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Exact solutions to the unsteady equations of perfect gases through Lie group analysis and substitution principles

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Dedicated to the memory of Prof. A. Donato, unforgettable friend and colleague

Abstract

In this paper, we consider the unsteady equations that govern two- and three-dimensional flows of a perfect gas. We explicitly characterize various classes of exact solutions by introducing some invertible transformations suggested by the invariance with respect to Lie groups of point symmetries and using suitable transformations known in literature as substitution principles. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The explicit determination of exact solutions to systems of partial differential equations (PDEs) of physical relevance is of great interest; besides its own intrinsic interest, these solutions (especially, when they contain arbitrary functions) may be used for modelling asymptotic limits of more complicated solutions, or for testing numerical procedures, or for solving special initial and/or boundary value problems.

One of the most powerful methods in order to determine particular solutions to PDEs is based upon the study of their invariance with respect to one-parameter Lie group of point transformations (see [1–7]). Once the Lie groups that leave assigned PDEs invariant are known, usually one tries to determine the corresponding similarity solutions, by solving the overdetermined system given by the original system augmented by the invariant surface conditions.

Moreover, important classes of exact solutions may be recovered by investigating the compatibility of a given system of PDEs with some differential and/or algebraic constraints [8,9]. Alternatively, one may look at the constraints to be imposed to a given set of PDEs in order to have their invariance with respect to an assigned family of transformations; for instance, this is the case of the result known in literature as Substitution Principle and originally introduced for the steady equations of ideal gas-dynamics [10,11] and ideal magneto-gas-dynamics [12]. Substitution Principles have

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been also given for the unsteady equations of ideal gas-dynamics with a separable equation of state having the pressure steady [13], and for the unsteady equations of ideal magneto-gas-dynamics having the total magnetic pressure steady [14].

Quite recently Oliveri [15,16] proved that the Substitution Principle for unsteady gas-dynamics can be obtained within the context of Lie group analysis and found a Substitution Principle for a class of solutions governing unsteady three-dimensional flows of a perfect gas, with adiabatic index equal to $\frac{5}{3}$, having also the pressure unsteady. Furthermore, in [17], the Substitution Principle to the unsteady equations of ideal magneto-gas-dynamics has been slightly generalized and a Substitution Principle for planar flows with a transverse magnetic field and adiabatic index equal to 2 has been found by means of Lie group analysis. Also, in [18] it has been established a Substitution Principle for the solutions of Galilean systems. Finally, in [19–21], Substitution Principles for n-dimensional flows of perfect fluids with adiabatic index $\Gamma = (n+2)/n$ have been given, as well as some examples of application.

Besides the search of similarity solutions, another relevant use of Lie point symmetries admitted by given PDEs consists in introducing some invertible point transformations that map the original system to an equivalent one (see for instance [22–26]), that can be managed (or for which exact solutions may be determined) more easily.

In the present paper, we shall consider the equations that govern the unsteady motion of an anisentropic perfect gas subject to no extraneous force and determine explicitly various classes of exact solutions by introducing some transformations that map the equations at hand to an equivalent autonomous form, and using the transformation referred to as Substitution Principle; in [27] the same techniques used in this paper have been successfully applied to the steady equations of perfect gases.

The scope of the paper is twofold: first of all, we want to show that Lie group analysis may be used not only for determining special symmetry reductions, but also for transforming the equations at hand into an equivalent form whose "simple" solutions provide non-trivial solutions when expressed in terms of the original variables; furthermore, we want to exhibit physically admissible solutions of perfect gases that verify the constraints to be required in order the substitution principles apply (to the best of authors' knowledge there are no relevant unsteady solutions to the equations of perfect gases suitable to be used with the substitution principle). The plan of the paper is as follows. In Section 2, we present the Lie group of point transformations that leave the equations at hand invariant. In Section 3 we report, for the sake of clarity, the theorems that will be used through the rest of paper. In Sections 4 and 5 we consider the unsteady two- and three-dimensional equations, respectively, and determine explicitly various classes of exact solutions. Finally, in Section 6 new (substituted) solutions containing up to two arbitrary functions are generated by applying the Substitution Principle.

2. Lie group analysis

The equations of a perfect gas subject to no extraneous force are

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p + \Gamma p \nabla\cdot\mathbf{v} = 0,$$

$$p^{1/\Gamma} s\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) + \nabla p = 0,$$

$$\frac{\partial s}{\partial t} + \mathbf{v}\cdot\nabla s = 0, \tag{2.1}$$

where $p(t,\mathbf{x})$ is the pressure, $s(t,\mathbf{x})$ (a function of) the entropy, $\mathbf{v}(t,\mathbf{x}) = (v_1(t,\mathbf{x}),\ldots,v_n(t,\mathbf{x}))$ the velocity vector, t the time and $\mathbf{x} = (x_1,\ldots,x_n)$ the spatial Cartesian rectangular coordinates, $\nabla = (\partial/\partial x_1,\ldots,\partial/\partial x_n)$ ($n = 2,3$ in the sequel), and Γ the adiabatic index; moreover, $\rho = p^{1/\Gamma}s$ is the mass density.

By straightforward analysis, it is found that the Lie groups of point transformations that leave system (2.1) invariant constitute a $((n^2+3n+8)/2)$-dimensional Lie algebra generated by the following infinitesimal operators (see also [2]):

$$\Xi_1 = \frac{\partial}{\partial t}, \quad \Xi_{k+1} = \frac{\partial}{\partial x_k},$$

$$\Xi_{n+2} = t\frac{\partial}{\partial t} + \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k},$$

$$\Xi_{n+3} = -t\frac{\partial}{\partial t} + \sum_{k=1}^{n} v_k \frac{\partial}{\partial v_k} + \frac{2\Gamma}{\Gamma - 1} p \frac{\partial}{\partial p},$$

$$\Xi_{n+4} = \frac{\Gamma}{\Gamma - 1} p \frac{\partial}{\partial p} + s\frac{\partial}{\partial s},$$

$$\Xi_{n+4+k} = t\frac{\partial}{\partial x_k} + \frac{\partial}{\partial v_k},$$

$$\Xi_{2n+4+l} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} + v_i \frac{\partial}{\partial v_j} - v_j \frac{\partial}{\partial v_i}, \qquad (2.2)$$

where $i, j, k = 1, \ldots, n$, $j > i$, and $l = 1, \ldots, n(n-1)/2$. The operators $\Xi_1, \ldots, \Xi_{n+1}$ characterize time and space translations, Ξ_{n+2}, Ξ_{n+3} and Ξ_{n+4} stretching transformations, $\Xi_{n+5}, \ldots, \Xi_{2n+4}$ the Galilean transformations; finally, the remaining operators characterize spatial rotations.

If $\Gamma = (n+2)/n$ [2] then we also have the invariance with respect to the so-called projective group that is generated by the infinitesimal operator

$$\Xi^{\star} = t^2\frac{\partial}{\partial t} + \sum_{k=1}^{n} x_k t \frac{\partial}{\partial x_k} - (n+2)pt\frac{\partial}{\partial p} + \sum_{k=1}^{n} (x_k - v_k t)\frac{\partial}{\partial v_k}. \qquad (2.3)$$

The use of the projective group enables us to generate of unsteady solutions starting from steady solutions (see [27]).

3. The theorems

As remarked above, the knowledge of the Lie point symmetries admitted by a system of PDEs may be employed to characterize classes of invariant solutions. But, one may look for the introduction of suitable invertible point transformations allowing one to map the given system of PDEs to an equivalent form for which classes of exact solutions may be found more simply. The latter task may be accomplished by means of the following theorem (for the details of the proof see [23,25]).

Theorem 1. *The general first-order system of partial differential equations*

$$\Omega_A\left(x_i, u_B, \frac{\partial u_B}{\partial x_i}\right) = 0 \quad (i = 1, \ldots, n; A, B = 1, \ldots, N) \qquad (3.1)$$

can be transformed by the invertible point transformation

$$X_i = X_i(x_j, u_B), \quad U_A = U_A(x_j, u_B) \qquad (3.2)$$

to the autonomous form

$$\hat{\Omega}_A\left(U_B, \frac{\partial U_B}{\partial X_i}\right) = 0, \qquad (3.3)$$

if and only if it is left invariant by n independent Lie groups of point transformations whose infinitesimal operators Ξ_i $(i = 1, \ldots, n)$ satisfy the conditions:

$$[\Xi_i, \Xi_j] = 0 \quad (i,j = 1, \ldots, n), \qquad (3.4)$$

denoting $[\cdot,\cdot]$ the commutator of two operators. Conditions (3.4) *mean that the operators Ξ_i $(i = 1, \ldots, n)$ generate a n-dimensional Abelian Lie algebra.*

The invertible point transformation (3.2) is built by considering the canonical variables associated to the n infinitesimal operators. This theorem can be applied also when the original system (3.1) is autonomous: in this case we get an equivalent autonomous system. If we determine solutions (even simple, for example, constant) of the transformed system (3.3) then we obtain, via (3.2), solutions to the original system (3.1). In what follows, this theorem will be the first tool that we shall use in order to build the exact solutions.

Also, it is of great importance to determine solutions that contain some arbitrary functions in order to have more degrees of freedom when solving initial and/or boundary value problems. To do so, we shall make use of some results known in literature as Substitution Principles. Hence, let us give a brief overview to the involved theorems.

Munk and Prim [10] and Prim [11] derived the remarkable result named Substitution Principle and stated as follows:

Theorem 2. *The steady equations* (*i.e., no time dependence*) *of an inviscid, thermally non-conducting* *gas are invariant with respect to the following transformation*:

$$p^*(\mathbf{x}) = p(\mathbf{x}), \quad \mathbf{v}^*(\mathbf{x}) = \frac{\mathbf{v}(\mathbf{x})}{m(\mathbf{x})}, \quad s^*(\mathbf{x}) = m^2(\mathbf{x})s(\mathbf{x}),$$

where $m(\mathbf{x})$ *represents a smooth scalar function of the space variables subjected to the constraint*

$$\mathbf{v}\cdot\nabla m = 0,$$

that is $m(\mathbf{x})$ *is constant along each individual streamline.*

Smith [13] extended this result to the class of unsteady flows having steady the pressure. A slight generalization, that has been given in [21], can be stated with the following theorem.

