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Conditional Invariance and Exact Solutions of a Nonlinear System

Pavlo MYRONYK and Natalia BUBENCHIKOVA

Volyn University, 9 Potapov Street, Lutsk, Ukraïna

Abstract

The Lie and Q-conditional invariance of one nonlinear system of PDEs of the third-order is searched. The ansatze have been built which reduce the PDEs system to ODEs. The classes of exact solutions of the given system are obtained. The relation between the Korteweg-de Vries equation and Harry-Dym equation has been established.

Let us consider the third-order nonlinear system of partial differential equations (PDEs):

$$\begin{aligned} &u_0 + u_{111} = 6(uv^2)_1 + au_1 + bv_1, \\ &v_0 + v_{111} = 6v^2 v_1 + av_1, \end{aligned} \qquad (1)$$

where a, b are constants, $x_0 = t, u_0 = \partial u/\partial x_0, u_1 = \partial u/\partial x_1, u_{111} = \partial^3 u/\partial {x_0}^3$.

When $a = 0$, the second equation of system (1) is the Korteweg-de Vries equation. Let us seek a maximal invariance algebra (MIA) of system (1).

Theorem 1 *MIA of system (1) is given by the following set of basis operators:*

1) $P_0 = \partial_0, \quad P_1 = \partial_1,$

$D_1 = 3x_0\partial_o + (x_1 - 2ax_0)\partial_1 - u\partial_u, \quad D_2 = v\partial_v \quad if \quad b = 0;$

2) $P_0,\ P_1,\ D = D_1 + D_2 \quad if \quad b \neq 0.$

The proof of this theorem can be carried out by means of the Lie algorithm [1].

The Lie ansatze and reduced ODEs for system (1) with $b = 0$ are given as:

1. $\omega = {x_0}^{-1/3}(x_1 + ax_0), \quad u = {x_0}^{\alpha/3}\varphi(\omega), \quad v = {x_0}^{-1/3}\psi(\omega),$

$$\varphi''' + \frac{\alpha}{3}\omega\varphi' - 6(\varphi\psi^2)' = 0,$$

$$\psi''' - \frac{1}{3}\psi - \frac{1}{3}\omega\psi' - 2(\psi^3)' = 0;$$

2. $\omega = {x_0}^{-1/3}x_1, \quad u = {x_1}^{a}\varphi(\omega), \quad v = {x_1}^{-1}\psi(\omega),$

$$\omega^3\varphi''' + 3\alpha\omega^2\varphi'' + \left(\delta\omega - \frac{1}{3}\omega^4\right)\varphi' + \gamma\varphi - 6\psi^2(\alpha\varphi + \omega\varphi') + 12\psi\varphi(\psi - \omega\psi') = 0,$$

$$\omega^3\psi''' - 3\omega^2\psi'' + \left(6\omega - \frac{1}{3}\omega^4\right)\psi' - 6\psi - 6\psi^2(\omega\psi' - \psi) = 0;$$

3. $\omega = \alpha x_0 - \alpha_0 x_1, \quad u = \exp(\theta x_1)\varphi(\omega), \quad v = \psi(\omega),$
$\alpha_0{}^3\varphi''' - 3\alpha_0{}^2\theta\varphi'' + (3\alpha_0\theta^2 - \alpha_1)\varphi' + (\theta^3 - 6\theta\psi^2 - a\theta)\varphi - 6\alpha_0(\varphi\psi^2)' = 0,$
$\alpha_0{}^3\psi''' - 2\alpha_0(\psi^3)' - (\alpha_1 + a\alpha_0)\psi' = 0,$

where $\gamma = \alpha(\alpha-1)(\alpha-2),\ \delta = 3\alpha(\alpha-1),\ \theta = \alpha_0/\alpha_1.$

The Lie ansatze and reduced ODEs with $b \neq 0$ are:

1. $\omega = x_0{}^{-1/3}x_1 + ax_0{}^{2/3}, \quad u = x_0{}^{1/3}\varphi(\omega), \quad v = x_0{}^{-1/3}\psi(\omega),$
$\varphi''' + \frac{1}{3}\varphi - \frac{1}{3}\omega\varphi' - 6(\varphi\psi^2)' - b\psi' = 0,$
$\psi''' + \frac{1}{3}\psi - \frac{1}{3}\omega\psi' - 6\psi^2\psi' = 0;$

