

*Computational Aspects
of
Moving Frames*

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- Reid, Lisle
 - Mansfield, Hubert, Kogan, ...
 - Morosov
 - McLenaghan, Smirnov, The, ...
 - Hickman, Hann
 - Shakiban, Lloyd, ...

Applications of Moving Frames

- Differential geometry
- Equivalence
- Symmetry
- Differential invariants
- Rigidity
- Joint Invariants & Semi-Differential Invariants
- Invariant differential forms and tensors
- Identities and syzygies
- Classical invariant theory
- Computer vision
 - object recognition
 - symmetry detection
- Invariant numerical methods
- Relativity & separation of variables: Killing tensors, etc.
- Poisson geometry & solitons
- Lie pseudogroups
- Explicit solutions of PDEs (group splitting)

Equivariant Moving Frames

M — smooth m -dimensional manifold.

G — transformation group acting on M

- finite-dimensional Lie group
 - infinite-dimensional Lie pseudo-group
-

Definition.

A *moving frame* is a G -equivariant map

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

$$\rho_{\text{left}}(z) = \rho_{\text{right}}(z)^{-1}$$

The Main Theorem

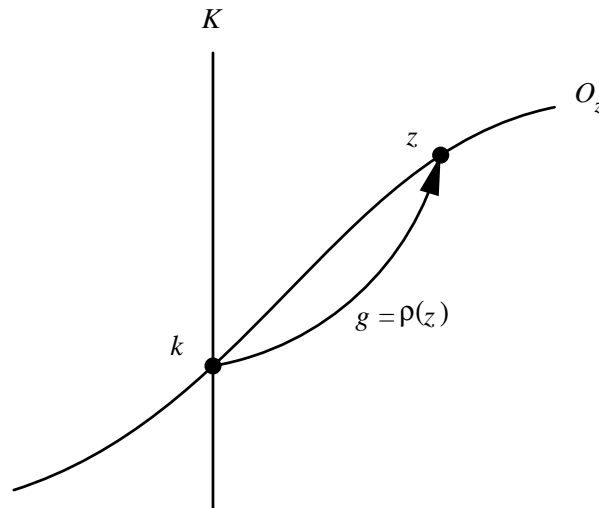
Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z .

$G_z = \{g \mid g \cdot z = z\} \implies$ Isotropy subgroup

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity:
 $\implies G_z = \{e\}$ for all $z \in M$.
- locally free — the orbits all have the same dimension as G :
 $\implies G_z$ is a discrete subgroup of G .
- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once
 $\not\approx$ irrational flow on the torus

Geometric Construction of Moving Frames

Normalization = choice of cross-section to the group orbits



K — cross-section to the group orbits

\mathcal{O}_z — orbit through $z \in M$

$k \in K \cap \mathcal{O}_z$ — unique point in the intersection

- k is the *canonical form* of z
- the (nonconstant) coordinates of k are the fundamental invariants

$g \in G$ — *unique* group element mapping k to z

\implies freeness

$\rho(z) = g$ left moving frame $\rho(h \cdot z) = h \cdot \rho(z)$

$$k = \rho^{-1}(z) \cdot z = \rho_{right}(z) \cdot z$$

Algebraic Construction of Moving Frames

$$r = \dim G \leq m = \dim M$$

Coordinate cross-section

$$K = \{ z_1 = c_1, \dots, z_r = c_r \}$$

left $w(g, z) = g^{-1} \cdot z$	right $w(g, z) = g \cdot z$	
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Choose $r = \dim G$ components to *normalize*:

$$w_1(g, z) = c_1 \quad \dots \quad w_r(g, z) = c_r$$

The solution

$$g = \rho(z)$$

is a (local) moving frame.

\implies Implicit Function Theorem

\implies Rational algebraic functions

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g, z)$ produces the fundamental invariants:

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \dots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

\implies These are the coordinates of the canonical form $k \in K$.

Theorem. Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \dots, I_{m-r}(z))$$

Invariantization

Definition. The *invariantization* of a function $F(z)$ with respect to a right moving frame ρ is the invariant function $I = \iota(F)$ defined by

$$I(z) = F(\rho(z) \cdot z).$$

$$\iota(z_j) = \begin{cases} c_j, & 1 \leq j \leq r, \\ I_{j-r}(z), & r+1 \leq j \leq m. \end{cases}$$

$$\iota [F(z_1, \dots, z_m)] = F(c_1, \dots, c_r, I_1(z), \dots, I_{m-r}(z))$$

Invariantization amounts to restricting F to the cross-section $I|K = F|K$ and then requiring that $I = \iota(F)$ be constant along the orbits.

In particular, if $I(z)$ is an invariant, then $\iota(I) = I$.

