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AN APPROACH TO EXACT SOLUTIONS FOR GRAVITATIONAL FIELD OF SPHERICAL SYMMETRY WITH CURVARURE AND TORSION

BY

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ABSTRACT: The compatibility of the gravitational field of spherical symmetry with the torsion of the U_4 spacetime is discussed. It is shown that exact solutions different from Schwarzschild spacetime can be found for the vacuum.

1. INTRODUCTION

The problem of the existence of exact solutions of the gravitational field equations in Riemann-Cartan spaces has been in our view due to necessity of explaining the physical nature of torsion as well as its implications regarding the characteristics of the gravitational field [1,2].

In the case of the spherical symmetry, the gravitational field in General Theory of Relativity is described by Schwarzschild metric. The open problem is whether, there is also another solution for the spherical symmetry in the case of vacuum, when one works in spaces with curvature and torsion.

In the following we shall deal with the study of the field equations of the U_4 theory, for the case of the spherical symmetry aiming at clarifying its compatibility with torsion.

2. THE CHOICE OF THE CONTORTION

Let the metric of spherical symmetry be [3]:

$$ds^2 = e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - e^{-\lambda}dt^2 \tag{1}$$

where λ is a function of r and t.
In the case of the U_4 theory, the connection is:

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + K_{jk}{}^i \tag{2}$$

Here $\left\{ {i \atop jk} \right\}$ represents the Christoffel connection for the metric (1) and $K_{jk}{}^i$, the contortion tensor suggested by the relations:

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$K_{11}{}^{1}$= -f, $K_{14}{}^{1}$= -h, $K_{44}{}^{1}$= -m
$K_{33}{}^{2}$= -u, $K_{32}{}^{3}$= -v (3)

These quantities enable the determination of the components of the Einstein tensor asnd the writing of the field equations of the U_4 theory [3]:

$$G^{ij} - (\nabla_k + K_{\alpha k}{}^{\alpha})(K^{ijk} + g^{ij}K^{mk}m - g^{ik}K^{mj}m) = -\chi T^{ij} \quad (4)$$

For the metric (1) with the contortion (3), the elements of the left side have been determined using the REDUCE algebraic computing system. For the vacuum, the following field equations have been got:

I $$\frac{2e^{-\lambda}\cot(\theta)v}{r^2} - \frac{e^{-\lambda}\cos(\theta)u}{2r^2\sin^3\theta} + f'e^{-2\lambda} - \dot{h} - \frac{\dot{\lambda}h}{2} - \frac{\lambda' m}{2} - \lambda' f e^{-2\lambda}$$
$$-\frac{\lambda' e^{-2\lambda}}{r} - \frac{m'}{2} + \frac{fm}{2} + 2h^2 - \frac{m}{r} - 2e^{-\lambda}\frac{v^2}{r^2} - \frac{e^{-\lambda}}{r^2} + \frac{e^{-\lambda}uv}{r^2\sin^2\theta}$$
$$-4f^2e^{-2\lambda} + \frac{2}{r}fe^{-2\lambda} + \frac{e^{-2\lambda}}{r^2} = 0$$

II $$v' - fv = 0$$

III $$-\dot{f} + h' - \frac{\dot{\lambda}}{r} + \lambda' h - hme^{2\lambda} + fh = 0$$

IV $$(fr+1)v=0$$

V $$\frac{e^{-\lambda}\cot(\theta)v}{r^2} - \frac{e^{-\lambda}\cos(\theta)u}{2r^2\sin^3\theta} + f'e^{-\lambda} - \dot{h} - \frac{\ddot{\lambda}}{2} - \frac{(\dot{\lambda})^2}{2} - \frac{3}{2}\dot{\lambda}h$$
$$-\frac{\lambda'' e^{-2\lambda}}{2} + \frac{(\dot{\lambda})^2 e^{-2\lambda}}{2} - \frac{\lambda' m}{2} - \lambda' fe^{-2\lambda} - \frac{\lambda'}{r}e^{-2\lambda} - \frac{m'}{2} + \frac{fm}{2}$$
$$+2h^2 - \frac{m}{r} - e^{-\lambda}\frac{v^2}{r^2} - 2f^2e^{-2\lambda} + \frac{2f^{-2\lambda}}{r} = 0$$

