_\nu)^\perp$.*

Definition 7. The affine subspace $\mathcal{A}_{\nu\tau} = \sigma + (\mathfrak{n} + \mathfrak{g}_\nu)^\perp$ contained in the coadjoint orbit $\mathcal{O}^\sigma(G) \subset \mathfrak{g}^*$ and denoted by $\mathcal{A}(\sigma, \mathfrak{n})$, will be called the isotropic affine subspace associated with the ideal \mathfrak{n} of \mathfrak{g} .

Remark 8. By relations (23), (25) and (31) for any $\sigma \in \mathfrak{g}^*$ the following conditions are equivalent: 1) $\mathcal{A}(\sigma, \mathfrak{n}) = \{\sigma\}$; 2) $\dim \mathcal{A}(\sigma, \mathfrak{n}) = 0$; 3) $\mathfrak{g}_\nu + \mathfrak{n} = \mathfrak{g}$; 4) $\mathfrak{g}_\sigma = \mathfrak{g}_{\nu\tau}$; 5) $\mathfrak{n}_\nu \subset \mathfrak{g}_\sigma$. Here, recall, $\nu = \sigma|_{\mathfrak{n}}$ and $\tau = \sigma|_{\mathfrak{g}_\nu}$.

Remark 9. If N is an affine algebraic Lie group, then its adjoint representation $N \rightarrow GL(\mathfrak{n})$, $n \mapsto \text{Ad}_n|_{\mathfrak{n}}$, is a \mathbb{F} -morphism. In this case the affine algebraic group N_ν always has a finite number of connected (irreducible) components, and consequently, $N_\nu^{\text{fin}} = N_\nu \subset G_{\nu\tau}$. Then by (31) $G_{\nu\tau} = G_\sigma \cdot N_\nu$ and, consequently, $G_{\nu\tau}/N_\nu \simeq G_\sigma/N_\sigma$. We obtain the exact sequence

$$e \rightarrow N_\sigma \rightarrow G_\sigma \rightarrow G_{\nu\tau}/N_\nu \rightarrow e,$$

which generalizes Rawnsley's exact sequence [1, Eq.(1)] in the case of semidirect products.

Remark 10. The dual space \mathfrak{n}^* is a Poisson manifold with the natural linear Poisson structure and with the coadjoint orbits as the corresponding symplectic leaves. Then the G -orbit $\mathcal{O}^\nu(G)$ as the union of such (isomorphic) leaves is a Poisson submanifold of \mathfrak{n}^* . The Poisson structure on $\mathcal{O}^\nu(G)$ has constant rank $\dim \mathcal{O}^\nu(N)$ and by Proposition 6 its corank equals $\dim \mathcal{A}_{\nu\tau} = \dim(\mathfrak{g}_\nu + \mathfrak{n})^\perp$.

2.3 Reduced-group orbits and index of a Lie algebra

We continue with the notation of the previous subsections. Here we consider the orbit $\mathcal{O}^\tau(G_\nu) \subset \mathfrak{g}_\nu^*$ in more details. We will show that this orbit is the union of disjoint coadjoint orbits of some reduced Lie algebra.

Indeed, as we remarked above (see (27)), the set $\mathcal{O}^\tau(G_\nu)$ consists of the restrictions $\text{Ad}_g^* \sigma|_{\mathfrak{g}_\nu}$, where $g \in G_\nu$. But by definition of the Lie group G_ν we have $\text{Ad}_g^* \sigma|_{\mathfrak{n}} = \nu$ for any $g \in G_\nu$, that is, all elements of the orbit vanish on the ideal $\mathfrak{n}_\nu^\natural$ of the Lie algebra \mathfrak{g}_ν (see (19)). Consider the quotient algebra $\mathfrak{b}_\nu = \mathfrak{g}_\nu / \mathfrak{n}_\nu^\natural$. Since the connected subgroup of G_ν corresponding to the subalgebra $\mathfrak{n}_\nu^\natural$ is not necessarily closed in G_ν , we will describe the coadjoint orbits of \mathfrak{b}_ν in terms of the Lie group G_ν .

Let $\pi_\nu : \mathfrak{g}_\nu \rightarrow \mathfrak{b}_\nu$ be the canonical homomorphism. The dual map $\pi_\nu^* : \mathfrak{b}_\nu^* \rightarrow \mathfrak{g}_\nu^*$ is a linear embedding and identifies the dual space \mathfrak{b}_ν^* naturally with the annihilator $(\mathfrak{n}_\nu^\natural)^\perp \subset \mathfrak{g}_\nu^*$ of $\mathfrak{n}_\nu^\natural$ in \mathfrak{g}_ν^* . By Lemma 2 and by relation (27) the set

$$\mathcal{O}^\tau = \{(\text{Ad}_g^* \sigma)|_{\mathfrak{g}_\nu}, g \in G_\nu\} \subset \mathfrak{b}_\nu^* \subset \mathfrak{g}_\nu^* \quad (34)$$

is a coadjoint orbit in $\mathfrak{b}_\nu^* = (\mathfrak{n}_\nu^\natural)^\perp$ passing through the element $\tau \in \mathfrak{b}_\nu^* \subset \mathfrak{g}_\nu^*$.

In particular, $\mathcal{O}^\tau(G_\nu)$ is the union of disjoint coadjoint orbits of the reduced Lie algebra \mathfrak{b}_ν . This orbit $\mathcal{O}^\tau(G_\nu)$ will be called a reduced-group orbit. Remark here that this group and this orbit are the analog of Rawnsley's the little-group and the little-group orbit in the case of semidirect products (see [1]). Our term "reduced" is motivated by the reduction by stages procedure of Marsden-Misiolek-Ortega-Perlmutter-Ratiu (see [10] and [11]), where the one-dimensional central extension of the quotient group G_ν^0/N_ν^0 (with the Lie algebra \mathfrak{b}_ν) is a natural symmetry group for the second step of the reduction procedure (see also Remark 20).

By (32) we can replace $\dim \mathfrak{n}_\nu - \dim \mathfrak{n}_\sigma$ in the left hand side of identity (17) by $\dim \mathfrak{g}_{\nu\tau} - \dim \mathfrak{g}_\sigma$. Therefore after simple rearrangements we obtain the identity

$$\begin{aligned} \dim \mathfrak{g}_\sigma &= [\dim \mathfrak{n} - \dim(\mathfrak{g}/\mathfrak{g}_\nu)] + \dim(\mathfrak{g}_{\nu\tau}/\mathfrak{n}_\nu) \\ &= [\dim \mathfrak{n} - \dim(\mathfrak{g}/\mathfrak{g}_\nu)] + [\dim(\mathfrak{g}_{\nu\tau}/\mathfrak{n}_\nu^\natural) - \dim(\mathfrak{n}_\nu/\mathfrak{n}_\nu^\natural)], \end{aligned} \quad (35)$$

where we recall that $\sigma \in \mathfrak{g}^*$ is an arbitrary element and $\nu = \sigma|_{\mathfrak{n}}$ and $\tau = \sigma|_{\mathfrak{g}_\nu}$.

Let $\mu \in \mathfrak{n}^*$. Because $\mu|_{\mathfrak{n}_\mu} = 0$ (i.e. $\mu \in \mathfrak{n}_\mu^\perp$) if and only if $\mu \in \text{ad}_\mathfrak{n}^* \mu$, and the function $\mu \mapsto \dim(\mathbb{F}\mu + \text{ad}_\mathfrak{n}^* \mu)$ is lower semi-continuous on $R(\mathfrak{n}^*)$, the set $R^\natural(\mathfrak{n}^*) = \{\mu \in R(\mathfrak{n}^*) : \dim(\mathfrak{n}_\mu/\mathfrak{n}_\mu^\natural) = 1\}$ is a Zariski open subset of \mathfrak{n}^* . Put $\delta^\natural(\mathfrak{n}) = 1$ if this set is not empty, and $\delta^\natural(\mathfrak{n}) = 0$ otherwise.

Remark 11. If $R^\natural(\mathfrak{n}^*) = \emptyset$ then $\mu \in \text{ad}_\mathfrak{n}^* \mu$ for all $\mu \in R(\mathfrak{n}^*)$ and, consequently, the each coadjoint orbit of \mathfrak{n} consisting of \mathfrak{n} -regular elements with arbitrary its element μ contains the set $\{z\mu\}$, where $z \neq 0$ if $\mathbb{F} = \mathbb{C}$ and $z > 0$ if $\mathbb{F} = \mathbb{R}$. This follows from the fact that the coadjoint orbits in $R(\mathfrak{n}^*) \subset \mathfrak{n}^*$ are defined uniquely by the integrable vector subbundle $\mu \mapsto \text{ad}_\mathfrak{n}^* \mu$, $\mu \in R(\mathfrak{n}^*)$ (of constant corank $\text{ind } \mathfrak{n}$) of the tangent bundle $TR(\mathfrak{n}^*)$. It is clear that $R^\natural(\mathfrak{n}^*) \neq \emptyset$ ($\delta^\natural(\mathfrak{n}) = 1$) if the algebra \mathfrak{n} is semisimple and $R^\natural(\mathfrak{n}^*) = \emptyset$ ($\delta^\natural(\mathfrak{n}) = 0$) if \mathfrak{n} is a Frobenius Lie algebra, i.e. $\text{ind } \mathfrak{n} = 0$.

Suppose that $\nu|_{\mathfrak{n}_\nu} \neq 0$, i.e. $\dim(\mathfrak{n}_\nu/\mathfrak{n}_\nu^\natural) = 1$. In this case the extension $\mathfrak{b}_\nu = \mathfrak{g}_\nu/\mathfrak{n}_\nu^\natural$

$$0 \rightarrow \mathfrak{n}_\nu/\mathfrak{n}_\nu^\natural \rightarrow \mathfrak{g}_\nu/\mathfrak{n}_\nu^\natural \rightarrow \mathfrak{g}_\nu/\mathfrak{n}_\nu \rightarrow 0 \quad (36)$$

is a one-dimensional central extension of the quotient algebra $\mathfrak{g}_\nu/\mathfrak{n}_\nu$. Let \mathcal{B}_ν be the image of the set $\mathcal{A}_\nu \subset \mathfrak{g}^*$ under the restriction map

$$\Pi_2^{\mathfrak{g}} : \mathfrak{g}^* \rightarrow \mathfrak{g}_\nu^*, \quad \beta \mapsto \beta|_{\mathfrak{g}_\nu}. \quad (37)$$

Put $\mathcal{B}_0 = \Pi_2^{\mathfrak{g}}(\mathfrak{n}^\perp)$. It is easy to see that $\mathcal{B}_\nu \subset (\mathfrak{n}_\nu^\natural)^{\perp\nu}$, where $(\mathfrak{n}_\nu^\natural)^{\perp\nu} = \mathfrak{b}_\nu^*$, and $\mathcal{B}_0 = (\mathfrak{n}_\nu)^{\perp\nu}$. By dimension arguments, $\dim \mathcal{B}_\nu = \dim(\mathfrak{g}_\nu/\mathfrak{n}_\nu)$, i.e. \mathcal{B}_ν is an affine subspace of codimension one in \mathfrak{b}_ν^* and therefore

$$\mathcal{B}_\nu = \{\tilde{\tau} \in (\mathfrak{n}_\nu^\natural)^{\perp\nu} : \tilde{\tau}|_{\mathfrak{n}_\nu} = \nu|_{\mathfrak{n}_\nu}\} = \tau + (\mathfrak{n}_\nu)^{\perp\nu}. \quad (38)$$

Remark 12. The restriction $\Pi_2^{\mathfrak{g}}|_{\mathcal{A}_\nu}$ of the linear map $\Pi_2^{\mathfrak{g}}$ is a bundle with the total space \mathcal{A}_ν , the affine space \mathcal{B}_ν as its base and the space $(\mathfrak{g}_\nu + \mathfrak{n})^\perp \subset \mathfrak{g}^*$ as its fibre.

