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G-STRUCTURES OF HIGHER ORDER

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Introduction.

One of the main problems of local differential geometry is to determine when a given geometric structure is integrable or flat. The problem of flatness for G-structures of first order and of finite type has been solved ([1], [7]), and we know that the vanishing of a finite number of cohomology classes implies the flatness of a given G-structure. This result covers the problem of flatness for Riemannian structures and conformal structures. In the paper we shall extend this result to G-structures of higher order, for example, projective structures.

For affine structures, projective structures or conformal structures, it is the classical result that if the dimension of the Lie algebra of infinitesimal automorphisms is maximal then the structure is flat. We shall also extend these results to more general G-structures of higher order and of finite type satisfying some appropriate conditions.

§1. Graded Lie algebras.

In general, by a *graded Lie algebra* we mean a Lie algebra $\sum_{p=-1}^{\infty}\mathfrak{g}_p$ which satisfies the following conditions:

$$(1)\qquad \dim \mathfrak{g}_p<\infty,$$

$$(2)\qquad [\mathfrak{g}_p, \mathfrak{g}_q]\subset\mathfrak{g}_{p+q},$$

in particular $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]=0$,

$$(3)\qquad \text{For every nonzero } t\in\mathfrak{g}_p,\ p\geqq 0,\ [t, \mathfrak{g}_{-1}]\neq 0.$$

The subalgebra $\sum_{p=0}^{\infty}\mathfrak{g}_p$ is called the *isotropy algebra* of $\sum_{p=-1}^{\infty}\mathfrak{g}_p$ and $\mathfrak{g}_0$ is called the *linear isotropy algebra.*

It is clear that if $\mathfrak{g}_k=0$ then $\mathfrak{g}_p=0$ for $p>k$. A graded Lie algebra $\sum\mathfrak{g}_p$ is said to be of *type* k if $\mathfrak{g}_{k-1}\neq 0$ and $\mathfrak{g}_k=0$. It is said to be of *infinite type* if $\mathfrak{g}_p\neq 0$ for all p.

Let $V=\mathbf{R}^n$ and V^* its dual. A Lie subalgebra $\sum_{p=-1}^{\infty}\mathfrak{g}_p$ of $V+V\otimes V^*+V\otimes S^2(V^*)+\cdots$, $\mathfrak{g}_{-1}=V$ and $\mathfrak{g}_i\subset V\otimes S^{i+1}(V^*)$, is a graded Lie algebra.

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Let W be a finite dimensional vector space and $\mathfrak{g}$ a subspace of $W\otimes V^*$. We set

$$\mathfrak{g}^{(p)}=\mathfrak{g}\otimes S^p(V^*)\cap W\otimes S^{p+1}(V^*).$$

$\mathfrak{g}^{(p)}$ is called the p-th *prolongation* of $\mathfrak{g}$.

Let $\sum \mathfrak{g}_p$ be a graded Lie algebra. The smallest integer r such that

$$\mathfrak{g}_{r-1+p}=\mathfrak{g}_{r-1}{}^{(p)} \qquad \text{for all } p$$

is called the *order* of the graded Lie algebra $\sum \mathfrak{g}_p$.

Let r and k be the order and the type of a graded Lie algebra respectively. Then in general $r\leqq k+1$.

EXAMPLES.

1. $V+\mathfrak{gl}(n)+\mathfrak{gl}(n)^{(1)}+\mathfrak{gl}(n)^{(2)}+\cdots$ is a graded Lie algebra of order 1 and of infinite type.
2. $V+\mathfrak{gl}(n)+\mathfrak{sl}(n)^{(1)}+\mathfrak{sl}(n)^{(2)}+\cdots$ is a graded Lie algebra of order 2 and of infinite type.
3. $V+\mathfrak{gl}(n)$ is a graded Lie algebra of order 2 and of type 1.
4. $V+\mathfrak{o}(n)$ is a graded Lie algebra of order 1 and of type 1.
5. $V+\mathfrak{co}(n)+\mathfrak{co}(n)^{(1)}\cong V+\mathfrak{co}(n)+V^*$ is a graded Lie algebra of order 1 and of type 2.
6. Let $\mathfrak{p}$ be the invariant complement to $\mathfrak{sl}(n)^{(1)}$ in $\mathfrak{gl}(n)^{(1)}$. Then $V+\mathfrak{gl}(n)+\mathfrak{p}\cong V+\mathfrak{gl}(n)+V^*$ is a graded Lie algebra of order 2 and of type 2.