Invariantization defines a canonical projection

$$\begin{array}{lcl} & \text{functions} & \longmapsto \text{invariants} \\ \iota: & \text{differential} & \\ & \text{forms} & \longmapsto \text{invariant forms} \end{array}$$

Prolongation

Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

An effective action can usually be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} : J^n(M, p) \longrightarrow J^n(M, p)$$

\implies differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

\implies joint invariants

- Prolonging to “multi-space”

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

\implies joint or semi-differential invariants

\implies invariant numerical approximations

Jet Space

$J^n = J^n(M, p)$ — n^{th} extended jet bundle for
 p -dimensional submanifolds $N \subset M$

Assume $N = \{u = f(x)\}$ is a graph (section).

$x = (x^1, \dots, x^p)$ — independent variables

$u = (u^1, \dots, u^q)$ — dependent variables

$$p + q = m = \dim M$$

Local coordinates on J^n :

$$z^{(n)} = (x, u^{(n)}) = (\dots x^i \dots u_J^\alpha \dots)$$

$$u_J^\alpha = \partial_J u^\alpha \quad 0 \leq \#J \leq n$$

— derivatives of dependent variables

= Taylor coefficients

Prolongation of Group Actions

G — transformation group acting on M

$G^{(n)}$ — prolonged action of G on the jet space J^n

Differential invariant: $I : J^n \rightarrow \mathbb{R}$

$$I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)})$$

\implies curvatures

Theorem. If G acts (locally) effectively on M , then G acts (locally) freely on a dense open subset $\mathcal{V}^n \subset J^n$ for $n \gg 0$.

The prolonged group formulae

$$w^{(n)} = (y, v^{(n)}) = g^{(n)} \cdot z^{(n)}$$

are obtained by implicit differentiation.

Moving Frames on Jet Space

Write out prolonged group action:

$$w^{(n)} = (y, v^{(n)}) = \begin{cases} g^{(n)} \cdot z^{(n)} & \text{right} \\ (g^{(n)})^{-1} \cdot z^{(n)} & \text{left} \end{cases}$$

Choose a coordinate cross-section $K \subset J^n$

$$z_1 = c_1 \quad \dots \quad z_r = c_r \quad r = \dim G$$
$$\implies z_\nu \text{ are jet coordinates } (x^i \text{ or } u_j^\alpha)$$

Solve the *Normalization Equations*

$$w_1(g, x, u^{(n)}) = c_1 \quad \dots \quad w_r(g, x, u^{(n)}) = c_r$$

for the group parameters

$$g = \rho^{(n)}(z^{(n)}) = \rho^{(n)}(x, u^{(n)})$$

The result is a moving frame of order n .

The Fundamental Differential Invariants

Invariantization of jet coordinates:

$$H^i(x, u^{(n)}) = \iota(x^i) \quad I_K^\alpha(x, u^{(k)}) = \iota(u_K^\alpha)$$

- phantom (constant) differential invariants
= normalization variables
- fundamental (nonconstant) differential invariants

Theorem. Every differential invariant can be (locally) uniquely written as a function of the fundamental differential invariants:

$$I = \Phi(\dots H^i \dots I_K^\alpha \dots)$$

Invariant Differentiation

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_{x^1}) \quad \dots \quad \mathcal{D}_p = \iota(D_{x^p})$$

Theorem. The higher order differential invariants are obtained by invariant differentiation of a finite number of fundamental differential invariants I_1, \dots, I_N with respect to $\mathcal{D}_1, \dots, \mathcal{D}_p$.

The Algebra of Differential Invariants

- Fundamental differential invariants

$$I_1, \dots, I_N \quad N = ???$$

- Invariant differential operators

$$\mathcal{D}_1, \dots, \mathcal{D}_p$$

- Recurrence Formulae:

$$\mathcal{D}_i I_J^\alpha = I_{Ji}^\alpha + R_{J,i}^\alpha$$

- Commutation formulae

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^p A_{ij}^k \mathcal{D}_k$$

- Classification of syzygies

$$F(\dots \mathcal{D}^J I^\alpha \dots) = 0$$

- Noncommutative differential
Gröbner bases

The Key Formula

$$d\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^r \gamma_{\kappa} \wedge \iota[\mathbf{v}_{\kappa}(\Omega)]$$

Ω — any function or differential form

$\mathbf{v}_1, \dots, \mathbf{v}_r$
— prolonged infinitesimal generators

$\gamma_1, \dots, \gamma_r$
— invariantized dual Maurer–Cartan forms

★ ★ ★ All recurrence formulae, syzygies, commutation formulae, etc. are found by applying the key formula with Ω replaced by the basic forms and functions!