By (8) and by G_ν -equivariance of the map $\Pi_2^{\mathfrak{g}}$ the space \mathcal{B}_ν is G_ν -invariant and is the union of coadjoint orbits of the Lie algebra \mathfrak{b}_ν and \mathfrak{g}_ν simultaneously (see expressions (27) and (34)). Moreover, the union of disjoint G_ν -invariant affine subspaces $\lambda\mathcal{B}_\nu = \mathcal{B}_{\lambda\nu}$, $\lambda \neq 0$ of \mathfrak{b}_ν^* is an open dense subset in \mathfrak{b}_ν^* :

$$\Pi_2^{\mathfrak{g}}\left(\bigcup_{\lambda \in \mathbb{F} \setminus \{0\}} \lambda\mathcal{A}_\nu\right) = \bigcup_{\lambda \in \mathbb{F} \setminus \{0\}} \lambda\mathcal{B}_\nu = \mathfrak{b}_\nu^* \setminus \mathcal{B}_0, \quad \dim \mathfrak{b}_\nu^* - \dim(\mathfrak{n}_\nu)^{\perp\nu} = 1. \quad (39)$$

Thus \mathcal{B}_ν and each of these affine spaces $\lambda\mathcal{B}_\nu$ contain coadjoint orbits of the Lie algebra \mathfrak{b}_ν of maximal dimension (as usual for one-dimensional central extensions). Now as an immediate consequence of identity (35) we obtain

Theorem 13. *Let \mathfrak{g} be a Lie algebra over the field \mathbb{F} and \mathfrak{n} be its non-zero ideal. Let ν be an element of $R^\natural(\mathfrak{n}^*)$ if $R^\natural(\mathfrak{n}^*) \neq \emptyset$ or an element of $R(\mathfrak{n}^*)$ if $R^\natural(\mathfrak{n}^*) = \emptyset$ and such that $\mathcal{A}_\nu \cap R(\mathfrak{g}^*) \neq \emptyset$. Then $\text{ind } \mathfrak{g} = \text{ind}(\mathfrak{g}, \mathfrak{n}) + (\text{ind } \mathfrak{b}_\nu - \delta^\natural(\mathfrak{n}))$, where $\mathfrak{b}_\nu = \mathfrak{g}_\nu/\mathfrak{n}_\nu^\natural$ and the ideal $\mathfrak{n}_\nu^\natural = \ker(\nu|_{\mathfrak{n}_\nu})$. For any $\sigma \in \mathfrak{g}^*$ and $\tau \in \mathfrak{g}_\nu^*$ such that $\sigma|_{\mathfrak{n}} = \nu$ and $\sigma|_{\mathfrak{g}_\nu} = \tau$ the element σ is \mathfrak{g} -regular if and only if the element τ is \mathfrak{b}_ν -regular.*

Remark 14. If $\nu|_{\mathfrak{n}_\nu} \neq 0$ then the one-dimensional algebra $\mathfrak{n}_\nu/\mathfrak{n}_\nu^\natural$ is a subalgebra of the center of $\mathfrak{g}_\nu/\mathfrak{n}_\nu^\natural$. Fix some element $z \in \mathfrak{n}_\nu$ such that $\nu(z) = 1$ and a splitting $\ker \tau = \mathfrak{s}_\nu \dot{+} \mathfrak{n}_\nu^\natural$ of the kernel of $\tau \in \mathfrak{g}_\nu^*$. It is clear that $\mathfrak{g}_\nu = \mathfrak{s}_\nu \dot{+} \mathbb{F}z \dot{+} \mathfrak{n}_\nu^\natural$ and for arbitrary $\xi, \eta \in \mathfrak{g}_\nu$ the $\mathbb{F}z$ -component of the commutator $[\xi, \eta]$ is the vector $\tau([\xi, \eta])z$. In other words, the central extension (36) is determined by the map $\mathfrak{g}_\nu \times \mathfrak{g}_\nu \rightarrow \mathbb{F}$, $(\xi, \eta) \mapsto \langle \tau, [\xi, \eta] \rangle$ on \mathfrak{g}_ν which factorizes to the cocycle γ_τ on $\mathfrak{g}_\nu/\mathfrak{n}_\nu$ (by (19) $[\mathfrak{g}_\nu, \mathfrak{n}_\nu] \subset \mathfrak{n}_\nu^\natural$ and by definition $\tau|_{\mathfrak{n}_\nu} = \nu|_{\mathfrak{n}_\nu}$). If $\nu|_{\mathfrak{n}_\nu} = 0$ then the map $(\xi, \eta) \mapsto \langle \tau, [\xi, \eta] \rangle$ on \mathfrak{g}_ν factorizes to the trivial cocycle γ_τ on the quotient algebra $\mathfrak{g}_\nu/\mathfrak{n}_\nu$. Note that the cocycle γ_τ coincides with the cocycle constructed by Panasyuk in [13] using direct calculations and is independent of the choice of the extension τ of the covector $\nu|_{\mathfrak{n}_\nu}$ [13, Lemma 2.1].

The dual space of a Lie algebra is a Poisson manifold with the natural linear Poisson structure induced by the commutator $[\cdot, \cdot]$. The coadjoint orbits in this space are the corresponding symplectic leaves of the Poisson structure. Since the affine subspace \mathcal{B}_ν is the union of the G_ν^0 -orbits in \mathfrak{b}_ν^* (the symplectic leaves), \mathcal{B}_ν is a Poisson submanifold of \mathfrak{b}_ν^* . Fixing the origin $\tau \in \mathcal{B}_\nu$ to identify \mathcal{B}_ν with the dual space $(\mathfrak{g}_\nu/\mathfrak{n}_\nu)^* = (\mathfrak{n}_\nu)^{\perp_\nu}$ (see (38)), we fix some ‘‘affine’’ Poisson structure η_τ on $(\mathfrak{g}_\nu/\mathfrak{n}_\nu)^*$. Remark that if the central extension (36) of the Lie algebra $\mathfrak{g}_\nu/\mathfrak{n}_\nu$ is trivial, this Poisson structure is equivalent to the natural linear Poisson structure η_{can} on the dual space to the Lie algebra $\mathfrak{g}_\nu/\mathfrak{n}_\nu$ (there will be a natural origin in \mathcal{B}_ν). If $\nu|_{\mathfrak{n}_\nu} = 0$ then the (trivial) cocycle γ_τ determines the trivial one-dimensional central extension of $\mathfrak{g}_\nu/\mathfrak{n}_\nu$ and, consequently, the new Poisson structure $\eta_\tau \simeq \eta_{\text{can}}$ on $(\mathfrak{g}_\nu/\mathfrak{n}_\nu)^*$.

Determine the index $\text{ind}(\mathfrak{g}_\nu/\mathfrak{n}_\nu, \eta_\tau)$ of the Poisson structure η_τ on $(\mathfrak{g}_\nu/\mathfrak{n}_\nu)^*$ as the codimension of the symplectic leaf of the maximal dimension in $(\mathfrak{g}_\nu/\mathfrak{n}_\nu)^*$. It is clear that $\text{ind}(\mathfrak{g}_\nu/\mathfrak{n}_\nu, \eta_\tau) = \text{ind } \mathfrak{b}_\nu - 1$ if $\nu|_{\mathfrak{n}_\nu} \neq 0$ because $\text{codim } \mathcal{B}_\nu$ in \mathfrak{b}_ν equals 1 and the symplectic leaves are coadjoint orbits of \mathfrak{b}_ν . Also $\text{ind}(\mathfrak{g}_\nu/\mathfrak{n}_\nu, \eta_\tau) = \text{ind } \mathfrak{b}_\nu$ if $\nu|_{\mathfrak{n}_\nu} = 0$ because in this case $\mathfrak{b}_\nu = \mathfrak{g}_\nu/\mathfrak{n}_\nu$ and $\eta_\tau \simeq \eta_{\text{can}}$. We obtain the following assertion of Panasyuk [13, Th. 2.7]:

Corollary 15 (Panasyuk’s formula). *Let \mathfrak{g} be a Lie algebra and \mathfrak{n} be its ideal. Then*

$$\text{ind } \mathfrak{g} = \text{ind}(\mathfrak{g}, \mathfrak{n}) + \text{ind}(\mathfrak{g}_\nu/\mathfrak{n}_\nu, \eta_\tau),$$

where $\nu \in \mathfrak{n}^*$ is a generic element.

Remark that Theorem 13 defines more precisely the notion of the set of ‘‘generic elements’’. This set is defined in [13] indirectly as an open dense subset in \mathfrak{n}^* on which the function $\nu \mapsto \text{ind}(\mathfrak{g}_\nu/\mathfrak{n}_\nu, \eta_\tau)$ is constant.