Given a graded Lie algebra $\sum \mathfrak{g}_p$, we define cohomology groups as follows:

$$C^{p,q}=\mathfrak{g}_{p-1}\otimes\wedge^q(V^*).$$

We define the coboundary operator

$$\partial\colon C^{p,q}\to C^{p-1,q+1}$$

by

$$(\partial t)(x_1,\cdots,x_{q+1})=\sum(-1)^i[t(x_1,\cdots,\hat{x}_i,\cdots,x_{q+1}),x_i] \qquad \text{for } x_1,\cdots,x_{q+1}\in V.$$

Then $\partial^2=0$.

We shall denote the cohomology groups of this complex by $H^{p,q}$.

§ 2. *G*-structures of higher order.

Let M be a differentiable manifold of dimension n and $F^p(M)$ the boundle of p-frames of M. The structure group of $F^p(M)$ is denoted by $G^p(n)$. The Lie algebra $\mathfrak{g}^p(n)$ of $G^p(n)$ is $V\otimes V^*+V\otimes S^2(V^*)+\cdots+V\otimes S^p(V^*)$. As to the details we shall refer to [2].

Let G be a subgroup of $G^r(n)$ whose Lie algebra $\mathfrak{g}$ is the isotropy algebra of a graded Lie algebra of order r and of type k.

A *G-structure* $P_G(M)$ *of order* r *and of type* k *on* M is a reduction of $F^r(M)$ to the subgroup G.

EXAMPLES.

1. Let G be a subgroup of $G^2(n)$ with Lie algebra $\mathfrak{g}=\mathfrak{gl}(n)$. A G-structure $P_G(M)$ on M is called an *affine* structure. An affine structure is a G-structure of order 2 and of type 1.

2. Let G be a subgroup of $G^2(n)$ with Lie algebra $\mathfrak{g}=\mathfrak{gl}(n)+\mathfrak{p}$, (cf. Example 6, § 1). A G-structure $P_G(M)$ on M is called a *projective* structure. A projective structure is a G-structure of order 2 and of type 2.

3. Let G be a subgroup of $G^2(n)$ with Lie algebra $\mathfrak{g}=\mathfrak{co}(n)+\mathfrak{co}(n)^{(1)}$. A G-structure $P_G(M)$ on M is called a *conformal* structure. A conformal structure is a G-structure of order 1 and of type 2.

Let $M_0=\tilde{H}/H$ be a homogeneous space of dimension n. Suppose that $\tilde{H}$ is a finite dimensional Lie group whose Lie algebra $\tilde{\mathfrak{h}}$ is a graded Lie algebra. $\tilde{H}$ operates transitively on M_0 and H can be considered as the isotropy subgroup of $\tilde{H}$ at $x_0 \in M_0$ so that $M_0=\tilde{H}/H$. M_0 has a G-structure as follows:

We fix a local diffeomorphism λ of a neighborhood of the origin 0 of $\mathbf{R}^n$ onto a neighborhood of $x_0 \in M_0$, that is, a chart at x_0. Let j^p: $\tilde{H} \to F^p(M_0)$ be defined by

$$j^p(a)=j^p_x(a\circ\lambda) \qquad \text{where} \quad x=(a\circ\lambda)(0).$$

Let r be the smallest integer such that j^r is injective. Let $e_r=j^r_{x_0}(\mathrm{id.}\circ\lambda)$ be the distinguished element of $j^r(\tilde{H})$. Then the tangent space $T_{e_r}(j^r(\tilde{H}))$ to $j^r(\tilde{H})$ at e_r is a subspace of $V+V\otimes V^*+V\otimes S^2(V^*)+\cdots$:

$$T_{e_r}(j^r(\tilde{H}))=V+\mathfrak{g}_0+\mathfrak{g}_1+\cdots+\mathfrak{g}_{r-1}$$

where $\mathfrak{g}_i\subset V\otimes S^{i+1}(V^*)$.