2. $\omega = \alpha_1 x_0 - \alpha_0 x_1, \quad u = \varphi(\omega), \quad v = \psi(\omega),$
$\alpha_0{}^3\varphi''' - (\alpha_1 + a\alpha_0 + 6\alpha_0\psi^2)\varphi' - (12\alpha_0\varphi\psi + \alpha_0 b)\psi' = 0,$
$\alpha_0{}^3\psi''' - (\alpha_1 + a\alpha_0)\psi' - 2\alpha_0(\psi^3)' = 0.$

In [3], the relation was established between system (1) and the system

$$\begin{aligned} u_{111} + u_0 &= \left(3\frac{v_0}{v_1} - 2a\right)u_1 + b, \\ v_{111} + v_0 &= \frac{3}{2}\frac{v_{11}{}^2}{v_1} + av_1. \end{aligned} \tag{2}$$

Theorem 2 *The maximal invariant Lie algebra for system (2) is* $< P_0, P_1, D, T_1, T_2, T_3, T_4, T_5 >$, *where*

$$P_0 = \partial_0, \quad P_1 = \partial_1, \quad D = 3x_0\partial_0 + (x_1 - 2ax_0)\partial_1 - bx_0\partial_u,$$
$$T_1 = v^2\partial_v, \quad T_2 = v\partial_v, \quad T_3 = \partial_v, \quad T_4 = (u + bx_0)\partial_u, \quad T_5 = \partial_u.$$

Note 1 *In [4], the Lie invariance is observed of the second equation of system (2) if* $a = 0$. *This gave the opportunity to obtain the new rational solution of the modified Korteweg-de Vries equation:*

$$v = (x_1 + c)^{-1},$$

$$v = \frac{3(x_1 + c_0)^2}{12x_0 + (x_1 + c_0) + 12c_1} - \frac{1}{x_1 + c_0}.$$

Analogous results for $c = c_0 = 0$ *were obtained in [3]. Also we obtain the soliton-type solution*

$$v = 6\sqrt{-\alpha}\cosh^{-2}\left(\frac{\sqrt{-\alpha}}{2}(\alpha x_0 - x_1)\right).$$

Here we give some results of the investigation of Q-conditional symmetry [1] of system (2) in the class of operators

$$Q = \partial_1 + \eta\partial_v, \tag{3}$$

where $\partial_1 = \partial/\partial x_1,\ \partial_v = \partial/\partial_v, \eta = \eta(x_0, u, v)$.

Theorem 3 *System (2) is Q-conditionally invariant under operator (3) if the function $\eta(x_0, u, v)$ satisfies the equation*

$$\eta_0 + \eta^3\eta_{vvv} - b\eta_u = 0. \tag{4}$$

Note 2 *When $b = 0$, system (2) is Q–conditionally invariant under the operator (3) if the function η satisfies the equation of Harry–Dym. Lie symmetry of equation (4) for $b = 0$ was obtained in [2].*

The main result on group classification of equation (4) with $b \neq 0$ is:

Theorem 4 *The maximal invariance algebra (in the Lie sense) for equation (4) is the algebra, the basis operators of which are set by the coordinates:*

$$\xi^0 = \alpha_1 x_0 + \alpha_2,$$
$$\xi^1 = \alpha_3 {x_1}^2 + \alpha_4 x_1 + \alpha_5,$$
$$\xi^2 = \alpha_6 - b\alpha_1 x_0,$$
$$\overline{\eta} = \left(2\alpha_3 x_1 + \alpha_4 - \frac{1}{3}\alpha_1\right)\eta,$$

where $\alpha_i = \alpha(z),\ i = \overline{1,6},\ z = x_2 + bx_0,\ x_2 \equiv u,\ x_1 \equiv v.$

With the help of operator (3) we obtained such exact solutions of system (2)

$$u(x_0, x_1) = c - bx_0,$$
$$v(x_0, x_1) = c_1 \exp\left(\left(ac - \frac{3}{2}c^3\right) + cx_1\right).$$

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