Assume that the ideal \mathfrak{n} is Abelian and there exists a complementary to \mathfrak{n} subalgebra $\mathfrak{k} \subset \mathfrak{g}$, i.e. $\mathfrak{g} = \mathfrak{k} \dot{+} \mathfrak{n}$. Then the Lie algebra \mathfrak{g} is a semi-direct product of \mathfrak{k} and the Abelian ideal \mathfrak{n} . It is evident that $\mathfrak{n}_\nu = \mathfrak{n}$ for any non-zero $\nu \in \mathfrak{n}^*$ and since $\mathfrak{n}_\nu \subset \mathfrak{g}_\nu$, the isotropy subalgebra $\mathfrak{g}_\nu = \mathfrak{k}_\nu \dot{+} \mathfrak{n}$, where $\mathfrak{k}_\nu = \mathfrak{k} \cap \mathfrak{g}_\nu$. But by (19) $[\mathfrak{k}_\nu, \mathfrak{n}_\nu^\natural] \subset \mathfrak{n}_\nu^\natural$, where $\mathfrak{n}_\nu^\natural = \ker \nu$ and, consequently, the algebra $\mathfrak{b}_\nu = \mathfrak{g}_\nu/\mathfrak{n}_\nu^\natural = (\mathfrak{k}_\nu + \mathfrak{n}_\nu^\natural)/\mathfrak{n}_\nu^\natural \dot{+} \mathfrak{n}_\nu/\mathfrak{n}_\nu^\natural$ is a trivial one-dimensional central extension of $\mathfrak{k}_\nu \simeq \mathfrak{g}_\nu/\mathfrak{n}_\nu$. Therefore $\text{ind } \mathfrak{k}_\nu = \text{ind } \mathfrak{b}_\nu - 1$. Note that $\text{ind}(\mathfrak{g}, \mathfrak{n}) = \text{ind}(\mathfrak{k}, \mathfrak{n})$ because \mathfrak{n} is Abelian. Since an element $\nu \in \mathfrak{n}^*$ is \mathfrak{g} -regular if and only if this element is \mathfrak{k} -regular, we obtain the following assertion of Raïs [6]:

Corollary 16 (Raïs’ formula). *Let the Lie algebra \mathfrak{g} be a semi-direct product of \mathfrak{k} and the Abelian ideal \mathfrak{n} . Let $\nu \in \mathfrak{n}^*$ be a \mathfrak{k} -regular element for which there exists a \mathfrak{g} -regular element $\sigma \in \mathfrak{g}^*$ such that $\sigma|_{\mathfrak{n}} = \nu$. Then $\text{ind } \mathfrak{g} = \text{ind}(\mathfrak{k}, \mathfrak{n}) + \text{ind } \mathfrak{k}_\nu$.*

2.4 The bundle of reduced-group orbits

We return to the general case when \mathfrak{n} is an arbitrary ideal of \mathfrak{g} and σ is an arbitrary element of \mathfrak{g}^* . Any element $\hat{\sigma} \in \mathfrak{g}^*$ determines a pair $(\hat{\nu}, \hat{\tau})$, where $\hat{\nu} = \hat{\sigma}|_{\mathfrak{n}}$ and $\hat{\tau} = \hat{\sigma}|_{\mathfrak{g}_\nu}$. Such a pair is denoted by $\Pi_{12}^{\mathfrak{g}}(\hat{\sigma})$. By the definition, $\Pi_{12}^{\mathfrak{g}}(\hat{\sigma}_1) = \Pi_{12}^{\mathfrak{g}}(\hat{\sigma}_2)$ if and only if $\hat{\sigma}_1, \hat{\sigma}_2 \in \mathcal{A}_{\hat{\nu}\hat{\tau}}$ for some $\hat{\nu} \in \mathfrak{n}^*$ and $\hat{\tau} \in \mathfrak{g}_\nu^*$. In

this case the elements $\hat{\sigma}_1, \hat{\sigma}_2$ belong to the same $\text{Ad}^*(G)$ -orbit \mathcal{O} in \mathfrak{g}^* because the set $\mathcal{A}_{\hat{\nu}\hat{\tau}}$ is an orbit of the Lie subgroup $N_{\hat{\nu}}^0 \subset G$. Therefore the Ad^* -action of G on the coadjoint orbit \mathcal{O} induces the action of G on the set $\Pi_{12}^{\mathfrak{g}}(\mathcal{O})$. We will show that on the set $\Pi_{12}^{\mathfrak{g}}(\mathcal{O})$ there exists a structure of a smooth manifold such that the map $\Pi_{12}^{\mathfrak{g}}|_{\mathcal{O}}$ is a G -equivariant submersion. Remark also that for arbitrary $\hat{\tau}_0 \in \mathfrak{g}_{\hat{\nu}}^*$ there exists some $\hat{\sigma}_0 \in \mathfrak{g}^*$ such that $\Pi_{12}^{\mathfrak{g}}(\hat{\sigma}_0) = (\hat{\nu}, \hat{\tau}_0)$ iff $\hat{\tau}_0|_{\mathfrak{n}_{\hat{\nu}}} = \hat{\nu}|_{\mathfrak{n}_{\hat{\nu}}}$. In this case such an element $\hat{\tau}_0 \in \mathfrak{g}_{\hat{\nu}}^*$ is called a $\mathfrak{g}_{\hat{\nu}}^*$ -extension of $\hat{\nu} \in \mathfrak{n}^*$.

Let B be the G -orbit in \mathfrak{n}^* with respect to the action ρ^* . Now we construct a bundle of reduced-group orbits over the orbit B . This bundle is the bundle $p : P \rightarrow B$ such that the fibre $F_P(\hat{\nu}) = p^{-1}(\hat{\nu})$ is an orbit of $G_{\hat{\nu}}$ in $\mathfrak{g}_{\hat{\nu}}^*$ passing through some $\mathfrak{g}_{\hat{\nu}}^*$ -extension of $\hat{\nu} \in \mathfrak{n}^*$ and if g belongs to G and $\hat{\tau} \in F_P(\hat{\nu})$ then $g \cdot \hat{\tau} \in F_P(g \cdot \hat{\nu})$ (a right action) is defined by $\langle g \cdot \hat{\tau}, \xi' \rangle = \langle \hat{\tau}, \text{Ad}_g \xi' \rangle$, where $\xi' \in \mathfrak{g}_{g \cdot \hat{\nu}}$. It follows that G acts transitively on P . We prove below that this bundle and this action are smooth.

The bundle of reduced-group orbits may be described in another way. Consider the (smooth) bundle $P_{\nu\tau} = G \times_{G_{\nu}} (G_{\nu}/G_{\nu\tau})$, the bundle associated to the principal bundle with base G/G_{ν} , total space G and fibre $(G_{\nu}/G_{\nu\tau})$. Here ν denotes some element of the orbit B and $\tau \in F_P(\nu)$. Then $F_P(\nu) \simeq G_{\nu}/G_{\nu\tau}$. The elements of $P_{\nu\tau}$ are orbits of G_{ν} (on $G \times (G_{\nu}/G_{\nu\tau})$), where the action on the right is given by $(g, [h]).h' = (gh', [h'^{-1}h])$ with $g \in G$, $h, h' \in G_{\nu}$ ($[h] = hG_{\nu\tau} \in G_{\nu}/G_{\nu\tau}$). The element $(g, [h]).G_{\nu}$ of $P_{\nu\tau}$, is identified with the point $(gh)^{-1} \cdot \tau$ in $F_P((gh)^{-1} \cdot \nu)$. Defining p by $p((g, [h]).G_{\nu}) = (gh)^{-1} \cdot \nu$ and the right action of G on P by $g' \cdot (g, [h]).G_{\nu} = (g'^{-1}g, [h]).G_{\nu}$ makes $p : P \rightarrow B$ a smooth bundle of reduced-group orbits over $B = \mathcal{O}^{\nu}(G)$. The following proposition generalizes Proposition 1 from [1].

Proposition 17. *There is a bijection between the set of bundles of reduced-group orbits and the set of coadjoint orbits of G on \mathfrak{g}^* .*

Proof. Let $p : P \rightarrow B$ be a bundle of reduced-group orbits, take $\nu \in B$, $\tau \in F_P(\nu)$ and choose some extension $\sigma \in \mathfrak{g}^*$ with $\sigma|_{\mathfrak{n}} = \nu \in \mathfrak{n}^*$ and $\sigma|_{\mathfrak{g}_{\nu}} = \tau \in \mathfrak{g}_{\nu}^*$. If \mathcal{O}^{σ} is the $\text{Ad}^*(G)$ -orbit through σ in \mathfrak{g}^* then it depends only on $p : P \rightarrow B$ but not of the choices made because all extensions of (ν, τ) are elements of this orbit (see (30)).

Conversely, let \mathcal{O} be an $\text{Ad}^*(G)$ -orbit in \mathfrak{g}^* and σ a point of \mathcal{O} with $\sigma|_{\mathfrak{n}} = \nu$ and $\sigma|_{\mathfrak{g}_{\nu}} = \tau$. Construct the bundle of reduced-group over B , the orbit of ν in \mathfrak{n}^* , with fibre $F_P(\nu)$, the G_{ν} -orbit of $\tau \in \mathfrak{g}_{\nu}^*$ in \mathfrak{g}_{ν}^* . This gives a bundle depending only on \mathcal{O} and not of the choices made. These two constructions are the inverses of each other and set up the required bijection. \square

If we have an orbit $\mathcal{O}^{\sigma} = \mathcal{O}^{\sigma}(G)$ in \mathfrak{g}^* and the associated bundle $p : P \rightarrow B$, then the following diagram (on the left) of G -equivariant maps is commutative. Recall that, by the definition, $\Pi_{12}^{\mathfrak{g}}(\hat{\sigma}) = (\hat{\sigma}|_{\mathfrak{n}}, \hat{\sigma}|_{\mathfrak{g}_{\nu}}) = (\hat{\nu}, \hat{\tau})$ and $\Pi_1^{\mathfrak{g}}(\hat{\sigma}) = \hat{\sigma}|_{\mathfrak{n}}$.

$$\begin{array}{ccc}
\mathcal{O}^{\sigma} & & G \times_{G_{\nu}} (G_{\nu}/G_{\sigma}) \\
\Pi_{12}^{\mathfrak{g}} \downarrow & \searrow \Pi_1^{\mathfrak{g}} & \Pi_{12}^{\mathfrak{g}} \downarrow \\
P & \xrightarrow{p} B & G \times_{G_{\nu}} (G_{\nu}/G_{\nu\tau}) \xrightarrow{p} G/G_{\nu}
\end{array} \quad (40)$$

As we remarked above, the fibres of $\Pi_{12}^{\mathfrak{g}}$ are affine subspaces of \mathfrak{g}^* whose associated vector space is $(\mathfrak{n} + \mathfrak{g}_{\hat{\nu}})^{\perp}$ (in general there will be no natural origin in $\Pi_{12}^{\mathfrak{g}}{}^{-1}(\hat{\nu}, \hat{\tau}) = \mathcal{A}_{\hat{\nu}\hat{\tau}}$). Thus the fibres of $\Pi_{12}^{\mathfrak{g}}$ are the orbits on \mathcal{O}^{σ} of the groups conjugated to $G_{\hat{\nu}\hat{\tau}}$.

