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Differential Hyperforms I.

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Abstract

A theory of higher order differential forms, called differential hyperforms, based on the theory of Schur functors, is constructed over Euclidean space. Generalizations of the deRham complex lead to the notions of a hypercomplex and hypercohomology theories based on higher order derivatives. Applications include the systematic derivation of higher order divergence identities for hyperjacobians and a wide variety of interesting higher order Pfaffian systems with integrability criteria. New, explicit formulae in the algebraic theory of Schur functors are also presented.

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1. Introduction

This is the first of two papers in which a new theory of higher order differential forms, or "hyperforms," is introduced, based on the recently developed and, in my view, fundamentally important theory of Schur functors. This paper is devoted to the construction of hyperforms over Euclidean space, but with no consideration of their behavior under changes of coordinates. The sequel will do the more difficult constructions over arbitrary smooth manifolds. Although a regrettably extensive amount of algebraic machinery must be developed before these hyperforms can be properly considered, I hope that the intrinsic beauty of the identities which can be found, as well as the range of potential applications in both differential geometry and partial differential equations will make the reader's effort in assimilating the material worthwhile.

When I first learned about differential forms and their applications in differential geometry, especially deRham's theorem, [28], I was always struck by the fact that these constructions only involved first order derivatives - either of the coefficients of the forms, or the functions giving changes of coordinates. Indeed, the only places in which higher order derivatives make their appearance in differential geometry are a) in jet bundles and overdetermined systems of differential equations, [14], [26], b) the more or less equivalent, but far less sophisticated theory of extensors, cf. [21] and references therein, c) the theory of higher order frame bundles or tangent vectors, eg. [16], [28, § 1.26]. Although extremely interesting and useful, none of the above mentioned theories has any of the flavor of differential forms with differentials, exact sequences and cohomology in any truly higher order sense. I propose that the theory of differential hyperforms fills this long neglected gap, and provides a correct setting for the systematic development of a differential geometry based on higher order derivatives with many potential applications. In particular, higher order cohomology, characteristic classes, etc., are among the ideas that remain to be developed, all of which must be dealt with in later investigations.

Be that as it may, the original motivation for my development of this theory came from a question raised in the study of variational problems of interest in nonlinear elasticity, [3], on what homogeneous differential polynomials could be written as higher order divergences, the first non-trivial example being the curvature identity

$$u_{xx}u_{yy} - u_{xy}^2 = - D_x^2\left(\frac{1}{2}u_y^2\right) + D_xD_y(u_xu_y) - D_y^2\left(\frac{1}{2}u_x^2\right) , \qquad (1.1)$$

the D's demoting derivatives. This problem was solved in [23]; the most general such polynomial is a linear combination, of polynomials I called hyperjacobians as they were higher order analogues of the classical Jacobian determinants - see section 15. It was also of interest to actually construct the explicit identities for these hyperjacobians, and to do this a some what unsophisticated version of differential hyperforms was proposed.

At the same time as [3], [23] were being written the concept of a Schur functor, or shape functor having been introduced by Towber, [27], and Lascoux, [17], was being developed by Akin, Buchsbaum and Weyman,

[1], [2], [30], for the purpose of resolving certain determinantal ideals. Although results in [3], [23] required detailed knowledge of certain properties of these determinatal ideals I was unaware of the existence of Schur functors until I heard David Buchsbaum lecture in Minnesota in January, 1981. I realized that my primitive algebraic constructions of hyperforms in [23] were just a special case of a much grander theory of Schur functors, thereby considerably gaining in power and range of applicability. This paper is the first fruit of this marriage of the powerful new algebraic tenchiques to differential geometric ideas. It is, in my opinion, just the beginning of the application of these fundamentally important functors to a wide range of geometric, topological and even physical problems.

The basic idea behind differential hyperforms is as follows. On the Euclidean space $M = \mathbb{R}^p$, corresponding to each shape (Young diagram) λ is a hyperform bundle Ξ_λ obtained by applying the Schur functor L_λ pointwise to the cotangent bundle T^*M; smooth sections of Ξ_λ are called λ-hyperforms. For each shape $\mu \supset \lambda$ there is a differential d^μ_λ taking λ-hyperforms to μ-hyperforms, so that the coefficients are differentiated $|\mu/\lambda|$ times. The differentials d^μ_λ commute in the obvious sense, and, moreover $d^\mu_\lambda \equiv 0$ if μ/λ has two or more boxes in any column. For $\lambda = 1^k$ a single column, $\Xi_\lambda \simeq \Lambda_k T^*$, and d^μ_λ for $\mu = 1^{k+1}$ is the ordinary differential. Thus the deRham complex forms a small part of the much larger <u>differential hypercomplex</u> formed by the hyperform bundles and differentials. In this paper, since d^μ_λ only involves $|\mu/\lambda|$ order differentiations, and no lower order, it cannot be invariant under changes of coordinates, which is why only Euclidean spaces are considered here.

The differential hypercomplex is also exact in a certain sense. The simplest manifestation of exactness occurs for shapes $\lambda \subset \mu \subset \nu$ with μ/λ and ν/μ each consisting of a single row of boxes, in consecutive rows, and which overlap in precisely one column; for instance $\lambda = (3,1)$, $\mu = (3,3)$, $\nu = (3,3,2)$. Then a given μ-hyperform η satisfies

$$d_\lambda^\mu \xi = \eta \tag{1.2}$$

for some λ-hyperform ξ if and only if

$$d_\mu^\nu \eta = 0 \ . \tag{1.3}$$

(Note $d_\mu^\nu d_\lambda^\mu \xi = d_\lambda^\nu \xi = 0$ for all ξ .) If $|\mu/\lambda| = k$, $|\nu/\mu| = \ell$, then (1.2) forms a large system of k-th order nonhomogeneous linear constant coefficient partial differential equations in the coefficient functions of ξ , and (1.2) constitute the full set of ℓ-th order integrability conditions for their solution. By suitable choice of λ , μ , ν one can find systems of any desired order k with integrability conditions of some other predetermined order ℓ . Often these systems are nontrivial - see section 16. For more general $\lambda \subset \mu$, the system (1.2) will still be of order $|\mu/\lambda|$, but the integrability conditions will consist of several systems of the form (1.3) corresponding to different $\nu_i \supset \mu$ with possibly different orders $|\nu_i/\mu|$. Again, any desired orders can be found by a suitable choice of λ , μ . The exactness also includes systems of mixed order with integrability conditions of various orders. These systems can be viewed as true, nontrivial higher order generalizations of the classical Pfaff systems, [12], [25]. They could be valuable as specific, nontrivial examples for further investigations into the Spencer-Goldschmidt theory of overdetermined systems of differential equations, [14], [26]. For manifolds, the lack of exactness of the hypercomplex will introduce new types of cohomology, but we will defer this to the second paper. (A similar type of hypercomplex has been introduced by Delong, [7], in his study of Killing tensors. It is of great interest to study the connection between our hyperforms and his differentials, which require a Riemannian connection.)

The Schur functors required for these constructions only involve real or complex vector spaces. This restriction to characteristic zero throughout leads to several simplifications in the underlying Schur functor theory, the most notable being the lack of distinction between the symmetric and divided power algebras and hence between Schur and co-Schur functors, cf. [2]. Actually, our definitions are a slightly hybridized version of co-Schur functors, but for simplicity the extraneous co- has been dropped.

For a vector space V, the Schur spaces $L_\lambda V$ have been around for a long time; they are just the irreducible representation modules of the general linear group $GL(V)$, [4], [20], [29]. Towber, [27], uses a second definition in terms of tensor powers of the symmetric algebra of V modulo an ideal of relations. Yet a third definition, used by Akin, Buchsbaum and Weyman, [2], is as the image of certain maps on these tensor powers. Each definition has its own particular advantage - the first, in conjuction with functoriality and Schur's lemma reduces otherwise impossible identities to the computation of a single constant; the second is best for specific computational examples; the last has the advantage in the development of the underlying theory. All three are used at one point or another here, and I see no reason to prefer one over the other.

Although a substantial amount of vital theoretical groundwork has been laid by Lascoux, [17], and Akin, Bucksbaum and Weyman, [2], I have, for the most part, found their constructions of maps between Schur spaces far too unwieldy to work with when it comes to specific computations, many of which are vital in the development of the theory. This is especially true in the so-called Pieri formula for decomposing certain tensor products. Sections 5 and 6 redo this important result with explicit, easily computable expressions for the relevant maps which have not appeared so far. (This redevelopment is one explanation for the somewhat inordinate length of this paper.) In essence, the Pieri formula provides a product between a vector $v \in V$ and a hyperform $\omega \in L_\lambda V$ to give a hyperform $v * \omega \in L_\mu V$ for any $\mu \supset \lambda$ with $|\mu / \lambda| = 1$. This Pieri product lends to an algebraic version of the differential hypercomplex, which we call the <u>Schur hypercomplex</u>, and which has similar properties of commutativity, closure and exactness - see Section 8.

It is also possible to define products between hyperforms, but here one runs into the problem that, according to the Littlewood-Richardson rule, the Schur space $L_\nu V$ may occur with multiplicity greater than one in the tensor product $L_\lambda V \otimes L_\mu V$, and hence there may be more than one type of invariantly defined product between a λ-hyperform and a μ-hyperform giving a ν-hyperform. For this reason, except for one special case, we have not investigated the complete "shape algebra " in any great detail.

If, however, λ,μ,ν are all of the form $n^q r$ for fixed n, $0 \leq r < n$, which we call "n-fat shapes", then there is a uniquely defined product, up to constant multiple. These fat hyperforms and products are exactly the hyperforms that were introduced in [23] for constructing hyperjacobian identities. As in [23], these products are nonassociative unless certain restrictions on the remainders r are met; this nonassociativity is another manifestation of the appearance of Schur spaces with multiplicity greater than one, this time in triple tensor products. Finally, the hyperjacobian identities are constructed using a version of Leibnitz' rule for hyperforms, which in turn relies on the notion of the algebraic differential of a functorial product between hyperforms; see sections 11, 12, 14. Thus, for instance, the identity (1.1) is just the coefficient of $dx^2 \otimes dy^2$ in the hyperform identity

$$d^2(du * du) = C\ d^2u * d^2u\ ,$$

where C is a constant, depending on the explicit definitions of the above $*$ products. See section 15 for details. The Schur functor-hyperform theory becomes a powerful tool for constructing these complicated, but beautiful identities.

Some words of advice on how to approach the paper might help the reader. One needs a certain familiarity with the concept of a Schur functor, although all necessary definitions are provided in section 3. The reader completely unfamiliar with these objects micht need to consult the basic references given in that section before proceeding. Much of sections 2-4 consists of technical definitions and results which are best left until required in subsequent parts. The first important concept is the Pieri map, introduced in section 5. It is helpful to play with some of the simpler examples, e.g. example 5.3, rather than on first reading trying to fathom the most general case. Skipping section 6, the key concept of a hypercomplex is introduced in section 7, of which only the first 3 pages need be read initially, followed by the algebraic Schur hypercomplex in section 8. Here, analysis of the examples and further computations of simple cases are essential to gain familiarity. At this stage, it is advisable to skip ahead to section 13 to understand the corresponding differential hypercomplex, whose exactness properties are in section 16, including a number of examples. The hyperjacobian identities, dicussed in section 15, then rely on the omitted material in sections 9-12, 14. With this overview complete, the more complicated proofs and properties can now be properly appreciated.