Theorem 3. *If*

$$\{p(t,\mathbf{x}),\ \mathbf{v}(t,\mathbf{x}),\ s(t,\mathbf{x})\}$$

represents a solution to the equations of perfect gases, then another solution is

$$\{[m(\mathbf{x})]^{-\beta}p(mt + h(m),\mathbf{x}),\ m(\mathbf{x})\mathbf{v}(mt + h(m),\mathbf{x}),$$

$$[m(\mathbf{x})]^{-\gamma}s(mt + h(m),\mathbf{x})\}$$

with

$$\gamma = \frac{(\Gamma - 1)\beta}{\Gamma} + 2,$$

provided that

$$\mathbf{v}\cdot\nabla m = 0, \quad \frac{\partial p}{\partial t} - \frac{\beta p}{t + h'(m)} = 0,$$

where $m(\mathbf{x})$ *is a smooth scalar function of* $\mathbf{x}$, *and* $h(m)$ *an arbitrary function of its argument if* $\beta = 0$, *or* $h(m) = h_0 m - h_0$ (*with* h_0 *constant*) *if* $\beta \neq 0$.

A further "mixed" Substitution Principle, enabling us to provide when $\Gamma = (n+2)/n$ unsteady solutions from steady solutions, has been also obtained (see [16,21,27]) by using the special form of the reduced system that is found by looking for the solutions invariant with respect to the projective group and the transformation given in Theorem 2.

4. Unsteady equations in 2D

Here let us focus our attention to the unsteady two-dimensional equations. By specializing the results reported in Section 2, we have that the Lie groups of invariance constitute a nine-dimensional Lie algebra. Moreover, if $\Gamma = 2$, we also have the infinitesimal operator of the projective group corresponding to (2.3).

It is easy to verify that the non-commuting operators are

$$[\Xi_1,\Xi_4] = \Xi_1, \quad [\Xi_1,\Xi_5] = -\Xi_1, \quad [\Xi_1,\Xi_7] = \Xi_2,$$
$$[\Xi_1,\Xi_8] = \Xi_3, \quad [\Xi_1,\Xi_{10}] = \Xi_5 - \Xi_4,$$
$$[\Xi_2,\Xi_4] = \Xi_2, \quad [\Xi_2,\Xi_9] = \Xi_3, \quad [\Xi_2,\Xi_{10}] = -\Xi_7,$$
$$[\Xi_3,\Xi_4] = \Xi_1, \quad [\Xi_3,\Xi_9] = -\Xi_2, \quad [\Xi_3,\Xi_{10}] = -\Xi_8,$$
$$[\Xi_5,\Xi_7] = -\Xi_7, \quad [\Xi_5,\Xi_8] = -\Xi_8, \quad [\Xi_5,\Xi_{10}] = \Xi_{10},$$
$$[\Xi_7,\Xi_9] = \Xi_8, \quad [\Xi_8,\Xi_9] = -\Xi_7, \quad [\Xi_4,\Xi_{10}] = \Xi_{10}.$$

Since we have three independent variables, in order to apply Theorem 1, we need three independent commuting operators that we build by taking three independent linear combinations of the operators $\Xi_1, \ldots, \Xi_{10}$:

$$\Xi_A = \sum_{i=1}^{10} \alpha_i \Xi_i, \quad \Xi_B = \sum_{i=1}^{10} \beta_i \Xi_i, \quad \Xi_C = \sum_{i=1}^{10} \gamma_i \Xi_i,$$

where α_i, β_i, γ_i, $(i = 1, \ldots, 10)$ are constants. Of course, we shall take $\alpha_{10} = \beta_{10} = \gamma_{10} = 0$ unless we have $\Gamma = 2$.

In order the operators Ξ_A, Ξ_B and Ξ_C generate a three-dimensional Abelian Lie algebra, we have to satisfy the following conditions:

$$\alpha_{[2}\beta_{4]} - \alpha_{[3}\beta_{9]} + \alpha_{[1}\beta_{7]} = 0,$$

$$\alpha_{[2}\beta_{9]} + \alpha_{[3}\beta_{4]} + \alpha_{[1}\beta_{8]} = 0,$$

$$\alpha_{[8}\beta_{9]} + \alpha_{[5}\beta_{7]} + \alpha_{[2}\beta_{10]} = 0,$$

$$\alpha_{[8}\beta_{5]} + \alpha_{[7}\beta_{9]} + \alpha_{[3}\beta_{10]} = 0,$$

$$\alpha_{[1}\beta_{4]} - \alpha_{[1}\beta_{5]} = 0,$$

$$\alpha_{[10}\beta_{4]} - \alpha_{[10}\beta_{5]} = 0,$$

$$\alpha_{[1}\beta_{10]} = 0,$$

$$\alpha_{[2}\gamma_{4]} - \alpha_{[3}\gamma_{9]} + \alpha_{[1}\gamma_{7]} = 0,$$

$$\alpha_{[2}\gamma_{9]} + \alpha_{[3}\gamma_{4]} + \alpha_{[1}\gamma_{8]} = 0,$$

$$\alpha_{[8}\gamma_{9]} + \alpha_{[5}\gamma_{7]} + \alpha_{[2}\gamma_{10]} = 0,$$

$$\alpha_{[8}\gamma_{5]} + \alpha_{[7}\gamma_{9]} + \alpha_{[3}\gamma_{10]} = 0,$$

$$\alpha_{[1}\gamma_{4]} - \alpha_{[1}\gamma_{5]} = 0,$$

$$\alpha_{[10}\gamma_{4]} - \alpha_{[10}\gamma_{5]} = 0,$$

$$\alpha_{[1}\gamma_{10]} = 0,$$

$$\beta_{[2}\gamma_{4]} - \beta_{[3}\gamma_{9]} + \beta_{[1}\gamma_{7]} = 0,$$

$$\beta_{[2}\gamma_{9]} + \beta_{[3}\gamma_{4]} + \beta_{[1}\gamma_{8]} = 0,$$

$$\beta_{[8}\gamma_{9]} + \beta_{[5}\gamma_{7]} + \beta_{[2}\gamma_{10]} = 0,$$

$$\beta_{[8}\gamma_{5]} + \beta_{[7}\gamma_{9]} + \beta_{[3}\gamma_{10]} = 0,$$

$$\beta_{[1}\gamma_{4]} - \beta_{[1}\gamma_{5]} = 0,$$

$$\beta_{[10}\gamma_{4]} - \beta_{[10}\gamma_{5]} = 0,$$

$$\gamma_{[10}\beta_{1]} = 0,$$

where we have used the notation $w_{[i}z_{j]} = w_i z_j - w_j z_i$.

The application of Theorem 1 leads us to distinguish various cases according to the choices of the parameters α_i, β_i and γ_i ($i = 1,\ldots,10$). In general, we will be able to introduce a variable transformation having the form

$$T = \sum_{i=1}^{3} \frac{a_i\phi_i}{\Delta}, \quad X_1 = \sum_{i=1}^{3} \frac{b_i\phi_i}{\Delta}, \quad X_2 = \sum_{i=1}^{3} \frac{c_i\phi_i}{\Delta},$$

$$v_1 = \phi_4 V_1 - \phi_5 V_2 + \chi_1(\alpha_{10}x_1 + e_1) + \chi_2 e_2,$$

$$v_2 = \phi_5 V_1 + \phi_4 V_2 + \chi_1(\alpha_{10}x_2 + e_3) + \chi_2 e_4,$$

$$p = \left(\chi_3^2 \exp\left(\sum_{i=1}^{3} \frac{d_i\phi_i}{\Delta} + 2f_1\phi_2 - 2f_2\phi_3\right)\right)^{\Gamma/(\Gamma-1)} P,$$

$$s = \exp\left(\sum_{i=1}^{3} \frac{d_i\phi_i}{\Delta}\right) S, \tag{4.1}$$

where a_i, b_i, c_i, $d_i = \alpha_6 a_i + \beta_6 b_i + \gamma_6 c_i$ ($i = 1,2,3$), e_j ($j = 1,2,3,4$) and Δ (that cannot be vanishing) are suitable constants linked to the constants α_k, β_k and γ_k ($k = 1,\ldots,10$). Moreover, ϕ_i ($i = 1,\ldots,5$) are suitable functions of x_1, x_2 and t, whereas, χ_1, χ_2 and χ_3 are suitable functions of t; finally, f_1 and f_2 are constants belonging to the set $\{-1,0,1\}$ according to the various cases.

The explicit form of the constants and the functions involved in this transformation will be given in the sequel. Finally, T, X_1, X_2 represent the new independent variables, whereas, V_1, V_2, P and S represent the new dependent variables.