The map $\Pi_{12}^{\mathfrak{g}} : \mathcal{O}^{\sigma}(G) \rightarrow P$ is smooth because the map $\Pi_1^{\mathfrak{g}} : \mathcal{O}^{\sigma}(G) \rightarrow B$ is a submersion and the left diagram is commutative. This fact can be established also by identifying G -equivariantly the bundle P with $P_{\nu\tau} = G \times_{G_{\nu}} (G_{\nu}/G_{\nu\tau})$. But by the definition, $\mathcal{O}^{\sigma} \simeq G/G_{\sigma}$ and $B \simeq G/G_{\nu}$. Consider the space $G \times_{G_{\nu}} (G_{\nu}/G_{\sigma})$, where the action on the right is given by $(g, hG_{\sigma}) \cdot h' = (gh', h'^{-1}hG_{\sigma})$ with g in G , h, h' in G_{ν} . The standard map

$$G \times_{G_{\nu}} (G_{\nu}/G_{\sigma}) \rightarrow G/G_{\sigma}, \quad [(g, hG_{\sigma})]_{G_{\nu}} \mapsto ghG_{\sigma}$$

is a G -equivariant diffeomorphism with respect to the natural left actions of G . Therefore, using this identification, we obtain the following expressions for the G -equivariant maps $p, \Pi_1^{\mathfrak{g}}, \Pi_{12}^{\mathfrak{g}}$: $p([(g, hG_{\nu\tau})]_{G_{\nu}}) = ghG_{\nu}$,

$$\Pi_1^{\mathfrak{g}}([(g, hG_{\sigma})]_{G_{\nu}}) = ghG_{\nu}, \quad \text{and} \quad \Pi_{12}^{\mathfrak{g}}([(g, hG_{\sigma})]_{G_{\nu}}) = [(g, hG_{\nu\tau})]_{G_{\nu}}.$$

It is clear that the diagram above (on the right) is also commutative and these two diagrams are equivalent. Remark also that by Proposition 6 the fibre $\mathcal{A}_{\nu\tau}$ is an isotropic submanifold of the coadjoint orbit $\mathcal{O}^{\sigma}(G)$. We have proved

Theorem 18. *The map $\Pi_{12}^{\mathfrak{g}} : \mathcal{O}^{\sigma}(G) \rightarrow P$ is a G -equivariant submersion of the coadjoint orbit \mathcal{O}^{σ} onto the bundle P of reduced-group orbits. This map is a bundle with the total space \mathcal{O}^{σ} , the base P and the affine space $\mathcal{A}_{\nu\tau}$ (the isotropic submanifold of $\mathcal{O}^{\sigma}(G)$) as its fibre. The commutative diagrams (40) are equivalent.*

2.5 Isotropic affine subspaces of coadjoint orbits

As follows from Proposition 6 each coadjoint orbit \mathcal{O} of the Lie algebra \mathfrak{g} contains the isotropic affine subspace associated with its ideal \mathfrak{n} . We will show below that if this affine subspace is trivial then any isotropic affine subspace of the corresponding coadjoint orbit in $\mathfrak{b}_{\nu}^* \subset \mathfrak{g}_{\nu}^*$ determines some isotropic affine subspace of \mathcal{O} .

Let $\mathcal{O}^{\sigma} = \mathcal{O}^{\sigma}(G)$, where $\sigma \in \mathfrak{g}^*$, be a coadjoint orbit in \mathfrak{g}^* . Consider also the coadjoint orbit \mathcal{O}^{τ} in \mathfrak{g}_{ν}^* (34) passing through the element $\tau \in \mathfrak{g}_{\nu}^*$, where $\tau = \sigma|_{\mathfrak{g}_{\nu}}$. To simplify the notation, this orbit $\mathcal{O}^{\tau} = \mathcal{O}^{\tau}(G_{\nu}^{\bullet})$ of the connected Lie group G_{ν}^{\bullet} will be considered as an orbit of the (closed) Lie subgroup $G_{\nu}^{\bullet} = G_{\nu}^0 \cdot G_{\nu\tau}$ of G_{ν} , containing the whole isotropy subgroup $G_{\nu\tau}$. The σ -orbit $\mathcal{O}^{\sigma}(G_{\nu}^{\bullet}) \subset \mathcal{O}^{\sigma}(G_{\nu})$ in \mathfrak{g}^* is also connected because, by (31), $G_{\nu}^{\bullet} = G_{\nu}^0 \cdot G_{\sigma}$. Recall that $\Pi_2^{\mathfrak{g}}$ denotes the natural G_{ν} -equivariant projection $\mathfrak{g}^* \rightarrow \mathfrak{g}_{\nu}^*$, $\beta \mapsto \beta|_{\mathfrak{g}_{\nu}}$ defined by (37).

Proposition 19. *Let $\sigma \in \mathfrak{g}^*$ be an arbitrary element and $\nu = \sigma|_{\mathfrak{n}}$, $\tau = \sigma|_{\mathfrak{g}_{\nu}}$. The restriction $p_2 = \Pi_2^{\mathfrak{g}}|_{\mathcal{O}^{\sigma}(G_{\nu}^{\bullet})}$ of the projection $\Pi_2^{\mathfrak{g}}$ is a G_{ν}^{\bullet} -equivariant submersion of the orbit $\mathcal{O}^{\sigma}(G_{\nu}^{\bullet})$ onto the coadjoint orbit \mathcal{O}^{τ} in \mathfrak{g}_{ν}^* . This map*

$p_2 : \mathcal{O}^\sigma(G_\nu^\bullet) \rightarrow \mathcal{O}^\tau$ is a bundle with the total space $\mathcal{O}^\sigma(G_\nu^\bullet)$, the coadjoint orbit \mathcal{O}^τ as its base and the affine space $\mathcal{A}_{\nu\tau} \simeq G_{\nu\tau}/G_\sigma$ as its fibre. Moreover, $p_2^*(\omega') = \omega|_{\mathcal{O}^\sigma(G_\nu^\bullet)}$, where ω' and ω are the canonical Kirillov-Kostant-Souriau symplectic 2-forms on the coadjoint orbits $\mathcal{O}^\tau \subset \mathfrak{g}_\nu^*$ and $\mathcal{O}^\sigma(G) \subset \mathfrak{g}^*$ respectively.

Proof. To prove the first part of the proposition it is sufficient to remark that

$$\mathcal{O}^\sigma(G_\nu^\bullet) \simeq G_\nu^\bullet/G_\sigma, \quad \mathcal{O}^\tau = \mathcal{O}^\tau(G_\nu^\bullet) \simeq G_\nu^\bullet/G_{\nu\tau}, \quad \mathcal{A}_{\nu\tau} \simeq G_{\nu\tau}/G_\sigma$$

and $G_\nu^\bullet/G_\sigma = G_\nu^\bullet \times_{G_{\nu\tau}} (G_{\nu\tau}/G_\sigma)$ with the standard right action of $G_{\nu\tau}$.

By G_ν^\bullet -equivariance of the map p_2 , we have $p_{2*}(\sigma)(\widehat{\text{ad}}_\xi^* \sigma) = \widehat{\text{ad}}_\xi^* \tau$ for $\xi, \eta \in \mathfrak{g}_\nu$. Then, by the definition, of the form ω' ,

$$(p_2^* \omega')(\sigma)(\widehat{\text{ad}}_\xi^* \sigma, \widehat{\text{ad}}_\eta^* \sigma) = \omega'(\tau)(\widehat{\text{ad}}_\xi^* \tau, \widehat{\text{ad}}_\eta^* \tau) = \tau([\xi, \eta]) = \sigma([\xi, \eta]).$$

Taking into account the expression (10) for ω at the point σ and G_ν^\bullet -invariance of the forms ω and ω' , we complete the proof. \square

Remark 20. The proposition above admits the following moment map interpretation which is motivated by Panasyuk's approach [13]. Indeed, the identity map $J_G : \mathcal{O}^\sigma(G) \rightarrow \mathfrak{g}^*$, $\hat{\sigma} \mapsto \hat{\sigma}$ is an equivariant moment map for Ad^* -action of G on $\mathcal{O}^\sigma(G)$. Since \mathfrak{n} is a subalgebra of \mathfrak{g} , the map $J_N = \Pi_1^{\mathfrak{g}} \circ J_G$ of $\mathcal{O}^\sigma(G)$ into \mathfrak{n}^* , $\hat{\sigma} \mapsto \hat{\sigma}|_{\mathfrak{n}}$ is an equivariant moment map for the restricted action of $N \subset G$ on $\mathcal{O}^\sigma(G)$. Then by (6) the set $J_N^{-1}(\nu) = \mathcal{A}_\nu \cap \mathcal{O}^\sigma(G) = \mathcal{O}^\sigma(G_\nu)$ is a submanifold of $\mathcal{O}^\sigma(G)$. If $N_\nu^{\text{fin}} = N_\nu$, then by Proposition 6 the quotient space $J_N^{-1}(\nu)/N_\nu \simeq \Pi_2^{\mathfrak{g}}(\mathcal{O}^\sigma(G_\nu))$ is a reduced symplectic manifold. This manifold is the orbit $\mathcal{O}^\tau(G_\nu) \subset \mathfrak{b}_\nu^* \subset \mathfrak{g}_\nu^*$, a union of disjoint coadjoint orbits (connected components) in the reduced Lie algebra \mathfrak{b}_ν^* . The reduced symplectic structure on $\mathcal{O}^\tau(G_\nu)$ coincides with the canonical Kirillov-Kostant-Souriau symplectic form on each connected component of $\mathcal{O}^\tau(G_\nu)$.

Proposition 21. *We retain the notation of Proposition 19. Suppose that the coadjoint orbit $\mathcal{O}^\tau \subset \mathfrak{g}_\nu^*$ contains the isotropic affine subspace $\mathcal{I}(\tau)$ passing through the point τ . If $\dim \mathcal{A}_{\nu\tau} = 0$, then the projection $p_2 : \mathcal{O}^\sigma(G_\nu^0) \rightarrow \mathcal{O}^\tau$ is a bijection and the preimage $\mathcal{I}(\sigma) = p_2^{-1}(\mathcal{I}(\tau))$, $\mathcal{I}(\sigma) \subset \mathcal{O}^\sigma(G_\nu^0) \subset \mathcal{A}_\nu$, is an isotropic affine subspace of the coadjoint orbit $\mathcal{O}^\sigma(G)$ passing through $\sigma \in \mathfrak{g}^*$.*

Proof. It is an immediate consequence of Proposition 19 that the map p_2 is a bijection and $\mathcal{I}(\sigma) = p_2^{-1}(\mathcal{I}(\tau))$ is an isotropic submanifold of the coadjoint orbit $\mathcal{O}^\sigma(G)$. Let us show that the set $\mathcal{I}(\sigma)$ is an affine subspace of \mathfrak{g}^* .

Since $\mathcal{I}(\tau)$ is an affine subspace of \mathfrak{g}_ν^* , this space contains any line $\tau + t\alpha$, where $\alpha \in \mathfrak{g}_\nu^*$ is a non-zero tangent vector to $\mathcal{I}(\tau)$ at τ . There exists a unique tangent vector $\alpha' \in T_\sigma \mathcal{O}^\sigma(G_\nu^0)$ such that $p_{2*}(\sigma)(\alpha') = \alpha$ because p_2 is a bijection. Moreover, $\alpha' \in \mathfrak{n}^\perp$ because $\mathcal{O}^\sigma(G_\nu^0) \subset \mathcal{A}_\nu$ and $\alpha = \Pi_2^{\mathfrak{g}}(\alpha')$ because p_2 is a restriction of the linear map $\Pi_2^{\mathfrak{g}}$ (26).