Finally I would like to thank P. Delong, J. Eagon, L. Green, R. Gulliver, S. Johnson, E. Kalnins, W. Miller, Jr., J. Roberts, and J. Thompson for patiently attending a seminar in which these ideas were first presented, and offering numerous helpful comments and suggestions.

2. Shapes and tableaux

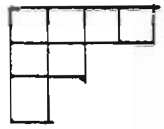
Let $\mathbb{N}^{\infty}$ denote the set of infinite sequences of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \ldots)$ with only finitely many nonzero terms. Finite sequences $\alpha = (\alpha_1, \ldots, \alpha_m)$ are viewed as elements of $\mathbb{N}^{\infty}$ by appending zeros: $\alpha = (\alpha_1, \ldots, \alpha_m, 0, 0, \ldots)$. A sequence $\alpha \in \mathbb{N}^{\infty}$ is identified with an array of boxes, called a (Young) diagram, with α_j boxes in the j-th row of the array (counting from the top down). A diagram $\lambda = (\lambda_1, \lambda_2, \ldots)$ is called a shape if the corresponding sequence is nonincreasing, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots$. Unless specifically mentioned otherwise, the symbols λ, μ, ν will always denote shapes, whereas α, β will denote arbitrary diagrams.

The weight of a diagram α is $|\alpha| = \alpha_1 + \alpha_2 + \ldots$; the number of rows is $m(\alpha) = \max\{i \mid \alpha_i > 0\}$. Given α, $\beta = \alpha + j$ denotes the diagram obtained by adding a single box to the j-th row of α, so $\beta_i = \alpha_i$, $i \neq j$, $\beta_j = \alpha_j + 1$. In the case of shapes, the relation $\mu = \lambda + j$ presumes that the resulting diagram is actually a shape, i.e. $\lambda_{j-1} > \lambda_j$ if $j > 1$. Given (j,k) positive, let $\beta = \alpha/(j,k+)$ be the diagram obtained by deleting all boxes in the j-th row of α in all columns beyond (and including) the k th column, so $\beta_i = \alpha_i$ for $i \neq j$, $\beta_j = \min\{\alpha_j, k-1\}$.

The diagrams, and hence the shapes, are partially ordered so $\alpha \subseteq \beta$ if $\alpha_j \leq \beta_j$ for all j, i.e. the diagram α is contained in the diagram β. If $\alpha \subseteq \beta$, $\alpha \neq \beta$, we write $\alpha \subset \beta$. If $\lambda \subset \mu$ are shapes, the skew-shape μ/λ denotes the array obtained by deleting the boxes in λ from μ. (This is not in general a diagram any longer.) If λ is a shape, the dual shape $\tilde{\lambda}$ is obtained by interchanging rows and columns, so $\tilde{\lambda}_j = \# \{i \mid \lambda_i \geq j\}$.

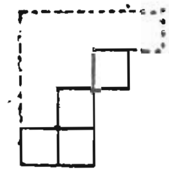
Shape $\lambda = (4,2,1)$

$m(\lambda) = 3$



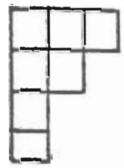
Skew-shape μ/λ ,

$\mu = (4,3,2,2)$



Dual shape:

$\tilde{\lambda} = (3,2,1,1)$



Hook h_{13} in μ

(filled in boxes)

$\ell_{13}(\mu) = 5$

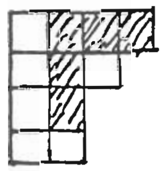
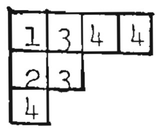


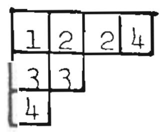
Tableau of shape λ

$S = \{1,2,3,4\}$



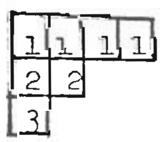
Standard tableau of

shape λ , same S .

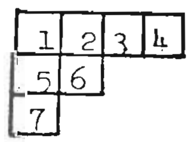


Simple and canonical

tableaux of shape λ



simple



canonical

Figure 1.

Given a shape μ , define

$$\mathcal{S}(\mu) = \{\lambda | \lambda \subset \mu\} ,$$

$$\mathcal{S}_k(\mu) = \{\lambda | \lambda \subset \mu , |\mu/\lambda| = k\} ,$$

$$\mathcal{S}^1(\mu) = \{\lambda | \lambda \subset \mu , \mu/\lambda \text{ has no column with two or more boxes}\}$$

$$\mathcal{S}^2(\mu) = \mathcal{S}(\mu) \sim \mathcal{S}^1(\mu)$$

$$\mathcal{S}_k^i(\mu) = \mathcal{S}^i(\mu) \cap \mathcal{S}_k(\mu) , \quad i = 1,2,$$

and

$$\mathcal{J}(\mu) = \{\nu | \mu \subset \nu\} ,$$

$$\mathcal{J}_k(\mu) = \{\nu | \mu \subset \nu , |\nu/\mu| = k\} ,$$

$$\mathcal{J}^1(\mu) = \{\nu | \mu \subset \nu , \nu/\mu \text{ has no column with two or more boxes}\} ,$$

$$\mathcal{J}^2(\mu) = \mathcal{J}(\mu) \sim \mathcal{J}^1(\mu) ,$$

$$\mathcal{J}_k^i(\mu) = \mathcal{J}^i(\mu) \cap \mathcal{J}_k(\mu) , \quad i = 1,2 .$$

Given $\underset{\sim}{\nu} = \{\nu^1 \ldots, \nu^k\} \subset \mathcal{J}^1(\mu)$, let

$$\mathcal{S}^1(\mu, \underset{\sim}{\nu}) = \mathcal{S}^1(\mu) \cap \mathcal{S}^2(\nu^1) \cap \ldots \cap \mathcal{S}^2(\nu^k) .$$

Note $\mathcal{S}^1(\mu, \underset{\sim}{\nu}) \neq \emptyset$. A shape $\lambda \in \mathcal{S}^1(\mu, \underset{\sim}{\nu})$ is <u>maximal</u> if no other shape in $\mathcal{S}^1(\mu, \underset{\sim}{\nu})$ contains λ ; the set of maximal shapes is denoted $\mathcal{S}^0(\mu, \underset{\sim}{\nu})$. Similarly, given $\lambda = \{\lambda^1, \ldots, \lambda^j\} \subset \mathcal{S}^1(\mu)$, let

$$\mathcal{J}^1(\underset{\sim}{\lambda}, \mu) = \mathcal{J}^1(\mu) \cap \mathcal{J}^2(\lambda^1) \cap \ldots \cap \mathcal{J}^2(\lambda^j) .$$

A shape $\nu \in \mathcal{J}^1(\underset{\sim}{\lambda}, \mu)$ is minimal if no other shape in $\mathcal{J}^1(\underset{\sim}{\lambda}, \mu)$ is contained in ν ; $\mathfrak{T}^0(\underset{\sim}{\lambda},\mu)$ denotes the set of all minimal shapes.

Note that for $\underset{\sim}{\lambda}\subset\mathfrak{S}^1(\mu)$,

$$\mathfrak{S}^0(\mu,\mathfrak{T}^0(\underset{\sim}{\lambda},\mu)) = \underset{\sim}{\lambda}$$

but, in general, for $\underset{\sim}{\nu}\subset\mathfrak{T}^1(\mu)$

$$\mathfrak{T}^0(\mathfrak{S}^0(\mu,\underset{\sim}{\nu}),\mu) \neq \underset{\sim}{\nu} .$$

However, equality will hold if $\nu_1 = \mu_1$ for all $\nu \in \underset{\sim}{\nu}$. The sets $\underset{\sim}{\lambda},\mu,\underset{\sim}{\nu}$ will be called <u>min-max related</u> if

$$\underset{\sim}{\lambda} = \mathfrak{S}^0(\mu,\underset{\sim}{\nu}) \quad \text{and} \quad \underset{\sim}{\nu} = \mathfrak{T}^0(\underset{\sim}{\lambda},\mu) .$$

If $\underset{\sim}{\lambda} = \{\lambda\}$ or $\underset{\sim}{\nu} = \{\nu\}$ consists of a single shape, then we will drop the bold-face and write $\mathfrak{S}^1(\mu,\nu)$, etc. In particular, shapes λ,μ,ν are min-max related if and only if μ/λ and ν/μ both are single rowed, lying in consecutive rows and overlapping in a single column; in other words,

$$\lambda = \mu/(i,\nu_{i+1}+) , \mu = \nu/(i,j+),$$

for some $i>0$, $j>0$.

<u>Example 2.1</u>

Let $\mu = (3,2,1)$. If $\nu = (3,3,2)$, then

$$\underset{\sim}{\lambda} = \mathfrak{S}^0(\mu,\nu) = \{(2,2,1) , (3,1,1)\}$$

and $\underset{\sim}{\lambda}, \mu, \nu$ are min-max related. If $\underset{\sim}{\nu} = \{(3,3,1,1), (3,2,2,1)\}$, then

$$\underset{\sim}{\lambda} = \mathcal{S}^0(\mu, \underset{\sim}{\nu}) = \{(3,2), (2,1,1)\}$$

and again $\underset{\sim}{\lambda}, \mu, \underset{\sim}{\nu}$ are min-max related.

Lemma 2.2 Let $\underset{\sim}{\lambda}, \mu, \underset{\sim}{\nu}$ be min-max related. If $\rho \in \mathcal{T}^1(\mu)$ with $\rho \not\supset \nu$ for all $\nu \in \underset{\sim}{\nu}$, then there exists $\lambda \in \underset{\sim}{\lambda}$ with $\rho \in \mathcal{T}^1(\lambda)$.

Proof

If ρ / μ consists only of boxes in the first row, then any λ will do. Otherwise $\mathcal{S}^1(\mu, \underset{\sim}{\nu} \cup \{\rho\}) \subset \mathcal{S}^1(\mu, \underset{\sim}{\nu})$ is nonempty, so let λ_o be an element. Let $\lambda \in \underset{\sim}{\lambda}$ contain λ_o. If $\rho \notin \mathcal{T}^1(\lambda)$, then $\lambda \in \mathcal{S}^1(\mu, \underset{\sim}{\nu} \cup \{\rho\})$, so λ cannot be $\mu, \underset{\sim}{\nu}$-maximal. Q.E.D.

Lemma 2.3 Let $\underset{\sim}{\lambda}, \mu, \underset{\sim}{\nu}$ be min-max related. Then

$$\text{i)} \quad \mathcal{T}^1(\mu) \subset \bigcup_{\lambda \in \underset{\sim}{\lambda}} \mathcal{T}^1(\lambda) \cup \bigcup_{\nu \in \underset{\sim}{\nu}} \mathcal{T}^1(\nu), \tag{2.1}$$

and

$$\text{ii)} \quad \mathcal{T}^1(\lambda) \cap \mathcal{T}^1(\nu) = \emptyset. \quad \text{For } \lambda \in \underset{\sim}{\lambda}, \ \nu \in \underset{\sim}{\nu}. \tag{2.2}$$

Proof

The second formula is obvious since if $\rho \supset \nu$ for $\nu \in \underset{\sim}{\nu}$ then $\rho \in \mathcal{T}^2(\lambda)$ for all $\lambda \in \underset{\sim}{\lambda}$. To prove (2.1), if $\rho \in \mathcal{T}^1(\mu)$, then either $\rho \in \mathcal{T}^1(\nu)$ for some $\nu \in \underset{\sim}{\nu}$ (ρ can't be in $\mathcal{T}^2(\nu)$ for any ν), or

$\rho \not\supseteq \nu$ for all $\nu \in \underset{\sim}{\nu}$, so, by Lemma 2.2, $\rho \in \mathcal{J}^{1}(\lambda)$ for some $\lambda \in \underset{\sim}{\lambda}$.

