The Superposition Principle for the Lie Type First Order PDEs

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Abstract

In this paper the superposition principle for the Lie type first order partial differential equations is constructed. Some examples of applications of that principle are given.

Résumé

Dans cet article, le principe de superposition de Lie pour les équations aux dérivées partielles du premier ordre est étudié. Quelques illustrations de ce principe sont présentées.

1 Introduction

The starting point for our considerations is the theorem formulated by Sophus Lie [1, 2] that addresses the problem of conditions under which systems of ODEs

$$\frac{du^\alpha}{dx} = \varphi^\alpha(x,u), \quad \alpha = 1,\dots,m \quad u^\alpha, \varphi^\alpha \in \mathbb{R} \quad or \quad \mathbb{C}. \tag{1.1}$$

admit use of superposition principle. By superposition principle for (1.1), one means the formula

$$u^\alpha(t) = J^\alpha(u_1(t),\dots,u_k(t),c_1,\dots,c_N), \quad \alpha = 1,\dots,m \tag{1.2}$$

expressing functionally any solution $u^\alpha(t)$ of system (1.1) in terms of a finite number k of particular solutions $u_1 = u_1(t),\dots,u_k = u_k(t)$ and a finite number N of arbitrary constant parameters $c_1, c_2, \dots, c_N$. A replacement of a given set of particular solutions $u_1 = u_1(t),\dots,u_k = u_k(t)$ with another one $u'_1 = u'_1(t),\dots,u'_k = u'_k(t)$ preserves the form of a mapping J^α, $\alpha = 1,\dots,m$ and entails only an exchange of constants $c_1,\dots,c_N$ for $c'_1,\dots,c'_N$.

Lie's theorem states that system of equations (1.1) admits a superposition principle (in the above sense) if and only if the right-hand side of (1.1) can be (possibly after a change of variables) presented in a form

$$\varphi^\alpha(x,u)\frac{\partial}{\partial u^\alpha} = a^i(x)X_i,$$

where the vector fields

$$X_i = y_i^\alpha(u)\frac{\partial}{\partial u^\alpha}, \quad i = 1,\dots,r$$

generate a finite-dimensional Lie algebra

$$[X_i, X_j] = C_{ij}^a X_a, \quad 1 \le i,j,a \le r.$$

The number k of particular solutions in expression (1.2) satisfies the following condition

$$kN \ge r.$$

The objective of this paper is an adaptation of Lie's theorem to nonlinear PDEs of the form

$$\frac{\partial u^\alpha}{\partial x^\mu}(x) = \varphi_\mu^\alpha(x,u), \quad \alpha = 1,\dots,m \quad \mu = 1,\dots,n. \tag{1.3}$$

in which all derivatives of unknown functions are expressible in terms of some functions of independent and dependent variables only. We demonstrate that Lie's theorem can be modified to comprise this case. A similar attempt was undertaken by E. Vessiot earlier this century [3, 4]. In our analysis we are using the apparatus of modern differential geometry (see [5, 6]). In its language, equations (1.3) can be written as equations on cross-section parallel with respect to connection satisfying zero curvature conditions (Section 3). The superposition formula (1.2) can also be expressed in terms of geometric invariants (as we show in Section 2). This approach provides us with explicit rules for constructing superposition formulae for PDEs of the type (1.3). We illustrate them with a simple example (Section 4). Interestingly, PDEs of the type (1.3) which admit superposition principle can be used, as differential constraints, in the process of constructing Auto-Bäcklund transformations for certain classes of PDEs. We demonstrate this connection using an example of the KdV equation.

2 The Superposition Formula for the Parallel Cross-Sections

Let us fix a principle bundle $(P, M, G, \pi\colon P \to M)$ in bundle space P and base space M. By G we denote here the structural group of the bundle and $\pi\colon P \to M$ is the canonical projection of P on the base space M.

Given the action

$$L\colon G \to \mathrm{Diff}\, F \tag{2.1}$$

of the structural group G on some manifold F one can associate the principle bundle $(P, M, G, \pi\colon P \to M)$ with the fibre bundle

$$\begin{array}{ccc} F & \longrightarrow & B \\ & & \Big\downarrow {\scriptstyle \pi_B} \\ & & M. \end{array} \tag{2.2}$$

For this bundle the manifold F is the typical fibre and the bundle space B is defined as the space of the orbits of the action

$$P \times F \ni (p, f) \to (pR(g), L(g^{-1})f) \tag{2.3}$$

where $g \in G$ and R denotes the right-side action of G on P.