★ ★ ★ To determine $\gamma_1, \dots, \gamma_r$, let Ω range over the phantom differential invariants

Infinitesimal Generators

- G — transformation group
- $G^{(n)}$ — prolonged action on J^n
- \mathfrak{g} — Lie algebra of
infinitesimal generators \mathbf{v}
- $\mathbf{v}^{(n)}$ — Prolonged inf. gens.
-

The Prolongation Formula

$$\mathbf{v}^{(n)} = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha, J} \varphi_J^\alpha(x, u^{(j)}) \frac{\partial}{\partial u_J^\alpha}$$
$$\varphi_J^\alpha = D_J Q^\alpha + \sum_{i=1}^p \xi^i u_{J,i}^\alpha$$

Characteristic

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha - \sum_{i=1}^p \xi^i \frac{\partial u^\alpha}{\partial x^i}$$

Euclidean group — SE(2)

$$(x, u) \longmapsto (x \cos \phi - u \sin \phi + a, x \sin \phi + u \cos \phi + b)$$

Transformed function $v = \bar{f}(y)$:

$$y = x \cos \phi - f(x) \sin \phi,$$

$$v = x \sin \phi + f(x) \cos \phi,$$

Second prolongation

$$(x, u, u_x, u_{xx}) \longmapsto \left(x \cos \phi - u \sin \phi, x \sin \phi + u \cos \phi, \frac{\sin \phi + u_x \cos \phi}{\cos \phi - u_x \sin \phi}, \frac{u_{xx}}{(\cos \phi - u_x \sin \phi)^3} \right)$$

Infinitesimal generators

$$\mathbf{v}_1 = \frac{\partial}{\partial x} \quad \mathbf{v}_2 = \frac{\partial}{\partial u} \quad \mathbf{v}_3 = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u}$$

Second prolongation

$$\mathbf{v}_3^{(2)} = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u} + (1 + u_x^2) \frac{\partial}{\partial u_x} + 3u_x u_{xx} \frac{\partial}{\partial u_{xx}}$$

$$Q = x + uu_x$$

$$\varphi^x = D_x Q + \xi u_{xx} = D_x(x + uu_x) - uu_{xx} = 1 + u_x^2$$

$$\varphi^{xx} = D_x^2 Q + \xi u_{xxx} = D_x^2(x + uu_x) - uu_{xxx} = 3u_x u_{xx}$$

Generating Differential Invariants

$$\rho^{(n)} : J^n \longrightarrow G$$

\implies moving frame of order n

Recurrence Formulae:

$$\mathcal{D}_i I_J^\alpha = I_{Ji}^\alpha + R_{J,i}^\alpha$$

$$\text{order}(R_{J,i}^\alpha) \leq \max\{\text{order } J + 1, n + 1\}$$

Theorem. The non-phantom differential invariants of order $\leq n + 1$ generate all differential invariants through invariant differentiation.

$$\mathcal{D}^K H^i, \quad \mathcal{D}^K I_J^\alpha,$$

for $\#J \leq n + 1$, $\alpha = 1, \dots, q$, $i = 1, \dots, p$, $\#K \geq 0$.

\implies Not necessarily a minimal system!

Commutation formulae

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^p A_{ij}^k \mathcal{D}_k$$

The coefficients A_{ij}^k can be obtained from the key formula by choosing $\Omega = dx^1, \dots, dx^p$.

Syzygies

Theorem. All syzygies among the differentiated invariants are differential consequences of the following three fundamental types:

$$\boxed{\mathcal{D}_j H^i = \delta_j^i + M_j^i}$$

— H^i non-phantom

$$\boxed{\mathcal{D}_J I_K^\alpha = c_\nu + M_{K,J}^\alpha}$$

— I_K^α generating

— $I_{J,K}^\alpha = c_\nu$ phantom

$$\boxed{\mathcal{D}_J I_{LK}^\alpha - \mathcal{D}_K I_{LJ}^\alpha = M_{LK,J}^\alpha - M_{LJ,K}^\alpha}$$

— $I_{LK}^\alpha, I_{LJ}^\alpha$ generating, $K \cap J = \emptyset$

\implies Not necessarily a minimal system!

Euclidean Plane Curves

Lifted invariants

$$y = x \cos \phi - u \sin \phi + a$$

$$v = x \cos \phi + u \sin \phi + b$$

$$v_y = \frac{\sin \phi + u_x \cos \phi}{\cos \phi - u_x \sin \phi}$$

$$v_{yy} = \frac{u_{xx}}{(\cos \phi - u_x \sin \phi)^3}$$

$$v_{yyy} = \frac{(\cos \phi - u_x \sin \phi)u_{xxx} - 3u_{xx}^2 \sin \phi}{(\cos \phi - u_x \sin \phi)^5}$$

$$dy = (\cos \phi - u_x \sin \phi) dx$$

$$D_y = \frac{1}{\cos \phi - u_x \sin \phi} D_x$$

Normalization

$$y = 0 \quad v = 0 \quad v_y = 0$$

Right moving frame $\rho: J^1 \longrightarrow \text{SE}(2)$

$$\phi = -\tan^{-1} u_x \quad a = -\frac{x + uu_x}{\sqrt{1 + u_x^2}} \quad b = \frac{xu_x - u}{\sqrt{1 + u_x^2}}$$

Fundamental normalized differential invariants

$$\left. \begin{aligned} \iota(x) &= H = 0 \\ \iota(u) &= I_0 = 0 \\ \iota(u_x) &= I_1 = 0 \end{aligned} \right\} \text{phantom diff. invs.}$$
$$\begin{aligned} \iota(u_{xx}) &= I_2 = \kappa \\ \iota(u_{xxx}) &= I_3 = \kappa_s \\ \iota(u_{xxxx}) &= I_4 = \kappa_{ss} + 3\kappa^3 \end{aligned}$$