Sometimes the exponential notation

$$\lambda = k_1^{m_1} \dots k_p^{m_p} ,$$

$k_1 > k_2 > \dots > k_p > 0$ will be used for a shape with m_1 rows of length k_1, m_2 rows of length k_2, etc.

A shape is <u>n-bounded</u> if $\lambda_j \leq n$ for all j, i.e. there are at most n columns in the diagram. A shape is <u>n-fat</u> if it is of the special form

$$\lambda = n^q r$$

for $0 \leq r < n$. Note that there is precisely one n-fat shape of a given weight; here $q = q(\lambda)$ and $r = r(\lambda)$ are the quotient and remainder respectfully obtained by dividing $|\lambda|$ by n.

The i,j <u>hook</u>, $h_{ij}(\lambda)$, is the set of boxes consisting of a single row and single column of boxes, contained in λ, which connects the last box in the i-th row of λ (called the <u>initial box</u> of the hook) to the last box in the j-th row (called the <u>terminal</u> box of the hook.) The <u>hook length</u>, the number of boxes in the hook, is given by the formula

$$\ell_{ij}(\lambda) = \lambda_i - \lambda_j + j - i + 1 , \quad i < j .$$

The total hook length of λ is just the product of all the hook lengths:

$$L(\lambda) = \prod_{i<j} \ell_{ij}(\lambda) . \tag{2.3}$$

If $\mu \supset \lambda$, the total hook length of μ mod λ, denoted $L(\mu,\lambda)$ is the product of the lengths of all hooks $h_{ij}(\mu)$ whose terminal box is in μ/λ.

A tableau T of shape λ with values in a set S is given by a map $T: \lambda \to S$, or, equivalently, by filling in the boxes in λ with elements of S. If S is ordered, the tableau is called row standard if the rows are nondecreasing, column standard if the columns are strictly increasing and standard if both row and column standard. The content of a tableau T with values in S is the map $c: S \to \mathbb{N}$ with $c(s)$ denoting the number of occurrences of s in T. The simple tableau of shape λ is the standard tableau all of whose entries in the j-th row are the integer j. The canonical tableau of shape λ is the standard tableau with entries $1,2,\ldots,|\lambda|$ arranged in increasing order, left to right, top to bottom.

The number of standard tableau $T: \lambda \to S$ can be calculated by the following formulae, [4], [29].

Theorem 2.4 Suppose $\#S = p$. Then the number of standard tableau of shape $\lambda = (\lambda_1,\ldots,\lambda_m)$, $m \leq p$, with values in S is

$$N_\lambda^p = \frac{\Delta(\ell_1,\ell_2,\ldots,\ell_p)}{\Delta(p-1,p-2,\ldots,0)}, \qquad (2.4)$$

where

$$\ell_i = \lambda_i + p - i, \quad (\lambda_j = 0 \quad j > m),$$

and Δ is the difference product

$$\Delta(x_1, \ldots x_p) = \prod_{i<j} (x_i - x_j) \ .$$

(If $m > p$, $N_\lambda^p = 0$.)

<u>Theorem 2.5</u> Let $|\lambda| = n$. The number of standard tableau of shape λ with entries in $\{1, \ldots, n\}$, each entry occurring precisely once, is

$$M_\lambda = \frac{n! \ \Delta(\ell_1, \ldots, \ell_m)}{\ell_1! \ \ell_2! \ \ldots \ \ell_m!} \quad , \tag{2.5}$$

where

$$\ell_i = \lambda_i + n - i \ .$$

3. Schur functors

In this section the basic theory of Schur functors in characteristic zero is outlined. Proofs of the statements can be found in the basic references [2], [17], [27]. Let V be a fixed (real or complex) vector space. The tensor, symmetric and exterior algebras of V are denoted by $\otimes_* = \otimes_* V$, $\Theta_* = \Theta_* V$, $\wedge_* = \wedge_* V$ respectively. Each of these is a graded algebra, so $\otimes_* V = \oplus_{i \geq 0} \otimes_i V$, etc.

Given a graded vector space $W = \oplus W_i$, and an element $\alpha \in \mathbb{N}^\infty$, define the tensor product

$$\otimes_\alpha W = W_{\alpha_1} \otimes W_{\alpha_2} \otimes \cdots .$$

For a shape λ, define the map.

$$\delta_\lambda : \otimes_\lambda \Theta_* \to \otimes_{\tilde{\lambda}} \wedge_*$$

by composing the tensor product of diagonal maps

$$\Theta_{\lambda_j} \to \otimes_{\lambda_j}$$

with the tensor products of projections

$$\otimes_{\tilde{\lambda}_j} \to \wedge_{\tilde{\lambda}_j} ,$$

remembering that the various copies of V occurring in the intermediate tensor product are labelled according to their positions in the diagram λ. The Schur space $L_\lambda = L_\lambda V$ is then defined to be the image of δ_λ. This is the definition of Schur functor favored by Lascoux, [17], Nielsen, [22] and Akin, Buchsbaum and Weyman, [1], [2], [30].

A different construction was proposed by Towber, [27]. Let $\mathcal{J}$ denote the two-sided ideal in $\otimes_* \Theta_*$ generated by relations of the form

$$\sum_{i=1}^{p+1} x_{\hat{i}} \otimes (x_i \odot z) = 0 , \tag{3.1}$$

called Young symmetries, for $x_1, \ldots, x_{p+1} \in V$, $z \in \Theta_{q-1} V$, and $p \geq q$, where $x_{\hat{i}} = x_1 \odot \ldots \odot x_{i-1} \odot x_{i+1} \odot \ldots \odot x_{p+1}$. Then

$$\ker \delta_\lambda = \mathcal{J}_\lambda = \mathcal{J} \cap \otimes_\lambda \Theta_* ,$$

hence the Schur space

$$L_\lambda = \otimes_\lambda \Theta_* / \mathcal{J}_\lambda$$

is the quotient space. For computational purposes it is useful to know the following consequential relations, cf. [27].

Theorem 3.1 Let $x = x_1 \odot \ldots \odot x_p \in \Theta_p$, $y \in \Theta_q$, $z \in \Theta_r$ with $p \geq q + r$. Then the relations

$$x \otimes (y \odot z) = (-1)^q \sum (x_I \odot y) \otimes (x_J \odot z) \tag{3.2}$$

are in the ideal $\mathcal{J}$. In (3.2) the sum is over all multi-indices I, J with $I \cup J = \{1, \ldots, p\}$, $I \cap J = \emptyset$ and $|J| = q$, so $x_J = x_{j_1} \odot \ldots \odot x_{j_q}$, etc.

The Schur space L_λ can be identified with the representation space of the irreducible representation ρ_λ of the general linear group $GL(V)$ corresponding to the shape λ. This interpretation, which can be inferred from the standard works on representation theory, e.g.

[4], [20], [29], leads to yet another definition of L_λ in terms of Young symmetrizers.

If $|\lambda| = n$, let T_o be the canonical tableau of shape λ . The Young symmetrizer for the shape λ is, cf. (2.5),

$$\sigma_\lambda = \frac{M_\lambda}{n!} \Sigma \operatorname{sgn}(\pi)\pi\rho \ , \tag{3.3}$$

the sum being over all permutations π and ρ of $\{1,\ldots,n\}$ with π preserving the columns and ρ the rows of T_o , cf. [4; page 101], [29; page 124]. This induces a map

$$\sigma_\lambda : \otimes_n V \to \otimes_n V \ ,$$

where the permutations π,ρ in σ_λ permute the various copies of V in the tensor product. With the above normalization, σ_λ is a projection:

$$\sigma_\lambda \circ \sigma_\lambda = \sigma_\lambda \ ,$$

and

$$\operatorname{im} \sigma_\lambda \cong L_\lambda \ .$$

In fact, σ_λ clearly restricts to a projection

$$\sigma_\lambda : \otimes_\lambda \Theta_* \to L_\lambda \ ,$$

for which we use the same notation.

A decomposable element of $\otimes_\lambda \Theta_*$ is of the form

$$x = (x_1^1 \odot \ldots \odot x_{\lambda_1}^1) \otimes \ldots \otimes (x_1^m \odot \ldots \odot x_{\lambda_m}^m) \ ,$$

where each $x^i_j \in V$, and can thus be identified with a tableau of shape λ with values in V, the entries being the x^i_j. This in turn defines a decomposable element of L_λ, which we also write as $\underset{\sim}{x}$, from now on omitting the maps δ_λ or σ_λ when there is no possibility of confusion. In particular, if V has basis $\{e_1,\ldots,e_p\}$, then for each tableau of shape λ with values in $\{1,\ldots,p\}$, there is a corresponding decomposable element e_T of $\otimes_\lambda \odot_*$, and hence L_λ, obtained by substituting the vector e_i for the integer i wherever it occurs in λ. (For the standard tableau of Figure 1, $e_T = (e_1 \odot e_2 \odot e_2 \odot e_4) \otimes (e_3 \odot e_3) \otimes e_4$.) The row standard tableaux T provide a basis e_T of $\otimes_\lambda \odot_*$, but these elements are no longer independent when viewed as elements of L_λ. The following theorem, [2], [8] provides a basis for the Schur space L_λ.

<u>Theorem</u> 3.2 Let $\dim V = p$, and $\{e_1,\ldots,e_p\}$ be a basis. Then the elements e_T for T standard tableaux of shape λ form a basis for L_λ. Therefore the dimension of $L_\lambda V$ is N^p_λ, as given in (2.4).

Given a nonstandard tableau T, it is useful to know how to rewrite the element e_T as a linear combination of basis elements of L_λ; this is known as a <u>straightening law</u>. Rearranging the symmetric products in e_T we may assume that T is at least row standard. Let the last non-standard entry of T occur at position (i,j). In other words, $t_{i,j} \geq t_{i-1,j}$, but $t_{i',j'} < t_{i'-1,j'}$ for all (i',j') with $i' > i$, or $i' = i$ and $j' > j$. Then use theorem 3.1 on the $i-1$ st and i-th rows of e_T with x being the symmetric product of the e's from the (i-1)st row of T, y the product of the first j entries of the i-th row and z any remaining entries in the i-th row. For instance, for the nonstandard tableau of figure 1, $(i,j) = (2,2)$, so we re-express e_T as a linear combination of elements corresponding to the tableaux

$$\begin{array}{|c|c|c|c|}\hline 2&2&3&4\\\hline 1&2\\\cline{1-2} 4\\\cline{1-1}\end{array}\ ,\qquad \begin{array}{|c|c|c|c|}\hline 2&2&3&4\\\hline 1&4\\\cline{1-2} 4\\\cline{1-1}\end{array}\ ,\qquad \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 2&4\\\cline{1-2} 4\\\cline{1-1}\end{array}\ ,\qquad \begin{array}{|c|c|c|c|}\hline 1&2&2&3\\\hline 4&4\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

the first and last with coefficients $+1$, the other two with coefficients $+2$. (For tableau with repeated entries, this step must sometimes be modified slightly by rearranging the equation if the original tableau appears on both sides of the identity.) Of the resulting tableaux, those which are not standard are again subjected to this procedure. The entire process is guarenteed to terminate in a finite number of steps, since at each stage the lexicographic order, obtained by writing out the entries of the tableau in reverse order, i.e. right to left, bottom to top, increases! See Doubillet - Rota - Stein, [8], or DeConcini - Eisenbud - Procesi, [6], for details.