Such a transformation allows us to transform system (2.1), specialized in 2 space dimensions, to the following equivalent autonomous form:

$$\begin{aligned}
&\frac{\partial}{\partial T}\Big((a_2V_1 + a_1V_2 + a_3)P^{1/\Gamma})\\
&\quad + \frac{\partial}{\partial X_1}((b_2V_1 + b_1V_2 + b_3)P^{1/\Gamma}\Big)\\
&\quad + \frac{\partial}{\partial X_2}((c_2V_1 + c_1V_2 + c_3)P^{1/\Gamma})\\
&\quad + \frac{d_2V_1 + d_1V_2 + d_3 + 2\Delta(f_1\Gamma V_1 - f_2)}{\Gamma - 1}P^{1/\Gamma}\\
&\quad = 0,
\end{aligned}$$

$$\begin{aligned}
&P^{1/\Gamma}S\Big((a_2V_1 + a_1V_2 + a_3)\frac{\partial V_1}{\partial T}\\
&\quad + (b_2V_1 + b_1V_2 + b_3)\frac{\partial V_1}{\partial X_1}\\
&\quad + (c_2V_1 + c_1V_2 + c_3)\frac{\partial V_1}{\partial X_2}\\
&\quad + \Delta(f_1(V_1^2 - V_2^2) - f_2V_1 + e_2) + \alpha_1\alpha_{10}\Big)\\
&\quad + a_2\frac{\partial P}{\partial T} + b_2\frac{\partial P}{\partial X_1} + c_2\frac{\partial P}{\partial X_2}\\
&\quad + \frac{\Gamma(d_2 + 2f_1\Delta)}{\Gamma - 1}P = 0,
\end{aligned}$$

$$\begin{aligned}
&P^{1/\Gamma}S\Big((a_2V_1 + a_1V_2 + a_3)\frac{\partial V_2}{\partial T}\\
&\quad + (b_2V_1 + b_1V_2 + b_3)\frac{\partial V_2}{\partial X_1}
\end{aligned}$$

$$+ (c_2 V_1 + c_1 V_2 + c_3)\frac{\partial V_2}{\partial X_2}$$

$$+ \Delta((2f_1 V_1 - f_2)V_2 + e_4)\Big)$$

$$+ a_1\frac{\partial P}{\partial T} + b_1\frac{\partial P}{\partial X_1} + c_1\frac{\partial P}{\partial X_2} + \frac{d_1\Gamma}{\Gamma - 1}P = 0,$$

$$(a_2 V_1 + a_1 V_2 + a_3)\frac{\partial S}{\partial T}$$

$$+ (b_2 V_1 + b_1 V_2 + b_3)\frac{\partial S}{\partial X_1}$$

$$+ (c_2 V_1 + c_1 V_2 + c_3)\frac{\partial S}{\partial X_2}$$

$$+ (d_2 V_1 + d_1 V_2 + d_3)S = 0. \tag{4.2}$$

$$\Delta = \frac{b_2 c_3 - b_3 c_2}{\alpha_9},$$

$$a_1 = \beta_{[4}\gamma_{5]}, \quad a_2 = \beta_{[9}\gamma_{4]} - \beta_{[9}\gamma_{5]}, \quad a_3 = \beta_{[4}\gamma_{9]},$$

$$b_1 = \alpha_{[5}\gamma_{4]}, \quad b_2 = \alpha_{[4}\gamma_{9]} - \alpha_{[5}\gamma_{9]}, \quad b_3 = \alpha_{[9}\gamma_{4]},$$

$$c_1 = \alpha_{[4}\beta_{5]}, \quad c_2 = \alpha_{[9}\beta_{4]} - \alpha_{[9}\beta_{5]}, \quad c_3 = \alpha_{[4}\beta_{9]}.$$

4.1. The case Γ arbitrary

First of all, let us consider the case in which $\alpha_9 \neq 0$ and $\alpha_4 \neq \alpha_5$; moreover, we begin by assuming $\Gamma \neq 2$ (that implies $\alpha_{10} = \beta_{10} = \gamma_{10} = 0$). In the sequel, if not otherwise stated, we shall take $\alpha_i = \beta_i = \gamma_i = 0$ $(i = 1,2,3,7,8)$, i.e., we neglect time and space translations and the Galilean transformations. These assumptions will not determine any loss of generality to the results but provide significant simplifications in the calculations: of course, we may include time and space translations and the Galilean transformations directly in the solutions we shall find by simply substituting t with $t - k_0$, x_1 with $x_1 - k_1 t - k_2$, x_2 with $x_2 - k_3 t - k_4$, V_1 with $V_1 - k_1$, and V_2 with $V_2 - k_3$ where k_0, k_1, k_2, k_3 and k_4 are arbitrary constants.

Under these hypotheses, transformation (4.1) and system (4.2) specialize along with the conditions:

$$\phi_1 = \arctan\frac{x_2}{x_1}, \quad \phi_2 = \ln\sqrt{x_1^2 + x_2^2}, \quad \phi_3 = \ln t,$$

$$\phi_4 = \frac{x_1}{t}, \quad \phi_5 = \frac{x_2}{t}, \quad \chi_1 = \chi_2 = 0, \quad \chi_3 = 1,$$

$$e_1 = e_2 = e_3 = e_4 = 0,$$

$$f_1 = f_2 = 1,$$

Now, let us start by considering the constant solutions of the transformed system we obtain. Essentially, two cases must be distinguished according to the possible choices of the constants therein involved. In the first one, we get a solution that, expressed in terms of the original variables, is

$$v_1 = \frac{x_1}{\Gamma t} - \frac{b + (\Gamma - 1)c}{a\Gamma}\frac{x_2}{t},$$

$$v_2 = \frac{x_2}{\Gamma t} + \frac{b + (\Gamma - 1)c}{a\Gamma}\frac{x_1}{t},$$

$$p = p_0\left(\exp(a\phi_1)\frac{r^{c-b+2}}{t^{c+2}}\right)^{\Gamma/(\Gamma-1)},$$

$$s = \frac{a^2\Gamma^3 p_0^{1-1/\Gamma}}{(\Gamma - 1)(\Gamma - 2)(b + (\Gamma - 1)c)}\exp(a\phi_1)\frac{r^{c-b}}{t^c}, \tag{4.3}$$

where p_0, a, b and c are arbitrary constants, $r = \exp(\phi_2)$, along with the condition

$$\frac{(a^2 + b^2 + c^2)(\Gamma - 1) + (2 + \Gamma(\Gamma - 2))bc}{2(\Gamma - 2)(b + (\Gamma - 1)c)} - 1 = 0.$$

The second solution (always expressed in terms of the original variables) is:

$$v_1 = \frac{x_1}{\Gamma t}, \quad v_2 = \frac{x_2}{\Gamma t},$$

$$p = \frac{p_0}{t^2}\left(\frac{r^\Gamma}{t}\right)^{(2+\Gamma c)/(\Gamma-1)},$$

$$s = \frac{(2 + \Gamma c)\Gamma^3 p_0^{1-1/\Gamma}}{(\Gamma - 1)^2}\left(\frac{r^\Gamma}{t}\right)^c. \tag{4.4}$$

Further classes of solutions may be found by assuming V_1 to be constant, $V_2 = 0$ and P, S non-constant.

After straightforward integrations and substitutions we are able to recover the solution

$$v_1 = \frac{x_1}{\Gamma t}, \quad v_2 = \frac{x_2}{\Gamma t}, \quad p = \frac{\Psi(\omega)}{t^2},$$

$$s = \frac{\Gamma^3 \omega^{1-2/\Gamma}}{(\Gamma - 1)\Psi^{1/\Gamma}} \frac{\mathrm{d}\Psi}{\mathrm{d}\omega}, \tag{4.5}$$

where $\Psi(\omega)$ is an arbitrary function of $\omega = r^\Gamma/t$. It is immediate to note that solution (4.4) is a particular instance of this solution when we take $\Psi = p_0 \omega^{(2+\Gamma c)/(\Gamma-1)}$.

If we assume $\alpha_4 = \alpha_5$, then we must have also $\beta_4 = \beta_5$ and $\gamma_4 = \gamma_5$; in order to have a map in which the new independent variables depend only on the old independent variables, we must suppose $\alpha_1 \neq 0$. Moreover, we take $\alpha_i = \beta_i = \gamma_i = 0$ $(i = 2, 3, 7, 8)$: this means that we neglect space translations and the Galilean transformations but include time translation.

In this case, transformation (4.1) and system (4.2) specialize along with the following conditions:

$$\phi_1 = \arctan\frac{x_2}{x_1}, \quad \phi_2 = \ln\sqrt{x_1^2 + x_2^2}, \quad \phi_3 = t,$$

$$\phi_4 = x_1, \quad \phi_5 = x_2, \quad \chi_1 = \chi_2 = 0, \quad \chi_3 = 1,$$

$$e_1 = e_2 = e_3 = e_4 = 0,$$

$$f_1 = 1, \quad f_2 = 0,$$

$$\Delta = \frac{b_2 c_1 - b_1 c_2}{\alpha_1},$$

$$a_1 = \beta_{[4}\gamma_{1]}, \quad a_2 = \beta_{[1}\gamma_{9]}, \quad a_3 = \beta_{[9}\gamma_{4]},$$

$$b_1 = \alpha_{[1}\gamma_{4]}, \quad b_2 = \alpha_{[9}\gamma_{1]}, \quad b_3 = \alpha_{[4}\gamma_{9]},$$

$$c_1 = \alpha_{[4}\beta_{1]}, \quad c_2 = \alpha_{[9}\beta_{1]}, \quad c_3 = \alpha_{[9}\beta_{4]}.$$

The constant solutions admitted by the transformed system provide, in terms of the original variables, the solution

$$v_1 = -v_0 x_2, \quad v_2 = v_0 x_1,$$

$$p = p_0 r^{\Gamma(2+d)/(\Gamma-1)},$$

$$s = \frac{\Gamma(2+d) p_0^{1-1/\Gamma}}{(\Gamma - 1) v_0^2} r^d, \tag{4.6}$$

where v_0, p_0 and d are arbitrary constants.

Other solutions can be obtained by assuming $V_1 = 0$, $V_2 = v_0$ (v_0 arbitrary constant), P and S non-constant. In terms of the original variables we have the solution

$$v_1 = -v_0 x_2, \quad v_2 = v_0 x_1,$$

$$p = \Psi(r), \quad s = \frac{1}{v_0^2 r \Psi^{1/\Gamma}} \frac{\mathrm{d}\Psi}{\mathrm{d}r}, \tag{4.7}$$

where $\Psi(r)$ is an arbitrary function of r. It is easy to ascertain that solution (4.6) is a particular instance of this one when we assume $\Psi(r) = p_0 r^{\Gamma(2+d)/(\Gamma-1)}$. Solutions (4.6) and (4.7) are steady and have been obtained in [27].