One can treat the cross-sections of the bundle (2.2) as the maps $\varphi\colon P \to F$ which satisfy the following equivalence condition

$$\begin{array}{ccc} P & \xrightarrow{\ \varphi\ } & F \\ {\scriptstyle R(g)}\Big\downarrow & & \Big\downarrow{\scriptstyle L(g)} \\ P & \xrightarrow[\ \varphi\ ]{} & F \end{array} \ \forall g \in G. \tag{2.4}$$

Below, we shall denote the space of all sections of the bundle (2.2) by $\mathcal{F}(P, F, L)$, and the space of the connections defined on the principal bundle P by $\mathcal{L}(P, G)$. For each $\gamma \in \mathcal{L}(P, G)$, cross-section $\varphi \in \mathcal{F}(P, F, L)$, and vector field $X \in C^\infty(TM)$, one defines the covariant derivative by

$$\nabla_X \varphi = d\varphi(H^\gamma X), \tag{2.5}$$

where $H^\gamma X \in C^\infty(TP)$ is the horizontal lift of the vector field X taken with respect to the connection γ. The following diagram

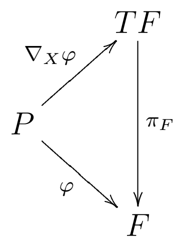
$$\begin{array}{ccc} P & \xrightarrow{\ \nabla_X\varphi\ } & TF \\ {\scriptstyle R(g)}\Big\downarrow & & \Big\downarrow{\scriptstyle L(g)} \\ P & \xrightarrow{\ \nabla_X\varphi\ } & TF \end{array} \ \forall g \in G. \tag{2.6}$$

commutes, since $d\varphi\colon TP \to TF$ and γ have the G-equivariance property. Therefore, the covariant derivative $\nabla_X\varphi\colon P \to TF$ of the cross-section φ with respect to the vector field X is a cross-section of the fibre bundle

$$\begin{array}{ccc} TF & \longrightarrow & VTB \\ & & \Big\downarrow {\scriptstyle \pi_B} \\ & & M \end{array} \tag{2.7}$$

which has TF as the typical fibre. Here, VTB denotes tangent bundle vertical to the bundle $B \to M$.

The following diagram takes place



(2.8)

holds.

Summarizing the above, for any fixed connection γ one has the map

$$\nabla\colon C^\infty(TM) \times \mathcal{F}(P, F, L) \to \mathcal{F}(P, TF, dL)$$

which, for given $\varphi \in \mathcal{F}(P, F, L)$, is a $C^\infty(M)$- linear map of the module of the vector fields $C^\infty(TM)$, in the module of the cross-sections which are covariant derivatives of the cross-section φ. From (2.8) it follows that the cross-sections $\nabla_X\varphi$, $X \in C^\infty(TM)$ form a module.

Now, if one takes any two fibre bundles $\pi_i\colon B_i \to M$, $i = 1, 2$ on the manifold M, with the typical fibre spaces $(F_i, L_i\colon G \to \mathrm{Diff}\, F_i)$, $i = 1, 2$ and a G-morphism $\Phi\colon F_1 \to F_2$ between them, then one has

$$\nabla_X \Phi \circ \varphi = d\Phi \circ \nabla_X \varphi \tag{2.9}$$

for each $X \in C^\infty(TM)$ and $\varphi \in \mathcal{F}(M, F_1, L_1)$. The property (2.9) shows the functorial character of the covariant derivative. It allows one to express the covariant derivative of the cross-section $\Phi \circ \varphi \in \mathcal{F}(M, F_2, L_2)$ by the covariant derivative of the cross section φ.

Let us consider a finite number N of G-spaces $(F_i, L_i\colon G \to \mathrm{Diff}\, F_i)$, $i = 1, \ldots, N$. Then one can define the product of the G-spaces $(F_1 \times \cdots \times F_N, L_1 \times \cdots \times L_N\colon G \to \mathrm{Diff}(F_1 \times \cdots \times F_N))$ and consider the G-morphism $\Phi\colon F_1 \times \ldots F_N \to F_1 \times \ldots F_N$ of those spaces

$$\Phi \circ (L_1(g) \times \cdots \times L_N(g)) = (L_1(g) \times \cdots \times L_N(g)) \circ \Phi, \quad \forall g \in G \tag{2.10}$$

Making use of Φ and projections $\mathrm{pr}_j\colon L_1 \times \cdots \times L_N \to L_j$ on the j-th component of the product one can construct the maps

$$\Phi_i^*(\varphi_1, \ldots, \varphi_N) = \mathrm{pr}_j \circ \Phi \circ (\varphi_1 \times \cdots \times \varphi_N) \tag{2.11}$$

from $\mathcal{F}(P, F_1, L_1) \times \cdots \times \mathcal{F}(P, F_N, L_N)$ into $\mathcal{F}(P, F_i, L_i)$. Here, one assumes that $\varphi_k \in \mathcal{F}(P, F_k, L_k)$. So, the formula (2.11) relates the superposition rule for the cross-sections $\varphi_1, \ldots, \varphi_N$ to each G-morphism Φ.