Finally, we remark that if $\lambda = k$ consists of one row, then L_λ is the symmetric power $\odot_k$. If $\lambda = 1^m$ consists of one column, then L_λ is just the wedge power $\wedge_m$.

More generally, the same methods can be used to define Schur functors $L_{\mu/\lambda}V$ for skew shapes μ/λ, cf. [2]. Our only interest in these more general functors is in the following decomposition theorem for direct sums.

Theorem 3.3 Suppose $V = W \oplus Z$. Then

$$L_\mu V = \bigoplus_k L_\mu^{(k)}(W,Z) \ , \tag{3.5}$$

where each $L_\mu^{(k)}(W,Z)$ has a natural filtration, with associated graded vector space

$$L_\mu^{(k)}(W,Z) \simeq \bigoplus_{\lambda \in S_k(\mu)} L_\lambda W \otimes L_{\mu/\lambda} Z \ , \tag{3.6}$$

but the direct isomorphism is not natural.

The proof of this theorem can be inferred from [2] or [17]. We are primarily interested in the special case $\dim Z = 1$, when

$$L_{\mu/\lambda} Z = \begin{cases} \bigcirc_k Z & \lambda \in S_k^1(\mu) \ , \\ \{0\} & \lambda \in S_k^2(\mu) \ . \end{cases}$$

Since each symmetric power $\bigcirc_k Z$ is also one-dimensional, we get the following corollary

Corollary 3.4 Suppose $V = W \oplus Z$, and $\dim Z = 1$. Then

$$L_\mu V = \bigoplus L_\mu^{(k)} W \tag{3.7}$$

where each $L_\mu^{(k)} W$ is filtered, with associated graded vector space

$$L_\mu^{(k)} W \simeq \bigoplus_{\lambda \in S_k^1(\mu)} L_\lambda W \ . \tag{3.8}$$

This corollary will be of key importance in proving an algebraic version of the Poincaré lemma for hypercomplexes.

4. Functorial Maps.

The category $\mathcal{C}$ is that of finite dimensional (real or complex) vector spaces, the morphisms being linear maps between vector spaces. A functorial map, or natural transformation, between two functors $\mathfrak{F}$ and $\mathcal{G}$ on $\mathcal{C}$ is a linear map

$$\varphi\colon \mathfrak{F}V \to \mathcal{G}V$$

for V any vector space, which commutes with the morphisms: if

$$A\colon V \to W$$

is linear then

$$\varphi \circ \mathfrak{F}A = \mathcal{G}A \circ \varphi. \tag{4.1}$$

For a given pair of functors $\mathcal{G},\mathfrak{F}$, the set of all functorial maps $\varphi\colon \mathfrak{F} \to \mathcal{G}$ forms a vector space itself under the obvious operations. (See [24].)

Of particular interest are functors $\mathfrak{F}$ consisting of combinations of tensor, symmetric or wedge powers, or, more generally, Schur functors. Any such algebraic functor applied to V, $\mathfrak{F}V$, decomposes into a direct sum of irreducible representations of $GL(V)$, i.e. a direct sum of Schur spaces $L_\lambda V$. This decomposition will be "functorial" : independent of the particular vector space V provided $\dim V$ is sufficiently large, so no $L_\lambda V$ is zero. In particular, the multiplicities of the various $L_\lambda V$ in $\mathfrak{F}V$ are independent of the particular vector space V. Schur's lemma then proves the following basic result.

<u>Theorem</u> 4.1 Let $\mathfrak{F}$ be a combination of tensor, symmetric, wedge powers, or Schur functors. Then the vector space of functorial maps $\varphi: \mathfrak{F} \to L_\mu$ for any given Schur functor L_μ has dimension equal to the multiplicity of L_μ in $\mathfrak{F}$, i.e. the multiplicity of $L_\mu V$ in $\mathfrak{F}V$ for any V with $L_\mu V \neq 0$.

These multiplicities are only known in general for fairly simple $\mathfrak{F}$'s, e.g. the tensor product of two Schur functors, see section 10. More general functors lead into the difficult area of <u>plethysm</u> of which only limited results are thus far known, [15], [18], [30], [32]. (all except the third are written in terms of Schur functions).

5. The Pieri Maps

For a specific instance of the previous theorem, we consider the Pieri Formula, [17], [30]

$$\mathbb{S}_\ell \otimes L_\lambda = \bigoplus_{\mu \in \mathcal{J}_\ell^{1}(\lambda)} L_\mu \tag{5.1}$$

decomposing the tensor product of a symmetric power with a Schur functor. Combining with projections onto the various summands we obtain functorial maps

$$\varphi_\lambda^\mu : \mathbb{S}_\ell \otimes L_\lambda \to L_\mu , \tag{5.2}$$

which we name Pieri maps. As a corollary of (5.1) and theorem 4.1 we have

Theorem 5.1 The Pieri maps are uniquely determined up to a constant multiple (independent of any particular vector space).

It is helpful to have explicit formulae for the Pieri maps, but, for simplicity, we restrict attention to the case $\ell = 1$. Let λ be fixed, and let $\mu = \lambda + k$ be the shape obtained by adding a single box to the k-th row of λ. The Pieri map

$$\varphi_\lambda^\mu : V \otimes L_\lambda \to L_\mu$$

is then constructed as follows:

First consider the linear map

$$\tau : \mathbb{S}_p \otimes \mathbb{S}_q \to \mathbb{S}_{p-1} \otimes \mathbb{S}_{q+1}$$

given by

$$\tau(x\otimes y) = \Sigma\, x_{\hat{i}}\otimes(y \odot x_i)$$

for $x = x_1 \odot \ldots \odot x_q$ decomposible. The (i,j) transposition τ_{ij} is the linear map on $\otimes_\lambda \Theta_*$ which is the identity on all factors except the ith and jth, where it agrees with τ:

$$\tau\colon \Theta_{\lambda_i} \otimes \Theta_{\lambda_j} \to \Theta_{\lambda_{i-1}} \otimes \Theta_{\lambda_{j+1}} .$$

(If $i > j$ the order of the factors is reversed). If $\lambda_i = 0$, $\tau_{ij} = 0$. Note that $\operatorname{im}\tau_{ij} \subset \otimes_{\lambda'_{ij}} \Theta_*$, where λ'_{ij} is the diagram obtained from λ by taking one box from the i-th row and adding onto the j-th row. (λ'_{ij} is not necessarily a shape.) Multiple transpositions corresponding to $J = (j_1,\ldots,j_p)$ are defined by

$$\tau_J = \tau_{j_1 j_2} \circ \tau_{j_2 j_3} \circ \cdots \circ \tau_{j_{p-1} j_p} . \tag{5.4}$$

Given $k < m$, define

$$A_m^k = \{J = (j_1,\ldots,j_p) \mid m = j_1 > j_2 > \ldots > j_p = k\} ,$$

where $p = \#J$ can range from 2 to $m-k+1$.

Let

$$\iota\colon V \otimes \otimes_\lambda \Theta_* \to \otimes_\lambda \Theta_* \otimes V$$

be the interchange map

$$\iota(v \otimes \omega) = \omega \otimes v ,$$

and identify the latter space with $\otimes_{\lambda_+} \Theta_*$, where $\lambda_+ = (\lambda_1,\ldots,\lambda_n,1)$. Finally define

$$\varphi_\lambda^\mu \colon V \otimes \otimes_\lambda \Theta_* \to \otimes_\mu \Theta_*$$

by

$$\varphi_\lambda^\mu = (\Sigma\, c_J^{-1}\, \tau_J) \circ \imath \ , \qquad (5.5)$$

the sum being over all $J \in A_{n+1}^k$. The coefficients are

$$c_J = c_J(\lambda) = \prod_{q=2}^{p-1} \ell_{kj_q}(\lambda) \ , \qquad (5.6)$$

$p = \# J$, i.e. the product of the hook lengths of all hooks whose initial box is in row k , and whose terminal box is in a row indexed by J (except for $j_p = n+1$).

<u>Theorem 5.2</u> Let $\mu = \lambda + k$. If φ_λ^μ is as defined in (5.6), then

$$\varphi_\lambda^\mu (V \otimes \mathcal{J}_\lambda) \subset \mathcal{J}_\mu \ .$$

Therefore φ_λ^μ induces the Pieri map

$$\varphi_\lambda^\mu \colon V \otimes L_\lambda \to L_\mu$$

(the two "different" meanings of φ_λ^μ should not cause any confusion.)

Before proving this theorem, it is helpful to consider some examples.

Example 5.3

Suppose $\lambda = (p,q)$ is a two-rowed shape. For $\mu = (p+1,q)$,

$$\varphi^{\mu}_{\lambda}(v \otimes (x \otimes y)) = (x \odot v) \otimes y + \frac{1}{p-q+2} \sum_{i=1}^{q} (x \odot y_i) \otimes (y_{\hat{i}} \odot v) \ , \qquad (5.7)$$

for $x \in \odot_p$, $y = y_1 \odot \dots \odot y_q \in \odot_q$, $v \in V$. The verification of theorem 5.2 is fairly routine in this example.

Even simpler are the cases $\mu = (p,q+1)$ or $\mu = (p,q,1)$, where

$$\varphi^{\mu}_{\lambda}(v \otimes (x \otimes y)) = \begin{cases} x \otimes (y \odot v) & \mu = (p,q+1) \\ x \otimes y \otimes v & \mu = (p,q,1) \ . \end{cases}$$

In fact, if $\mu = \lambda + k$, then φ^{μ}_{λ} only affects the i-th rows of $x \in \otimes_{\lambda} \odot_{*}$ for $i \geq k$.

For $\lambda = (p,q,r)$, $\mu = (p+1,q,r)$, we have the more complicated formula

$$\varphi^{\mu}_{\lambda}(v \otimes (x \otimes y \otimes z)) = (x \odot v) \otimes y \otimes z + \frac{1}{p-q+2} \sum_{i} (x \odot y_i) \otimes (y_{\hat{i}} \odot v) \otimes z \ +$$

$$+ \frac{1}{p-r+3} \sum_{i} (x \odot z_i) \otimes y \otimes (z_{\hat{i}} \odot v) + \frac{1}{(p-q+2)(p-r+3)} \sum_{i,j} (x \odot y_i) \otimes (y_{\hat{i}} \odot z_j) \otimes (z_{\hat{j}} \odot v)$$

The general formula (5.5) can now be fathomed.

Lemma 5.4 Let $i \neq j$, $k \neq \ell$. For $\zeta \in \otimes_{\lambda} \odot_{*}$,

$$[\tau_{ij}, \tau_{k\ell}](\zeta) = \begin{cases} (\lambda_i - \lambda_j)\zeta & i = \ell \ , \ j = k, \\ \tau_{kj}(\zeta) & i = \ell \ , \ j \neq k, \\ -\tau_{i\ell}(\zeta) & j = k \ , \ i \neq \ell, \\ 0 & \text{otherwise} \ . \end{cases}$$

(Here $[f,g] = f\circ g - g\circ f$ for linear f,g .).