Now, let us consider the case in which $\alpha_4 = \alpha_9 = 0$; it also has to be $\beta_4 = \gamma_4 = \beta_9 = \gamma_9 = 0$. As a first subcase, we take α_5, β_5, γ_5 non-vanishing and as before we continue to neglect the coefficients related to Galilean transformation; transformation (4.1) and system (4.2) then specialize according to the positions

$$\phi_1 = x_1, \quad \phi_2 = x_2, \quad \phi_3 = \ln(\alpha_1 - \alpha_5 t),$$

$$\phi_4 = \frac{1}{\alpha_5 t - \alpha_1}, \quad \phi_5 = 0, \quad \chi_1 = \chi_2 = 0, \quad \chi_3 = 1,$$

$$e_1 = e_2 = e_3 = e_4 = 0,$$

$$f_1 = 0, \quad f_2 = 1,$$

$$\Delta = c_1 b_2 - c_2 b_1,$$

$$a_1 = \beta_{[5}\gamma_{3]}, \quad a_2 = \beta_{[2}\gamma_{5]}, \quad a_3 = \beta_{[3}\gamma_{2]},$$

$$b_1 = \alpha_{[3}\gamma_{5]}, \quad b_2 = \alpha_{[5}\gamma_{2]}, \quad b_3 = \alpha_{[2}\gamma_{3]},$$

$$c_1 = \alpha_{[5}\beta_{3]}, \quad c_2 = \alpha_{[2}\beta_{5]}, \quad c_3 = \alpha_{[3}\beta_{2]}.$$

Unfortunately, the transformed system we find does not possess physically meaningful constant solutions; nevertheless, we are able to find a non-constant solution by assuming $V_1 = -(c_2/c_1)V_2$; what we finally get in terms of the original variables is

$$v_1 = -\frac{c_2}{c_1}V(\omega), \quad v_2 = V(\omega),$$

$$p = p_0, \quad s = S(\omega), \tag{4.8}$$

$V(\omega)$ and $S(\omega)$ being arbitrary functions of $\omega = c_1 x_1 + c_2 x_2$, whereas, p_0 is a constant. This is a steady solution that becomes unsteady by using the invariance with respect to the Galilean transformations.

If we take, besides $\alpha_4 = \alpha_9 = \beta_4 = \beta_9 = \gamma_4 = \gamma_9 = 0$, also $\alpha_5 = \beta_5 = \gamma_5 = 0$ but include the Galilean transformation, i.e., we take α_i, β_i and γ_i $(i = 7, 8)$ non-vanishing, we have that transformation (4.1) and system (4.2) specialize according to the conditions

$$\phi_1 = x_1 - \frac{\alpha_7}{2\alpha_1}t^2, \quad \phi_2 = x_2 - \frac{\alpha_8}{2\alpha_1}t^2, \quad \phi_3 = t,$$

$$\phi_4 = 1, \quad \phi_5 = 0, \quad \chi_1 = 0, \quad \chi_2 = t, \quad \chi_3 = 1,$$

$$e_1 = e_3 = f_1 = f_2 = 0,$$

$$e_2 = \frac{\alpha_7}{\alpha_1}, \quad e_4 = \frac{\alpha_8}{\alpha_1},$$

$$\Delta = b_2 c_1 - b_1 c_2,$$

$$a_1 = \beta_{[1}\gamma_{3]}, \quad a_2 = \beta_{[2}\gamma_{1]}, \quad a_3 = \beta_{[2}\gamma_{3]},$$

$$b_1 = \alpha_{[3}\gamma_{1]}, \quad b_2 = \alpha_{[1}\gamma_{2]}, \quad b_3 = \alpha_{[2}\gamma_{3]},$$

$$c_1 = \alpha_{[1}\beta_{3]}, \quad c_2 = \alpha_{[2}\beta_{1]}, \quad c_3 = \alpha_{[3}\beta_{2]}.$$

By considering the constant solutions admitted by the transformed system we have to require the conditions

$$d_2 V_{1_0} + d_1 V_{2_0} + d_3 = 0, \quad \alpha_7 d_1 - \alpha_8 d_2 = 0,$$

where we have assumed $V_1 = v_{1_0}$ and $V_2 = v_{2_0}$.

Two cases may be distinguished. If $\alpha_8 = 0$ (and $d_1 = 0$), we have, in terms of the original variables, the solution

$$v_1 = \frac{d_3}{d_2} + \frac{\alpha_7}{\alpha_1}t, \quad v_2 = v_{2_0},$$

$$p = p_0 \exp\left(\frac{a\Gamma\omega}{\Gamma - 1}\right),$$

$$s = -\frac{\alpha_1 a\Gamma p_0^{1-1/\Gamma}}{\alpha_7(\Gamma - 1)}\exp(a\omega), \tag{4.9}$$

where

$$\omega = x_1 - \frac{d_3}{d_2}t - \frac{\alpha_7}{2\alpha_1}t^2,$$

p_0 and a being arbitrary constants such that the field variables s and p result positive. On the contrary, if $\alpha_8 \neq 0$ $(d_2 = (\alpha_7/\alpha_8)d_1)$, then the solution we recover is

$$v_1 = v_{1_0} + \frac{\alpha_7}{\alpha_1}t, \quad v_2 = \frac{d_3}{d_1} - \frac{\alpha_7}{\alpha_8}v_{1_0} + \frac{\alpha_8}{\alpha_1}t,$$

$$p = p_0 \exp\left(\frac{a\Gamma\omega}{\Gamma - 1}\right),$$

$$s = -\frac{\alpha_1 a\Gamma p_0^{1-1/\Gamma}}{\Gamma - 1}\exp(a\omega), \tag{4.10}$$

where

$$\omega = \alpha_7 x_1 + \alpha_8 x_2 - \alpha_8\frac{d_3}{d_1}t - \frac{\alpha_7^2 + \alpha_8^2}{2\alpha_1}t^2$$

and p_0, a are arbitrary constants such that p and s are positive.

Now we look for non-constant solutions. By assuming $\alpha_7 V_1 + \alpha_8 V_2 = c$, with c constant, and taking $\alpha_8 \neq 0$ we get the solution

$$v_1 = \Phi(\omega) + \frac{\alpha_7}{\alpha_1}t, \quad v_2 = \frac{c - \alpha_7\Phi(\omega)}{\alpha_8} + \frac{\alpha_8}{\alpha_1}t,$$

$$p = \Psi(\omega), \quad s = -\frac{\alpha_1}{\Psi^{1/\Gamma}}\frac{d\Psi}{d\omega}, \tag{4.11}$$

where $\Phi(\omega)$ and $\Psi(\omega)$ are arbitrary functions of

$$\omega = \alpha_7 x_1 + \alpha_8 x_2 - ct - \frac{\alpha_7^2 + \alpha_8^2}{2\alpha_1}t^2.$$

In particular, when $c = \alpha_8 d_3/d_1$, $\Psi = p_0 \exp(a\Gamma\omega/(\Gamma - 1))$ and $\Phi = v_{1_0}$ we obtain solution (4.10).

4.2. The case $\Gamma = 2$

Now we consider the case $\Gamma = 2$; therefore, we take in the expression of Ξ_A, Ξ_B and Ξ_C the constants α_{10}, β_{10} and γ_{10} different from zero.

Also, to simplify the calculations, we neglect space translations, and the Galilean transformations, i.e., we take $\alpha_i = \beta_i = \gamma_i = 0$ $(i = 2, 3, 7, 8)$. Transformation (4.1) and System (4.2) specialize along with the conditions

$$\phi_1 = \arctan\frac{x_2}{x_1},$$

$$\phi_2 = \ln\left(\sqrt{\frac{x_1^2 + x_2^2}{\alpha_{10}t^2 + (\alpha_4 - \alpha_5)t + \alpha_1}}\right) - \tau,$$

$$\phi_3 = \tau,$$

$$\phi_4 = \frac{(\alpha_5 - \alpha_4)x_1}{\alpha_{10}t^2 + (\alpha_4 - \alpha_5)t + \alpha_1},$$

$$\phi_5 = \frac{(\alpha_5 - \alpha_4)x_2}{\alpha_{10}t^2 + (\alpha_4 - \alpha_5)t + \alpha_1},$$

$$\chi_2 = 0,$$

$$\chi_1 = \frac{t}{\alpha_{10}t^2 + (\alpha_4 - \alpha_5)t + \alpha_1},$$

$$\chi_3 = \frac{\alpha_4 - \alpha_5}{\sqrt{\alpha_{10}t^2 + (\alpha_4 - \alpha_5)t + \alpha_1}},$$

$$e_1 = e_2 = e_3 = e_4 = 0,$$

$$f_1 = 1, \quad f_2 = -1,$$

$$\Delta = \frac{b_2c_3 - b_3c_2}{\alpha_9},$$

$$a_1 = \beta_{[4}\gamma_{5]}, \quad a_2 = \beta_{[9}\gamma_{4]} - \beta_{[9}\gamma_{5]}, \quad a_3 = 2\beta_{[4}\gamma_{9]},$$

$$b_1 = \alpha_{[5}\gamma_{4]}, \quad b_2 = \alpha_{[4}\gamma_{9]} - \alpha_{[5}\gamma_{9]}, \quad b_3 = 2\alpha_{[4}\gamma_{9]},$$

$$c_1 = \alpha_{[4}\beta_{5]}, \quad c_2 = \alpha_{[9}\beta_{4]} - \alpha_{[9}\beta_{5]}, \quad c_3 = 2\alpha_{[4}\beta_{9]},$$

where

$$\tau = \frac{\alpha_4 - \alpha_5}{\sqrt{4\alpha_1\alpha_{10} - (\alpha_4 - \alpha_5)^2}}\arctan \times\left(\frac{2\alpha_1\alpha_{10}t + \alpha_4 - \alpha_5}{\sqrt{4\alpha_1\alpha_{10} - (\alpha_4 - \alpha_5)^2}}\right),$$

along with the constraints $\alpha_{10} > 0$, $4\alpha_1\alpha_{10} - (\alpha_4 - \alpha_5)^2 > 0$.