By definition ([3, 4]), the cross-section $\varphi \in \mathcal{F}(M, F, L)$ is parallel to the vector field $X \in C^\infty(TM)$ if

$$\nabla_X \varphi = 0. \tag{2.12}$$

Let us prove the following proposition.

Proposition 1. *If $\varphi_i \in \mathcal{F}(M, F_i, L_i)$, $i = 1, \ldots, N$ are the X-parallel sections then for any G-morphism*

$$\Phi\colon L_1 \times \ldots L_N \to L_1 \times \cdots \times L_N$$

we have

$$\nabla_X \Phi_j^*(\varphi_1, \ldots, \varphi_N) = 0, \quad \textit{where} \quad j = 1, \ldots, N \tag{2.13}$$

Proof. Let us consider the decomposition

$$T(F_1 \times \cdots \times F_N) = T_1 + \cdots + T_N \tag{2.14}$$

of the tangent bundle $T(F_1 \times \cdots \times F_N)$, consistent with the product structure of the manifold $F_1 \times \cdots \times F_N$. Let

$$(\pi_i)_p \colon T_p(F_1 \times \cdots \times F_N) \to (T_i)_p$$

be the projection defined by the decomposition (2.14). From equations (2.9) and (2.11) one has

$$\begin{aligned}
\nabla_X \Phi_j^*(\varphi_1, \ldots, \varphi_N) &= \nabla_X(\mathrm{pr}_j \circ \Phi \circ (\varphi_1 \times \cdots \times \varphi_N)) \\
&= d(\mathrm{pr}_j \circ \Phi) \circ \nabla_X(\varphi_1 \times \cdots \times \varphi_N) \\
&= d(\mathrm{pr}_j \circ \Phi)(\pi_1 \circ d(\varphi_1 \times \cdots \times \varphi_N) + \cdots + \pi_N \circ d(\varphi_1 \times \cdots \times \varphi_N))(H^\gamma X).
\end{aligned} \tag{2.15}$$

But

$$\pi_i \circ d(\varphi_1 \times \cdots \times \varphi_N)(H^\gamma X) = 0$$

if and only if

$$\nabla_X \varphi_i = d\varphi_i(H^\gamma X) = 0. \tag{2.16}$$

Hence, one finds that

$$\nabla_X \Phi_j^*(\varphi_1, \ldots, \varphi_N) = d(\mathrm{pr}_j \circ \Phi)(0) = 0 \tag{2.17}$$

if $\nabla_X \varphi_i = 0$ for all $i = 1, \ldots, N$. □

Proposition 1 implies that the superposition formula (2.11) maps the X-parallel cross-sections onto the X-parallel ones.

Let us now consider the submanifold $N \to M$. If one assumes the zero curvature condition

$$\operatorname{curv} \gamma|_{TN} = 0, \tag{2.18}$$

where $TN \to TM$ is the distribution of subspaces $T_m N \to T_m M$ which are tangent to N and $\operatorname{curv} \gamma$ is a curvature 2-form of the connection γ, then one can define cross-sections parallel to N such that $\nabla_X \varphi = 0$ for $X \in C^\infty(TN)$. In other words, Proposition 1 leads to the conclusion that the set of N-parallel cross-sections is invariant with respect to the superposition rule defined by (2.11).

3 Application of the Superposition Formula to First Order Partial Differential Equations

The geometric objects and constructions discussed in the previous section, after having been presented in local coordinates, assume the form of differential expressions or differential equations. So, in this section, we can apply them to some problems appearing in the theory of first order *PDEs*.