Let $\mathfrak{g}_p=\mathfrak{g}_{r-1}^{(p-r+1)}$ for $p\geqq r$. Then there exists an integer k such that $\mathfrak{g}_p=0$ for $p\geqq k$ since $\tilde{H}$ is of finite dimension. $V+\mathfrak{g}_0+\cdots+\mathfrak{g}_{r-1}+\mathfrak{g}_r+\cdots+\mathfrak{g}_{k-1}$ is a graded Lie algebra of order r and of type k. We shall denote $j^r(H)$ by G. G can be considered as a subgroup of $G^r(n)$. We set $P_G(M_0)=j^r(\tilde{H})$, then $P_G(M_0)$ is a subbundle of $F^r(M_0)$ with structure group G, that is, a G-structure of order r and of type k on M_0. The G-structure $P_G(M_0)$ can also be considered as follows: Let Γ be the *pseudogroup* of transformations of M_0 induced by the action of $\tilde{H}$. Then M_0 has a Γ-structure and $P_G(M_0)$ is the G-structure determined by the Γ-structure. The G-structure is called the *standard flat* G-structure. Let $s=\mathrm{Max}\{r,k\}$. Then we can associate canonically a subbundle $\widetilde{P_G(M_0)}$ of $F^s(M_0)$ with the standard flat G-structure $P_G(M_0)$. Let $e_s=j^s_{x_0}(\mathrm{id.}\circ\lambda)$ be the distinguished element of $\widetilde{P_G(M_0)}$. Then the tangent space to $\widetilde{P_G(M_0)}$ at e_s is $V+\mathfrak{g}_0+\cdots+\mathfrak{g}_{r-1}+\mathfrak{g}_r+\cdots+\mathfrak{g}_{k-1}$.

A G-structure is said to be *flat* if it is locally isomorphic with the standard flat G-structure.

Let $P_G(M)$ be a G-structure of order r and of type k on M. Let θ be the canonical form of $F^r(M)$ restricted to $P_G(M)$. Then θ is a $V+\mathfrak{g}_0+\cdots+\mathfrak{g}_{r-2}$-valued 1-form on $P_G(M)$. Let ω_i be the $\mathfrak{g}_i$-component of θ, then $\theta=(\omega_{-1}, \omega_0, \omega_1, \cdots, \omega_{r-2})$. For each $u\in P_G(M)$, let G_u be the subspace of $T_u(P_G(M))$ consisting of vectors tangent to the fibre through u. Then G_u is isomorphic with $\mathfrak{g}$, the Lie algebra of G. A complement to G_u in $T_u(P_G(M))$ on which the forms $\omega_0, \omega_1, \cdots, \omega_{r-2}$ all vanish is called a *horizontal space* at u.

Let H be a horizontal space at u, then $H\cong V$.

Now let ξ and η be elements of V, and X and Y the corresponding elements in H. We define

$$C_H\in \mathrm{Hom}\,(V\wedge V,\ V+\mathfrak{g}_0+\cdots+\mathfrak{g}_{r-2})$$

by

$$C_H(\xi, \eta)=d\theta(X, Y).$$

We shall denote the $\mathrm{Hom}\,(V\wedge V, \mathfrak{g}_i)$-component of C_H by C_H^i.

PROPOSITION 2. 1. $\quad C_H^i=0 \quad$ *for* $\; i<r-2$.