The constant solution of the transformed system provide the following solution to the original system:

$$v_1 = \frac{(2at + b)x_1 - 2v_0x_2}{2(at^2 + bt + c)},$$

$$v_2 = \frac{2v_0x_1 + (2at + b)x_2}{2(at^2 + bt + c)},$$

$$p = \frac{b^4p_0}{(at^2 + bt + c)^2}\left(\frac{r^2}{at^2 + bt + c}\right)^{2(d+1)},$$

$$s = \frac{16b^2(1 + d)\sqrt{p_0}}{b^2(4v_0^2 + 1) - 4ac}\left(\frac{r^2}{at^2 + bt + c}\right)^{d}, \tag{4.12}$$

where $a = \alpha_{10}$, $b = \alpha_4 - \alpha_5$, $c = \alpha_1$, v_0 and p_0 are arbitrary constants, whereas $r = \sqrt{at^2 + bt + c}\exp(\phi_2)$. Also, the solution is physically meaningful (i.e., the entropy s and the pressure p are positive) provided that $(1 + d)/(b^2(4V_0^2 + 1) - 4ac) > 0$.

Another more general class of solutions may be obtained by assuming V_1 and V_2 to be constant, while P and S non-constant. The solution we get, when written in terms of the original variables, is

$$v_1 = \frac{(2at + b)x_1 - 2v_0x_2}{2(at^2 + bt + c)},$$

$$v_2 = \frac{2v_0x_1 + (2at + b)x_2}{2(at^2 + bt + c)},$$

$$p = \frac{b^4\Psi(\omega)}{(at^2 + bt + c)^2},$$

$$s = \frac{8b^2}{b^2(4v_0^2 + 1) - 4ac}\frac{1}{\sqrt{\Psi}}\frac{d\Psi}{d\omega}, \tag{4.13}$$

where $\Psi(\omega)$ is an arbitrary function of

$$\omega = \frac{r^2}{at^2 + bt + c}$$

and $(b^2(4v_0^2 + 1) - 4ac)$ with the same sign as $d\Psi/d\omega$. It is noticed that solution (4.12) is recovered as a particular case of solution (4.13) when $\Psi = p_0\omega^{2(d+1)}$.

Moreover, if all the field variables are non-constant, we find the following solution to the original system:

$$v_1 = \frac{(2at+b)x_1 - 2x_2\Phi(\omega)}{2(at^2+bt+c)},$$

$$v_2 = \frac{(2at+b)x_2 + 2x_1\Phi(\omega)}{2(at^2+bt+c)},$$

$$p = \frac{b^4\Psi(\omega)}{(at^2+bt+c)^2},$$

$$s = \frac{8b^2}{b^2(4\Phi^2+1)-4ac}\frac{1}{\sqrt{\Psi(\omega)}}\frac{d\Psi}{d\omega}, \qquad (4.14)$$

where $\Phi(\omega)$, $\Psi(\omega)$ are arbitrary functions of the argument

$$\omega = \frac{r^2}{at^2+bt+c}.$$

A subcase of the previous may be obtained when we also assume $\alpha_1 = \beta_1 = \gamma_1 = 0$, i.e., we also neglect time translations. Transformation (4.1) and System (4.2) specialize along with the conditions

$$\phi_1 = \arctan\frac{x_2}{x_1},$$

$$\phi_2 = \ln\left(\frac{\sqrt{x_1^2+x_2^2}}{t}\right),$$

$$\phi_3 = \ln\left(\frac{t}{\alpha_{10}t+\alpha_4-\alpha_5}\right),$$

$$\phi_4 = \frac{\alpha_4-\alpha_5}{\alpha_{10}t+\alpha_4-\alpha_5}\frac{x_1}{t},$$

$$\phi_5 = \frac{\alpha_4-\alpha_5}{\alpha_{10}t+\alpha_4-\alpha_5}\frac{x_2}{t},$$

$$\chi_1 = \frac{1}{\alpha_{10}t}, \quad \chi_2 = 0, \quad \chi_3 = \frac{\alpha_4-\alpha_5}{t},$$

$$e_1 = e_2 = e_3 = e_4 = 0,$$

$$f_1 = 1, \quad f_2 = -1,$$

$$\Delta = \frac{b_2c_3 - b_3c_2}{\alpha_9},$$

$$a_1 = \beta_{[4}\gamma_{5]}, \quad a_2 = \beta_{[9}\gamma_{4]} - \beta_{[9}\gamma_{5]}, \quad a_3 = \beta_{[5}\gamma_{9]},$$

$$b_1 = \alpha_{[5}\gamma_{4]}, \quad b_2 = \alpha_{[4}\gamma_{9]} - \alpha_{[5}\gamma_{9]}, \quad b_3 = \alpha_{[9}\gamma_{5]},$$

$$c_1 = \alpha_{[4}\beta_{5]}, \quad c_2 = \alpha_{[9}\beta_{4]} - \alpha_{[9}\beta_{5]}, \quad c_3 = \alpha_{[9}\beta_{5]}.$$

If we assume that all the field variables involved in the transformed system are non-constant, we find the following solution to the original system:

$$v_1 = \frac{(2at+b)x_1 - 2x_2\Phi(\omega)}{2t(at+b)},$$

$$v_2 = \frac{2x_1\Phi(\omega) + (2at+b)x_2}{2t(at+b)},$$

$$p = \frac{b^4\Psi(\omega)}{t^2(a+bt)^2},$$

$$s = \frac{8}{(4\Phi^2+1)\sqrt{\Psi}}\frac{d\Psi}{d\omega}, \qquad (4.15)$$

where $\Phi(\omega)$, $\Psi(\omega)$ are arbitrary functions of the argument $\omega = r^2/(t(a+bt))$. This solution is recovered from (4.14) when $c = 0$.

Different results are obtained when $\alpha_4 = \alpha_5$; if this is the case, in order to have an explicit transformation, we have to choose also $\beta_4 = \beta_5$ and $\gamma_4 = \gamma_5$; moreover, we include time translation. Transformation (4.1) and system (4.2) specialize according to the positions (provided that $\alpha_1\alpha_{10} > 0$):

$$\phi_1 = \arctan\frac{x_2}{x_1},$$

$$\phi_2 = \ln\sqrt{\frac{x_1^2+x_2^2}{\alpha_1+\alpha_{10}t^2}},$$

$$\phi_3 = \frac{\arctan(\sqrt{\alpha_{10}/\alpha_1}\,t)}{\sqrt{\alpha_1\alpha_{10}}},$$

$$\phi_4 = \frac{x_1}{\alpha_1+\alpha_{10}t^2},$$

$$\phi_5 = \frac{x_2}{\alpha_1+\alpha_{10}t^2},$$

$$\chi_1 = \frac{t}{\alpha_1+\alpha_{10}t^2}, \quad \chi_2 = 0,$$

$$\chi_3 = \frac{1}{\sqrt{\alpha_1+\alpha_{10}t^2}},$$

$$e_1 = e_2 = e_3 = e_4 = 0,$$

$$f_1 = 1, \quad f_2 = 0,$$

$$\Delta = \frac{b_3 c_2 - b_2 c_3}{\alpha_9},$$

$$a_1 = \beta_{[4}\gamma_{10]}, \quad a_2 = \beta_{[9}\gamma_{10]}, \quad a_3 = \beta_{[9}\gamma_{4]},$$

$$b_1 = \alpha_{[10}\gamma_{4]}, \quad b_2 = \alpha_{[10}\gamma_{9]}, \quad b_3 = \alpha_{[4}\gamma_{9]},$$

$$c_1 = \alpha_{[4}\beta_{10]}, \quad c_2 = \alpha_{[9}\beta_{10]}, \quad c_3 = \alpha_{[9}\beta_{4]}.$$

By searching for the constant solutions to the transformed system we obtain the following solution to the original equations:

$$v_1 = \frac{ax_1 t - v_0 x_2}{at^2 + b}, \quad v_2 = \frac{v_0 x_1 + ax_2 t}{at^2 + b},$$

$$p = \frac{p_0}{(at^2 + b)^2}\left(\frac{r^2}{at^2 + b}\right)^{(2+d)},$$

$$s = \frac{2(2+d)\sqrt{p_0}}{v_0^2 - ab}\left(\frac{r}{t}\right)^d \tag{4.16}$$

with $r = \sqrt{\alpha_1 + \alpha_{10} t^2}\exp(\phi_2)$. Another class of solutions may be found by assuming $V_1 = 0$, $V_2 = v_0$ (v_0 constant) and P and S non-constant; in this case we obtain, in terms of the original variables, the following solution:

$$v_1 = \frac{ax_1 t - x_2 v_0}{at^2 + b}, \quad v_2 = \frac{x_1 v_0 + ax_2 t}{at^2 + b},$$

$$p = \frac{p_0 \Psi(\omega)}{(at^2 + b)^2}, \quad s = \frac{2}{v_0^2 - ab}\frac{1}{\sqrt{\Psi}}\frac{\mathrm{d}\Psi}{\mathrm{d}\omega}, \tag{4.17}$$

where $\Psi(\omega)$ is an arbitrary function of $\omega = r^2/(at^2 + b)$. We can note that solution (4.16) is an particular instance of (4.17) when we choose $\Psi = p_0 \omega^{(2+d)}$.

