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PROLONGATIONS OF PSEUDOGROUP STRUCTURES TO TANGENT BUNDLES

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1. Introduction. Recently, K.Yano and S.Kobayashi [6] defined the notion of the prolongations of tensor fields to tangent bundle and A.Morimoto [3] studied the prolongations of G-structures to tangent bundle.

The purpose of the present note is to give some remarks on the prolongations of pseudogroup structures and almost structures on a manifold to its tangent bundle.

We summarize basic notations which will be used in the present note.

$T(M)$: tangent bundle of M
$T_x(M)$: tangent space of M at x
Tf : differential of a differentiable mapping f
$F^r(M)$: bundle of r-frames of M
$G^r(n)$: structure group of $F^r(M)$ ($n = \dim M$).

2. Prolongations of pseudogroups to tangent bundle. Let Γ be a pseudogroup of differentiable transformations of $\boldsymbol{R}^n$.

Let $i_x : T_x(\boldsymbol{R}^n) \to \boldsymbol{R}^n$ for $x \in \boldsymbol{R}^n$ be the canonical identification of $T_x(\boldsymbol{R}^n)$ with $\boldsymbol{R}^n$. For an element φ of Γ, we set $\overline{\varphi}_x = i_x^{-1} \circ \varphi \circ i_x$. Then $\overline{\varphi}_x$ is a differentiable transformation of a neighborhood of a point of $T_x(\boldsymbol{R}^n)$ into $T_x(\boldsymbol{R}^n)$. Let U be the domain of $\varphi \in \Gamma$. We define $\overline{\varphi} : T(U) \to T(U)$ as follows: For $(x, \dot{x}) \in T(U) = U \times \boldsymbol{R}^n$, $\overline{\varphi}(x, \dot{x}) = (x, \overline{\varphi}_x(\dot{x}))$. Then $\overline{\varphi}$ is a differentiable transformation of a subset of $T(U)$ into $T(U)$.

Let $\widetilde{\Gamma} = \{T\varphi \circ \overline{\psi} \mid \varphi, \psi \in \Gamma\}$. Then $\widetilde{\Gamma}$ is a pseudogroup of differentiable transformations of $T(\boldsymbol{R}^n)$. It is clear that *if Γ is transitive, so is $\widetilde{\Gamma}$.*

Let $\mathcal{L}$ be the sheaf of germs of all Γ-vector fields on $\boldsymbol{R}^n$ and $\widetilde{\mathcal{L}}$ the sheaf of germs of all vector fields on $T(\boldsymbol{R}^n)$, each of which is a sum of a complete lift and a vertical lift of Γ-vector fields on $\boldsymbol{R}^n$. Then $\widetilde{\mathcal{L}}$ is the sheaf of germs of all $\widetilde{\Gamma}$-vector fields on $T(\boldsymbol{R}^n)$. Let L be the stalk of $\mathcal{L}$ at the origin $0 \in \boldsymbol{R}^n$ and $\widetilde{L}$ the stalk of $\widetilde{\mathcal{L}}$ at the point $\overline{0} = (0, 0) \in T(\boldsymbol{R}^n)$. *If Γ is a transitive pseudogroup, then L and $\widetilde{L}$ are transitive filtered Lie algebras.*

Let $\sum_{p=-1}^{\infty} \mathfrak{g}_p$ with $\mathfrak{g}_{-1}=\boldsymbol{R}^n$ and $\sum_{p=-1}^{\infty} \widetilde{\mathfrak{g}}_p$ with $\widetilde{\mathfrak{g}}_{-1}=\boldsymbol{R}^{2n}$ be the associated graded Lie algebras of L and $\widetilde{L}$, respectively. Let $x^1, \cdots, x^n$ be a coordinate system in $\boldsymbol{R}^n$ and $x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n$ the canonically induced coordinate system in $T(\boldsymbol{R}^n)$. Let $X \in \sum \mathfrak{g}_p$. Then X can be written as $X=\sum x^i(\partial/\partial x^i)$ with

$$X^i = a^i + \sum a^i_j x^j + \frac{1}{2} \sum \Sigma a^i_{jk}\, x^j x^k + \cdots,$$

where a^i, a^i_j, $a^i_{jk}, \cdots$ are real numbers. It is clear that $(a^i) \in \boldsymbol{R}^n$, $(a^i_j) \in \mathfrak{g}_0$, $(a^i_{jk}) \in \mathfrak{g}_1, \cdots$.