Invariant horizontal one-form

$$\iota(dx) = \sqrt{1 + u_x^2} dx = ds$$

Invariant differentiation

$$\iota(D_x) = \frac{1}{\sqrt{1 + u_x^2}} D_x = D_s$$

Prolonged infinitesimal generators

$$\begin{aligned}\mathbf{v}_1 &= \partial_x & \mathbf{v}_2 &= \partial_u \\ \mathbf{v}_3 &= -u \partial_x + x \partial_u + (1 + u_x^2) \partial_{u_x} + 3u_x u_{xx} \partial_{u_{xx}} + \cdots\end{aligned}$$

$$dI = D_s I \cdot ds$$

Horizontal recurrence formula

$$d\iota(F) = \iota(dF) + \iota(\mathbf{v}_1(F)) \gamma^1 + \iota(\mathbf{v}_2(F)) \gamma^2 + \iota(\mathbf{v}_3(F)) \gamma^3$$

Use phantom invariants

$$0 = dH = \iota(dx) + \sum \iota(\mathbf{v}_\kappa(x)) \gamma^\kappa = ds + \gamma^1,$$

$$0 = dI_0 = \iota(du) + \sum \iota(\mathbf{v}_\kappa(u)) \gamma^\kappa = \gamma^2,$$

$$0 = dI_1 = \iota(du_x) + \sum \iota(\mathbf{v}_\kappa(u_x)) \gamma^\kappa = \kappa ds + \gamma^3,$$

to solve for

$$\gamma^1 = -ds \quad \gamma^2 = 0 \quad \gamma^3 = -\kappa ds$$

$$\gamma^1 = -ds \quad \gamma^2 = 0 \quad \gamma^3 = -\kappa ds$$

Recurrence formulae

$$\begin{aligned} \kappa_s ds = d\kappa = d(I_2) &= \iota(du_{xx}) + \iota(\mathbf{v}_3(u_{xx})) \gamma^3 \\ &= \iota(u_{xxx} dx) - \iota(3u_x u_{xx}) \kappa ds = I_3 ds \end{aligned}$$

$$\begin{aligned} \kappa_{ss} ds = d(I_3) &= \iota(du_{xxx}) + \iota(\mathbf{v}_3(u_{xxx})) \gamma^3 \\ &= \iota(u_{xxxx} dx) - \iota(4u_x u_{xxx} + 3u_{xx}^2) \kappa ds = (I_4 - 3I_2^3) ds \end{aligned}$$

$$\begin{array}{ll} \kappa = I_2 & I_2 = \kappa \\ \kappa_s = I_3 & I_3 = \kappa_s \\ \kappa_{ss} = I_4 - 3I_2^3 & I_4 = \kappa_{ss} + 3\kappa^3 \\ \kappa_{sss} = I_5 - 19I_2^2 I_3 & I_4 = \kappa_{sss} + 19\kappa^2 \kappa_s \end{array}$$

Example

$$(x^1, x^2, u) \in M = \mathbb{R}^3$$

$$G = \text{GL}(2)$$

$$(x^1, x^2, u) \longmapsto (\alpha x^1 + \beta x^2, \gamma x^1 + \delta x^2, \lambda u)$$

$$\lambda = \alpha\delta - \beta\gamma$$

\implies Classical invariant theory

Prolongation (lifted differential invariants):

$$y^1 = \lambda^{-1}(\delta x^1 - \beta x^2) \quad y^2 = \lambda^{-1}(-\gamma x^1 + \alpha x^2)$$

$$v = \lambda^{-1}u$$

$$v_1 = \frac{\alpha u_1 + \gamma u_2}{\lambda} \quad v_2 = \frac{\beta u_1 + \delta u_2}{\lambda}$$

$$v_{11} = \frac{\alpha^2 u_{11} + 2\alpha\gamma u_{12} + \gamma^2 u_{22}}{\lambda}$$

$$v_{12} = \frac{\alpha\beta u_{11} + (\alpha\delta + \beta\gamma)u_{12} + \gamma\delta u_{22}}{\lambda}$$

$$v_{22} = \frac{\beta^2 u_{11} + 2\beta\delta u_{12} + \delta^2 u_{22}}{\lambda}$$

Normalization

$$y^1 = 1 \quad y^2 = 0 \quad v_1 = 1 \quad v_2 = 0$$

Nondegeneracy

$$x^1 \frac{\partial u}{\partial x^1} + x^2 \frac{\partial u}{\partial x^2} \neq 0$$

First order moving frame

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} x^1 & -u_2 \\ x^2 & u_1 \end{pmatrix}$$