On the contrary, if we neglect time translation, transformation (4.1) and system (4.2) specialize according to the positions

$$\phi_1 = \arctan\frac{x_2}{x_1},$$

$$\phi_2 = \ln\left(\frac{\sqrt{x_1^2 + x_2^2}}{t}\right),$$

$$\phi_3 = -\frac{1}{t},$$

$$\phi_4 = \frac{x_1}{t^2}, \quad \phi_5 = \frac{x_2}{t^2},$$

$$\chi_1 = \frac{1}{\alpha_{10} t}, \quad \chi_2 = 0, \quad \chi_3 = \frac{1}{t},$$

$$e_1 = e_2 = e_3 = e_4 = 0,$$

$$f_1 = 1, \quad f_2 = 0,$$

$$\Delta = \frac{b_3 c_2 - b_2 c_3}{\alpha_9},$$

$$a_1 = \beta_{[4}\gamma_{10]}, \quad a_2 = \beta_{[9}\gamma_{10]}, \quad a_3 = \beta_{[9}\gamma_{4]},$$

$$b_1 = \alpha_{[10}\gamma_{4]}, \quad b_2 = \alpha_{[10}\gamma_{9]}, \quad b_3 = \alpha_{[4}\gamma_{9]},$$

$$c_1 = \alpha_{[4}\beta_{10]}, \quad c_2 = \alpha_{[9}\beta_{10]}, \quad c_3 = \alpha_{[9}\beta_{4]}.$$

A class of solutions may be found by assuming $V_1 = 0$, whereas, V_2, P and S non-constant; in this case we obtain, in terms of the original variables, the following solution:

$$v_1 = \frac{x_1 t - x_2 \Phi(\omega)}{t^2}, \quad v_2 = \frac{x_1 \Phi(\omega) + x_2 t}{t^2},$$

$$p = \frac{\Psi(\omega)}{t^4}, \quad s = \frac{1}{\omega \Phi^2 \sqrt{\Psi}}\frac{\mathrm{d}\Psi}{\mathrm{d}\omega}, \tag{4.18}$$

where $\Phi(\omega)$ and $\Psi(\omega)$ are arbitrary functions of $\omega = \exp(\phi_2)$.

Now we look for the exact solutions when we assume $\alpha_9 = \alpha_4$ and $\alpha_5 \neq 0$, $\beta_5 \neq 0$. In this case, if we want an explicit transformation we need to choose $\alpha_1 = \beta_1 = \gamma_1 = 0$, $\alpha_i \neq 0$, and $\beta_i \neq 0$ ($i = 2, 3$). Transformation (4.1) and system (4.2) specialize according to the positions

$$\phi_1 = \frac{\alpha_5 \alpha_7 + \alpha_{10}(\alpha_5 x_1 + \alpha_2)}{\alpha_{10}(\alpha_{10} t - \alpha_5)},$$

$$\phi_2 = \frac{\alpha_5 \alpha_8 + \alpha_{10}(\alpha_5 x_2 + \alpha_3)}{\alpha_{10}(\alpha_{10} t - \alpha_5)},$$

$$\phi_3 = \ln\left(\frac{\alpha_{10} t - \alpha_5}{t}\right), \quad \phi_4 = \frac{1}{t}, \quad \phi_5 = 0,$$

$$\chi_1 = \frac{1}{\alpha_{10}t - \alpha_5}, \quad \chi_2 = 0, \quad \chi_3 = \frac{1}{\alpha_{10}t - \alpha_5},$$

$$e_1 = \alpha_7 + \frac{\alpha_2\alpha_{10}}{\alpha_5}, \quad e_2 = 0,$$

$$e_3 = \alpha_8 + \frac{\alpha_3\alpha_{10}}{\alpha_5}, \quad e_4 = 0,$$

$$f_1 = 0, \quad f_2 = -1, \quad \Delta = c_1 b_2 - c_2 b_1,$$

$$a_1 = \beta_{[5}\gamma_{3]}, \quad a_2 = \beta_{[2}\gamma_{5]}, \quad a_3 = \beta_{[2}\gamma_{3]},$$

$$b_1 = \alpha_{[3}\gamma_{5]}, \quad b_2 = \alpha_{[2}\gamma_{5]}, \quad b_3 = \alpha_{[2}\gamma_{3]},$$

$$c_1 = \alpha_{[5}\beta_{3]}, \quad c_2 = \alpha_{[5}\beta_{2]}, \quad c_3 = \alpha_{[3}\beta_{2]}.$$

Like in the case $\Gamma \neq 2$, the transformed system does not admit physically acceptable constant solution. Nevertheless, we may find the following solution obtained by assuming the velocity non-constant whose components are linked by $V_1 = -(c_2/c_1)V_2$ (c_1 and c_2 constants) in terms of the original variables, we find

$$v_1 = \frac{-c_2 V(\omega) + c_1(\alpha_{10}x_1 + \alpha_7)}{c_1(\alpha_{10}t - \alpha_5)},$$

$$v_2 = \frac{V(\omega) + (\alpha_{10}x_2 + \alpha_8)}{\alpha_{10}t - \alpha_5},$$

$$p = \frac{p_0}{(\alpha_{10}t - \alpha_5)^4}, \quad s = S(\omega), \tag{4.19}$$

where p_0 is a constant and $V(\omega)$ and $S(\omega)$ are arbitrary functions of

$$\omega = \frac{c_1(\alpha_{10}x_1 + \alpha_7) + c_2(\alpha_{10}x_2 + \alpha_8)}{\alpha_{10}t - \alpha_5}.$$

The subcase of the previous one that we must analyze to cover all the situations is the case in which $\alpha_5 = \beta_5 = \gamma_5 = 0$. Transformation (4.1) and system (4.2) specialize according to the positions

$$\phi_1 = \frac{2(\alpha_{10}x_1 + \alpha_7) + \alpha_2/t}{2\alpha_{10}t},$$
$$\phi_2 = \frac{2(\alpha_{10}x_2 + \alpha_8) + \alpha_3/t}{2\alpha_{10}t},$$
$$\phi_3 = -\frac{1}{t},$$

$$\phi_4 = \frac{1}{t}, \quad \phi_5 = 0,$$

$$\chi_1 = \frac{1}{\alpha_{10}t}, \quad \chi_2 = -\frac{1}{t^2}, \quad \chi_3 = \frac{1}{t},$$

$$e_1 = \alpha_7, \quad e_2 = -\frac{\alpha_2}{\alpha_{10}},$$

$$e_3 = \alpha_8, \quad e_4 = -\frac{\alpha_3}{\alpha_{10}},$$

$$f_1 = f_2 = 0, \quad \Delta = \frac{b_2 c_1 - b_1 c_2}{a_{10}},$$

$$a_1 = \beta_{[10}\gamma_{8]}, \quad a_2 = \beta_{[7}\gamma_{10]}, \quad a_3 = \beta_{[8}\gamma_{7]},$$
$$b_1 = \alpha_{[10}\gamma_{8]}, \quad b_2 = \alpha_{[7}\gamma_{10]}, \quad b_3 = \alpha_{[7}\gamma_{8]},$$
$$c_1 = \alpha_{[8}\beta_{10]}, \quad c_2 = \alpha_{[10}\beta_{7]}, \quad c_3 = \alpha_{[8}\beta_{7]}.$$

The constant solutions ($V_1 = v_{1_0}$, $V_2 = v_{2_0}$ and $P = p_0$, v_{1_0}, v_{2_0}, p_0 being arbitrary constants) to the recovered transformed system have to satisfy the following conditions:

$$d_2 v_{1_0} + d_1 v_{2_0} + d_3 = 0, \quad \alpha_2 d_1 - \alpha_3 d_2 = 0.$$

Two cases must be distinguished. If $\alpha_3 = 0$ ($d_1 = 0$), in terms of the original variables we have

$$v_1 = \frac{d_3}{d_2 t} + \frac{(\alpha_{10}x_1 + \alpha_7)t + \alpha_2}{\alpha_{10}t^2},$$

$$v_2 = \frac{v_{2_0}}{t} + \frac{\alpha_{10}x_2 + \alpha_8}{\alpha_{10}t},$$

$$p = \frac{p_0}{t^4}\exp(2a\omega),$$

$$s = -\frac{2\alpha_{10}a\sqrt{p_0}}{\alpha_2}\exp(a\omega), \tag{4.20}$$

where

$$\omega = \frac{x_1}{t} + \frac{\alpha_2}{2\alpha_{10}t^2} + \frac{d_3}{d_2 t}$$

and a, p_0 and v_{2_0} arbitrary constants such that s and p are positive.

If $\alpha_3 \neq 0$ ($d_2 = -d_1\alpha_2/\alpha_3$) it is found

$$v_1 = \frac{v_{1_0}}{t} + \frac{(\alpha_{10}x_1 + \alpha_7)t + \alpha_2}{\alpha_{10}t^2},$$

$$v_2 = \frac{d_3}{d_1 t} - \frac{\alpha_2 v_{1_0}}{\alpha_3 t} + \frac{(\alpha_{10}x_2 + \alpha_8)t + \alpha_3}{\alpha_{10}t^2},$$

$$p = \frac{p_0}{t^4}\exp(2a\omega),$$

$$s = -2\alpha_{10}a\sqrt{p_0}\exp(a\omega), \tag{4.21}$$

where

$$\omega = \frac{\alpha_2 x_1 + \alpha_3 x_2}{t} + \frac{\alpha_2\alpha_7 + \alpha_3\alpha_8}{\alpha_{10}t} + \frac{\alpha_3 d_3}{d_1 t} + \frac{\alpha_2^2 + \alpha_3^2}{2\alpha_{10}t^2}$$

and a, v_{1_0}, p_0 arbitrary constants such that s and p are positive.