Let X^c (resp. X^v) be the complete (resp. vertical) lift of X. Then we can easily see that

$$X^c = \sum (a^i + \sum a^i_j x^j + \frac{1}{2} \sum a^i_{jk} x^j x^k + \cdots) \frac{\partial}{\partial x^i} + \sum (a^i_j + \sum a^i_{jk} x^k + \cdots) \dot{x}^j \frac{\partial}{\partial \dot{x}^i}$$

and

$$X^v = \sum (a^i + \sum a^i_j x^j + \frac{1}{2} \sum a^i_{jk} x^j x^k + \cdots) \frac{\partial}{\partial \dot{x}^i}.$$

On the other hand, the associated graded Lie algebra $\sum \widetilde{\mathfrak{g}}_p$ of $\widetilde{L}$ is generated by $X^c + Y^v$ for $X, Y \in \sum \mathfrak{g}_p$. Let

$$X = \sum (a^i + \sum a^i_j x^j + \frac{1}{2} \sum a^i_{jk} x^j x^k + \cdots) \frac{\partial}{\partial x^i}$$

and

$$Y = \sum (b^i + \sum b^i_j x^j + \frac{1}{2} \sum b^i_{jk} x^j x^k + \cdots) \frac{\partial}{\partial x^i}.$$

Then we have

$$X^c + Y^v = \sum (a^i + \sum a^i_j x^j + \frac{1}{2} \sum a^i_{jk} x^j x^k + \cdots) \frac{\partial}{\partial x^i}$$

$$+ \sum \Big\{ (b^i + \sum b^i_j x^j + \frac{1}{2} \sum b^i_{jk} x^j x^k + \cdots)$$

$$+\sum(a_j^i+\sum a_{jk}^i x^k+\cdots)\dot{x}^j\Big\}\frac{\partial}{\partial\dot{x}^i}\,.$$

This implies

PROPOSITION 2.1. *Let $\sum\mathfrak{g}_p$ and $\sum\widetilde{\mathfrak{g}}_p$ be the associated graded Lie algebras of L and $\widetilde{L}$, respectively. Then we have*

$$\widetilde{\mathfrak{g}}_0=\left\{(a_\beta^\alpha)\in \boldsymbol{R}^{2n}\otimes(\boldsymbol{R}^{2n})^*\,\middle|\,(a_j^i)=(a_{j+n}^{i+n})\in\mathfrak{g}_0,\ (a^{i+n}_{j})\in\mathfrak{g}_0,\ (a_{j+n}^i)=0\right\}^{1)}$$

$$=\left\{\begin{pmatrix}A & 0\\ B & A\end{pmatrix}\in \boldsymbol{R}^{2n}\otimes(\boldsymbol{R}^{2n})^*\,\middle|\,A,\,B\in\mathfrak{g}_0\right\}$$

$$\cong\mathfrak{g}_0\times\mathfrak{g}_0\,,$$

$$\widetilde{\mathfrak{g}}_1=\Big\{(a_{\beta\gamma}^\alpha)\in \boldsymbol{R}^{2n}\otimes S^2(\boldsymbol{R}^{2n})^*\,\Big|\,(a_{jk}^i)=(a_{j+n,k}^{i+n})\in\mathfrak{g}_1,\,(a_{jk}^{i+n})\in\mathfrak{g}_1,$$

$$\text{all other components are zero}\Big\}$$

$$\cong\mathfrak{g}_1\times\mathfrak{g}_1,$$

$$\widetilde{\mathfrak{g}}_2=\Big\{(a_{\beta\gamma\delta}^\alpha)\in \boldsymbol{R}^{2n}\otimes S^3(\boldsymbol{R}^{2n})^*\,\Big|\,(a_{jkl}^i)=(a_{j+n,kl}^{i+n})\in\mathfrak{g}_2,\,(a_{jkl}^{i+n})\in\mathfrak{g}_2,$$

$$\text{all other components are zero}\Big\}$$

$$\cong\mathfrak{g}_2\times\mathfrak{g}_2\,,$$

$$\cdots\cdots.$$

COROLLARY 2.2. *If $\sum\mathfrak{g}_p$ is a graded Lie algebra of order r and of type k, so is $\sum\widetilde{\mathfrak{g}}_p$.*