Proof. Let h be the horizontal projection determined by the horizontal space H. Then our assertion is equivalent to

$$d\omega_i\circ h=0 \qquad \text{for} \quad i<r-2.$$

This follows from the identities

$$d\omega_i\circ h=d\omega_i+\frac{1}{2}([\omega_{-1}, \omega_{i+1}]+[\omega_0, \omega_i]+\cdots+[\omega_{i+1}, \omega_{-1}])$$

for $i<r-2$ and the fact that $\theta=(\omega_{-1}, \omega_0, \cdots, \omega_{r-2})$ is the restriction of the canonical form of $F^r(M)$ to the subbundle $P_G(M)$. Q.E.D.

From the proof of Proposition 2. 1 we have

$$d\omega_{r-3}+\frac{1}{2}([\omega_{-1}, \omega_{r-2}]+[\omega_0, \omega_{r-3}]+\cdots+[\omega_{r-2}, \omega_{-1}])=0.$$

By applying exterior differentiation to this equation and composing with the horizontal projection h we have

$$[d\omega_{r-2}\circ h,\ \omega_{-1}]=0.$$

This implies that C_H^{r-2} is a cocycle.

Let H and H' be two horizontal spaces at u. It is easily seen that

$$C_{H'}^{r-2}-C_H^{r-2}\in\partial\,\mathrm{Hom}\,(V, \mathfrak{g}_{r-1}).$$

Hence the cohomology class C^{r-2} of C_H^{r-2} is independent of the choice of the horizontal space H.

C^{r-2} is an element of the cohomology group $H^{r-1,2}$ associated with the graded Lie algebra $V+\mathfrak{g}_0+\mathfrak{g}_1+\cdots+\mathfrak{g}_{r-1}$.

We call $C=C^{r-2}$ the *structure tensor* of the G-structure $P_G(M)$. C is a $H^{r-1,2}$-

valued function on $P_G(M)$.

$P_G(M)$ is said to be *r-flat* if $C=0$.

Then problem of flatness for G-structure $P_G(M)$ has following three cases:

(I) If $r=k+1$, it is clear that *$P_G(M)$ is flat if and only if $C=0$.*

EXAMPLE. Affine structure.

(II) (i) If $r=k$ and $P_G(M)$ is k-flat, then there exists a canonically associated subbundle $P'_G(M)$ of $F^{k+1}(M)$. The structure group of $P'_G(M)$ is isomorphic with G. Let C' be the structure tensor of $P'_G(M)$.

Then *$P_G(M)$ is flat if and only if it is $(k+1)$-flat,* that is, $C'=0$. In other words, let θ' be the canonical form of $F^{k+1}(M)$ restricted to $P'_G(M)$. θ' is considered as a 1-form on $P_G(M)$ with values in $V+\mathfrak{g}_0+\cdots+\mathfrak{g}_{k-1}$ and it defines a Cartan connection on $P_G(M)$. The condition $C'=0$ is equivalent to the flatness of the Cartan connection.

EXAMPLE. Riemannian structure.

(ii) If $r=k$ and it is possible to associate canonically a Cartan connection with $P_G(M)$, then *$P_G(M)$ is flat if and only if the Cartan connection is flat.*

EXAMPLE. Projective structure.

(III) If $r<k$ and $P_G(M)$ is r-flat, then there exists a canonically associated subbundle $P_{G^{(1)}}(M)$ of $F^{r+1}(M)$, where $G^{(1)}$ is the semidirect product of G and the nilpotent Lie group generated by $\mathfrak{g}_r+\mathfrak{g}_{r+1}+\cdots/\mathfrak{g}_{r+1}+\cdots$. $P_G(M)$ is said to be $(r+1)$-flat if the structure tensor of $P_{G^{(1)}}(M)$ vanishes.

If $P_G(M)$ is $(r+1)$-flat, then there exists a canonically associated subbundle $P_{G^{(2)}}(M)$ of $F^{r+2}(M)$, where $G^{(2)}$ is the semidirect product of $G^{(1)}$ and the nilpotent Lie group generated by $\mathfrak{g}_{r+1}+\mathfrak{g}_{r+2}+\cdots/\mathfrak{g}_{r+2}+\cdots$. $P_G(M)$ is said to be $(r+2)$-flat if the structure tensor of $P_{G^{(2)}}(M)$ vanishes.