Let $\Omega \subset M$ be the domain of the coordinate system $(x^1, \ldots, x^n)$ such that $\pi_p^{-1}(\Omega) \cong G \times \Omega$, i.e. the principal bundle P is trivial over Ω. Let $(g_1, \ldots, g_r)$ be a system of coordinates on the Lie group G defined in some neighbourhood of the identity element and $(u^1, \ldots, u^m)$ be a system of coordinates on F. Let us also fix the system of the left invariant vector fields

$$\Gamma_i = \Gamma_i^j \frac{\partial}{\partial g^j} \quad i = 1, \ldots, r \tag{3.1}$$

on the group G which form the basis of the group Lie algebra

$$[\Gamma_i, \Gamma_j] = C^k_{ij}\Gamma_k. \tag{3.2}$$

By the group action on the fibre one can transport the vector fields Γ_i, $i = 1, \ldots, r$ on the vector fields

$$Y_i = y^\alpha_i(u)\frac{\partial}{\partial u^\alpha} \tag{3.3}$$

defined on F. The vector fields Y_i, $i = 1, \ldots, r$ satisfy also the relation (3.2).

The γ-connection one-form a^γ in the local trivialization $\pi_P^{-1}(\Omega) \cong G \times \Omega$ is given by

$$a^\gamma(y, x) = -\Gamma^j_i(g)\frac{\partial}{\partial g^j} \otimes a^i_\mu(x)\, dx^\mu \tag{3.4}$$

and thus, one finds the horizontal lift

$$H^\gamma(x) = -\Gamma^j_i(g)a^i_\mu(x)\theta^\mu(x)\frac{\partial}{\partial g^j} + \theta^\mu(x)\frac{\partial}{\partial x^\mu} \tag{3.5}$$

of the vector field

$$X = \theta^\mu(x)\frac{\partial}{\partial x^\mu}. \tag{3.6}$$

Now, after expressing (2.12) in the above coordinates we come to the following differential equations

$$[\frac{\partial u^\alpha}{\partial x^\mu} - a^i_\mu(x)y^\alpha_i(u)]\theta^\mu(x) = 0, \tag{3.7}$$

where we put

$$u^\alpha = \varphi^\alpha(g, x)$$

and made use of the relation

$$\Gamma^j_i(g)\frac{\partial \varphi^\alpha}{\partial g^j}(g, x) = y^\alpha_i(\varphi(g, x)) \tag{3.8}$$

which follows from (2.4). It is also a consequence of the equation (2.4) that one can identify locally the cross-sections φ with functions

$$u^\alpha = u^\alpha(x) = \varphi^\alpha(x).$$

Thus, one can consider (3.7) as the system of partial differential equations for the function u^α.

In order to discuss the parallel cross-sections over a submanifold $N \to M$ we require that the map $u^\alpha = \varphi^\alpha(x)$ satisfies equation (3.7) for any vector field X tangent to N. In the special case, when $N = M$, the system (3.7) becomes

$$\frac{\partial u^\alpha}{\partial x^\mu} = a^i_\mu(x)y^\alpha_i(u), \quad \mu = 1, \ldots, n \quad \alpha = 1, \ldots, m. \tag{3.9}$$

If $\dim N = 1$ then (3.7) is reduced to the system of ordinary differential equations which was studied by S. Lie [1, 2]. He was the first to investigate the problem of the superposition principle for 1st order systems of ODEs. He proved that for $\dim N = 1$, equations (3.9) are exactly those systems of ODEs which allow a functional superposition principle. We show later that the superposition principle for PDEs (3.9) can be treated as a special form of the superposition formula (2.11).

Summarizing our considerations on the coordinate presentation, let us mention, for the sake of completeness, that zero curvature conditions (2.18) constitute consistency conditions for the system of PDEs (3.7). In the case given by (3.9) the conditions (2.18) are reduced to

$$\frac{\partial a_\mu^k}{\partial x^\nu} - \frac{\partial a_\nu^k}{\partial x^\mu} + \frac{1}{2} C_{ij}^k a_\mu^i a_\nu^j = 0. \tag{3.10}$$

From now on we restrict our attention to the subcase $N = M$. This restriction does not limit the generality of our analysis.

In the above considerations we have treated the cross-sections as the maps from P into F, satisfying the equivariance condition (2.4). The equivalent treatment is to consider them as the maps $\varphi\colon M \to B$ such that $\pi_B \circ \varphi = id$.