COROLLARY 2.3. *If $\sum\mathfrak{g}_p$ is a graded Lie algebra of order 1, then $\sum\mathfrak{g}_p$ is involutive if and only if $\sum\widetilde{\mathfrak{g}}_p$ is involutive.*

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1) $\alpha,\beta,\gamma,\delta=1,2,\cdots,n,n+1,\cdots,2n.$
$i,j,k,l=1,2,\cdots,n.$

PROOF. Let $e_1, \cdots, e_n$ and $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{2n}$ be the canonical bases for $\boldsymbol{R}^n$ and $\boldsymbol{R}^{2n}$, respectively. Let

$$d_k = \dim \{t \in \mathfrak{g}_0 \mid [t, e_1] = \cdots = [t, e_k] = 0\}$$

and

$$\tilde{d}_\alpha = \dim \{t \in \tilde{\mathfrak{g}}_0 \mid [t, e_1] = \cdots = [t, e_\alpha] = 0\}.$$

Since $\tilde{\mathfrak{g}}_0 = \left\{ \begin{pmatrix} A & 0 \\ B & A \end{pmatrix} \middle| A, B \in \mathfrak{g}_0 \right\}$, we have

$$\tilde{d}_k = 2d_k \ (1 \leqq k < n)$$

and

$$\tilde{d}_\alpha = 0 \ (n \leqq \alpha < 2n).$$

This, together with Proposition 2.1, implies

$$\dim \tilde{\mathfrak{g}}_1 - \dim \tilde{\mathfrak{g}}_0 - \sum_{\alpha=1}^{2n-1} \tilde{d}_\alpha = 2\left\{\dim \mathfrak{g}_1 - \dim \mathfrak{g}_0 - \sum_{k=1}^{n-1} d_k\right\}.$$

Hence $\sum \mathfrak{g}_p$ is involutive if and only if $\sum \tilde{\mathfrak{g}}_p$ is involutive.

Q.E.D.

3. Prolongations of pseudogroup structures to tangent bundles. Let Γ be a pseudogroup of differentiable transformations of $\boldsymbol{R}^n$ and let M be a differentiable manifold of dimension n. A Γ-*atlas* on M is a collection of local diffeomorphisms $\{\lambda_i, U_i\}$ of M into $\boldsymbol{R}^n$ which satisfies $\cup U_i = M$ and $\lambda_i \circ \lambda_j^{-1} \in \Gamma$ for all i and j such that $U_i \cap U_j \neq \phi$. Two Γ-atlases are said to be *equivalent* if their union is a Γ-atlas. An equivalence class of Γ-atlases is called a Γ-*structure* on M.

First of all we prove the following

PROPOSITION 3.1. *If* $\{\lambda_i, U_i\}$ *is a* Γ*-atlas on* M, *then* $\{\bar{\varphi} \circ T\lambda_i, T(U_i)\}$ *is a* $\tilde{\Gamma}$*-atlas on* $T(M)$.

PROOF. If $\lambda_i : U_i \to \boldsymbol{R}^n$ and $\varphi \in \Gamma$, then $\bar{\varphi} \circ T\lambda_i : T(U_i) \to T(\boldsymbol{R}^n)$. Furthermore if $U_i \cap U_j \neq \phi$, then $(\bar{\varphi} \circ T\lambda_i) \circ (\bar{\psi} \circ T\lambda_j)^{-1}$ is a differentiable transformation of $(\bar{\psi} \circ T\lambda_j)(T(U_i \cap U_j))$ into $(\bar{\varphi} \circ T\lambda_i)(T(U_i \cap U_j))$.
Since

$$(\bar{\varphi}\circ T\lambda_i)\circ(\bar{\psi}\circ T\lambda_j)^{-1}=\bar{\varphi}\circ T\lambda_i\circ(T\lambda_j)^{-1}\circ\bar{\psi}^{-1}$$

$$=\bar{\varphi}\circ T\lambda_i\circ T\lambda_j^{-1}\circ\bar{\psi}^{-1}=\bar{\varphi}\circ T(\lambda_i\circ\lambda_j^{-1})\circ\bar{\psi}^{-1}$$

and $\lambda_i\circ\lambda_j^{-1}\in\Gamma$, we have

$$(\bar{\varphi}\circ T\lambda_i)\circ(\bar{\psi}\circ T\lambda_j)^{-1}\in\tilde{\Gamma}.$$

Hence $\{\bar{\varphi}\circ T\lambda_i,\ T(U_i)\}$ is a $\tilde{\Gamma}$-atlas on $T(M)$. Q.E.D.