(i) Assume $P_G(M)$ is k-flat. Then there exists a canonically associated subbundle $P_{G^{(k+1-r)}}(M)$ of $F^{k+1}(M)$. The structure group $G^{(k+1-r)}$ is isomorphic with $G^{(k-r)}$. *$P_G(M)$ is flat if and only if the structure tensor of $P_{G^{(k+1-r)}}(M)$ vanishes.*

(ii) Assume $P_G(M)$ is $(k-1)$-flat. Then there exists a canonically associated subbundle $P_{G^{(k-r)}}(M)$ of $F^k(M)$. We also assume that it is possible to associate canonically a Cartan connection with $P_{G^{(k-r)}}(M)$. Then *$P_G(M)$ is flat if and only if the Cartan connection is flat.*

EXAMPLE. Conformal structure.

These results may be stated as follows.

THEOREM. *A G-structure $P_G(M)$ of order r and of type k is flat if and only if it is $(k+1)$-flat.*

§ 3. Infinitesimal automorphisms of a G-structure.

Let $P_G(M)$ be a G-structure of order r and of type k on M. Let $s=\mathrm{Max}\{r, k\}$.

We can associate canonically a subbundle $\widetilde{P_G(M)}$ of $F^s(M)$ under the assumption that $P_G(M)$ is $(k-1)$-flat if $r<k$.

Let $\tilde{\theta}$ be a $V+\mathfrak{g}_0+\cdots+\mathfrak{g}_{k-1}$-valued 1-form on $\widetilde{P_G(M)}$ defined as follows (cf. § 2):

(I) If $s=r=k+1$, then $\tilde{\theta}$ is the canonical form of $F^{k+1}(M)$ restricted to $\widetilde{P_G(M)}=P_G(M)$.

(II–i and III–i) If $s=k$ and $P_G(M)$ is k-flat, then there exists a canonically associated subbundle of $F^{k+1}(M)$. The canonical form of $F^{k+1}(M)$ restricted to the subbundle can be considered as a 1-form on $\widetilde{P_G(M)}$. We shall take the 1-form as $\tilde{\theta}$.

(II–ii and III–ii) If there exists a canonical Cartan connection on $\widetilde{P_G(M)}$, let $\tilde{\theta}$ be the 1-form defining the Cartan connection.

In every case $\tilde{\theta}$ gives rise to a complete parallelism on $\widetilde{P_G(M)}$. Let ω_i be the $\mathfrak{g}_i$-component of $\tilde{\theta}$, then

$$\tilde{\theta}=(\omega_{-1}, \omega_0, \omega_1, \cdots, \omega_{k-1}).$$

For each $\tilde{u}\in\widetilde{P_G(M)}$, there exists a unique complement H to the vertical space at $\tilde{u}$ on which the forms $\omega_0, \omega_1, \cdots, \omega_{k-1}$ all vanish. We call H a *horizontal space* at $\tilde{u}$.

Let H be a horizontal space at $\tilde{u}$, then $H\cong V$. Let h be the horizontal projection determined by H. With each element $\xi\in V$, we can associate a unique vector field ξ^* of $\widetilde{P_G(M)}$ satisfying

$$\tilde{\theta}(\xi^*)=\xi.$$

We call ξ^* the *standard horizontal vector field* corresponding to ξ.

PROPOSITION 3. 1. $$d\tilde{\theta}\circ h=d\tilde{\theta}+\frac{1}{2}[\tilde{\theta}, \tilde{\theta}],$$

that is,

$$d\tilde{\theta}(hX, hY)=d\tilde{\theta}(X, Y)+\frac{1}{2}[\tilde{\theta}(X), \tilde{\theta}(Y)]$$

for any vectors X and Y at $\tilde{u}$.