Finally, looking for the non-constant solutions and assuming $\alpha_3 V_1 + \alpha_2 V_2 = c$, we find

$$v_1 = \frac{\Phi(\omega)}{t} + \frac{(\alpha_{10}x_1 + \alpha_7)t + \alpha_2}{\alpha_{10}t^2},$$

$$v_2 = \frac{c - \alpha_2\Phi(\omega)}{\alpha_3 t} + \frac{(\alpha_{10}x_2 + \alpha_8)t + \alpha_3}{\alpha_{10}t^2},$$

$$p = \frac{\Psi(\omega)}{t^4}, \quad s = \frac{\alpha_{10}}{\sqrt{\Psi}}\frac{d\Psi}{d\omega}, \tag{4.22}$$

where $\Phi(\omega)$ and $\Psi(\omega)$ are arbitrary functions of

$$\omega = \frac{\alpha_2 x_1 + \alpha_3 x_2}{t} + \frac{\alpha_2\alpha_7 + \alpha_3\alpha_8}{\alpha_{10}t} + \frac{c}{t} + \frac{\alpha_2^2 + \alpha_3^2}{2\alpha_{10}t^2}.$$

This solution, for particular choices of $\Psi(\omega)$, $\Phi(\omega)$ and c, contains as a particular case solution (4.20).

5. Unsteady equations in 3D

In this case, the field equations (2.1) are left invariant by 13 Lie groups of point transformations whose operators are given by specifying with $n = 3$ in (2.2). Moreover, if $\Gamma = \frac{5}{3}$, we have also the infinitesimal operator of the projective group corresponding to (2.3). By following the same procedure of the last section, i.e., by applying Theorem 1, we need to consider four linear independent combinations of the operators Ξ_i $(i = 1,\dots,14)$ admitted by our equations and then require that these four operators (say, Ξ_A, Ξ_B, Ξ_C and Ξ_D) generate a four-dimensional Abelian Lie algebra.

Unfortunately, the constants α_i, β_i, γ_i and δ_i $(i = 1,\dots,14)$ involved in the operators Ξ_A, Ξ_B, Ξ_C and Ξ_D, do not allow the operators to be such that the 4×4 matrix with entries given by the infinitesimal generators of the independent variables has maximal rank: this implies that the transformation of variables cannot be explicit.

Nevertheless, it is possible to construct classes of solutions by simply extending to the three-dimensional case the solutions found in the previous section.

5.1. The case Γ arbitrary

The first solution is obtained by extending solution (4.5); what we get is

$$v_i = \frac{2}{3\Gamma - 1}\frac{x_i}{t} \quad (i = 1,2,3)$$

$$p = \frac{\Psi(\omega)}{t^{6\Gamma/(3\Gamma-1)}},$$

$$s = \frac{(3\Gamma-1)^3\omega^{1-4/(3\Gamma-1)}}{12(\Gamma-1)}\frac{1}{\Psi^{1/\Gamma}}\frac{d\Psi}{d\omega}, \tag{5.1}$$

where $\Psi(\omega)$ is an arbitrary function of $\omega = (x_1^2 + x_2^2 + x_3^2)^{(3\Gamma-1)/4}/t$.

By extending solution (4.7) we find

$$v_1 = -v_0 x_2 + w_0 x_3,$$

$$v_2 = v_0 x_1 + z_0 x_3,$$

$$v_3 = -w_0 x_1 - z_0 x_2,$$

$$p = \Psi(r), \quad s = \frac{1}{r\Psi^{1/\Gamma}}\frac{d\Psi}{dr}, \tag{5.2}$$

where $\Psi(r)$ is an arbitrary function of

$$r = ((v_0^2 + w_0^2)x_1^2 + (v_0^2 + z_0^2)x_2^2 + (w_0^2 + z_0^2)x_3^2 + 2w_0 z_0 x_1 x_2 + 2v_0 z_0 x_1 x_3 - 2v_0 w_0 x_2 x_3)^{1/2}.$$

In the same way, starting with solution (4.11), we obtain

$$v_1 = -\frac{c_2\Phi_1(\omega) + c_3\Phi_2(\omega)}{c_1},$$

$$v_2 = \Phi_1(\omega), \quad v_3 = \Phi_2(\omega),$$

$$p = p_0, \quad s = S(\omega), \tag{5.3}$$

where $\Phi_1(\omega)$, $\Phi_2(\omega)$ and $S(\omega)$ are arbitrary functions of $\omega = c_1x_1 + c_2x_2 + c_3x_3$.

Finally, from solution (4.11) it is found

$$v_1 = \Phi_1(\omega) + \frac{\alpha_7}{\alpha_1}t,$$

$$v_2 = \frac{c - \alpha_7\Phi_1(\omega) - \alpha_9\Phi_2(\omega)}{\alpha_8} + \frac{\alpha_8}{\alpha_1}t,$$

$$v_3 = \Phi_2(\omega) + \frac{\alpha_9}{\alpha_1}t, \quad p = \Psi(\omega),$$

$$s = -\frac{\alpha_1}{\Psi^{1/\Gamma}}\frac{d\Psi}{d\omega}, \tag{5.4}$$

where $\Phi_1(\omega)$, $\Phi_2(\omega)$ and $\Psi(\omega)$ are arbitrary functions of

$$\omega = \alpha_7x_1 + \alpha_8x_2 + \alpha_9x_3 - ct - \frac{\alpha_7^2 + \alpha_8^2 + \alpha_9^2}{2\alpha_1}t^2$$

with c constant, where we assumed $\alpha_8 \neq 0$.

5.2. The case $\Gamma = \frac{5}{3}$

When $\Gamma = \frac{5}{3}$ (which is a physical case) other solutions can be obtained. By extending solution (4.13) (taking $v_0 = 0$) to the three-dimensional case, we get the solution

$$v_i = \frac{(2at + b)x_i}{2(at^2 + bt + c)} \quad (i = 1, 2, 3),$$

$$p = \frac{b^5\Psi(\omega)}{(at^2 + bt + c)^{5/2}}, \quad s = \frac{2b^2}{b^2 - 4ac}\frac{1}{\Psi^{3/5}}\frac{d\Psi}{d\omega}, \tag{5.5}$$

where $\Psi(\omega)$ is an arbitrary functions of the argument $\omega = (x_1^2 + x_2^2 + x_3^2)/(at^2 + bt + c)$; moreover, by extending solution (4.15) (with $\Phi = v_0$) we obtain the solution

$$v_1 = \frac{(2at + b)x_1 - 2bv_0x_2 + 2bw_0x_3}{2t(at + b)},$$

$$v_2 = \frac{2bv_0x_1 + (2at + b)x_2 + 2bz_0x_3}{2t(at + b)},$$

$$v_3 = \frac{-2bw_0x_1 - 2bz_0x_2 + (2at + b)x_3}{2t(at + b)},$$

$$p = \frac{b^5\Psi(\omega)}{t^{5/2}(a + bt)^{5/2}}, \quad s = \frac{8}{\Psi^{3/5}}\frac{d\Psi}{d\omega}, \tag{5.6}$$

where $\Psi(\omega)$ is an arbitrary function of $\omega = r^2/(t(a + bt))$, where

$$r = ((1 + 4v_0^2 + 4w_0^2)x_1^2 + (1 + 4v_0^2 + 4z_0^2)x_2^2 + (1 + 4w_0^2 + 4z_0^2)x_3^2 + 8w_0z_0x_1x_2 + 8v_0z_0x_1x_3 - 8v_0w_0x_2x_3)^{1/2}.$$

Another solution (that generalizes solution (4.18) is

$$v_1 = \frac{x_1t - v_0x_2 + x_3w_0}{t^2},$$

$$v_2 = \frac{v_0x_1 + x_2t + x_3z_0}{t^2},$$

$$v_3 = \frac{-w_0x_1 - x_2z_0 + x_3t}{t^2},$$

$$p = \frac{\Psi(\omega)}{t^5}, \quad s = \frac{1}{\omega\Psi^{3/5}}\frac{d\Psi}{d\omega}, \tag{5.7}$$

where $\Psi(\omega)$ is an arbitrary function of $\omega = r/t$ with

$$r = ((v_0^2 + w_0^2)x_1^2 + (v_0^2 + z_0^2)x_2^2 + (w_0^2 + z_0^2)x_3^2 + 2w_0z_0x_1x_2 + 2v_0z_0x_1x_3 - 2v_0w_0x_2x_3)^{1/2}.$$

Furthermore, we build the following solution as extension of (4.18):

$$v_1 = \frac{-x_2\Phi(\omega) + x_3\Phi(\omega) + x_1t}{t^2},$$

$$v_2 = \frac{x_1\Phi(\omega) + x_3\Phi(\omega) + x_2t}{t^2},$$

$$v_3 = \frac{-x_1\Phi(\omega) - x_2\Phi(\omega) + x_3t}{t^2},$$

$$p = \frac{\Psi(\omega)}{t^5}, \quad s = \frac{1}{2\omega\Phi^2\Psi^{3/5}}\frac{d\Psi}{d\omega}, \tag{5.8}$$

where $\Phi(\omega)$ and $\Psi(\omega)$ are arbitrary functions of $\omega = (x_1^2 + x_2^2 + x_3^2)/t$.