THEOREM 3.2. *If M has a Γ-structure, then $T(M)$ has a $\tilde{\Gamma}$-structure.*

PROOF. Let $\{\lambda_i, U_i\}$ and $\{\lambda'_\alpha, U'_\alpha\}$ be two Γ-atlases on M. Then $\{\bar{\varphi}\circ T\lambda_i, T(U_i)\}$ and $\{\bar{\varphi}'\circ T\lambda'_\alpha, T(U'_\alpha)\}$ are $\tilde{\Gamma}$-atlases on $T(M)$. It suffices to prove that if $\{\lambda_i, U_i\}$ and $\{\lambda'_\alpha, U'_\alpha\}$ are equivalent, then $\{\bar{\varphi}\circ T\lambda_i, T(U_i)\}$ and $\{\varphi'\circ T\lambda'_\alpha, T(U'_\alpha)\}$ are equivalent. $\{\lambda_i, U_i\}$ and $\{\lambda'_\alpha, U'_\alpha\}$ are equivalent if and only if $\lambda'_\alpha\circ\lambda_i^{-1}\in\Gamma$ for all i and α such that $U_i\cap U'_\alpha\neq\phi$.

Suppose $\{\lambda_i, U_i\}$ and $\{\lambda'_\alpha, U'_\alpha\}$ are equivalent. Then we have

$$(\bar{\psi}'\circ T\lambda'_\alpha)(\bar{\varphi}\circ T\lambda_i)^{-1}=\bar{\psi}'\circ T\lambda'_\alpha\circ(T\lambda_i)^{-1}\circ\bar{\varphi}^{-1}$$
$$=\bar{\psi}'\circ T\lambda'_\alpha\circ T\lambda_i^{-1}\circ\bar{\varphi}^{-1}=\bar{\psi}'\circ T(\lambda'_\alpha\circ\lambda_i^{-1})\circ\bar{\varphi}^{-1}\in\tilde{\Gamma}$$

for all i and α such that $U_i\cap U'_\alpha\neq\phi$. This implies that $\{\bar{\varphi}\circ T\lambda_i, T(U_i)\}$ and $\{\bar{\varphi}'\circ T\lambda'_\alpha, T(U'_\alpha)\}$ are equivalent. Q.E.D.

4. **Prolongations of almost Γ-structures**. Following the notations of §2 let $\sum\mathfrak{g}_p$ and $\sum\tilde{\mathfrak{g}}_p$ be the associated graded Lie algebras of L and $\tilde{L}$, respectively. By Corollary 2.2 we can assume that both $\sum\mathfrak{g}_p$ and $\sum\tilde{\mathfrak{g}}_p$ are of order r.

Let G_0 (resp. $\tilde{G}_0$) be the Lie subgroup of $G^1(n)$ (resp. $G^1(2n)$) whose Lie algebra is $\mathfrak{g}_0$ (resp. $\tilde{\mathfrak{g}}_0$). Let G_1 (resp. $\tilde{G}_1$) be the semidirect product of G_0 (resp. $\tilde{G}_0$) and the nilpotent Lie group generated by $\mathfrak{g}_1+\mathfrak{g}_2+\cdots/\mathfrak{g}_2+\mathfrak{g}_3+\cdots$ (resp. $\tilde{\mathfrak{g}}_1+\tilde{\mathfrak{g}}_2+\cdots/\tilde{\mathfrak{g}}_2+\tilde{\mathfrak{g}}_3+\cdots$). Then G_1 (resp. $\tilde{G}_1$) is a Lie subgroup of $G^2(n)$ (resp. $G^2(2n)$). Inductively let G_{r-1} (resp. $\tilde{G}_{r-1}$) be the semidirect product of G_{r-2} (resp. $\tilde{G}_{r-2}$) and the nilpotent Lie group generated by $\mathfrak{g}_{r-1}+\mathfrak{g}_r+\cdots/\mathfrak{g}_r+\mathfrak{g}_{r+1}+\cdots$ (resp. $\tilde{\mathfrak{g}}_{r-1}+\tilde{\mathfrak{g}}_r+\cdots/\tilde{\mathfrak{g}}_r+\tilde{\mathfrak{g}}_{r+1}+\cdots$). Then G_{r-1} (resp. $\tilde{G}_{r-1}$) is a Lie subgroup of $G^r(n)$ (resp. $G^r(2n)$) whose Lie algebra is $\mathfrak{g}_0+\mathfrak{g}_1+\cdots/\mathfrak{g}_r+\mathfrak{g}_{r+1}+\cdots$ (resp. $\tilde{\mathfrak{g}}_0+\tilde{\mathfrak{g}}_1+\cdots/\tilde{\mathfrak{g}}_r+\tilde{\mathfrak{g}}_{r+1}+\cdots$). It is easily seen that $\tilde{G}_{r-1}$ is isomorphic with $T(G_{r-1})$. Let j^r_n: $T(G^r(n))\to G^r(2n)$ be the injective homomorphism so that