Proof. Every vector of $\widetilde{P_G(M)}$ is a sum of a vertical vector and a horizontal vector. Since both sides of the above equality are bilinear and skew-symmetric, it is sufficient to verify the equality in the following three special cases:

(1) X and Y are horizontal.

Let $\xi, \eta\in V$ and ξ^*, η^* the corresponding elements in H. Then

$$[\tilde{\theta}(\xi^*), \tilde{\theta}(\eta^*)]=[\xi, \eta]=0$$

since $[V, V]=0$. Thus the equality clearly holds.

(2) X and Y are vertical.

Let $X=A^*$ and $Y=B^*$ where $A, B\in\mathfrak{g}_0+\cdots+\mathfrak{g}_{k-1}$. Here A^* and B^* are the vertical vectors corresponding to A and B respectively. We have

$$2d\tilde{\theta}(A^*, B^*)=A^*\cdot\tilde{\theta}(B^*)-B^*\cdot\tilde{\theta}(A^*)-\tilde{\theta}([A^*, B^*])$$

$$=-[A, B]$$

$$=-[\tilde{\theta}(A^*), \tilde{\theta}(B^*)]$$

since $\tilde{\theta}(A^*)=A$, $\tilde{\theta}(B^*)=B$ and $[A^*, B^*]=[A, B]^*$. On the other hand

$$d\tilde{\theta}(hA^*, hB^*)=0.$$

(3) X is vertical and Y is horizontal.

Let $X=A^*$ where $A\in\mathfrak{g}_0+\cdots+\mathfrak{g}_{k-1}$ and $Y=\xi^*$ where $\xi\in V$. We extend ξ^* to a horizontal vector field which will be also denoted by ξ^*.

We have

$$2d\tilde{\theta}(A^*, \xi^*)=A^*\cdot\tilde{\theta}(\xi^*)-\xi^*\cdot\tilde{\theta}(A^*)-\tilde{\theta}([A^*, \xi^*])$$

$$=-\tilde{\theta}([A^*, \xi^*])=-[A, \xi]$$

and

$$[\tilde{\theta}(A^*), \tilde{\theta}(\xi^*)]=[A, \xi].$$

Hence the both sides of the equality vanish. Q.E.D.

Let $\Omega=d\tilde{\theta}\circ h$. Then we have

PROPOSITION 3. 2. $$d\Omega=[\Omega, \tilde{\theta}].$$

Proof. From $\Omega=d\tilde{\theta}+(1/2)[\tilde{\theta}, \tilde{\theta}]$ we have

$$d\Omega=[d\tilde{\theta}, \tilde{\theta}]$$

$$=[\Omega, \tilde{\theta}]-\frac{1}{2}[[\tilde{\theta}, \tilde{\theta}], \tilde{\theta}].$$

This, together with the Jacobi identity, implies

$$d\Omega=[\Omega, \tilde{\theta}].$$ Q.E.D.

Let Ω^p be the $\mathfrak{g}_p$-component of Ω. Then we have

PROPOSITION 3. 3. *If $\mathfrak{g}_0$ contains the identity element E, then $L_{E^*}\Omega^p=p\Omega^p$, where E^* denotes the vertical vector field on $\widetilde{P_G(M)}$ corresponding to E.*

Proof. We have

$$L_{E^*}\Omega=(\iota_{E^*}\circ d+d\circ\iota_{E^*})\Omega$$

$$=\iota_{E^*}d\Omega=\iota_{E^*}[\Omega, \tilde{\theta}]$$

by Proposition 3. 2. This implies

$$(L_{E^*}\Omega)(X, Y)=[\Omega(X, Y), E]$$

for every X and Y.

On the other hand, from the definition of the bracket operation of the graded Lie algebra[1] we have

$$[A, E]=pA \qquad \text{for} \quad A\in\mathfrak{g}_p.$$

Hence

$$L_{E^*}\Omega^p=p\Omega^p. \qquad \text{Q.E.D.}$$

Every vector field X on M generates a 1-parameter local group of local transformations.