Since $\mathcal{I}(\tau) \subset \mathcal{O}^\tau$, for any $t \in \mathbb{F}$ there exists $g \in G_\nu^0$ such that $\tau + t\alpha = \widehat{\text{Ad}}_g^* \tau$. Now to complete the proof it is sufficient to show that the point $\sigma' = \text{Ad}_{g^{-1}}^* \sigma + t \text{Ad}_{g^{-1}}^* \alpha'$ coincides with σ . Indeed, by G_ν -equivariance of the linear map $\Pi_2^{\mathfrak{g}}$

$$\Pi_2^{\mathfrak{g}}(\sigma') = \Pi_2^{\mathfrak{g}}(\text{Ad}_{g^{-1}}^* \sigma + t \text{Ad}_{g^{-1}}^* \alpha') = \widehat{\text{Ad}}_{g^{-1}}^* \tau + t \widehat{\text{Ad}}_{g^{-1}}^* \alpha = \tau.$$

But $\sigma' \in \mathcal{A}_\nu$ because $\text{Ad}^*(G_\nu)(\mathcal{A}_\nu) = \mathcal{A}_\nu$ and $\sigma + t\alpha' \in \mathcal{A}_\nu$. Therefore σ' belongs to the one-point set $\mathcal{A}_{\nu\tau} = \{\sigma\}$. \square

Proposition 22. *We retain the notation of Proposition 19. Suppose that $\dim \mathcal{A}_{\nu\tau} = 0$ and the quotient algebra $\mathfrak{b}_\nu = \mathfrak{g}_\nu / \mathfrak{n}_\nu^\natural$, $\mathfrak{n}_\nu^\natural = \ker(\nu|_{\mathfrak{n}_\nu})$ is Abelian. Then*

- 1) $\mathcal{O}^\sigma(G) = \mathcal{O}^\sigma(N)$ and $\mathcal{O}^\nu(G) = \mathcal{O}^\nu(N)$, where $\mathcal{O}^\sigma(G)$ and $\mathcal{O}^\nu(N)$ are the coadjoint orbits of the Lie algebras \mathfrak{g} and \mathfrak{n} respectively;
- 2) the projection $p_1 : \mathcal{O}^\sigma(G) \rightarrow \mathcal{O}^\nu(N)$, $\sigma' \mapsto \sigma'|_{\mathfrak{n}}$, is a symplectic G -equivariant covering map with the discrete fiber $\simeq N_\nu/N_\sigma$ and $\mathfrak{g}_\sigma = \mathfrak{g}_\nu$, $\mathfrak{n}_\sigma = \mathfrak{n}_\nu$;
- 3) if $N_\nu = N_\nu^{\text{fin}}$, then p_1 is a diffeomorphism, and, in particular, $G_\nu = G_\sigma$, $N_\nu = N_\sigma$.

Proof. Since N is a normal subgroup of G , the G -orbit $\mathcal{O}^\nu(G)$ is a disjoint union of isomorphic N -orbits. These N -orbits are open subsets because $\dim \mathcal{O}^\nu(G) - \dim \mathcal{O}^\nu(N) = \dim \mathcal{A}_{\nu\tau} = 0$. Then $\mathcal{O}^\nu(G) = \mathcal{O}^\nu(N)$ because G is connected.

By Proposition 19 $\dim \mathcal{O}^\sigma(G_\nu) = \dim \mathcal{O}^\tau(G_\nu)$ because $\mathcal{A}_{\nu\tau} = \{\sigma\}$. Since each connected component of $\mathcal{O}^\tau(G_\nu)$ is a coadjoint orbit of the Lie algebra \mathfrak{b}_ν which is Abelian, $\dim \mathcal{O}^\tau(G_\nu) = 0$. Thus by Lemma 4 the G -equivariant map $p_1 : \mathcal{O}^\sigma(G) \rightarrow \mathcal{O}^\nu(G)$ is a bundle with the discrete fibre $\mathcal{O}^\sigma(G_\nu)$. Taking into account the identity $\mathcal{O}^\nu(G) = \mathcal{O}^\nu(N)$ and the N -equivariance of the local diffeomorphism p_1 we obtain that $T_\sigma \mathcal{O}^\sigma(G) = T_\sigma \mathcal{O}^\sigma(N)$, i.e. the orbit $\mathcal{O}^\sigma(N)$ is an open subset of $\mathcal{O}^\sigma(G)$. Using the same arguments as above, we obtain that $\mathcal{O}^\sigma(G) = \mathcal{O}^\sigma(N)$. Since $\mathcal{O}^\sigma(G) \simeq N/N_\sigma$ and $\mathcal{O}^\nu(G) \simeq N/N_\nu$, the fiber $\mathcal{O}^\sigma(G_\nu) \simeq N_\nu/N_\sigma$. Since $\dim \mathcal{O}^\sigma(G) = \dim \mathcal{O}^\nu(G)$, $\dim \mathfrak{g}_\sigma = \dim \mathfrak{g}_\nu$. Thus $\mathfrak{g}_\sigma = \mathfrak{g}_\nu$ because $\mathfrak{g}_\sigma \subset \mathfrak{g}_\nu$.

The local diffeomorphism p_1 is symplectic with respect to the canonical symplectic structures on the both coadjoint orbits. To prove this fact it is sufficient to observe that $T_\sigma \mathcal{O}^\sigma(G) = \text{ad}_\mathfrak{n}^* \sigma$, $p_{1*}(\sigma)(\text{ad}_\xi^* \sigma) = \widetilde{\text{ad}}_\xi^* \nu$ and $\sigma([\xi, \eta]) = \nu([\xi, \eta])$ for any $\xi, \eta \in \mathfrak{n}$ (by N -equivariance of p_1), and to use definition (10) of the canonical symplectic form.

If $N_\nu = N_\nu^{\text{fin}}$, then by Proposition 6 the group $\text{Ad}^*(N_\nu)$ preserves the one-point set $\mathcal{A}_{\nu\tau} = \{\sigma\}$ and, consequently, $N_\nu = N_\sigma$. Hence p_1 is a diffeomorphism. \square

2.6 Coadjoint orbits in general position

In the previous subsection we considered arbitrary coadjoint orbits of \mathfrak{g} . Now we consider the structure of the orbits in general position. To this end put

$$\text{co}(\mathfrak{g}, \mathfrak{n}) = \text{ind } \mathfrak{n} - \text{ind}(\mathfrak{g}, \mathfrak{n}). \quad (41)$$

Taking into account that $\dim \mathcal{O}^\nu(G) - \dim \mathcal{O}^\nu(N) = \text{codim}_{\mathfrak{n}^*} \mathcal{O}^\nu(N) - \text{codim}_{\mathfrak{n}^*} \mathcal{O}^\nu(G)$ for any $\nu \in \mathfrak{n}^*$, we can interpret the number $\text{co}(\mathfrak{g}, \mathfrak{n})$ as a ‘‘complexity’’ of the action of $N \subset G$ on homogeneous spaces of G in general position. Then by (23)

$$\text{co}(\mathfrak{g}, \mathfrak{n}) = \dim(\mathfrak{n} + \mathfrak{g}_\nu)^\perp = \dim \mathcal{A}(\sigma, \mathfrak{n}) \quad (42)$$

for all ν from some dense subset of \mathfrak{n}^* containing the non-empty Zariski open set of all \mathfrak{g} -regular and \mathfrak{n} -regular points of \mathfrak{n}^* . Here $\sigma \in \mathcal{A}_\nu = (\Pi_1^{\mathfrak{g}})^{-1}(\nu)$ and $\mathcal{A}(\sigma, \mathfrak{n}) = \sigma + \dim(\mathfrak{n} + \mathfrak{g}_\nu)^\perp$ is the isotropic affine subspace of the coadjoint orbit $\mathcal{O}^\sigma \subset \mathfrak{g}^*$.

The case when $\text{co}(\mathfrak{g}, \mathfrak{n}) = 0$ we consider in more detail.

Lemma 23. *Suppose that $\text{co}(\mathfrak{g}, \mathfrak{n}) = 0$. Let $\nu \in \mathfrak{n}^*$ be any \mathfrak{n} -regular point. Then*

- 1) $\nu \in \mathfrak{n}^*$ is a \mathfrak{g} -regular point;
- 2) $\mathcal{O}^\nu(G) = \mathcal{O}^\nu(N)$ or, equivalently, $\mathfrak{g}_\nu + \mathfrak{n} = \mathfrak{g}$;
- 3) if $\mathcal{A}_\nu \cap R(\mathfrak{g}^*) \neq \emptyset$ and the quotient algebra $\mathfrak{b}_\nu = \mathfrak{g}_\nu / \mathfrak{n}_\nu^\natural$ is Abelian then for each \mathfrak{n} -regular point $\nu_1 \in \mathfrak{n}^*$ (i) the algebra $\mathfrak{b}_{\nu_1} = \mathfrak{g}_{\nu_1} / \mathfrak{n}_{\nu_1}^\natural$ is Abelian; (ii) $\mathcal{A}_{\nu_1} \subset R(\mathfrak{g}^*)$; (iii) $\mathfrak{g}_{\sigma_1} = \mathfrak{g}_{\nu_1}$, where $\sigma_1 \in \mathcal{A}_{\nu_1}$; (iv) the Lie algebra \mathfrak{g}_{ν_1} is Abelian; (v) there exists an Abelian Lie algebra $\mathfrak{a} \subset \mathfrak{g}_{\nu_1}$ such that the Lie algebra \mathfrak{g} is a semidirect product of \mathfrak{a} and the ideal \mathfrak{n} , i.e. $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{n}$.

Proof. Since $\text{co}(\mathfrak{g}, \mathfrak{n}) = 0$, $\dim \mathcal{O}^{\nu_0}(G) = \dim \mathcal{O}^{\nu_0}(N)$ for some point $\nu_0 \in R(\mathfrak{n}^*)$ which is \mathfrak{g} -regular. Hence $\mathfrak{g}_{\nu_0} + \mathfrak{n} = \mathfrak{g}$. But for each $\nu_1 \in R(\mathfrak{n}^*)$ the isotropy algebra $\mathfrak{n}_{\nu_1} = \mathfrak{g}_{\nu_1} \cap \mathfrak{n}$ has constant dimension $\text{ind } \mathfrak{n}$ and $\dim \mathfrak{g}_{\nu_1} \geq \dim \mathfrak{g}_{\nu_0}$. Therefore $\mathfrak{g}_{\nu_1} + \mathfrak{n} = \mathfrak{g}$, i.e. $\dim \mathfrak{g}_{\nu_1} = \dim \mathfrak{g}_{\nu_0}$. In particular, $\mathfrak{g}_\nu + \mathfrak{n} = \mathfrak{g}$.