The proof is trivial.

Proof of theorem 5.2

Note first that the relations (3.1) defining the ideal $\mathcal{J}_\lambda$ span the image of $\tau_{\ell,\ell+1}$ in $\otimes_\lambda \Theta_*$, for $1\leq \ell < n$, hence it suffices to prove that

$$[\varphi^{\mu}_{\lambda} , \tau_{\ell,\ell+1}] = 0 \tag{5.8}$$

for all such ℓ . There are three cases depending on the relative magnitudes of k and ℓ .

Consider the case $k=\ell$. Indices in A^k_{n+1} split into two classes:

$$J = (k,K) \ , \ J' = (k,k+1,K) \ ,$$

where $K \in A^m_{n+1}$ for some $m>\ell+2$. Then by lemma 5.4 ,

$$[\tau_J \ , \tau_{k,k+1}] = \tau_K \circ [\tau_{mk},\tau_{k,k+1}] = -\tau_K \circ \tau_{m,k+1} \ ,$$

whereas

$$[\tau_{J'},\tau_{k,k+1}] = \tau_K \circ [\tau_{m,k+1} \circ \tau_{k+1,k},\tau_{k,k+1}] = (\lambda_k - \lambda_{k+1} + 2)\tau_K\circ\tau_{m,k+1} \ .$$

The latter coefficient is just the $(k,k+1)$ hook length of λ , hence using (5.6) ,

$$[c_J^{-1} \ \tau_J + c_{J'}^{-1} \ \tau_{J'} \ , \ \tau_{k,k+1}] = 0 \ .$$

Summing over K proves (5.8).

Next consider the case $k < \ell$. Now the indices in A^k_{n+1} split into three classes: $J = (I,\ell,K)$, $J' = (I,\ell+1,K)$, $J'' = (I,\ell,\ell+1,K)$. for $I \in A^k_j$, $j < \ell$, $K \in A^m_{n+1}$, $m > \ell+1$. Again by lemma 5.4,

$$[\tau_J, \tau_{\ell,\ell+1}] = \tau^* ,$$

$$[\tau_{J'}, \tau_{\ell,\ell+1}] = -\tau^* ,$$

$$[\tau_{J''}, \tau_{\ell,\ell+1}] = (\lambda_\ell - \lambda_{\ell+1} + 1)\tau^* ,$$

where

$$\tau^* = \tau_K \circ \tau_{m,\ell+1} \circ \tau_{\ell j} \circ \tau_I .$$

Therefore

$$[C_J^{-1}\, \tau_J + C_{J'}^{-1}\, \tau_{J'} + C_{J''}^{-1}\, \tau_{J''} \, , \, \tau_{\ell,\ell+1}] = 0 ,$$

since

$$C_{J'}^{-1} = C_J^{-1} + (\lambda_\ell - \lambda_{\ell+1} + 1) C_{J''}^{-1} ,$$

as can easily be verified.

The remaining case $\ell < k$ is treated similarly, and is left to the reader.

It remains to check that the functorial maps $\varphi^\mu_\lambda \colon V \otimes L_\lambda \to L_\mu$ so constructed are not identically zero for all vector spaces V. By functoriality, it suffices to prove this when V is of sufficiently large dimension, and we need only check that $\varphi^\mu_\lambda(v \otimes \underset{\sim}{x})$ is nonzero for one element $v \otimes \underset{\sim}{x}$. It is easy to see that if $e_1, \ldots, e_{m+1}$ are linearly independent in V , setting $v = e_{m+1}$, $\underset{\sim}{x} = e_{T_o}$ for T_o the simple tableau of shape λ , then each term in $\varphi^{\mu}_{\lambda}(v \otimes \underset{\sim}{x})$ is a (nonzero) multiple of a standard tableau of shape μ . Thus by the basis theorem 3.2, $\varphi^{\mu}_{\lambda}(v \otimes \underset{\sim}{x}) \notin \mathcal{J}_{\mu}$, which completes the proof. (See also (9.10) for a more explicit example.)

6. Polarization Maps.

In his thesis, [30], Weyman introduced and studied the functorial maps

$$\chi_\mu^\lambda : L_\mu \to \Theta_\ell \otimes L_\lambda ,$$

reversing the direction of the Pieri maps. (See also [31].) In the special case that L_μ (and hence L_λ) are symmetric powers, these agree with the classical partial polarizations of symmetric polynomials, hence the χ_μ^λ will be called polarization maps. It is easily shown that χ_μ^λ is uniquely determined up to a constant multiple, and, moreover, when composed with φ_λ^μ gives a multiple of the identity map $1\!\!1$ on L_μ.

Weyman bases the construction of the χ_μ^λ on the composition of maps

$$\otimes_\mu \Theta_* \to \otimes_{\tilde{\mu}} \wedge_* \overset{\Delta}{\to} \Theta_\ell \otimes \otimes_{\tilde{\lambda}} \Theta_* ,$$

where Δ is a tensor product of diagonal maps $\wedge_{j+1} \to V \otimes \wedge_j$ on the factors corresponding to columns with boxes in μ/λ, followed by symmetric multiplication on the ℓ extra copies of V. Needless to say, this formulation is not of great help when one is interested in readily computable formulae.

Again, for simplicity we restrict attention to the case $\mu = \lambda + k$, first define

$$\chi_\mu^\lambda : \otimes_\mu \Theta_* \to V \otimes \otimes_\lambda \Theta_*$$

by

$$\chi_\mu^\lambda = \Sigma(-1)^p \tilde{c}_J^{-1} \tau_J , \tag{6.1}$$

the sum being over all $J \in A_k^o$, $p = \#J$, the copy of V in the image of χ_μ^λ being given row index 0 . The modified constants are

$$\tilde{c}_J = \tilde{c}_J(\mu) = \prod_{q=2}^{p-1} (\ell_{j_q k}(\mu) - 1) . \tag{6.2}$$

Note that

$$\ell_{jk}(\mu) - 1 = \mu_j - \mu_k + k - j .$$

Theorem 6.1 Let $\mu = \lambda + k$. With χ_μ^λ defined as in (6.1) we have

$$\chi_\mu^\lambda(\mathcal{I}_\mu) \subset V \otimes \mathcal{I}_\lambda ,$$

hence χ_μ^λ induces the partial polarization

$$\chi_\mu^\lambda : L_\mu \to V \otimes L_\lambda .$$

The proof is very similar to that of theorem 5.2, so we leave it to the reader and content ourselves with a couple of easy examples.

Example 5.2

Let $\mu = (p,q)$ be a two rowed shape. If $p > q$, so $\lambda = (p-1,q)$ is a shape, then

$$\chi_\mu^\lambda(x \otimes y) = \sum_{i=1}^{p} x_i \otimes x_{\hat{i}} \otimes y$$

for $x = x_1 \odot \dots \odot x_p \in \Theta_p$, $y = y_1 \odot \dots \odot y_q \in \Theta_q$, whereas for $\lambda = (p,q-1)$,

$$\chi^\lambda_\mu(x\otimes y) = \sum_{i=1}^{q} y_i \otimes x \otimes y_{\hat{i}} - \frac{1}{p-q+1} \sum_{i,j} x_i \otimes (x_{\hat{i}} \odot y_j) \otimes y_{\hat{j}} .$$

Similarly, if $\mu=(p,q,r)$, $\lambda=(p,q,r-1)$,

$$\chi^\lambda_\mu(x\otimes y\otimes z) = \sum_k z_k \otimes x\otimes y\otimes z_{\hat{k}} - \frac{1}{p-r+2} \sum_k x_i \otimes (x_{\hat{i}} \odot z_k) \otimes y \otimes z_{\hat{k}}$$

$$- \frac{1}{q-r+1} \sum_{j,k} y_j \otimes x \otimes (y_{\hat{j}} \odot z_k) \otimes z_{\hat{k}} + \frac{1}{(p-r+2)(q-r+1)} \sum_{i,j,k} x_i \otimes (x_{\hat{i}} \odot y_j) \otimes (y_{\hat{j}} \odot z_k) \otimes z_{\hat{k}}$$

<u>Theorem 6.2</u> Let $\lambda+k=\mu$. Then

$$\varphi^\mu_\lambda \circ \chi^\lambda_\mu = a^\mu_\lambda \mathbb{1} \quad \text{on } L_\mu , \tag{6.3}$$

where

$$a^\mu_\lambda = (\mu_k+n-k) \prod_{i<k} \frac{\ell_{ik}(\mu)}{\ell_{ik}(\mu)-1} \prod_{j>k} \frac{\ell_{kj}(\mu)-2}{\ell_{kj}(\mu)-1} . \tag{6.4}$$

The proof requires an easy lemma:

<u>Lemma 6.3</u> Let $\zeta \in \otimes_\mu \mathcal{O}_*$. If $J \in A^o_m$, $m\leq n$, $p=\#J$, then

$$\tau_{om} \circ \tau_J(\zeta) = (-1)^p \mu_m \zeta + \xi$$

for some $\xi \in \mathcal{I}_\mu$.

For $J \in A^o_k$, $\#J=p$, $K \in A^k_{n+1}$, $k_2=m<n$, set $L=(k_2,k_3,\ldots,k_{q-1},k,j_2,\ldots,j_q) \in A^o_m$. Then

$$\tau_k \circ z \circ \tau_J(\zeta) = \tau_{om} \circ \tau_L(\zeta)$$

$$= (-1)^{p+q} \mu_m \zeta + \xi$$

for some $\xi \in \mathcal{J}_\mu$. Substituting (5.5), (6.2) into (6.3), we immediately see that

$$a_\lambda^\mu = \sum_{J,K} (-1)^p \mu_{k_2} c_K(\lambda)^{-1} \tilde{c}_J(\mu)^{-1} ,$$

the sum being over all $J \in A_k^o$ and all $K \in A_{n+1}^k$, $p = \#K$. The proof of (6.4) therefore reduces to the following series of easy combinatorial lemmas.

Lemma 6.4 The following formulae hold:

i) $$\sum_{J \in A_k^o} \tilde{c}_J(\mu)^{-1} = \prod_{i<k} \frac{\ell_{ik}(\mu)}{\ell_{ik}(\mu) - 1} , \tag{6.5}$$

ii) $$\sum_{K \in A_m^k} (-1)^p c_K(\lambda)^{-1} = \prod_{k<j<m} \frac{\ell_{kj}(\lambda) - 1}{\ell_{kj}(\lambda)} , \tag{6.6}$$

where $p = \#K$.

These reduce to the expansion of the products

$$\Pi(x_i \pm 1) / x_i$$

for appropriate x_i .

Lemma 6.5 For $k \leq m \leq n$,

$$\sum_{k_2 \leq m} (-1)^p \mu_{k_2} c_K(\lambda)^{-1} = (\mu_k + m - k) \prod_{k<j\leq m} \frac{\ell_{kj}(\lambda) - 1}{\ell_{kj}(\lambda)} \tag{6.7}$$

the sum being over $K \in A_{n+1}^k$, $\#K = p$, $k_2 \leq m$.

Proof

Use induction on m . By (5.6), (6.6), the induction step from $m-1$ to m reduces to the formula

$$(\mu_k + m - k - 1) - \frac{\mu_m}{\ell_{km}(\lambda)} = \frac{(\mu_k + m - k)(\ell_{km}(\lambda) - 1)}{\ell_{km}(\lambda)} ,$$

which is easily checked, since

$$\mu_m = \lambda_m , \qquad m > k ,$$

and

$$\ell_{km}(\lambda) = \lambda_k - \lambda_m + m - k + 1 .$$

From (6.5), (6.7) it is easy to verify (6.4).