Normalized differential invariants

$$\begin{aligned} J^1 &= 1 & J^2 &= 0 \\ I &= \frac{u}{x^1 u_1 + x^2 u_2} \\ I_1 &= 1 & I_2 &= 0 \\ I_{11} &= \frac{(x^1)^2 u_{11} + 2x^1 x^2 u_{12} + (x^2)^2 u_{22}}{x^1 u_1 + x^2 u_2} \\ I_{12} &= \frac{-x^1 u_2 u_{11} + (x^1 u_1 - x^2 u_2) u_{12} + x^2 u_1 u_{22}}{x^1 u_1 + x^2 u_2} \\ I_{22} &= \frac{(u_2)^2 u_{11} - 2u_1 u_2 u_{12} + (u_1)^2 u_{22}}{x^1 u_1 + x^2 u_2} \end{aligned}$$

Phantom differential invariants

$$I_1 \quad I_2$$

Generating differential invariants

$$I \quad I_{11} \quad I_{12} \quad I_{22}$$

Invariant differential operators

$$\begin{aligned} \mathcal{D}_1 &= x^1 D_1 + x^2 D_2 && \text{— scaling process} \\ \mathcal{D}_2 &= -u_2 D_1 + u_1 D_2 && \text{— Jacobian process} \end{aligned}$$

Recurrence formulae

$$\mathcal{D}_1 J^1 = \delta_1^1 - 1 = 0$$

$$\mathcal{D}_2 J^1 = \delta_2^1 - 0 = 0$$

$$\mathcal{D}_1 J^2 = \delta_1^2 - 0 = 0$$

$$\mathcal{D}_2 J^2 = \delta_2^2 - 1 = 0$$

$$\mathcal{D}_1 I = I_1 - I(1 + I_{11}) = -I(1 + I_{11})$$

$$\mathcal{D}_2 I = I_2 - I I_{12} = -I I_{12}$$

$$\mathcal{D}_1 I_1 = I_{11} - I_{11} = 0$$

$$\mathcal{D}_2 I_1 = I_{12} - I_{12} = 0$$

$$\mathcal{D}_1 I_2 = I_{12} - I_{12} = 0$$

$$\mathcal{D}_2 I_2 = I_{22} - I_{22} = 0$$

$$\mathcal{D}_1 I_{11} = I_{111} + (1 - I_{11})I_{11}$$

$$\mathcal{D}_2 I_{11} = I_{112} + (2 - I_{11})I_{12}$$

$$\mathcal{D}_1 I_{12} = I_{112} - I_{11}I_{12}$$

$$\mathcal{D}_2 I_{12} = I_{122} + (1 - I_{11})I_{22}$$

$$\mathcal{D}_1 I_{22} = I_{122} + (I_{11} - 1)I_{22} - 2I_{12}^2$$

$$\mathcal{D}_2 I_{22} = I_{222} - I_{12}I_{22}$$

\implies Use I to generate I_{11} and I_{12}

Syzygies

$$\mathcal{D}_1 I_{12} - \mathcal{D}_2 I_{11} = -2I_{12}$$

$$\mathcal{D}_1 I_{22} - \mathcal{D}_2 I_{12} = 2(I_{11} - 1)I_{22} - 2I_{12}^2$$

$$(\mathcal{D}_1)^2 I_{22} - (\mathcal{D}_2)^2 I_{11} =$$

$$= 2I_{22}\mathcal{D}_1 I_{11} + (5I_{12} - 2)\mathcal{D}_1 I_{12} + (3I_{11} - 5)\mathcal{D}_1 I_{22} - \\ - (2I_{11} - 5)(I_{11} - 1)I_{12} + 4(I_{11} - 1)I_{12}^2$$

Commutation formulae

$$[\mathcal{D}_1, \mathcal{D}_2] = -I_{12}\mathcal{D}_1 + (I_{11} - 1)\mathcal{D}_2$$

Euclidean Surfaces

$S \subset M = \mathbb{R}^3$ coordinates: $z = (x, y, u)$

Group: $G = E(3)$

$$z \longmapsto Rz + a, \quad R \in O(3)$$

Normalization — coordinate cross-section

$$x = y = u = u_x = u_y = u_{xy} = 0.$$

Left moving frame

$$a = z \quad R = (\mathbf{t}_1 \ \mathbf{t}_2 \ \mathbf{n})$$

- $\mathbf{t}_1, \mathbf{t}_2 \in TS$ — Frenet frame
- \mathbf{n} — unit normal

Fundamental differential invariants

$$\kappa^1 = \iota(u_{xx}) \quad \kappa^2 = \iota(u_{yy})$$

\implies principal curvatures

$$H = \frac{1}{2}(\kappa^1 + \kappa^2) \text{ — mean curvature}$$

$$K = \kappa^1 \kappa^2 \text{ — Gaussian curvature}$$

Frenet coframe

$$\varpi^1 = \iota(dx^1) = \omega^1 + \eta^1 \quad \varpi^2 = \iota(dx^2) = \omega^2 + \eta^2$$

Invariant differential operators

$$\mathcal{D}_1 \quad \mathcal{D}_2$$

\implies Frenet differentiation

Fundamental Syzygy:

Use the recurrence formula to compare

$$\iota(u_{xxyy}) \quad \text{with} \quad \begin{aligned} \kappa_{,22}^1 &= \mathcal{D}_2^2 \iota(u_{xx}) \\ \kappa_{,11}^2 &= \mathcal{D}_1^2 \iota(u_{yy}) \end{aligned}$$

$$\kappa_{,22}^1 - \kappa_{,11}^2 + \frac{\kappa_{,1}^1 \kappa_{,1}^2 + \kappa_{,2}^1 \kappa_{,2}^2 - 2(\kappa_{,1}^2)^2 - 2(\kappa_{,2}^1)^2}{\kappa^1 - \kappa^2} - \kappa^1 \kappa^2 (\kappa^1 - \kappa^2) = 0$$

\implies Codazzi equations

Lie Pseudogroups

Sur la théorie, si importante sans doute, mais pour nous si obscure, des «groupes de Lie infinis», nous ne savons rien que ce qui trouve dans les mémoires de Cartan, première exploration à travers une jungle presque impénétrable; mais celle-ci menace de refermer sur les sentiers déjà tracés, si l'on ne procède bientôt à un indispensable travail de défrichage.

— André Weil, *Oeuvres Scientifiques*

Pseudogroups in Action

- Lie — Medolaghi — Vessiot
- Cartan ... Guillemin, Sternberg
- Kuranishi, Spencer, Goldschmidt, Kumpera, ...
- Relativity
- Gauge theory and field theories
 Maxwell, Yang–Mills, conformal, string, ...
- Noether’s Second Theorem
- Fluid Mechanics, Metereology
 Euler, Navier–Stokes,
 boundary layer, quasi-geostropic , ...
- Solitons (in $2 + 1$)
 K–P, Davey-Stewartson, ...
- Kac–Moody
- *Lie groups!*

⇒ with J. Pohjanpelto

Pseudo-groups

Let M be a manifold. A *pseudo-group* is a collection of local diffeomorphisms $\varphi: M \rightarrow M$ such that, for any $\varphi, \psi \in \mathcal{G}$,

- *Identity:* $\mathbf{1}_M \in \mathcal{G}$,
 - *Inverses:* $\varphi^{-1} \in \mathcal{G}$,
 - *Restriction:* $U \subset \text{dom } \varphi \implies \varphi|_U \in \mathcal{G}$,
 - *Composition:* $\text{im } \varphi \subset \text{dom } \psi \implies \psi \circ \varphi \in \mathcal{G}$.
-

Definition. A *Lie pseudo-group* \mathcal{G} is defined by an involutive system of partial differential equations:

$$F(z, \varphi^{(n)}) = 0.$$

\implies *Analytic (Cartan-Kähler)*

\implies Abstract object \mathcal{G} ???

Infinitesimal Generators

\mathfrak{g} — space of infinitesimal generators of \mathcal{G}

Locally defined vector fields:

$$\mathbf{v} = \sum_{a=1}^m \zeta^a(z) \frac{\partial}{\partial z^a}$$

Infinitesimal determining equations

$$L^{(n)}(z, \zeta^{(n)}) = 0 \tag{*}$$

\implies obtained by linearization

Remark: If \mathcal{G} is the symmetry group of a system of differential equations $\Delta(x, u^{(n)}) = 0$, then (*) is the (involutive completion of) the usual Lie determining equations for the symmetry group.

Groupoids

Define $G^{(n)}$ to be the space of Taylor polynomials (jets) of pseudo-group diffeomorphisms.

$G^{(n)}$ forms a *groupoid* since you can only compose two Taylor series (algebraically) if the target of the first matches the source of the second!

One-dimensional case:

$$\varphi : \quad y = a_0 + a_1(x - x_0) + \frac{a_2}{2} (x - x_0)^2 + \frac{a_3}{3!} (x - x_0)^3 + \cdots ,$$

$$\psi : \quad z = b_0 + b_1(y - y_0) + \frac{b_2}{2} (y - y_0)^2 + \frac{b_3}{3!} (y - y_0)^3 + \cdots ,$$

Composition:

$$\varphi \circ \psi : \quad z = c_0 + c_1(x - x_0) + \frac{c_2}{2} (x - x_0)^2 + \frac{c_3}{3!} (x - x_0)^3 + \cdots ,$$

where, **provided** $y_0 = a_0$ (i.e., source = target)

$$c_0 = b_0, \quad c_1 = b_1 a_1, \quad c_2 = b_2 a_1^2 + b_1 a_2,$$

$$c_3 = b_3 a_1^3 + 3b_2 a_1 a_2 + b_1 a_3, \quad \dots$$

\implies Faà-di-Bruno formula.

We will view the pseudo-group \mathcal{G} as the projective limit of its Taylor groupoids:

$$G^{(0)} \longleftarrow G^{(1)} \longleftarrow G^{(2)} \longleftarrow \dots$$

Moving frame of order n :

$$\rho^{(n)} : \mathbf{J}^n \longrightarrow G^{(n)}$$

\implies requires freeness!

\implies compatibility

The moving frame $\rho^{(n)}$ is fixed by the same basic normalization algorithm, using coordinates on $G^{(n)}$ provided by the independent Taylor coefficients.