The third solution (extension of (4.19)) is

$$v_1 = -\frac{c_2\Phi_1(\omega) + c_3\Phi_2(\omega)}{c_1(\alpha_{10}t - \alpha_5)} + \frac{\alpha_{10}x_1 + \alpha_7}{\alpha_{10}t - \alpha_5},$$

$$v_2 = \frac{\Phi_1(\omega) + \alpha_{10}x_2 + \alpha_8}{\alpha_{10}t - \alpha_5},$$

$$v_3 = \frac{\Phi_2(\omega) + \alpha_{10}x_3 + \alpha_9}{\alpha_{10}t - \alpha_5},$$

$$p = \frac{p_0}{(\alpha_{10}t - \alpha_5)^5}, \quad s = S(\omega), \tag{5.9}$$

where $\Phi_1(\omega)$, $\Phi_2(\omega)$ and $S(\omega)$ are arbitrary functions of

$$\omega = \frac{c_1(\alpha_{10}x_1 + \alpha_7) + c_2(\alpha_{10}x_2 + \alpha_8) + c_3(\alpha_{10}x_3 + \alpha_9)}{\alpha_{10}t - \alpha_5}.$$

Finally, starting from (4.22), another solution is

$$v_1 = \frac{\Phi_1(\omega)}{t} + \frac{(\alpha_{10}x_1 + \alpha_7)t + \alpha_2}{\alpha_{10}t^2},$$

$$v_2 = \frac{c - \alpha_2\Phi_1(\omega) - \alpha_4\Phi_2(\omega)}{\alpha_3 t} + \frac{(\alpha_{10}x_2 + \alpha_8)t + \alpha_3}{\alpha_{10}t^2},$$

$$v_3 = \frac{\Phi_2(\omega)}{t} + \frac{(\alpha_{10}x_3 + \alpha_9)t + \alpha_4}{\alpha_{10}t^2},$$

$$p = \frac{\Psi(\omega)}{t^5}, \quad s = \frac{\alpha_{10}}{\Psi^{5/3}}\frac{d\Psi}{d\omega}, \tag{5.10}$$

where $\Phi_1(\omega)$, $\Phi_2(\omega)$ and $\Psi(\omega)$ are arbitrary functions of

$$\omega = \frac{\alpha_2 x_1 + \alpha_3 x_2 + \alpha_4 x_3}{t} + \frac{\alpha_2\alpha_7 + \alpha_3\alpha_8 + \alpha_4\alpha_9}{\alpha_{10}t} + \frac{c}{t} + \frac{\alpha_2^2 + \alpha_3^2 + \alpha_4^2}{2\alpha_{10}t^2}.$$

6. New substituted solutions

In this section we construct new classes of exact solutions by means of the Substitution Principle established in Section 3.

6.1. The 2D case

By applying Theorem 3 to solution (4.7), provided that

$$-x_2\frac{\partial m}{\partial x_1} + x_1\frac{\partial m}{\partial x_2} = 0, \tag{6.1}$$

by which m is an arbitrary function of $r = \sqrt{x_1^2 + x_2^2}$, we find the new unsteady solution (where we make use also of the invariance with respect to the space translations and Galilean transformations):

$$v_1 = -(x_2 - k_3 t - k_4)\Phi(\hat{r}) + k_1$$

$$v_2 = (x_1 - k_1 t - k_2)\Phi(\hat{r}) + k_3,$$

$$p = \Psi(\hat{r}), \quad s = \frac{1}{\hat{r}\Phi^2\Psi^{1/\Gamma}}\frac{d\Psi}{d\hat{r}}, \tag{6.2}$$

where $\Phi(\hat{r})$ and $\Psi(\hat{r})$ are arbitrary function of

$$\hat{r} = \sqrt{(x_1 - k_1 t - k_2)^2 + (x_2 - k_3 t - k_4)^2}.$$

In the case $\Gamma = 2$, starting from (4.14) in which we choose $\Phi = 0$ and $\Psi = \Psi_0\omega^{-2}$, Ψ_0 being an arbitrary positive constant, we are able to use Theorem 3. The function m has to satisfy the constraint

$$x_1\frac{\partial m}{\partial x_1} + x_2\frac{\partial m}{\partial x_2} = 0,$$

whence $m = M(x_2/x_1)$ (M arbitrary function of its argument); the solution we obtain is

$$v_1 = \frac{(2a(Mt + H) + b)x_1 M}{2(a(Mt + H)^2 + b(Mt + H) + c)},$$

$$v_2 = \frac{(2a(Mt + H) + b)x_2 M}{2(a(Mt + H)^2 + b(Mt + H) + c)},$$

$$p = \frac{b^4\Psi_0}{r^4},$$

$$s=\frac{16b^2\sqrt{\Psi_0}}{4ac-b^2}\frac{(a(Mt+H)^2+b(Mt+H)+c)^2}{r^4M^2}, \tag{6.3}$$

where also $H(x_2/x_1)$ is an arbitrary function of its argument. Of course, we may include space translations and the Galilean transformations and get the solution

$$v_1=\frac{(2a(\widehat{M}t+\widehat{H})+b)(x_1-k_1t-k_2)}{2(a(\widehat{M}t+\widehat{H})^2+b(\widehat{M}t+\widehat{H})+c)}\widehat{M}+k_1,$$

$$v_2=\frac{(2a(\widehat{M}t+\widehat{H})+b)(x_2-k_3t-k_4)}{2(a(\widehat{M}t+\widehat{H})^2+b(\widehat{M}t+\widehat{H})+c)}\widehat{M}+k_3,$$

$$p=\frac{b^4\Psi_0}{\hat{r}^4},$$

$$s=\frac{16b^2\sqrt{\Psi_0}}{4ac-b^2}\frac{(a(\widehat{M}t+\widehat{H})^2+b(\widehat{M}t+\widehat{H})+c)^2}{\hat{r}^4\widehat{M}^2}, \tag{6.4}$$

in which also the pressure p is unsteady, where $\hat{r}$ has the same expression of the previous solution, and $\widehat{M}$ and $\widehat{H}$ are arbitrary functions of $(x_2-k_3t-k_4)/(x_1-k_1t-k_2)$.

6.2. The 3D case

By applying Theorem 3 to solution (5.2), provided that

$$(-v_0x_2+w_0x_3)\frac{\partial m}{\partial x_1}+(v_0x_1+z_0x_3)\frac{\partial m}{\partial x_2}-(w_0x_1+z_0x_2)\frac{\partial m}{\partial x_3}=0, \tag{6.5}$$

by which m is an arbitrary function of $q_1=\sqrt{x_1^2+x_2^2+x_3^2}$, and $q_2=z_0x_1-w_0x_2+v_0x_3$, we find the new unsteady solution (where we make use also of the invariance with respect to the space translations and Galilean transformations):

$$v_1=-v_0(x_2-k_3t-k_4)\Phi(\hat{q}_1,\hat{q}_2)+w_0(x_3-k_5t-k_6)\Phi(\hat{q}_1,\hat{q}_2)+k_1,$$

$$v_2=v_0(x_1-k_1t-k_2)\Phi(\hat{q}_1,\hat{q}_2)+z_0(x_3-k_5t-k_6)\Phi(\hat{q}_1,\hat{q}_2)+k_3,$$

$$v_3=-w_0(x_1-k_1t-k_2)\Phi(\hat{q}_1,\hat{q}_2)-z_0(x_2-k_3t-k_4)\Phi(\hat{q}_1,\hat{q}_2)+k_5,$$

$$p=\Psi(\hat{r}),\quad s=\frac{1}{2\hat{r}\Phi^2\Psi^{1/\Gamma}}\frac{d\Psi}{d\hat{r}},$$

where $\Phi(\hat{q}_1,\hat{q}_2)$ and $\Psi(\hat{r})$ are arbitrary functions, respectively, of

$$\hat{q}_1=((x_1-k_1t-k_2)^2+(x_2-k_3t-k_4)^2+(x_3-k_5t-k_6)^2)^{1/2},$$

$$\hat{q}_2=z_0(x_1-k_1t-k_2)-w_0(x_2-k_3t-k_4)+v_0(x_3-k_5t-k_6),$$

$$\hat{r}=((v_0^2+w_0^2)(x_1-k_1t-k_2)^2+(v_0^2+z_0^2)(x_2-k_3t-k_4)^2+(w_0^2+z_0^2)(x_3-k_5t-k_6)^2+2w_0z_0(x_1-k_1t-k_2)(x_2-k_3t-k_4)+2v_0(x_3-k_5t-k_6)(z_0(x_1-k_1t-k_2)-w_0(x_2-k_3t-k_4)))^{1/2}.$$

When $\Gamma=\frac{5}{3}$, starting from the solution (5.5) where we assume $\Psi=\Psi_0\omega^{-5/2}$, we may apply Theorem 3. The function m has to satisfy the constraint

$$x_1\frac{\partial m}{\partial x_1}+x_2\frac{\partial m}{\partial x_2}+x_3\frac{\partial m}{\partial x_3}=0,$$

whereupon, it follows $m=M(x_2/x_1,x_3/x_1)$, M being an arbitrary function of its arguments. The new substituted solution arises:

$$v_i=\frac{(2a(Mt+H)+b)x_iM}{2(a(Mt+H)^2+b(Mt+H)+c)}\quad (i=1,2,3),$$

$$p=\frac{b^5\Psi_0}{r^5},$$

$$s=\frac{20b^2\Psi_0^{2/5}}{(4ac-b^2)M^2}\frac{(a(Mt+H)^2+b(Mt+H)+c)^2}{r^4},$$

where also $H(x_2/x_1, x_3/x_1)$ is an arbitrary function of the indicated arguments. Also in this case, we may include space translations and the Galilean transformations, and obtain a solution in which the pressure is unsteady

$$v_i = \frac{(2a(\hat{M}t + \hat{H}) + b)(x_i - k_{2i-1}t - k_{2i})}{2(a(\hat{M}t + \hat{H})^2 + b(\hat{M}t + \hat{H}) + c)}\hat{M} + k_{2i-1} \quad (i = 1, 2, 3),$$

$$p = \frac{b^5\Psi_0}{\hat{r}^5},$$

$$s = \frac{20b^2\Psi_0^{2/5}}{(4ac - b^2)\hat{M}^2}\frac{(a(\hat{M}t + \hat{H})^2 + b(\hat{M}t + \hat{H}) + c)^2}{\hat{r}^4},$$

where $\hat{r}$ has the same expression as above, whereas $\hat{M}$ and $\hat{H}$ are arbitrary functions of the arguments: $(x_2 - k_3t - k_4)/(x_1 - k_1t - k_2)$ and $(x_3 - k_5t - k_6)/(x_1 - k_1t - k_2)$.

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