VI $$hve^{\lambda} = 0$$

VII $$\frac{e^{-\lambda}\cos(\theta)u}{2r^2\sin^3\theta} - f'e^{-2\lambda} + \dot{h} + \frac{\ddot{\lambda}}{2} + \frac{(\dot{\lambda})^2}{2} + \frac{3}{2}\dot{\lambda}h + \frac{\lambda''}{2}e^{-2\lambda}$$
$$-\frac{(\lambda')^2e^{-2\lambda}}{2} + \frac{\lambda' m}{2} + \lambda' fe^{-2\lambda} + \frac{\lambda'}{r}e^{-2\lambda} + \frac{m'}{2} - \frac{fm}{2} - 2h^2 + \frac{m}{r} +$$
$$+2e^{-\lambda}\frac{v^2}{r^2} + \frac{3e^{-\lambda}uv}{r^2\sin^2\theta} + 2f^2e^{-2\lambda} - \frac{2fe^{-2\lambda}}{r} = 0$$

VIII $$2\dot{f} + e^{2\lambda}\dot{\lambda}m + \dot{\lambda}h - 2fh = 0 \quad (5)$$

IX $$\dot{v} - vh = 0$$

X $$\frac{2e^{\lambda}\cot(\theta)v}{r^2}-\frac{e^{\lambda}\cos(\theta)u}{2r^2\sin^3(\theta)}+f'-\frac{e^{2\lambda}\dot{\lambda}h}{2}-\frac{\lambda' me^{2\lambda}}{2}-\lambda' f-\frac{\lambda'}{r}-$$
$$-\frac{m'}{2}e^{2\lambda}+\frac{3fm}{2}e^{2\lambda}+h^2e^{2\lambda}-\frac{m}{r}e^{2\lambda}-2e^{\lambda}\frac{v^2}{r^2}-\frac{e^{\lambda}}{r^2}+\frac{e^{\lambda}uv}{r^2\sin^3\theta}$$
$$-2f^2+\frac{2f}{r}+\frac{1}{r^2}=0$$

The solving of the system (5) implies more possibilities of selecting the contortions, which result were not satisfactory. A choice which does not lead to a contradiction is v = 0. Moreover, u = 0 is also chosen taking into account the fact that u and v are linked to the angular side. For a much more simplification of the field equations h = 0 is chosen too.

If $\dot{\lambda} \neq 0$, from equations III and VIII we get:

$$\dot{f}=-\frac{\dot{\lambda}}{r} \tag{6}$$

and

$$\dot{f}=-\frac{\dot{\lambda}me^{2\lambda}}{2} \tag{7}$$

From I and X, the equation results:

$$fm+2e^{-2\lambda}f^2=0 \tag{8}$$

Thus for m, when f≠0, we get:

$$m=-2fe^{-2\lambda} \tag{9}$$

Replacing (9) in (7), for f the relation is found:f= -1/r which makes impossible the choice $\dot{\lambda}\neq 0$, thus $\dot{f}\neq 0$.

Considering then $\dot{\lambda}=0,\quad \dot{f}=0$, the system (5) and the relation (9) lead to the equation system:

$$-\frac{\lambda''}{2}+\frac{(\lambda')^2}{2}-\frac{\lambda'}{r}+2f'-2f\lambda'-3f^2+\frac{4f}{r}=0 \tag{10}$$
$$2f'-2f\lambda'-\frac{\lambda'}{r}-5f^2+\frac{4f}{r}-\frac{e^{\lambda}}{r^2}+\frac{1}{r^2}=0$$

The equation system (10) enable the finding out of the function λ which defines the metric and of the contortion f.

4. CONCLUSION

In the selected situation, the metric does not depend on time, thus the field is stationary. This makes the torsion be a quantity which depends only on distance.

The finding out of the solutions of the systems (10) is difficult. But it can be say that, in this case, the spherical symmetry is compatible with the torsion, and thus another solution may exist for vacuum, different from Schwarzschild metric. It is understood that for f = 0, thus in the case of canceling the torsion, the system (10) has as solution Schwarzschild metric.

In the system of equations (5) a separation of the variable t and r could not be done as in the case of the space with absolute parallelism [4] which shows that in the case of the space with curvature and torsion, some aspects connected to the properties of the gravitational field are modified.

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