Theorem. (O–P) For $n \gg 0$, if \mathcal{G} acts freely on \mathbf{J}^n then it acts freely on \mathbf{J}^k for all $k \geq n$.

\implies Bezout's theorem.

Differential Invariants for Pseudo-groups

Invariantization of jet coordinates:

$$H^i(x, u^{(n)}) = \iota(x^i) \quad I_K^\alpha(x, u^{(k)}) = \iota(u_K^\alpha)$$

- phantom (constant) differential invariants
 - fundamental (nonconstant) differential invariants
- \implies For pseudo-groups, phantom invariants appear at arbitrary order.

Theorem. Every differential invariant can be (locally) uniquely written as a function of the (non-phantom) fundamental differential invariants.

Invariant Differentiation

Invariant differential operators:

$$\mathcal{D}_j = \iota(D_{x^j})$$

Theorem. (Tresse–Kumpera)

The higher order differential invariants are obtained by invariant differentiation of a finite number of fundamental differential invariants I_1, \dots, I_N .

Gröbner Basis Approach

Identify the cross-section variables with the complementary monomials to a certain algebraic module \mathcal{J} , which is the pull-back of the symbol module of the pseudo-group under a certain explicit linear map.

\implies Compatible term ordering.

\implies Algebraic specification of compatible moving frames of all orders $n > n^*$.

Theorem. (O–P) For $n \gg 0$, the differential invariants are in one-to-one correspondence with the leading monomials in \mathcal{J} . The (higher order) generating differential invariants correspond to the Gröbner basis monomials.

Structure Equations

The full diffeomorphism pseudo-group:

$$d\mu[[H]] = \nabla_H \mu[[H]] \wedge (\mu[[H]] - dZ)$$

$$\mu^a[[H]] = \sum_J \frac{1}{J!} \mu_J^a H^J$$

μ_J^a — Maurer–Cartan forms
 = right-invariant contact forms on $\mathcal{D}^{(\infty)} \subset J^\infty(M, M)$.

\mathcal{G} — Lie pseudo-group acting on M

The Maurer–Cartan forms are obtained by restricting the μ_J^a to $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$.

Theorem. (O–P) The right-invariant Maurer–Cartan forms for \mathcal{G} satisfy the invariantized linear determining equations

$$L^{(n)}(Z, \mu^{(n)}) = 0, \tag{**}$$

Thus, the structure equations for the pseudo-group are obtained by restricting the diffeomorphism structure equations to the solution space of (**).

The Key Formula

$$d\iota(\Omega) = \iota \left[d\Omega + \widehat{\mathbf{v}}^{(\infty)}(\Omega) \right]$$

where
$$\widehat{\mathbf{v}}^{(\infty)} = \sum_{i=1}^p \eta^i \frac{\partial}{\partial x^i} + \sum_{\alpha, K} \psi_{\alpha}^K \frac{\partial}{\partial u_{\alpha}^K}$$

The coefficients of $\widehat{\mathbf{v}}^{(\infty)}$ are obtained from the coefficients φ_{α}^K of a prolonged vector field by replacing:

- source jet variables u_j^{α} by their invariantizations $I_j^{\alpha} = \iota(u_j^{\alpha})$
- derivatives of vector field coefficients ζ_j^a by the pull-backs ν_j^{α} of the Maurer–Cartan forms

- All recurrence formulae for differential invariants and invariant differential forms follow from specialization.
- In particular, the formulae for the pull-backs of the Maurer–Cartan forms ν_j^{α} are found by solving the recurrence formula for the phantom differential invariants.

First Example

$$\boxed{X = f(x), \quad Y = y, \quad U = \frac{u}{f'(x)}}$$

Horizontal coframe

$$d_H X = f_x dx, \quad d_H Y = dy,$$

Implicit differentiations

$$D_X = \frac{1}{f_x} D_x, \quad D_Y = D_y.$$

Prolonged pseudo-group transformations on surfaces $S \subset \mathbb{R}^3$

$$\begin{aligned} X = f & & Y = y & & U = \frac{u}{f_x} \\ U_X = \frac{u_x}{f_x^2} - \frac{u f_{xx}}{f_x^3} & & U_Y = \frac{u_y}{f_x} \\ U_{XX} = \frac{u_{xx}}{f_x^3} - \frac{3u_x f_{xx}}{f_x^4} - \frac{u f_{xxx}}{f_x^4} + \frac{3u f_{xx}^2}{f_x^5} \\ U_{XY} = \frac{u_{xy}}{f_x^2} - \frac{u_y f_{xx}}{f_x^3} & & U_{YY} = \frac{u_{yy}}{f_x} \end{aligned}$$

\implies action is free at every order.