Using equations (3.2) and (3.10) we find that the rank $\varphi(M) \subset B$ of the parallel cross-section φ is the maximal integral submanifold of the n-dimensional distribution, generated locally by the following fields

$$Z_\mu = \frac{\partial}{\partial x^\mu} - a_\mu^i(x) y_i^\alpha(u) \frac{\partial}{\partial u^\alpha}. \tag{3.11}$$

We assume here that M is a connected set. Thus, if one fixes the initial point $x_0 \in M$, the parallel cross-section φ is uniquely determined by its value $\varphi(x_0)$ at that point. Considering the parallel cross-section φ as a map $\varphi\colon P \to F$ one can uniquely determine it by choosing the point $f_0 = \varphi(p_0) \in F$, where $\pi_p(p_0) = x_0$.

Let us now reconstruct the superposition formula (2.11). Our main task will be the construction of some G-morphism Φ which is an ingredient entering in the formula (2.11). We start from the finite sequence of the parallel cross-sections $\varphi_i \in \mathcal{F}(M, F_i, L_i)$ with the given initial data $f_{0i} = \varphi_i(p_0) \in F_i$, $i = 1, \dots, N$. Additionally, we impose the condition

$$G_{f_{01}} \cap \dots \cap G_{f_{0N}} = \{e\} \tag{3.12}$$

where $G_{f_{0i}}$ is the $L_i(G)$-stabilizer of the point f_{0i}.

From this condition it follows that the action

$$L = L_1 \times \dots \times L_N \colon G \to \mathrm{Diff}\, O_N \tag{3.13}$$

of the group G on the orbit O_N of the element $(f_{01}, \dots, f_{0N})$ is free. Thus, any $L(G)$-morphism Φ of the orbits O_N will be determined by its value $(g_{01}, \dots, g_{0N}) = \Phi(f_{01}, \dots, f_{0N})$ at the point $(f_{01}, \dots, f_{0N}) \in O_N$. The equations

$$f_i = L_i(g) f_{0i}, \quad i = 1, \dots, N \tag{3.14}$$

allow us to identify the orbit O_N with the group G, $I\colon O_N \to G$. So, by virtue of $I(g_{01}, \dots, g_{0N}) = g$ one obtains the group anti-isomorphism

$$\begin{gathered} G \ni g \to \Phi_g \in Aut_G\, O_N, \\ \Phi_{g_1 g_2} = \Phi_{g_2} \circ \Phi_{g_1} \quad g_1, g_2 \in G. \end{gathered} \tag{3.15}$$

After the substitution of the group anti-isomorphism Φ_g into (2.11) one finds the superposition formula

$$\varphi_i' = \Phi_{ig}^*(\varphi_1, \dots, \varphi_N) = \mathrm{pr}_i \circ \Phi_g \circ (\varphi_1 \times \dots \varphi_N) \tag{3.16}$$

for the set of N solutions to the system of PDEs of the form (3.9). These solutions are the ones which are determined by the initial data

$$f_{0i}' = \mathrm{pr}_i \circ \Phi_g(f_{01}, \dots, f_{0N}).$$

The quantities pr_i and Φ_g are G-equivariant maps. The point f_{0i} belongs to the same orbit as the point f'_{0i}, that is, $f'_{0i} \in \mathrm{pr}_i(O_N)$.

Now, if one restricts equation (3.16) to the subcase when $F_i = F$ and F is the $L(G)$-homogeneous space, that is, when one considers N solutions of the same system of equations, then formula (3.16) gives the superposition principle

$$\varphi = \Phi_g^*(\varphi_1, \ldots, \varphi_N) \tag{3.17}$$

which expresses the general solution φ in terms of some fundamental system of solutions $\varphi_1, \ldots, \varphi_N$. It is so, since G acts transitively on F and any initial data $f_0 = \varphi(p_0) \in F$ can be obtained from the fixed ones by a group action. In order to obtain (3.17) we fixed some $i \in \{1, \ldots, N\}$ and put $\varphi = \varphi'_i$. The cross-section φ does not change if in (3.17) one takes gh instead of g with h belonging to the stabilizer G_f. Hence, one sees that the superposition principle (3.17) contains the parameters (constants of motion) which run over the homogeneous space $G/G_f = F$.

Concluding this section, let us say that (3.17) is the generalization of the Lie superposition principle to the case of 1st order PDEs.

4 Example of Application to First Order PDEs Based on SU(2) Group

In this section, we show how the presented approach to the superposition principle works in the case when the symmetry group G (the so called gauge group) is SU(2).