Let $P_G(M)$ be a G-structure of order r and of type k on M. We call X an *infinitesimal automorphism* of $P_G(M)$ if the local 1-parameter group of local transformations generated by X in a neighborhood of each point of M consists of local automorphism of $P_G(M)$. The local group generated by X, prolonged to $F^s(M)$, induces a vector field on $F^s(M)$, which will be denoted by $\tilde{X}$.

PROPOSITION 3. 4. *For a vector field X on M, the following conditions are mutually equivalent*:

(i) *X is an infinitesimal automorphism of $P_G(M)$*;

(ii) *$\tilde{X}$ is tangent to $\widetilde{P_G(M)}$ at every point of $\widetilde{P_G(M)}$*;

(iii) $L_{\tilde{X}}\tilde{\theta}=0$;

(iv) $L_{\tilde{X}}\xi^*=0$ *for every* $\xi\in V$.

Proof. (i)⇒(ii). Let φ_t and $\tilde{\varphi}_t$ be the local 1-parameter group of local transformations generated by X and $\tilde{X}$ respectively. If X is an infinitesimal automorphism of $P_G(M)$, then φ_t is a local automorphism and hence $\tilde{\varphi}_t$ maps $\widetilde{P_G(M)}$ into itself. Thus $\tilde{X}$ is tangent to $\widetilde{P_G(M)}$ at every point of $\widetilde{P_G(M)}$.

(ii)⇒(i). If $\tilde{X}$ is tangent to $\widetilde{P_G(M)}$ at every point of $\widetilde{P_G(M)}$, the integral curve of $\tilde{X}$ through each point of $\widetilde{P_G(M)}$ is contained in $\widetilde{P_G(M)}$ and hence each $\tilde{\varphi}_t$ maps $\widetilde{P_G(M)}$ into itself. This means that each φ_t is a local automorphism and X is an infinitesimal automorphism of $P_G(M)$.

(i)⇒(iii). Since $\tilde{\theta}$ is canonically associated with $\widetilde{P_G(M)}$, every automorphism,

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1) The bracket operation of the graded Lie algebra $\sum_{p=-1}^{\infty}\mathfrak{g}_p$ is defined as follows: If $A\in\mathfrak{g}_p$ and $B\in\mathfrak{g}_q$, then

$$[A, B](x_0, \cdots, x_{p+q})=\frac{1}{p!\,(q+1)!}\sum A(B(x_{j_0}, \cdots, x_{j_q}), x_{j_{q+1}}, \cdots, x_{j_{p+q}})$$
$$-\frac{1}{(p+1)!\,q!}\sum B(A(x_{k_0}, \cdots, x_{k_p}), x_{k_{p+1}}, \cdots, x_{k_{p+q}}),$$

for $x_0, \cdots, x_{p+q}\in\mathfrak{g}_{-1}=V$.

prolonged to $\widetilde{P_G(M)}$, leaves $\tilde{\theta}$ invariant.

(iii)$\Rightarrow$(iv). If $L_{\tilde{X}}\tilde{\theta}=0$, then

$$0=\tilde{X}\cdot\tilde{\theta}(\xi^*)=(L_{\tilde{X}}\tilde{\theta})(\xi^*)+\tilde{\theta}(L_{\tilde{X}}\xi^*)$$
$$=\tilde{\theta}(L_{\tilde{X}}\xi^*).$$

On the other hand, $\tilde{\theta}$ defines a complete parallelism on $\widetilde{P_G(M)}$. Hence we have $L_{\tilde{X}}\xi^*=0$.

(iv)$\Rightarrow$(i). Let $P(\tilde{u}_0)$ be the set of points in $\widetilde{P_G(M)}$ which can be joined to $\tilde{u}_0$ by an integral curve of a standard horizontal vector field. Then

$$\bigcup_{\tilde{u}_0\in\widetilde{P_G(M)}} P(\tilde{u}_0)=\widetilde{P_G(M)}.$$

From $L_{\tilde{X}}\xi^*=0$, we see that $\tilde{\varphi}_t$ leaves each $P(\tilde{u}_0)$ invariant and hence leaves $\widetilde{P_G(M)}$ invariant, that is, φ_t is a local automorphism of $P_G(M)$. Hence X is an infinitesimal automorphism. Q.E.D.