Coordinate cross-section

$$X = 0, \quad U = 1, \quad U_X = 0, \quad U_{XX} = 0.$$

Moving frame

$$f = 0, \quad f_x = u, \quad f_{xx} = u_x, \quad f_{xxx} = u_{xx}.$$

Differential invariants

$$U_Y \mapsto J = \frac{u_y}{u}$$

$$U_{XY} \mapsto J_1 = \frac{uu_{xy} - u_x u_y}{u^3} \quad U_{YY} \mapsto J_2 = \frac{u_{yy}}{u}$$

Invariant coframe

$$dX \mapsto u dx, \quad dY \mapsto dy,$$

Invariant differentiations

$$\mathcal{D}_1 = \frac{1}{u} D_x \quad \mathcal{D}_2 = D_y$$

Higher order differential invariants: $\mathcal{D}_1^m \mathcal{D}_2^n J$

$$J_{,1} = \mathcal{D}_1 J = \frac{uu_{xy} - u_x u_y}{u^3} = J_1,$$

$$J_{,2} = \mathcal{D}_2 J = \frac{uu_{yy} - u_y^2}{u^2} = J_2 - J^2.$$

$$\boxed{X = f(x), \quad Y = y, \quad U = \frac{u}{f'(x)}}$$

Prolonged infinitesimal generator $a = a(x)$:

$$\begin{aligned} \mathbf{v}^{(\infty)} = & a \partial_x - u a_x \partial_u - (u a_{xx} + 2 u_x a_x) \partial_{u_x} - u_y a_x \partial_{u_y} - \\ & - (u a_{xxx} + 3 u_x a_{xx} + 3 u_{xx} a_x) \partial_{u_{xx}} - \\ & - (u_y a_{xx} + 2 u_{xy} a_x) \partial_{u_{xy}} - u_{yy} a_x \partial_{u_{yy}} - \dots \end{aligned}$$

Normalization

$$X = 0, \quad U = 1, \quad U_X = 0, \quad U_{XX} = 0, \quad \dots$$

Fundamental differential invariants:

$$\begin{aligned} \iota(x) = F = 0, \quad \iota(y) = y, \quad \iota(u) = I_{00} = 1, \\ \iota(u_x) = I_{10} = 0, \quad \iota(u_y) = I_{10} = J, \\ \iota(u_{xx}) = I_{20} = 0, \quad \iota(u_{xy}) = I_{11} = J_1, \quad \iota(u_{yy}) = I_{02} = J_2, \end{aligned}$$

Invariant horizontal forms:

$$\varpi^1 = \iota(dx) = u dx, \quad \varpi^2 = \iota(dy) = dy,$$

Pulled-back Maurer–Cartan forms:

$$\nu, \quad \nu_1, \quad \nu_2, \quad \dots$$

$$\widehat{\mathbf{v}}^{(\infty)} = \nu \partial_x - u \nu_1 \partial_u - (u \nu_2 + 2 u_x \nu_1) \partial_{u_x} - u_y \nu_1 \partial_{u_y} - \dots$$

Phantom invariants:

$$\begin{aligned} 0 = dH &= \varpi^1 + \nu, & 0 = dI_{10} &= J_1 \varpi^2 + \vartheta_1 - \nu_2, \\ 0 = dI_{00} &= J \varpi^2 + \vartheta - \nu_1, & 0 = dI_{20} &= J_3 \varpi^2 + \vartheta_3 - \nu_3, \end{aligned}$$

Solve for pulled-back Maurer–Cartan forms:

$$\begin{aligned} \nu &= -\varpi^1, & \nu_2 &= J_1 \varpi^2 + \vartheta_1, \\ \nu_1 &= J \varpi^2 + \vartheta, & \nu_3 &= J_3 \varpi^2 + \vartheta_3, \end{aligned}$$

Recurrence formulae: $dy = \varpi^2$

$$dJ = J_1 \varpi^1 + (J_2 - J^2) \varpi^2 + \vartheta_2 - J \vartheta,$$

$$dJ_1 = J_3 \varpi^1 + (J_4 - 3 J J_1) \varpi^2 + \vartheta_4 - J \vartheta_1 - J_1 \vartheta,$$

$$dJ_2 = J_4 \varpi^1 + (J_5 - J J_2) \varpi^2 + \vartheta_5 - J_2 \vartheta,$$

$$\mathcal{D}_1 J = J_1, \quad \mathcal{D}_2 J = J_2 - J^2, \quad d_{\mathcal{V}} J = \vartheta_2 - J \vartheta,$$

$$\mathcal{D}_1 J_1 = J_3, \quad \mathcal{D}_2 J_1 = J_4 - 3 J J_1, \quad d_{\mathcal{V}} J_1 = \vartheta_4 - J \vartheta_{10} - J_1 \vartheta,$$

$$\mathcal{D}_1 J_2 = J_4, \quad \mathcal{D}_2 J_2 = J_5 - J J_2, \quad d_{\mathcal{V}} J_2 = \vartheta_5 - J_2 \vartheta,$$

\implies All higher order differential invariants are obtained from J by invariant differentiation

Invariant horizontal forms

$$d\varpi^1 = -J \varpi^1 \wedge \varpi^2 + \vartheta \wedge \varpi^1, \quad d\varpi^2 = 0.$$

Commutation formula

$$[\mathcal{D}_1, \mathcal{D}_2] = J \mathcal{D}_1.$$