We assume that the base space M and the connection ∇ are not subject to any restrictions. Let us consider two Lie type systems (3.9) for which typical fiber manifolds F_1 and F_2 are $\mathbb{C}P(1)$ and $\mathbb{C}P(2)$, respectively. The left action of the group SU(2) is determined on $\mathbb{C}P(1)$ by a fractional map

$$L_1(g)(z) = \frac{az+b}{-\bar{b}z+\bar{a}}, \tag{4.1}$$

where

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SU}(2)$$

and z is a homogeneous coordinate of $[\begin{pmatrix} 1 \\ z \end{pmatrix}] \in \mathbb{C}P(1)$. The left action L_2 on $\mathbb{C}P(2)$ is a cannonical action which, in the homogeneous coordinate system $[\begin{pmatrix} 1 \\ z_1 \\ z_2 \end{pmatrix}] \in \mathbb{C}P(2)$, assumes the form

$$L_2(g)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \dfrac{a^2 z_1 - b^2 z_2 - \sqrt{2}ab}{\sqrt{2}a\bar{b}z_1 + \sqrt{2}b\bar{a}z_2 + |a|^2 - |b|^2} \\ \dfrac{-\bar{b}^2 z_1 + \bar{a}^2 z_2 - \sqrt{2}\bar{a}\bar{b}}{\sqrt{2}a\bar{b}z_1 + \sqrt{2}b\bar{a}z_2 + |a|^2 - |b|^2} \end{pmatrix}. \tag{4.2}$$

Fixing the base $\Gamma_k = i\sigma_k$, $k = 1, 2, 3$, where σ_k are Pauli matrices, in the Lie algebra SU(2) and applying theoretical considerations carried out in the previous section, we obtain two following Lie type systems of PDEs

$$\frac{\partial z}{\partial x_\mu} = -a^1_\mu(x)i(1-z^2) + a^2_\mu(x)(1+z^2) - 2ia^3_\mu(x)z, \quad c.c. \tag{4.3}$$

for the action $L_1(g)\colon \mathbb{C}P(1) \to \mathbb{C}p(1)$; and

$$\begin{aligned}\frac{\partial z_1}{\partial x_\mu} &= a^1_\mu(x) i\sqrt{2}(1 + \frac{i}{\sqrt{2}} z_2 - (z_1 + \frac{1}{2}z_2)(z_1 - z_2)) \\ &\quad - a^2_\mu(x)\sqrt{2}(1 + \frac{1}{\sqrt{2}} z_2 + (z_1 + \frac{1}{2}z_2)(z_1 + z_2)) - 2ia^3_\mu(x) z_1 \\ \frac{\partial z_2}{\partial x_\mu} &= -a^1_\mu(x) i\sqrt{2}(1 - \frac{i}{\sqrt{2}} z_1 + (z_2 - \frac{1}{2}z_1)(z_1 - z_2)) \\ &\quad - a^2_\mu(x)\sqrt{2}(1 - \frac{1}{\sqrt{2}} z_1 + (z_2 - \frac{1}{2}z_1)(z_1 + z_2)) + 2ia^3_\mu(x) z_2, \quad c.c.\end{aligned} \tag{4.4}$$

for the action $L_2(g)\colon \mathbb{C}P(2) \to \mathbb{C}P(2)$.

Let us fix the solution $z = z(t)$ of equation (3.3) with the initial condition $z(0) = 0$ and the solutions $z_1 = z_1(t)$, $z_2 = z_2(t)$ of equations (4.4) with the initial conditions $z_1(0) = 1$, $z_2(0) = 0$. After performing the procedure described by transformations (3.12)–(3.17) we obtain the following superposition formula which provides the general solutions $u = u(t)$ and $u_1 = u_1(t)$, $u_2 = u_2(t)$ of systems (4.3) and (4.4), respectively

$$u(t) = \frac{\sqrt{2}z_1 + (\sqrt{2}\bar{z}_1 - 1) - \lambda(\sqrt{2}z_1^{-1} + 2) + \theta(\sqrt{2}\bar{\lambda} z_1(\sqrt{2}\bar{z}_1 - 1) + \sqrt{2}z_1 + 2)}{\sqrt{2}\bar{z}_1 - 1 + \sqrt{2}\bar{z}_1(\sqrt{2}z_1 + 2) - \theta(\bar{\lambda}(\sqrt{2}\bar{z}_1 - 1) + \sqrt{2}\bar{z}_1(\sqrt{2}z_1 + 2))} \tag{4.5}$$

and

$$u_1(t) = \frac{\lambda(\sqrt{2}z_1 + z_2) - 2(z_1)^2 - \sqrt{2}z_1}{1 - 2(\bar{z}_1 + \sqrt{2}\lambda|z_1|^2 + \sqrt{2}\lambda\bar{z}_1 z_2)}, \quad u_2(t) = -\frac{1}{2u_1(t)}, \tag{4.6}$$

where $\lambda \in \mathbb{C}$ and $\theta \in S^1$ are constant complex parameters and $z = z(t)$ and $z_1 = z_1(t)$, $z_2 = z_2(t)$ are the particular solutions of systems (4.3) and (4.4), respectively.