Let $\mathcal{L}$ be the sheaf of germs of infinitesimal automorphisms of $P_G(M)$. Let $\mathcal{L}_x$ be the stalk of $\mathcal{L}$ at $x\in M$. Then

$$\dim \mathcal{L}_x\leqq\dim \widetilde{P_G(M)}.$$

PROPOSITION 3. 5. *If* $\mathfrak{g}_0$ *contains the identity element and* $\dim\mathcal{L}_x=\dim\widetilde{P_G(M)}$ *at every point* x *of* M. *Then* $\Omega=0$.

Proof. Let E be the identity element in $\mathfrak{g}_0$ and E^* the vertical vector field on $\widetilde{P_G(M)}$ corresponding to E. Let ξ^* and η^* be the standard horizontal vector fields on $\widetilde{P_G(M)}$. Then we have

$$[E^*,\xi^*]=\xi^* \qquad\text{and}\qquad [E^*,\eta^*]=\eta^*.$$

This, together with Proposition 3. 3, implies

$$E^*\cdot\Omega^p(\xi^*,\eta^*)=(L_{E^*}\Omega^p)(\xi^*,\eta^*)+\Omega^p([E^*,\xi^*],\eta^*)+\Omega^p(\xi^*,[E^*,\eta^*])$$
$$=p\Omega^p(\xi^*,\eta^*)+\Omega^p(\xi^*,\eta^*)+\Omega^p(\xi^*,\eta^*)$$
$$=(p+2)\Omega^p(\xi^*,\eta^*).$$

On the other hand, if $\tilde{X}$ is the vector field of $\widetilde{P_G(M)}$ induced by an infinitesimal automorphism X of $P_G(M)$, then from Proposition 3. 4 we have

$$L_{\tilde{X}}\Omega=L_{\tilde{X}}\left(d\tilde{\theta}+\frac{1}{2}[\tilde{\theta},\tilde{\theta}]\right)=0.$$

This, together with Proposition 3. 4, implies

$$\tilde{X}\cdot\Omega(\xi^*,\eta^*)=(L_{\tilde{X}}\Omega)(\xi^*,\eta^*)+\Omega([\tilde{X},\xi^*],\eta^*)+\Omega(\xi^*,[\tilde{X},\eta^*])=0.$$

Since $\dim \mathcal{L}_x = \dim \widetilde{P_G(M)}$, for every point $\tilde{u}$ of $\widetilde{P_G(M)}$ there exists an infinitesimal automorphism X of $P_G(M)$ such that $\tilde{X}_{\tilde{u}} = E^*_{\tilde{u}}$. We have therefore

$$(p+2)\Omega^p(\xi^*, \eta^*)_{\tilde{u}} = (E^* \cdot \Omega^p(\xi^*, \eta^*))_{\tilde{u}}$$
$$= (\tilde{X} \cdot \Omega^p(\xi^*, \eta^*))_{\tilde{u}} = 0.$$

Since $\tilde{u}$ is an arbitrary point of $\widetilde{P_G(M)}$, we have $\Omega^p = 0$ and hence $\Omega = 0$. Q.E.D.

Thus we have the following

THEOREM. *Let $P_G(M)$ be a G-structure of order r and of type k on M. Let $s = \text{Max}\{r, k\}$. If it is possible to associate canonically a subbundle $\widetilde{P_G(M)}$ of $F^s(M)$ and if the linear isotropy algebra $\mathfrak{g}_0$ contains the identity element, then $P_G(M)$ is flat if and only if* $\dim \mathcal{L}_x = \dim \widetilde{P_G(M)}$ *at every point x of M.*

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