7. Hypercomplexes.

The Pieri maps

$$\varphi_\lambda^\mu(v\otimes .) : L_\lambda \to L_\mu , \qquad \mu \supset \lambda , \ |\mu/\lambda| = 1 ,$$

lead one to the general notion of a hypercomplex. In this section, the basic general results on hypercomplexes are presented, in preparation for a more detailed discussion of the Pieri maps in the following section.

Definition 7.1 A hypercomplex is given by a collection of vector spaces W_λ indexed by shapes λ , and linear maps

$$f_\lambda^\mu : W_\lambda \to W_\mu$$

defined whenever $\mu \supset \lambda$, $|\mu/\lambda| = 1$ subject to the two conditions:

a) Commutativity: The resulting collection of spaces and maps forms a commutative diagram. This allows one to unambiguiously define maps

$$f_\lambda^\nu : W_\lambda \to W_\nu$$

for any $\nu \supset \lambda$ by iterating the above maps f_λ^μ in any convenient order so as to reach W_ν from W_λ .

b) Closure The resulting maps f_λ^ν are identically zero whenever ν/λ has two or more boxes in any column, i.e. $\lambda \in \mathcal{S}^2(\nu)$ in the notation of section 2.

Within any hypercomplex, there are a number of interesting "subcomplexes". First we generalize the notion of a complex as used in homological algebra, [5].

Definition 7.2 An n-complex is given by a set of vector spaces W_j indexed by nonnegative integers j, and linear maps

$$f_j : W_j \to W_{j+1}$$

subject to the condition that the n+1-fold iteration

$$f_{j+n} \circ \cdots \circ f_j : W_j \to W_{j+n+1}$$

vanishes identically for all j.

A complex in the ordinary sense is thus a 1-complex. Of course, any n-complex is also an n+1-complex. To exclude this trivial case, the term n-complex will usually imply that n-fold iterations $f_{j+n-1} \circ \cdots \circ f_j$ are not all identically zero.

Within any hypercomplex, there are distinguished n-complexes. Namely, consider only the n-fat shapes, ordered by their weight, and the maps $f_\lambda^\mu : W_\lambda \to W_\mu$ for λ, μ n-fat, $|\mu/\lambda| = 1$. The closure condition on the hypercomplex implies that these are indeed n-complexes.

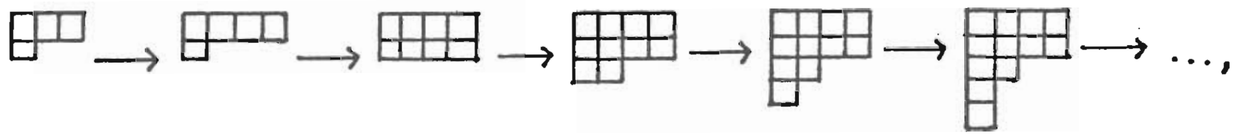
There are also a large number of ordinary complexes. For any shape λ, define shapes μ^j, with $\mu^o = \lambda$, $\mu^1 = \lambda + 1$,

$$\mu_i^j = \begin{cases} \lambda_{i-1} + 1 & i \leq j, \\ \lambda_i & i \geq j, \end{cases} \qquad j \geq 2.$$

Then it is easy to see that

$$f_j = f^{\mu^{j+1}}_{\mu^j} : W_{\mu^j} \to W_{\mu^{j+1}}$$

form a complex - $f_{j+1} \circ f_j = 0$ - called the λ - <u>subcomplex</u> of the given hypercomplex. For example, if $\lambda = (3,1)$, then the λ - subcomplex is



where we have suppressed the W's .

We now turn to the definition of exactness for a hypercomplex.

<u>Definition 7.3</u> A hypercomplex is exact if for every shape $\mu \neq 0$, every $\underset{\sim}{\lambda} \subset \mathcal{S}^1(\mu)$, $\underset{\sim}{\nu} \subset \mathcal{T}^1(\mu)$ with $\underset{\sim}{\lambda},\mu,\underset{\sim}{\nu}$ min-max related (see section 2)

$$\bigcap_{\nu \in \underset{\sim}{\nu}} \ker f^{\nu}_{\mu} = \sum_{\lambda \in \underset{\sim}{\lambda}} \operatorname{im} f^{\mu}_{\lambda} . \tag{7.1}$$

Note: the sum is not necessarily direct.

The motivation for this definition comes from the exactness of the Schur hypercomplex, to be discussed in the following section. Note that in (7.1), by closure, the left hand side always contains the right hand side. Thus we can define the cohomology of a hypercomplex:

<u>Definition 7.4</u> Given a hypercomplex, and $\underset{\sim}{\lambda},\mu,\underset{\sim}{\nu}$ min-max related, the $\underset{\sim}{\lambda},\mu,\underset{\sim}{\nu}$ - cohomology is

$$H_{\underset{\sim}{\lambda},\mu,\underset{\sim}{\nu}} = \bigcap_{\nu \in \underset{\sim}{\nu}} \ker f^{\nu}_{\mu} \Big/ \sum_{\lambda \in \underset{\sim}{\lambda}} \operatorname{im} f^{\mu}_{\lambda} .$$

We will not pursue the investigation of this cohomology theory here in any detail.

To prove exactness of a hypercomplex, it actually suffices to check exactness of the λ - subcomplexes, and in fact, only special cases of those. This is a vital simplification in the exactness proofs discussed subsequently.

Theorem 7.5 Given a hypercomplex $f^{\mu}_{\lambda}: W_{\lambda} \to W_{\mu}$, suppose that for all min-max related shapes λ,μ,ν with $|\nu/\mu| = 1$ (or, alternatively, with $|\mu/\lambda| = 1$)

$$\ker f^{\nu}_{\mu} = \operatorname{im} f^{\mu}_{\lambda} . \tag{7.2}$$

Then the entire hypercomplex is exact.

Proof

We do the case $|\nu/\mu| = 1$, the other case being similar. Let $\underset{\sim}{\lambda},\mu,\underset{\sim}{\nu}$ be min-max related. Note that no shape in $\underset{\sim}{\nu}$ contains any other shape therein. Let ν^* be the smallest shape containing all the shapes in $\underset{\sim}{\nu}$. Let $r = m(\nu^*/\mu) > 1$ be the last row in which ν^* contains boxes not in μ , and let $k = \nu^*_r - \mu_r$ be the number of boxes in this last row. (Note $r \neq 1$ for the min-max relation to hold.) The proof proceeds by induction, first on k , then r .

For the induction on k , if $k \geq 2$, let $c = \nu^*_r$. For each $\nu^i \in \underset{\sim}{\nu}$, let σ^i be the shape obtained by deleting the box (r,c) from ν^i , if it occurs. The σ^i are all different. Let $\underset{\sim}{\sigma} = \{\sigma^i\}$, so $r(\sigma^*/\mu) = r$ $\sigma^*_r - \mu_r = k-1$. Let $\underset{\sim}{\rho} = \mathcal{S}^o(\mu,\sigma)$, and note that $\underset{\sim}{\rho},\mu,\underset{\sim}{\sigma}$ are min-max related and there is a one-to-one correspondence between $\lambda^j \in \underset{\sim}{\lambda}$ and $\rho^j \in \underset{\sim}{\rho}$ with $\rho^j \subset \lambda^j$.

Consider the sequence of maps

$$\oplus W_{\rho^j} \xrightarrow{F_1} \oplus W_{\lambda^j} \xrightarrow{F_2} W_{\mu} \xrightarrow{F_3} \oplus W_{\sigma^i} \xrightarrow{F_4} \oplus W_{\nu^i} \, .$$

Here F_1, F_4 are direct sums of $f^{\lambda^i}_{\rho^i}$, $f^{\nu^i}_{\sigma^i}$ respectively, $F_2(\oplus x_j) = \Sigma f^{\mu}_{\lambda_j}(x_j)$, $x_j \in W_{\lambda^j}$, $F_3(y) = \oplus f^{\sigma^i}_{\mu}(y)$, $y \in W_{\mu}$. By the induction hypothesis,

$$\ker F_3 = \operatorname{im}(F_2 \cdot F_1),$$

$$\ker F_4 = \operatorname{im}(F_3 \circ F_2),$$

since

$$\mathcal{S}^o(\sigma^i, \nu^i) \subset \mathcal{S}^o(\mu, \sigma),$$

as can easily be checked. Therefore

$$\begin{aligned} \ker(F_4 \circ F_3) &= \operatorname{im} F_2 + \ker F_3 \\ &= \operatorname{im} F_2 + \operatorname{im}(F_2 \cdot F_1) = \operatorname{im} F_2 \, . \end{aligned}$$

But this is precisely (7.1) for $\underset{\sim}{\lambda}, \mu, \underset{\sim}{\nu}$.

To do the induction step from $r-1$ to r we need an easy "diagram-chasing" lemma:

<u>Lemma 7.6</u> Consider a commutative diagram of linear maps

$$\begin{array}{ccccc} W_1 & \xrightarrow{F_1} & W_2 & \xrightarrow{F_2} & W_3 \\ {\scriptstyle F_3}\downarrow & & {\scriptstyle F_4}\downarrow & & {\scriptstyle F_5}\downarrow \\ W_4 & \xrightarrow{F_6} & W_5 & \xrightarrow{F_7} & W_6 \end{array}$$

with the top and bottom rows exact. Then the conditions

i) $\ker(F_5 \cdot F_2) = \ker F_2 + \ker F_4$

ii) $\operatorname{im}(F_4 \cdot F_1) = \operatorname{im} F_4 \cap \operatorname{im} F_6$

are equivalent. In particular, if either

a) F_3 is surjective, or

b) F_5 is injective,

then both i) and ii) hold.

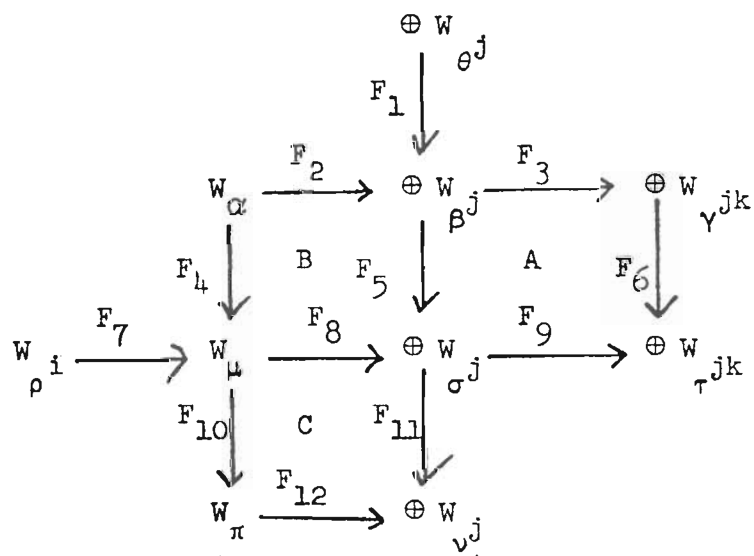
Now suppose ν^* has a single box in the last row $r > 2$ not in μ, in column $c = \nu_r^* = \mu_r + 1$. Let $\pi = \mu + r$, $\alpha = \mu/(r-1,c+)$ so $\{\alpha\} = \mathcal{S}^o(\mu,\pi)$. (In this proof, $\alpha,\beta,\gamma,\theta$ will all denote shapes.) Given $\nu^j \in \underset{\sim}{\nu}$, set $\sigma^j = \nu^j/(r,c+) \supset \mu$, so $\sigma_r^j = \mu_r^j$. Let $\underset{\sim}{\rho} = \mathcal{S}^o(\mu,\underset{\sim}{\sigma})$, $\underset{\sim}{\sigma} = \{\sigma^j\}$. Note that $\underset{\sim}{\lambda} = \underset{\sim}{\rho} \cup \{\alpha\}$. Let $\underset{\sim}{\tau}^j = \{\tau^{jk}\} = \mathcal{T}^o(\mu,\sigma^j)$. For $\sigma^j \neq \nu^j$, set $\beta^j = \sigma^j/(r-1,c+)$, so $\mathcal{S}^o(\sigma^j,\nu^j) = \{\beta^j\}$. Note that $\beta^j \supseteq \alpha$. For $r > 2$, set $\{\theta^j\} = \mathcal{S}^o(\beta^j,\sigma^j)$, so θ^j differs from β^j in the $(r-2)$nd row. Finally, if $\beta^j \neq \alpha$, let $\gamma^j = \{\gamma^{jk}\} = \mathcal{T}^o(\alpha,\beta^j)$. (See Figure 2 for illustrational purposes.)