$\widetilde{G}_{r-1}=j^r_n(T(G_{r-1}))$. For the sake of simplicity we denote G_{r-1} (resp. $\widetilde{G}_{r-1}$) by G (resp. $\widetilde{G}$).

Let M be a differentiable manifold of dimension n. Let P be a G-structure on M, that is, a reduction of the structure group $G^r(n)$ of $F^r(M)$ to the subgroup G.

Let $j^r_M: T(F^r(M)) \rightarrow F^r(T(M))$ be the injection determined by $j^r_n: T(G^r(n)) \rightarrow G^r(2n)$. Then $j^r_M(T(P))$ is a $\widetilde{G}$-structure on $T(M)$, that is, a reduction of the structure group $G^r(2n)$ of $F^r(T(M))$ to the subgroup $\widetilde{G}$. We shall call the $\widetilde{G}$-structure *the prolongation* of P and denote it by $\widetilde{P}$.

A (local) diffeomorphism of M (resp. $T(M)$) is a (local) *G-automorphism* (resp. *$\widetilde{G}$-automorphism*) if and only if it leaves the G-structure P (resp. $\widetilde{G}$-structure $\widetilde{P}$) invariant. A $\widetilde{G}$-automorphism f of a $\widetilde{G}$-structure $\widetilde{P}$ is said to be *fibre-preserving* if f maps a fibre of $T(M) \rightarrow M$ into a fibre.

THEOREM 4.1. *Let $\widetilde{P}$ be the prolongation of P. Then every (local) $\widetilde{G}$-automorphism is fibre-preserving.*

PROOF. Let $\pi^r: F^r(M) \rightarrow F^1(M)$ and $\widetilde{\pi}^r: F^r(T(M)) \rightarrow F^1(T(M))$ be the natural projections. We shall denote by the same letters the natural projections $\pi^r: G^r(n) \rightarrow G^1(n)$ and $\widetilde{\pi}^r: G^r(2n) \rightarrow G^1(2n)$ so that $\pi^r(G)=G_0$ and $\widetilde{\pi}^r(\widetilde{G})=\widetilde{G}_0$.

Let $P_0=\pi^r(P)$ and $\widetilde{P}_0=\widetilde{\pi}^r(\widetilde{P})$. If f is a G-automorphism (resp. $\widetilde{G}$-automorphism), then it is necessarily a G_0-automorphism (resp. $\widetilde{G}_0$-automorphism).

Let f be a local diffeomorphism of $T(M)$ and let $T(U)$ and $T(V)$ be open sets of $T(M)$ such that f maps $T(U)$ onto $T(V)$. Let $x \in T(U)$ and $y \in T(V)$ such that $f(x)=y$. Let $x^1, \cdots, x^n, x^{n+1}, \cdots, x^{2n}$ with $x^{n+i}=\dot{x}^i$ (resp. $y^1, \cdots, y^n, y^{n+1}, \cdots, y^{2n}$ with $y^{n+i}=\dot{y}^i$) be a local coordinate system at $x \in T(U)$ (resp. $y \in T(V)$). Furthermore we assume that $T(U)$ and $T(V)$ are so small that they admit local cross sections $\sigma: T(U) \rightarrow \widetilde{P}_0$ and $\tau: T(V) \rightarrow \widetilde{P}_0$, respectively. If f is a (local) $\widetilde{G}$-automorphism, then it is a (local) $\widetilde{G}_0$-automorphism and hence there is a mapping g of $T(U)$ into $\widetilde{G}_0$ such that