If the quotient algebra $\mathfrak{b}_\nu = \mathfrak{g}_\nu / \mathfrak{n}_\nu^\natural$ is Abelian then the coadjoint orbit \mathcal{O}^τ is a one-point set $\{\tau\}$, i.e. $\mathfrak{g}_{\nu\tau} = \mathfrak{g}_\nu$. Since $\mathcal{A}_\nu \cap R(\mathfrak{g}^*) \neq \emptyset$, there exists a \mathfrak{g} -regular element $\sigma \in \mathfrak{g}^*$ such that its restriction $\sigma|_{\mathfrak{n}} = \nu$. But by Remark 8, $\mathfrak{g}_{\nu\tau} = \mathfrak{g}_\sigma$ and $\mathfrak{g}_{\nu_1\tau_1} = \mathfrak{g}_{\sigma_1}$, where $\sigma_1 \in \mathcal{A}_{\nu_1}$ and $\tau_1 = \sigma_1|_{\mathfrak{g}_{\nu_1}}$. Hence

$$\dim \mathfrak{g}_{\sigma_1} = \dim \mathfrak{g}_{\nu_1\tau_1} \leq \dim \mathfrak{g}_{\nu_1} = \dim \mathfrak{g}_\nu = \dim \mathfrak{g}_{\nu\tau} = \dim \mathfrak{g}_\sigma = \text{ind } \mathfrak{g}$$

because $\mathfrak{g}_{\nu_1\tau_1} \subset \mathfrak{g}_{\nu_1}$. But by definition $\dim \mathfrak{g}_{\sigma_1} \geq \text{ind } \mathfrak{g}$. Therefore $\mathfrak{g}_{\sigma_1} = \mathfrak{g}_{\nu_1}$ and $\mathcal{A}_{\nu_1} \subset R(\mathfrak{g}^*)$ (all these points are \mathfrak{g} -regular). The Lie algebra \mathfrak{g}_{σ_1} is Abelian as an isotropy algebra of a \mathfrak{g} -regular element of the coadjoint representation (one can prove this fact differentiating the identity $\langle \sigma_t, [\mathfrak{g}_{\sigma_t}, \mathfrak{g}_{\sigma_t}] \rangle = 0$ using definition (2) of \mathfrak{g}_{σ_t}). Hence the algebra $\mathfrak{g}_{\nu_1} = \mathfrak{g}_{\sigma_1}$ is Abelian. Since $\mathfrak{g}_{\nu_1} + \mathfrak{n} = \mathfrak{g}$, there exists a subspace $\mathfrak{a} \subset \mathfrak{g}_{\nu_1}$ for which $\mathfrak{g} = \mathfrak{a} \dot{+} \mathfrak{n}$. This subspace is an Abelian subalgebra of \mathfrak{g} . \square

2.7 Integral orbits: a necessary but non sufficient condition

In this subsection we will use the notation of the previous subsections, but suppose in addition that the ground field \mathbb{F} is the field \mathbb{R} of real numbers.

First of all we will give an exposition of some results of Kostant [18, §§5.6, 5.7, Theorem 5.7.1] on the geometry of coadjoint orbits.

Let H be a connected Lie group with the Lie algebra \mathfrak{h} . Fix some covector $\varphi \in \mathfrak{h}^*$ and consider the coadjoint orbit $\mathcal{O}^\varphi = \mathcal{O}^\varphi(H) \simeq H/H_\varphi$ in \mathfrak{h}^* . We will say that the coadjoint orbit \mathcal{O}^φ in the dual space \mathfrak{h}^* is integral if its canonical symplectic form is integral, i.e. this form determines an integral cohomology class in $H^2(\mathcal{O}^\varphi, \mathbb{Z}) \subset H^2(\mathcal{O}^\varphi, \mathbb{R})$.

Denote by H_φ^\sharp the set (possibly empty) of all characters $\chi : H_\varphi \rightarrow \mathbb{S}^1 \subset \mathbb{C}$ such that $d\chi(e) = 2\pi i \cdot \varphi|_{\mathfrak{h}_\varphi}$, where \mathfrak{h}_φ is the Lie algebra of the isotropy group H_φ . For such a character $\chi \in H_\varphi^\sharp$,

$$\chi(\exp \xi) = \exp(2\pi i \cdot \langle \varphi, \xi \rangle) \quad \text{for all } \xi \in \mathfrak{h}_\varphi. \quad (43)$$

Since the identity component H_φ^0 of H_φ is generated by its neighborhood of the unity, the restriction $\chi|_{H_\varphi^0}$ is defined uniquely by equation (43). Therefore if H_φ^\sharp is not empty H_φ^\sharp is a $\pi_{H_\varphi/H_\varphi^0}^*$ -principal homogeneous space, where $\pi_{H_\varphi/H_\varphi^0}^*$ is the group of \mathbb{S}^1 -valued characters of the quotient group H_φ/H_φ^0 . In this case $|H_\varphi^\sharp| = |\pi_{H_\varphi/H_\varphi^0}^*|$ [18].

Let \tilde{H} be the connected simply connected Lie group with the Lie algebra \mathfrak{h} , the universal covering group of the connected Lie group H and $\tilde{p} : \tilde{H} \rightarrow H$ be the corresponding covering homomorphism. Then $\mathcal{O}^\varphi = \tilde{H}/\tilde{H}_\varphi$, where \tilde{H}_φ is the isotropy group of the element $\varphi \in \mathfrak{h}^*$. By definition $\tilde{H}_\varphi = \tilde{p}^{-1}(H_\varphi)$ and $H_\varphi \simeq \tilde{H}_\varphi/D$, where D is the kernel of the restricted homomorphism $\tilde{p}|_{\tilde{H}_\varphi}$. The following Kostant's theorem [18, Theorem 5.7.1] is crucial for the forthcoming considerations.

Theorem 24 (B.Kostant). *The orbit \mathcal{O}^φ in \mathfrak{h}^* is integral if and only if the character set \tilde{H}_φ^\sharp is not empty.*

Remark that one can not formulate the integrality condition for the orbit \mathcal{O}^φ only in terms of the connected Lie group H_φ (defining this orbit) because as it will be shown below (see Example 25) in the general case the characters $\chi \in \tilde{H}_\varphi^\sharp$ are not constant on the closed discrete subgroup D of the center of \tilde{H}_φ . In other words, it is possible that $H_\varphi^\sharp = \emptyset$ while $\tilde{H}_\varphi^\sharp \neq \emptyset$.

Example 25. Consider the connected Lie group $H = SO(3)$ and its universal covering group $\tilde{H} = SU(2)$ with the Lie algebra $\mathfrak{h} = su(2)$. Using the invariant scalar product $\langle \varphi_1, \varphi_2 \rangle = -\frac{1}{2} \text{Tr } \varphi_1 \varphi_2$ on \mathfrak{h} we can identify the spaces \mathfrak{h} and \mathfrak{h}^* . It is evident that for $\varphi = \text{diag}(ib, -ib) \in su(2)$ with $b \in \mathbb{R}$ the isotropy group $\tilde{H}_\varphi = \{\text{diag}(e^{ia}, e^{-ia}), a \in \mathbb{R}\}$ and the isotropy algebra $\mathfrak{h}_\varphi = \{\text{diag}(ia, -ia), a \in \mathbb{R}\}$. In particular, \tilde{H}_φ contains the element $-E = \text{diag}(-1, -1) \in SU(2)$ of

the kernel of the covering homomorphism $\tilde{p} : SU(2) \rightarrow SO(3)$. Under our identification of \mathfrak{h} with \mathfrak{h}^* the map (43) $\tilde{\chi} : \exp(\mathfrak{h}_\varphi) \rightarrow \mathbb{S}^1$, $\text{diag}(e^{ia}, e^{-ia}) \mapsto e^{2\pi iab}$, is well defined if and only if $2\pi b \in \mathbb{Z}$. Since the group \tilde{H}_φ is connected, by Theorem 24, the orbit \mathcal{O}^φ is integral if and only if the number $2\pi b$ is integer. For such a covector φ the set \tilde{H}_φ^\sharp contains a unique element, the character $\tilde{\chi}$. But if the number $2\pi b$ is odd then $\tilde{\chi}(-E) = -1$. For such a covector φ the set H_φ^\sharp is empty while $\tilde{H}_\varphi^\sharp \neq \emptyset$. Indeed, in the opposite case for $\chi \in H_\varphi^\sharp$ we have by definition that $\chi \circ \tilde{p} \in \tilde{H}_\varphi^\sharp$. Therefore $\chi \circ \tilde{p} = \tilde{\chi}$. But $(\chi \circ \tilde{p})(-E) = 1$ while $\tilde{\chi}(-E) = -1$, the contradiction.

The character $\chi|_{H_\varphi^0}$, $\chi \in H_\varphi^\sharp$ on H_φ^0 admits another interpretation in terms of differential forms. Choose a contractible neighborhood $U \subset H_\varphi^0$ of the unity for which all intersections $U \cap hU$, $h \in H_\varphi^0$ are also (smoothly) contractible (one uses, for instance, a convex set relative to any invariant Riemannian structure on H_φ^0). The left H_φ^0 -invariant one-form θ_φ with $\theta_\varphi(e) = \varphi|_{\mathfrak{h}_\varphi}$ on the Lie group H_φ^0 is closed because, by the definition (2) of an isotropy algebra, $\varphi([\mathfrak{h}_\varphi, \mathfrak{h}_\varphi]) = 0$. Therefore a character on H_φ^0 determined by (43) exists if and only if the one-form θ_φ is integral, i.e. $\theta_\varphi \in H^1(H_\varphi^0, \mathbb{Z})$. In this case there exists a family of local functions $\{f_h : hU \rightarrow \mathbb{R}, h \in H_\varphi^0\}$ such that $df_h = \theta_\varphi$ on the open subset hU and $f_{h_1} - f_{h_2} \in \mathbb{Z}$ if $h_1U \cap h_2U \neq \emptyset$, $h_1, h_2 \in H_\varphi^0$. By H_φ^0 -invariance of the form θ_φ the family $\{f_h\}$ determines the character on H_φ^0 if $f_e(e) = 0$. Then $\chi|_{hU} = \exp(2\pi i f_h)$ and

$$f_h - (l_{h^{-1}}^* f_e + (1/2\pi i) \ln \chi(h)) \in \mathbb{Z} \quad \text{on } hU, \text{ where } l_{h^{-1}}(h') = h^{-1}h'. \quad (44)$$

In this case we will say that the character $\chi|_{H_\varphi^0}$ is associated with the (integer) form θ_φ .