5 Comments on the Bäcklund Transformation in the Context of the Superposition Principle

Let us consider now an N-th order system of PDEs for the unknown functions $u^1 = u^1(x), \ldots, u^m = u^m(x)$ in many independent variables

$$\begin{gathered}\Delta^i(x, u^{(N)}) = 0, \quad i = 1, \ldots, M \\ x = (x^1, \ldots, x^n), \quad u = (u^1, \ldots, u^m).\end{gathered} \tag{5.1}$$

It is convenient to treat $(u^1, \ldots, u^m)$ as coordinates on a typical fiber bundle F and $(x^1, \ldots, x^n)$ as coordinates on the base manifold M. Consequently, the functions $u^1(x), \ldots, u^m(x)$ can be construed as coordinates of the section of the fibre bundle

$$\begin{array}{ccc} F & \longrightarrow & B \\ & & \downarrow \\ & & M \end{array} \tag{5.2}$$

parallel to certain connection ∇. In other words, we assume that the functions u^α satisfy Lie type system of PDEs

$$\frac{\partial u^\alpha}{\partial x^\mu} = a^i_\mu(x) y^\alpha_i(u) \quad \alpha = 1, \ldots, m \quad \mu = 1, \ldots, n \tag{5.3}$$

with the initial conditions

$$u^\alpha(0) = u_0^\alpha, \tag{5.4}$$

for which the compatibility conditions

$$\frac{\partial a_\mu^k}{\partial x^\nu} - \frac{\partial a_\nu^k}{\partial x^\mu} + \frac{1}{2} C_{ij}^k a_\mu^i a_\nu^j = 0, \tag{5.5}$$

are also satisfied. Using (5.3) we eliminate all partial derivatives of the functions u^α from equations (5.1). As a result we obtain a functional equation

$$\tilde{\Delta}(x, u, a_\mu^{k\,(N-1)}) = 0 \tag{5.6}$$

for the functions $u^\alpha(x)$. Equation (5.6) depends on the choosen connection $\{a_\mu^k(x)\}$ satisfying compatibility conditions (5.5). If equation (5.6) satisfies the assumptions of the implicit function theorem then, for a given $a_\mu^i(x)$, its solution $u^\alpha = u^\alpha(x)$ determines a local submanifold $\mathcal{N}$ of the bundle $\pi \colon B \to M$. If the intersection of $\mathcal{N}$ with the image of the section $\varphi \colon M \to B$ forms certain submanifold $\mathcal{N} \cap \varphi(M) \subset B$ for which the projection $\pi(\mathcal{N} \cap \varphi(M))$ has a nonzero dimension, then conditions (5.5) and (5.6) form a consistent system of PDEs on $\pi(\mathcal{N} \cap \varphi(M))$. Note that the existence of such intersection can always be ensured locally by an adequate choice of initial conditions for u_0^α.

Finally, let us pay attention to a special case when the system of equations composed of (5.5) and (5.6) has particular solutions $a_\mu^i = a_\mu^i(x)$ defined on the local manifold $\pi(N \cap \varphi(M))$ which are parametrized by at least one constant and by some function satisfying the original PDE (5.1). Under these conditions the system of PDEs (5.3) determines an Auto-Bäcklund transformation (Auto-BT) for (5.1). Let us note that for a system of equations to define a BT, it has to admit a superposition formula (the permutability theorem [7, 8]). Since the system of the type (5.3) satisfies this necessary condition, thus such a choice of the constraints to the original equation increases our chances of actually finding a BT [9].