$$\widetilde{f}(\sigma(x)) = \tau(f(x)) \cdot g(x), \tag{4.1}$$

where $\widetilde{f}$ denotes the prolongation of f to $F^1(T(M))$. The local cross sections σ and τ are expressed by

$$\sigma(x)=\left(x\,; \cdots, \sum \sigma^\alpha_\beta \left(\frac{\partial}{\partial x_\alpha}\right)_x, \cdots\right)$$

and

$$\tau(y)=(y;\cdots,\sum\tau_\beta^\alpha\left(\frac{\partial}{\partial y_\alpha}\right)_y,\cdots),$$

where σ_β^α and τ_β^α are differentiable functions on $T(U)$ and $T(V)$, respectively. Let $f=(f^\alpha)$ and $g=(g_\beta^\alpha)$. Then, from (4.1), we have

$$\sum\sigma_\beta^\gamma(x)\cdot\left(\frac{\partial f^\alpha}{\partial x^\gamma}\right)_x=\sum\tau_\gamma^\alpha(f(x))\cdot g_\beta^\gamma(x).$$

Since (τ_β^α) is non-singular, we denote by $(\overline{\tau_\beta^\alpha})$ the inverse matrix of (τ_β^α). Then we have

$$\sum\overline{\tau_\gamma^\alpha}(f(x))\cdot\sigma_\beta^\lambda(x)\cdot\left(\frac{\partial f^\gamma}{\partial x^\lambda}\right)_x=g_\beta^\alpha(x).$$

Since the matrix $((g_\beta^\alpha(x))$ belongs to $\widetilde{G}_0$, we have

$$\left(\sum\overline{\tau_\gamma^\alpha}(f(x))\cdot\sigma_\beta^\lambda(x)\cdot\left(\frac{\partial f^\gamma}{\partial x^\lambda}\right)_x\right)\in\widetilde{G}_0.$$

Since every element of $\widetilde{G}_0$ is of the form $\begin{pmatrix}a & 0\\ * & a\end{pmatrix}$ with $a\in G_0$, we have

$$\sum\overline{\tau_\gamma^i}(f(x))\cdot\sigma_{j+n}^\lambda(x)\cdot\left(\frac{\partial f^\gamma}{\partial x^\lambda}\right)_x=0\quad(i,j=1,2,\cdots,n). \tag{4.2}$$

We can take $\sigma: T(U)\to\widetilde{P}_0$ and $\tau: T(V)\to\widetilde{P}_0$ as follows: Let $\phi: U\to P_0$ (resp. $\psi: V\to P_0$) be local cross section and set $\sigma=j_M^1\circ T\phi$ (resp. $\tau=j_M^1\circ T\psi$). Then σ (resp. τ) is a local cross section of $T(U)$ (resp. $T(V)$) into $\widetilde{P}_0$ and

$$(\sigma_\beta^\alpha)=\begin{pmatrix}\phi_j^i & 0\\ \sum\frac{\partial\phi_j^i}{\partial x^k}x^{k+n} & \phi_j^i\end{pmatrix}\left(\text{resp. }(\tau_\beta^\alpha)=\begin{pmatrix}\psi_j^i & 0\\ \sum\frac{\partial\psi_j^i}{\partial y^k}y^{k+n} & \psi_j^i\end{pmatrix}\right)$$

where (ϕ_j^i) (resp. (ψ_j^i)) denotes the non-singular matrix which represents the local cross section ϕ (resp. ψ) ([3]). It is clear that the matrix $(\overline{\tau_\beta^\alpha})$ is of the form

$$(\overline{\tau_\beta^\alpha})=\begin{pmatrix}\overline{\psi_j^i} & 0\\ * & \overline{\psi^i}\end{pmatrix},$$

where $(\bar{\psi}^i_j)=(\psi^i_j)^{-1}$. If we take σ and τ as above, then, from (4.2), we have

$$\sum \overline{\tau^i_k}(f(x))\cdot\sigma^{l+n}_{j+n}(x)\cdot\left(\frac{\partial f^k}{\partial x^{l+n}}\right)_x = 0 .$$

Since $(\overline{\tau^i_k})=(\overline{\psi^i_k})$ and $(\sigma^{l+n}_{j+n})=(\phi^l_j)$ are non-singular, we have

$$\left(\frac{\partial f^k}{\partial x^{l+n}}\right)_x = 0 .$$

This implies that f is fibre-preserving. Q.E.D.

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