Proposition 26. *Let σ be an arbitrary element of \mathfrak{g}^* , $\nu = \sigma|_{\mathfrak{n}}$ and $\tau = \sigma|_{\mathfrak{g}_\nu}$. There is a bijection between the sets $G_{\nu\tau}^\sharp$ and G_σ^\sharp , where $G_{\nu\tau}^\sharp$ denotes the set of all characters $\chi : G_{\nu\tau} \rightarrow \mathbb{S}^1 \subset \mathbb{C}$ such that $d\chi(e) = 2\pi i \cdot \tau|_{\mathfrak{g}_\nu}$. This bijection is induced by the restriction map $\chi \mapsto \chi|_{G_\sigma}$.*

Proof. Note that $\tau = \sigma|_{\mathfrak{g}_\nu}$ and $G_\sigma \subset G_{\nu\tau}$. But $\mathfrak{g}_\sigma \subset \mathfrak{g}_{\nu\tau}$, thus $\tau|_{\mathfrak{g}_\sigma} = \sigma|_{\mathfrak{g}_\sigma}$ and by the definition for any $\chi \in G_{\nu\tau}^\sharp$ we have $\chi|_{G_\sigma} \in G_\sigma^\sharp$. Therefore, in order to prove the proposition it is sufficient to show that each character $\psi \in G_\sigma^\sharp$ admits an extension to some character $\chi \in G_{\nu\tau}^\sharp$. This extension is unique because by (33) the groups $G_{\nu\tau}/G_{\nu\tau}^0$ and G_σ/G_σ^0 are isomorphic and, in particular, $\pi_{G_{\nu\tau}/G_{\nu\tau}^0}^* \simeq \pi_{G_\sigma/G_\sigma^0}^*$.

Consider now a character $\psi \in G_\sigma^\sharp$. Since G_σ^0 is a closed subgroup of $G_{\nu\tau}^0$, we can choose a contractible neighborhood $U \subset G_{\nu\tau}^0$ of the unity such that all intersections $U \cap hU$, $h \in G_\sigma^0$, are also contractible and, in addition, $U \cap hU \neq \emptyset$ if and only if $U \cap hU \cap G_\sigma^0 \neq \emptyset$ (there exists a local cross section $S \subset G_{\nu\tau}^0$ such that the map $(s, g) \mapsto sg$, $S \times G_\sigma^0 \rightarrow SG_\sigma^0 \subset G_{\nu\tau}^0$ is a diffeomorphism). Let θ_τ be a $G_{\nu\tau}^0$ -invariant one-form on the Lie group $G_{\nu\tau}^0$ such that $\theta_\tau(e) = \tau|_{\mathfrak{g}_{\nu\tau}}$. Since the form θ_τ is closed, there exists a function $f_e : U \rightarrow \mathbb{R}$ such that $df_e = \theta_\tau|_U$, $f_e(e) = 0$. Put $f_h = l_{h^{-1}}^* f_e + (1/2\pi i) \ln \psi(h)$ for all $h \in G_\sigma^0 \setminus \{e\}$.

Then $df_h = \theta_\tau|_{hU}$ because the one-form θ_τ is G_σ^0 -invariant. Thus the difference $f_{h_1} - f_{h_2}$ on the set $h_1U \cap h_2U \neq \emptyset$ is a real constant.

On the other hand, the co-vector $\sigma|_{\mathfrak{g}_\sigma}$ determines the left G_σ^0 -invariant one-form θ_σ on the group G_σ^0 . By the definition, θ_σ coincides with the restriction $\theta_\tau|_{G_\sigma^0}$ and $\psi|_{G_\sigma^0}$ is the character associated with this form $\theta_\sigma \in H^1(G_\sigma^0, \mathbb{Z})$. Therefore from (44) it follows that the difference $f_{h_1} - f_{h_2}$ is an integer constant on some nonempty subset $h_1U \cap h_2U \cap G_\sigma^0$ and, consequently, on the whole open set $h_1U \cap h_2U$. In other words, the function $\chi_e : G_\sigma^0 \cdot U \rightarrow \mathbb{S}^1$ given by $\chi_e|_{hU} = \exp(2\pi i f_h)$ is a well defined extension of the function $\psi|_{G_\sigma^0}$ onto the open set $G_\sigma^0 \cdot U \supset G_\sigma^0$.

Put $\mathbf{U} = G_\sigma^0 \cdot U$. Considering the family of functions $\{l_{g^{-1}}^* f_h\}$ (for which $d(l_{g^{-1}}^* f_h) = \theta_\tau|_{ghU}$), we obtain that $l_{g^{-1}}^* \chi_e = s\chi_e$ on $g\mathbf{U} \cap \mathbf{U} \neq \emptyset$, where s is some constant factor from \mathbb{S}^1 . But by (30) and (33) the space $G_{\nu\tau}^0/G_\sigma^0 \simeq \mathcal{A}_{\nu\tau}$ is contactable. Therefore there exists a character χ^0 on $G_{\nu\tau}^0$ which is an extension of $\psi|_{G_\sigma^0}$ and which is associated with the one-form θ_τ . Moreover, $\chi^0(\tilde{g}g\tilde{g}^{-1}) = \chi^0(g)$ for any (fixed) $\tilde{g} \in G_{\nu\tau}$ and for all $g \in G_{\nu\tau}^0$. Indeed, putting $F(g) = \chi^0(\tilde{g}g\tilde{g}^{-1})$ and $a_{\tilde{g}} : g \mapsto \tilde{g}g\tilde{g}^{-1}$ on $G_{\nu\tau}^0$, we obtain that

$$\frac{1}{2\pi i} \cdot \frac{dF}{F} = \frac{1}{2\pi i} \cdot a_{\tilde{g}}^* \left(\frac{d\chi^0}{\chi^0} \right) = a_{\tilde{g}}^* \theta_\tau = \theta_\tau = \frac{1}{2\pi i} \cdot \left(\frac{d\chi^0}{\chi^0} \right)$$

because by (28) $\langle \tau, \text{Ad}_{\tilde{g}} \xi \rangle = \langle \sigma, \text{Ad}_{\tilde{g}} \xi \rangle = \langle \tau, \xi \rangle$ for all $\xi \in \mathfrak{g}_{\nu\tau} \subset \mathfrak{g}_\nu$. Since $F(e) = \chi^0(e)$, we have $F = \chi^0$.

Taking into account that $G_{\nu\tau} = G_\sigma \cdot G_{\nu\tau}^0$, $G_{\nu\tau}^0 \cap G_\sigma = G_\sigma^0$ (see (31)) and $\psi = \chi^0|_{G_\sigma^0}$ we obtain that the map $\chi : G_{\nu\tau} \rightarrow \mathbb{S}^1$, $\chi(hg) = \psi(h)\chi^0(g)$, where $h \in G_\sigma$ and $g \in G_{\nu\tau}^0$, is well defined. This map determines a character on $G_{\nu\tau}$ because $\chi^0(hgh^{-1}) = \chi^0(g)$ for all $h \in G_\sigma \subset G_{\nu\tau}$ and $g \in G_{\nu\tau}^0$. Finally, χ belongs to the set $G_{\nu\tau}^\#$ because $\chi|_{G_{\nu\tau}^0} = \chi^0$. \square

Remark that Proposition 26 generalizes Rawnley's Proposition 2 from [1].

Proposition 27. *Let $\sigma \in \mathfrak{g}^*$ and $\nu = \sigma|_{\mathfrak{n}}$. An integrality of the coadjoint orbit $\mathcal{O}^\tau \subset \mathfrak{g}_\nu^*$ is a necessary condition for an integrality of the coadjoint orbit $\mathcal{O}^\sigma \subset \mathfrak{g}^*$. In general, this condition is not sufficient for an integrality of \mathcal{O}^σ .*

Proof. If the form ω on $\mathcal{O}^\sigma = \mathcal{O}^\sigma(G)$ is integral, then its restriction $\omega|_{\mathcal{O}^\sigma(G_\nu^\bullet)}$ to the submanifold $\mathcal{O}^\sigma(G_\nu^\bullet) \subset \mathcal{O}^\sigma(G)$ is also integral. Since by Proposition 19 the map $p_2 : \mathcal{O}^\sigma(G_\nu^\bullet) \rightarrow \mathcal{O}^\tau$ is a locally trivial fibering with a contractible fibre, the affine space $\mathcal{A}_{\nu\tau}$, the map $p_2^* : \Lambda^2(\mathcal{O}^\tau) \rightarrow \Lambda^2(\mathcal{O}^\sigma(G_\nu^\bullet))$ induces an isomorphism $H^2(\mathcal{O}^\tau, \mathbb{Z}) \rightarrow H^2(\mathcal{O}^\sigma(G_\nu^\bullet), \mathbb{Z})$. Since by Proposition 19 $p_2^*(\omega') = \omega|_{\mathcal{O}^\sigma(G_\nu^\bullet)}$, the canonical symplectic form ω' on \mathcal{O}^τ is integral and we obtain the first assertion of the proposition.

Remark also that the first assertion of the proposition follows also from Proposition 26. Indeed, we can assume without restricting the generality that G is a connected and simply connected Lie group with the Lie algebra \mathfrak{g} . By Theorem 24, the character set $G_\sigma^\#$ is not empty. By Proposition 26, $G_{\nu\tau}^\# \neq \emptyset$. Let \widetilde{G}_ν^0 be the universal covering group of the connected group G_ν^0 (with the

Lie algebra \mathfrak{g}_ν). By Theorem 24 the coadjoint orbit \mathcal{O}^τ is integral if and only if $(\widetilde{G}_\nu^0)^\sharp \neq \emptyset$. However, the covering homomorphism $\widetilde{G}_\nu^0 \rightarrow G_\nu^0$ induces the homomorphism $(\widetilde{G}_\nu^0)_\tau \rightarrow (G_\nu^0)_\tau$ and, consequently, $(\widetilde{G}_\nu^0)^\sharp \neq \emptyset$ if $(G_\nu^0)_\tau \neq \emptyset$. Therefore $(\widetilde{G}_\nu^0)^\sharp \neq \emptyset$, because $G_{\nu\tau}^\sharp \neq \emptyset$ and $(G_\nu^0)_\tau$ is an open subgroup of $G_{\nu\tau}$.

The second assertion of the proposition will be proven in the next subsection showing that the converse is not necessarily true. More precisely, we will construct the Lie algebra \mathfrak{g} which is a semi-direct product of some Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ and the Abelian ideal \mathfrak{n} and choose two coadjoint orbits $\mathcal{O}^\tau \subset \mathfrak{g}_\nu^*$ and $\mathcal{O}^\sigma \subset \mathfrak{g}^*$ which are not integral simultaneously while $\tau = \sigma|_{\mathfrak{g}_\nu}$. \square

Remark 28. All connected components of the reduced-group orbit $\mathcal{O}^\tau(G_\nu)$ are coadjoint orbits of the Lie algebras \mathfrak{g}_ν and \mathfrak{b}_ν (under the identification of \mathfrak{b}_ν^* with $(\mathfrak{n}_\nu^\sharp)^{\perp_\nu} \subset \mathfrak{g}_\nu^*$, see (34) and Lemma 2). These orbits are simultaneously either integral or non-integral.