Note that each τ^{jk} differs from σ^j in at most one row, labelled $r_{jk} \leq r$. It is not too difficult to show that if j is such that $\sigma^j \neq \nu^j$ and $\beta^j \neq \alpha$, then $\underset{\sim}{\gamma}^j$ can be constructed from $\underset{\sim}{\tau}^j$ by setting

$$\gamma^{jk} = \begin{cases} \tau^{jk}/(r-1,c+)\,, & r_{jk} < r-1\,, \\ \tau^{jk} & r_{jk} = r-1\,, \end{cases}$$

and ignoring any τ^{jk} with $r_{jk} = r$. Let $\underset{\sim}{\hat{T}}^j = \{\tau^{jk} | r_{jk} \le r - 1\}$

Consider the commutative diagram, where the F_j are constructed as in the induction step on k :



(See figure 2 for an representative example.) The only nontrivial question of commutativity corresponds to τ^{jk} with $r_{jk} = r$, since no corresponding γ^{jk} exists. But $\tau^{jk} \in \mathcal{T}^2(\beta^j)$ since the r and $r - 1$st rows in τ^{jk}/β^j overlap, so that part of $F_9 \circ F_5$ vanishes by closure.

By the induction hypothesis, or (7.2), each short sequence consisting of two consecutive maps in the same row or column is exact, e.g. $\ker F_9 = \operatorname{im} F_8$ (For F_5, F_{11} , note that if $\sigma^j = \nu^j$, $F_5^j : \{0\} \to W_{\sigma^j}$ is injective. For F_2, F_3 , if $\beta^j = \alpha$, $F_3^j : W_{\beta^j} \to \{0\}$ is surjective.)

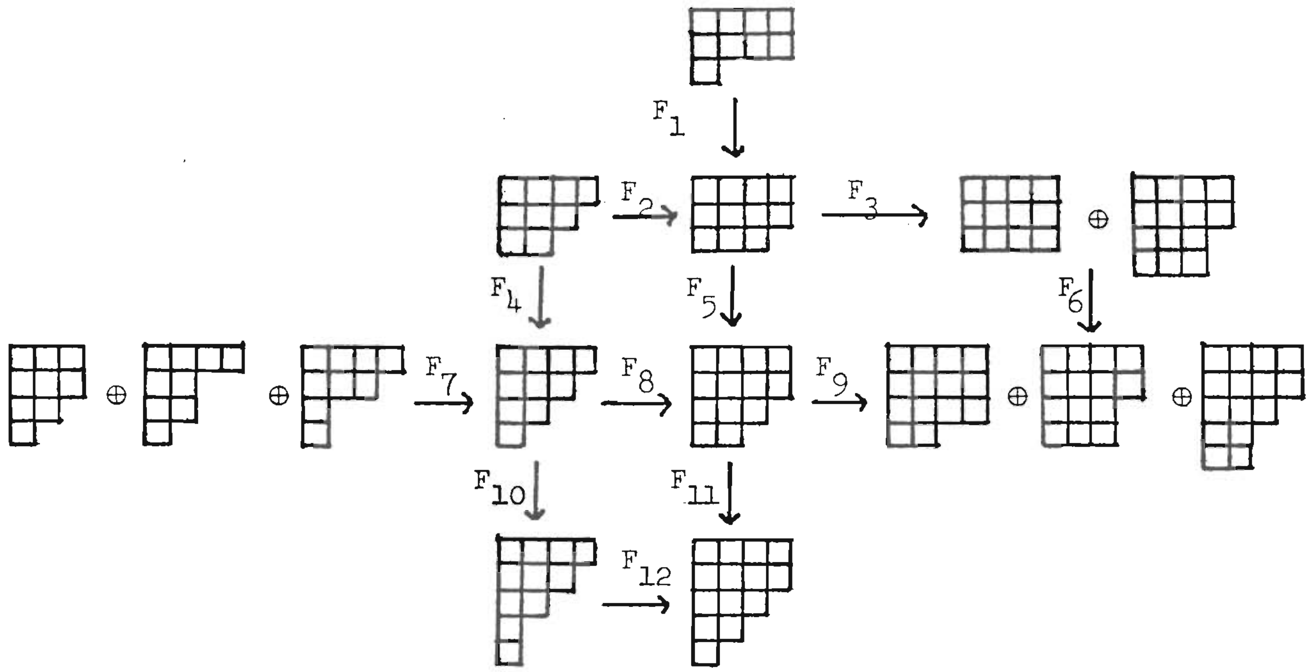


Figure 2 $\quad r = 5 \quad c = 1$

$\underset{\sim}{\lambda} = \{(3,3,2,1),\ (4,2,2,1),\ (4,3,1,1),\ (4,3,2)\}$

$\mu = (4,3,2,1)$

$\nu = (4,4,3,2,1)$

$\pi = (4,3,2,1,1)$

$\sigma = (4,4,3,2)$

$\underset{\sim}{\rho} = \{(3,3,2,1),\ (4,2,2,1),\ (4,3,1,1)\}$

$\underset{\sim}{\tau} = \{(4,4,4,2),\ (4,4,3,3),\ (4,4,3,2,2)\}$

$\alpha = (4,3,2)$

$\beta = (4,4,3)$

$\underset{\sim}{\gamma} = \{(4,4,4),\ (4,4,3,3)\}$

$\theta = (4,4,1)$

(The W's have been omitted leaving only the relevant shapes in the diagram.)

I claim that for each j the subdiagram

$$\begin{array}{ccc} W_{\beta^j} & \xrightarrow{F_3^j} & \oplus W_{\gamma^{jk}} \\ {\scriptstyle F_5^j}\downarrow & & \downarrow{\scriptstyle F_6^j} \\ W_{\sigma^j} & \xrightarrow{F_9^j} & \oplus W_{\tau^{jk}} \end{array}$$

satisfies condition i) of lemma 7.6, i.e.

$$\ker F_6^j \circ F_3^j = \ker F_3^j + \ker F_5^j \; . \tag{7.3}$$

If $\sigma^j = \nu^j$, or $\alpha = \beta^j$, this is trivial since $F_6^j : \{0\} \longrightarrow \oplus W_{\tau^{jk}}$ is injective. Otherwise, note that

$$\mathcal{S}^o(\beta^j, \hat{\underset{\sim}{I}}^j) = \{\alpha, \theta^j\} \; .$$

Moreover, the parts of F_6^j corresponding to $r_{jk} = r-1$ or r are injective, hence contribute nothing to the above kernels. Therefore, by the induction hypothesis

$$\ker(F_6^j \circ F_3^j) = \operatorname{im} F_2^j + \operatorname{im} F_1^j \, ,$$

so by exactness, the claim is proven.

Now use lemma 7.6 on boxes B and A in the large diagram. Since we have just shown condition i) holds for box A , by summing (7.3) over j , condition ii) holds for box B . A second application of the lemma, this time to boxes B and C shows that condition i) holds for box C . But this is just

$$\ker(F_{11} \circ F_8) = \ker F_8 + \ker F_{10}$$
$$= \operatorname{im} F_7 + \operatorname{im} F_4 ,$$

by exactness, which is precisely what we need to establish. This completes the induction step on r, and hence the proof of the theorem.

One point to note in the definition of exactness for hypercomplexes is that there is no compatible notion solely for n-complexes. For instance, $\mu = (3,3,1)$, $\nu = (3,3,2)$ are both 3-fat, but in an exact hypercomplex

$$\ker f_\mu^\nu = \operatorname{im} f_\lambda^\mu$$

where $\lambda = (3,2,1)$ is no longer fat. This indicates that one must consider the full hypercomplex, and not just n-subcomplexes, to get any meaningful cohomological results.

The notions of hypercomplex, etc. can be dualized:

<u>Definition 7.7</u>. A <u>cohypercomplex</u> is given by spaces W_λ^* indexed by shapes λ, and linear maps

$$g_\mu^\lambda : W_\mu^* \longrightarrow W_\lambda^*$$

defined whenever $\mu \supset \lambda$, $|\mu| = |\lambda| + 1$, subject to conditions of

a) Commutativity: The diagram of maps is commutative, which allows us to unambiguously define

$$g_\nu^\lambda : W_\nu^* \longrightarrow W_\lambda^*$$

for any $\nu \supset \lambda$ by iteration.

b) Closure: $g_\nu^\lambda \equiv 0$ whenever $\lambda \in S^2(\nu)$.

Note that the g_μ^λ , for μ,λ n-fat, again form an n-complex.

Definition 7.8

A cohypercomplex is exact if for every $\underset{\sim}{\lambda},\mu,\underset{\sim}{\nu}$ min-max related,

$$\sum_{\nu \in \underset{\sim}{\nu}} \text{im } g_\nu^\mu = \bigcap_{\lambda \in \underset{\sim}{\lambda}} \ker g_\mu^\lambda .$$

There is an analogous theorem to theorem 7.5 for checking exactness of cohypercomplex, which, for brevity, we do not state or prove explicitly. Similarly one can define the homology of a cohypercomplex.

Finally we note that for finite - dimensional vector spaces, the notions of hypercomplex and cohypercomplex are precisely dual to each other:

Lemma 7.9 If each W_λ in a hypercomplex is a finite - dimensional vector space, then the dual spaces W_λ^* and dual maps $g_\mu^\lambda = (f_\lambda^\mu)^*$ form a cohypercomplex, and vice-versa. Exactness of one of these implies exactness of the other.

8. The Schur Hypercomplex.

The Pieri maps of theorem 5.2 will now be used to define an algebraic hypercomplex between the Schur spaces of a vector space V . For each $v \in V$ the Pieri map $\varphi_\lambda^\mu(v \otimes .)$ defines a linear function from L_λ to L_μ for any $\mu \supset \lambda$, $|\mu| = |\lambda| + 1$, uniquely determined up to constant multiple by functoriality. We now fix this constant.