Let us apply the above considerations to the case of the KdV equation

$$u_t + u_{xxx} + 3u_x^2 = 0. \tag{5.7}$$

We assume $SL(2, \mathbb{R})$ as a gauge group, $\mathbb{RP}(1)$ as a typical fibre and $M = \mathbb{R}^2$ as a base manifold. Then equations (5.3) are reduced to the partial differential Riccati type equations

$$\begin{aligned} u_x &= a(x,t)u^2 + b(x,t)u + h(x,t), \\ u_t &= p(x,t)u^2 + q(x,t)u + k(x,t). \end{aligned} \tag{5.8}$$

The compatibility conditions (5.5) for system (5.8) are

$$\begin{aligned} a_t - p_x + aq - pb &= 0, \\ b_t - q_x + 2(ak - ph) &= 0, \\ h_t - k_x + kb - hq &= 0. \end{aligned} \tag{5.9}$$

Substituting (5.8) into (5.7) we obtain a polynomial expression of the fourth degree in the dependent variable u, of the form

$$\begin{aligned} &3a^2(2a+1)u^4 + 6ab(2a+1)u^3 \\ &\quad + (p + 3b^2 + 6ah + 4ab_x + 5ba_x + 4a^2h + 7ab^2 + a_{xx} + 4a)u^2 \\ &\quad + (q + 6bh + 2ah_x + 3b_xb + 8abh + b^3 + b_{xx} + 4a_xh)u \\ &\quad + k + 3h^2 + h_xb + b^2h + h_{xx} + 2b_xh + 2ah^2 = 0 \end{aligned} \tag{5.10}$$

The above condition corresponds to the functional equation (5.6).

We look for such a form of (5.10) which does not lead to the algebraic constraints on the function u. This demand ensures that the dimension of the submanifold $\pi(\mathcal{N} \cap \varphi(M))$ is nonzero since we have

$$\pi(\mathcal{N} \cap \varphi(M)) = \mathbb{R}^2.$$

Consequently, the systems (5.5) and (5.6) are consistent. So, we require that the coefficients of the successive powers of u in equation (5.10) vanish

$$\begin{aligned}
&a^2(2a+1) = 0,\\
&ab(2a+1) = 0,\\
&p + 3b^2 + 6ah + 4ab_x + 5ba_x + 4a^2h + 7ab^2 + a_{xx} + 4a = 0,\\
&q + 6bh + 2ah_x + 3b_xb + 8abh + b^3 + b_{xx} + 4a_xh = 0,\\
&k + 3h^2 + h_xb + b^2h + h_{xx} + 2b_xh + 2ah^2 = 0.
\end{aligned} \tag{5.11}$$

Thus, we obtain an overdetermined system of equations for the coefficients a, b, h, p, q and k. When solving this system together with zero curvature conditions (5.9), we have to consider two separate cases.

In the first case, when $a = 0$, the set of solutions of equations (5.12) is uniquely defined (with the freedom of some constants only). Then a BT does not exist and only particular solutions to (5.7) can be constructed. In the second case, when $a = -1/2$, the set of solutions of the overdetermined system (5.12) is parametrized by the function b satisfying the original KdV equation (5.7)

$$b_t + b_{xxx} + 3b_x^2 = 0 \tag{5.12}$$

and by some real constant of integration λ. A unique solution for system (5.11) is given by

$$\begin{aligned}
&a = -1/2 &&p = b_x + \lambda\\
& &&q = -2(b_{xx} + b(b_x + \lambda))\\
&h = (\lambda - \frac{1}{2}b^2 - b_x) &&k = b_{xxx} + 2bb_{xx} + b_x^2 + (b^2 + 2\lambda)b_x + \lambda b^2 - 2\lambda^2
\end{aligned}$$

Substituting (5.12) into DCs (5.8) we obtain the well known [7] Auto-BT for the KdV equation

$$\begin{aligned}
u_x &= -2\lambda - \frac{1}{2}(u-b)^2 - b_x \quad \lambda \in \mathbb{R}\\
u_t &= -2(u_x^2 + u_xb_x + b_x^2) + (u-b)(u-b)_{xx} - b_t
\end{aligned} \tag{5.13}$$

The change of variable

$$(u, b) \to (u, y = b - u) \tag{5.14}$$

transforms the equations (5.13) into the Riccati equations for y

$$\begin{aligned}
y_x &= 2\lambda - 2u_x - \frac{1}{2}y^2\\
y_t &= (4\lambda y + 2u_{xx} - 2u_xy)_x
\end{aligned} \tag{5.15}$$

Equation (5.2) is a compatibility condition for system (5.13). The function u, found by integrating system (5.13) gives a solution of KdV (5.7). This means that there exists an Auto-BT, since for any particular solution b of (5.12) the system (5.13) defines a specific transformation between solutions of equations (5.12) and (5.7). In other words, from each solution of (5.12) we can construct a solution of (5.7) and vice versa. Note that KdV equation (5.7) admits infinitely many compatible differential constraints (5.13) which are parametrized by function b and one arbitrary constant λ.

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