2.8 Split extensions using Abelian algebras (semidirect products)

In this subsection we will finish the proof of Proposition 27. To this end we construct a connected and simply connected Lie group G and construct some coadjoint orbit $\mathcal{O}^\sigma(G)$ in \mathfrak{g}^* such that the set $(G_\nu^0)^\sharp$ is empty while the coadjoint orbit $\mathcal{O}^\tau = \mathcal{O}^\tau(G_\nu^0)$ is integral. Then by Proposition 26 the set $(G_\sigma)^\sharp$ is also empty, i.e. the orbit $\mathcal{O}^\sigma(G)$ is not integral.

Let K be a connected and simply connected Lie group with the Lie algebra \mathfrak{k} , and for k in K and f in the dual \mathfrak{k}^* of \mathfrak{k} , let $\text{Ad}_k^* f$ denote the coadjoint action of k on f . If δ is a representation of K on a real, finite-dimensional space V , let $d\delta$ be the corresponding tangent representation of \mathfrak{k} .

We can form the semi-direct product $G = K \rtimes_\delta V$ using the representation δ and identifying V with its group of translations. Then the Lie group G can be taken as $K \times V$ with multiplication $(k_1, v_1)(k_2, v_2) = (k_1 k_2, v_1 + k_1 \cdot v_2)$ for $k_j \in K$, $v_j \in V$ and the algebra $\mathfrak{g} = \mathfrak{k} \rtimes_{d\delta} V$ of G can be taken as $\mathfrak{k} \dot{+} V$ with the Lie bracket

$$[(\zeta_1, y_1), (\zeta_2, y_2)] = ([\zeta_1, \zeta_2], \zeta_1 \cdot y_2 - \zeta_2 \cdot y_1)$$

for ζ_j in \mathfrak{k} and y_j in V . Here $k_j \cdot v_j = \delta(k_j)(v_j)$ and $\zeta_j \cdot y_j = d\delta(\zeta_j)(y_j)$. Since $(k, v)^{-1} = (k^{-1}, -k^{-1} \cdot v)$, the adjoint action of G on \mathfrak{g} is given by

$$\text{Ad}_{(k,v)}(\zeta, y) = (\text{Ad}_k \zeta, k \cdot y - (\text{Ad}_k \zeta) \cdot v). \quad (45)$$

The dual \mathfrak{g}^* of \mathfrak{g} can be identified with $\mathfrak{k}^* \times V^*$ and the coadjoint action of G on \mathfrak{g}^* is given by

$$\langle \text{Ad}_{(k,v)}^*(f', \nu'), (\zeta, y) \rangle = \langle \text{Ad}_k^* f', \zeta \rangle - \langle \nu', (\text{Ad}_k \zeta) \cdot v \rangle + \langle k^* \cdot \nu', y \rangle, \quad (46)$$

where f' is in \mathfrak{k}^* and ν' in V^* ; also, by the definition, $\langle k^* \cdot \nu', y \rangle = \langle \nu', k \cdot y \rangle$. Note that all above formulas for semidirect products are standard up to notation (see for example, [1, §2] or [2, §2]).

The subgroup $N = \{(e, v) \in G, v \in V\}$ is a normal commutative subgroup of G with the Lie algebra $\mathfrak{n} = \{(0, y), y \in V\}$. The Ad-action (45) of G on \mathfrak{n} induces the action of G on \mathfrak{n}^* : $(k, v) \cdot \nu = k^* \cdot \nu$. Therefore for $\nu \in \mathfrak{n}^* = V^*$

$$G_\nu = \{(k, v) \in K \times_\delta V, k^* \cdot \nu = \nu\} \quad \text{and} \quad \mathfrak{g}_\nu = \{(\zeta, y) \in \mathfrak{k} \times_{d\delta} V, \zeta^* \cdot \nu = 0\},$$

that is $G_\nu = K_\nu \times_\delta V$ and $\mathfrak{g}_\nu = \mathfrak{k}_\nu \times_{d\delta} V$, where K_ν is the isotropy group of $\nu \in V^*$ with Lie algebra $\mathfrak{k}_\nu = \{\zeta \in \mathfrak{k} : \zeta^* \cdot \nu = 0\}$. It is easy to verify using (46) that G_ν is the stabilizer of the affine subspace $\mathcal{A}_\nu = \{(f, \nu), f \in \mathfrak{k}^*\}$.

Putting $\sigma = (f, \nu)$ and $\tau = \sigma|_{\mathfrak{g}_\nu}$, we obtain that $\tau = (\varphi, \nu)$, where $\varphi = f|_{\mathfrak{k}_\nu}$. By definition (28), the Lie group

$$\begin{aligned} G_{\nu\tau} &= \{(k, v) \in K_\nu \times_\delta V : (\text{Ad}_{(k,v)}^*(f, \nu))|_{\mathfrak{g}_\nu} = (f, \nu)|_{\mathfrak{g}_\nu}\} \\ &= \{(k, v) \in K_\nu \times_\delta V : (\text{Ad}_k^* f)|_{\mathfrak{k}_\nu} = f|_{\mathfrak{k}_\nu}\}, \end{aligned} \quad (47)$$

because $k^* \cdot \nu = \nu$, $\zeta^* \cdot \nu = 0$ and $\text{Ad}_k \zeta \in \mathfrak{k}_\nu$ for all $k \in K_\nu$, $\zeta \in \mathfrak{k}_\nu$. In particular, $\zeta \cdot v \in \ker \nu$ if $v \in \ker \nu$. In other words, $G_{\nu\tau} = K_{\nu\varphi} \times_\delta V$, where

$$K_{\nu\varphi} = \{k \in K_\nu : \langle \varphi, \text{Ad}_k \zeta \rangle = \langle \varphi, \zeta \rangle, \forall \zeta \in \mathfrak{k}_\nu\}. \quad (48)$$

Suppose now that the group K_ν is connected. Let \widetilde{K}_ν be its universal covering group with the covering homomorphism $\tilde{p}_\nu : \widetilde{K}_\nu \rightarrow K_\nu$. Then $\widetilde{G}_\nu = \widetilde{K}_\nu \times_{\tilde{\delta}} V$ is the universal covering group of G_ν , where the semi-direct product is determined by the representation $\tilde{\delta} = \delta \circ \tilde{p}_\nu$. Since the group G_ν is connected, the coadjoint orbit $\mathcal{O}^\tau \subset \mathfrak{g}_\nu^*$ is the orbit $\mathcal{O}^\tau(G_\nu) \simeq G_\nu/G_{\nu\tau}$. But this orbit is also an orbit of \widetilde{G}_ν , that is $\mathcal{O}^\tau \simeq \widetilde{G}_\nu/(\widetilde{G}_\nu)_\tau$. It is easy to verify using expressions similar to (46) and (47) that $(\widetilde{G}_\nu)_\tau = (\widetilde{K}_\nu)_\varphi \times_{\tilde{\delta}} V$, where $(\widetilde{K}_\nu)_\varphi = \tilde{p}_\nu^{-1}(K_{\nu\varphi})$.

Now we will establish bijections between the sets $(\widetilde{G}_\nu)_\tau^\sharp$ and $(\widetilde{K}_\nu)_\varphi^\sharp$, $G_{\nu\tau}^\sharp$ and $K_{\nu\varphi}^\sharp$ using Rawnsley's formula [1, Eq.(2)]. Indeed, for any character $\psi \in K_{\nu\varphi}^\sharp$ the function $\chi(k, v) = \psi(k) \exp(2\pi i \langle \nu, v \rangle)$ on the group $G_{\nu\tau} = K_{\nu\varphi} \times_\delta V$ is a character because $k^* \cdot \nu = \nu$. By (43) this character χ is a unique extension of ψ such that $\chi \in G_{\nu\tau}^\sharp$. Thus there is a bijection between $G_{\nu\tau}^\sharp$ and $K_{\nu\varphi}^\sharp$. Using similar arguments one establishes a bijection between $(\widetilde{G}_\nu)_\tau^\sharp$ and $(\widetilde{K}_\nu)_\varphi^\sharp$ because $(\widetilde{G}_\nu)_\tau = (\widetilde{K}_\nu)_\varphi \times_{\tilde{\delta}} V$. By Proposition 26 and Theorem 24, the orbit $\mathcal{O}^\tau(G_\nu)$ is integral and the orbit $\mathcal{O}^\sigma(G)$ is not integral if and only if $(\widetilde{G}_\nu)_\tau^\sharp \neq \emptyset$ and $G_{\nu\tau}^\sharp = \emptyset$ or, equivalently, $(\widetilde{K}_\nu)_\varphi^\sharp \neq \emptyset$ and $K_{\nu\varphi}^\sharp = \emptyset$. Remark also that the coadjoint orbit \mathcal{O}^φ in \mathfrak{k}_ν^* passing through the point φ is isomorphic to the homogeneous spaces $K_\nu/K_{\nu\varphi}$ and $\widetilde{K}_\nu/(\widetilde{K}_\nu)_\varphi$ simultaneously.

Example 29. Now we consider a connected and simply connected algebraic Lie group $K = SU(3)$ and its representation $\delta : SU(3) \rightarrow \text{End}(\mathfrak{gl}(3, \mathbb{C}))$, $\delta(k)(v) = kvk^t$, in the space V of all complex matrices of order three (considered as a real space). Here k^t denotes the transpose of a matrix $k \in SU(3)$. Using the nondegenerate 2-form $\langle v_1, v_2 \rangle = \text{Re Tr } v_1 v_2$ on V we identify the space V with dual V^* . Under this identification the dual representation δ^* is given by $\delta^*(k)(v) = k^t v k$. It is clear that for the covector $\nu = E$, where E is the identity

matrix, the isotropy group $H = K_\nu$ is the group $SO(3) = SO(3, \mathbb{C}) \cap SU(3)$. Its universal covering group $\widetilde{H} = \widetilde{K}_\nu$ is isomorphic to $SU(2)$. But as we showed above (see Example 25) there is an element $\varphi \in \mathfrak{h}^* = \mathfrak{k}_\nu^*$ such that $\widetilde{H}_\varphi^\# = (\widetilde{K}_\nu)_\varphi^\# \neq \emptyset$ while $H_\varphi^\# = K_{\nu\varphi}^\# = \emptyset$. Thus, as we proved above, $(\widetilde{G}_\nu)_\tau^\# \neq \emptyset$ while $G_{\nu\tau}^\# = \emptyset$, that is the condition of Proposition 27 is not sufficient.

Remark 30. The Rawnsley's assertion [1, Corollary to Prop.2] claims that an arbitrary coadjoint orbit \mathcal{O}^σ in the dual space \mathfrak{g}^* of the semidirect product \mathfrak{g} is integral if and only if the coadjoint orbit $\mathcal{O}^\varphi \simeq K_\nu/K_{\nu\varphi}$ in \mathfrak{k}_ν^* is integral. From Example 29 it follows that in general this assertion is not true. The gap in the proof of this assertion [1, Corollary to Prop.2] consists in an illegal using of Kostant's theorem 24 (with the not necessary simply connected group $H = K_\nu$).

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