Definition 8.1 Let $\mu = \lambda + j$. The Pieri product

$$*_j : V \otimes L_\lambda \longrightarrow L_\mu$$

is defined by

$$v *_j \zeta = c_\lambda^\mu \, \varphi_\lambda^\mu(v \otimes \zeta) \ , \ v \in V \ , \ \zeta \in L_\lambda \ , \qquad (8.1)$$

where φ_λ^μ is the Pieri map in theorem 5.2, and

$$c_\lambda^\mu = \frac{\mu_j!}{L(\tilde{\mu},\tilde{\lambda})} \qquad (8.2)$$

where $L(\tilde{\mu},\tilde{\lambda})$ is the total hook length of $\tilde{\mu}$ mod $\tilde{\lambda}$, as defined in section 2. The subscript j on the Pieri product will be dropped when there is no ambiguity. Set $\psi_\lambda^\mu = c_\lambda^\mu \varphi_\lambda^\mu$.

The precise motivation for the choice (8.2) of the constants c_λ^μ will be discussed later: The fundametal result to be proved is

Theorem 8.2 Fix $0 \neq v \in V$. Then the Pieri products

$$v* : L_\lambda \longrightarrow L_\mu$$

form an exact hypercomplex, called the Schur hypercomplex associated with v .

The corresponding partial polarizations will be denoted

$$p = p_{\mu}^{\lambda} : L_{\mu} \longrightarrow V \otimes L_{\lambda}$$

with

$$p_{\mu}^{\lambda} = \frac{L(\tilde{\mu},\tilde{\lambda})}{\mu_j! \; a_{\lambda}^{\mu}} \chi_{\mu}^{\lambda} , \tag{8.3}$$

so that by theorem 6.2,

$$\psi_{\lambda}^{\mu} \circ p_{\mu}^{\lambda} = \mathbb{1} \quad \text{on} \quad L_{\mu} .$$

The remainder of this section is devoted to a proof of théorem 8.2. The three aspects of commutativity, closure and exactness are treated in turn. For the time being, define the Pieri product by formula (8.1), but leave the precise choice of the constants c_{λ}^{μ} open.

Lemma 8.3 Let the Pieri product $*_j$ be defined by (8.1). Let $\mu = \lambda + j$, $\mu' = \lambda + k$, $\nu = \mu + k = \mu' + j$ be shapes. Then for $v \in V$, $\zeta \in L_{\lambda}$,

$$v *_j (v *_k \zeta) = K[v *_k (v *_j \zeta)] , \tag{8.4}$$

where K is a constant depending only on λ, ν and the c_{λ}^{μ} .

<u>Proof</u>

It is easy to see that the compositions $v *_j (v *_k \zeta)$ or $v *_k (v *_j \zeta)$ induce functorial maps

$$\Theta_2 \otimes L_\lambda \longrightarrow L_\nu$$

since $\{v \odot v \mid v \in V\}$ spans Θ_2. By Pieri's formula and theorem 4.1, any such functorial map is uniquely determined up to constant multiple, hence, provided the left hand side of (8.4) is not identically zero (which we show subsequently) the lemma is proven.

To explicitly determine the constant K in (8.4), by functoriality we need only consider vector spaces V of sufficiently large dimension. Thus if $m = m(\lambda)$, let $e_1, \ldots, e_{m+1}$ be linearly independent in V, with $v = e_{m+1}$. Let $\zeta = e_{T_o}$ correspond to the simple tableau T_o of shape λ. From the formula (5.5) it is easy to check that every summand e_T occurring in $v *_j \zeta$ and $v *_k (v *_j \zeta)$ corresponds to a standard tableau T, hence we need not worry about cancellations among terms. Thus to determine K, we need only compare the coefficients of one of these terms in the two products, which, for ease of computation, we choose to be e_T, where T is the tableau of shape ν coinciding with T_o on λ, and with $m+1$ in the remaining two boxes in ν/λ. For definiteness, assume $j < k$, the case $j = k$ being trivial.

For $v *_k (v *_j \zeta)$, e_T arises only from the leading terms in (5.5), i.e. those corresponding to $J = (k, n+1)$ (in the case of $*_k$). Therefore e_T appears with coefficient $c_\lambda^\mu c_\mu^\nu$. However, in $v *_j (v *_k \zeta)$, there are two ways in which e_T can arise: In the formula for $*_k$, only the term in (5.5) with $J = (k,n+1)$ will contribute, but in $*_j$, both $J = (j,n+1)$ and $J = (j,k,n+1)$ will make contributions. Thus e_T appears in $v *_j(v *_k\zeta)$ with coefficient

$$c_\lambda^{\mu'} c_{\mu'}^{\nu}(1+\ell_{jk}(\mu')^{-1}) = c_\lambda^{\mu'} c_{\mu'}^{\nu}\ell_{jk}(\lambda)(\ell_{jk}(\lambda) - 1)^{-1} .$$

Thus we have proven

<u>Lemma 8.4</u> The constant K in lemma 8.3 has the value

$$K = \frac{c_\lambda^{\mu'} c_{\mu'}^{\nu}\ell_{jk}(\lambda)}{c_\lambda^{\mu} c_{\mu}^{\nu}(\ell_{jk}(\lambda) - 1)} . \tag{8.5}$$

The goal now is to choose the constants c_λ^μ in some consistent fashion so that the Pieri products actually commute, i.e. the constant K in (8.5) is always 1 . Clearly there are several ways of doing this, of which we mention four.

The first, and easiest to verify, is to define

$$c_\lambda^\mu = L(\mu,\lambda) = \prod_{i=1}^{k-1} (\lambda_i - \lambda_k + k - i) \tag{8.6}$$

as the total hook length of μ mod λ , for $\mu = \lambda + k$. The proof that $K = 1$ in (8.5) is simple. This choice has the undesirable feature that the Pieri product on the wedge powers

$$v *_n : \Lambda_{n-1} \longrightarrow \Lambda_n$$

is not $v \wedge$, but rather

$$v *_n \zeta = n! \ \zeta \wedge v$$

under the identification of $L_{1^n} \simeq \Lambda_n$.

This leads to the second choice

$$c^{\mu}_{\lambda} = L(\mu,\lambda) / k! \quad , \quad \mu = \lambda + k \ . \tag{8.7}$$

Now, if $\lambda = 1^{n-1}$, $\mu = 1^n$, $c^{\mu}_{\lambda} = 1$, so we do get the ordinary wedge product. Note also that in both of these choices the products corresponding to one-rowed Schur functors coincide with the symmetric product:

$$v *_1 \zeta = v \odot \zeta \ , \qquad \zeta \in L_n \simeq \odot_n \ .$$

Alternatively, we can "dualize" the diagram by transposing all the shapes and replacing the multiples by their reciprocals:

$$c^{\mu}_{\lambda} = L(\tilde{\mu},\tilde{\lambda})^{-1} \ . \tag{8.8}$$

The verification that $K = 1$ in (8.5) is slightly more tricky, but still straight - forward. This choice no longer leaves the Pieri product agreeing with the symmetric product on one-rowed Schur spaces, so to rectify this we are led to the final choice (8.2). Both (8.7) and (8.2) have the desirable features that the Pieri products agree with symmetric and wedge products on the appropriate spaces. There is little reason to prefer one over the other. The only motivation for our use of (8.2) is that the c^{μ}_{λ} are all less than 1, whereas in (8.7) they are greater than 1, and the coefficients of formulae involving the Pieri products rapidly become extremely large.

The choice (8.2) has the additional advantage that $c_\lambda^\mu = 1$ whenever $\lambda \subset \mu$ are n-fat shapes.

We have thus proved the commutativity of the Pieri products with our choice (8.2) of the constants c_λ^μ. More generally, polarizing (8.4) leads to the identity.

Corollary 8.5 Let $v, w \in V$. With the notation of lemma 8.2 we have

$$v *_j (w *_k \zeta) + w *_j (v *_k \zeta) = v *_k (w *_j \zeta) + w *_k (v *_j \zeta), \qquad (8.9)$$

for all $\zeta \in L_\lambda$.

Define the (modified) Pieri maps

$$\psi_\lambda^\mu = c_\lambda^\mu \varphi_\lambda^\mu : V \otimes L_\lambda \longrightarrow L_\mu$$

for $|\mu| = |\lambda| + 1$, $\mu \supset \lambda$, so

$$\psi_\lambda^\mu (v \otimes \zeta) = v * \zeta . \qquad (8.10)$$

We have shown that the resulting diagram is commutative, hence we can unambiguously define functorial maps.

$$\psi_\lambda^\nu : \Theta_k \otimes L_\lambda \longrightarrow L_\nu \qquad (8.11)$$

for $\nu \supset \lambda$, $|\nu| = |\lambda| + k$, by iterating the maps $\psi_\lambda^\nu (v \otimes \cdot)$ and using the fact that $\{v^k = v \odot \dots \odot v | v \in V\}$ spans $\Theta_k V$. By Pieri's formula and theorem 4.1, these maps must be identically zero whenever ν/λ has two or more boxes in any column. This, therefore proves the closure of the Schur hypercomplex.

Lemma 8.6 Suppose $\lambda \in S_k^2(\nu)$. Suppose $v_1,\ldots,v_k \in V$. Then

$$\Sigma\, v_{\pi 1} * v_{\pi 2} * \ldots * v_{\pi k} * \xi = 0,$$

for any $\xi \in L_\lambda$. The sum is over all permutations π of $\{1,\ldots,k\}$, and the Pieri products $*$ are taken over any series of shapes $\lambda = \mu_0 \subset \mu_1 \subset \mu_2 \subset \ldots \subset \mu_k = \nu$ with $|\mu_i/\mu_{i-1}| = 1$.

This follows immediately from the fact that L_ν does not appear in the Pieri decomposition (5.1) of $\Theta_k \otimes L_\lambda$.

We now turn to a proof of exactness. Fix $0 \neq v \in V$, let Z be the one-dimensional subspace of V spanned by v and let W be any complementary subspace, so $V = W \oplus Z$. We use the decomposition of $L_\lambda V$ given in corollary 3.4.

Lemma 8.7 Let $0 \neq v \in V$, $\lambda \in S_k^1(\mu)$. Then the linear map

$$\psi_\lambda^\mu(v^k \otimes \cdot) : L_\lambda W \longrightarrow L_\mu V$$

is an injection. Moreover, summing over $\lambda \in S_k^1(\mu)$,

$$\Sigma\, \psi_\lambda^\mu(v^k \otimes \cdot) : \oplus\, L_\lambda W \longrightarrow L_\mu^{(k)} W ,$$

is an isomorphism (cf. (3.8).)

Proof

Let $J = (j_1,\ldots,j_k)$ be a nondecreasing sequence of integers $1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq m = m(\mu)$, with $\#\{i \mid j_i = \ell\} \leq \mu_\ell$. Order these sequences lexicographically. For each J , let $\mu - J$ denote the diagram of boxes obtained by deleting boxes in rows $j_1, j_2, \ldots, j_k$ from μ .

For instance, if $\mu = (4,3,1)$, $J = (1,1,2,3)$, then $\mu - J = (2,2)$. We call J admissible if $\mu - J$ is a shape, and $\mu - J \in S^1(\mu)$. Conversely to each $\lambda \in S^1(\mu)$, there is an admissible $J = J(\lambda)$ with $\lambda = \mu - J$.

Given a decomposable $\underset{\sim}{y} \in L_\mu^{(k)} W \subset L_\mu V$, let $J(y)$ denote the row numbers of boxes in which the k copies of v appear in the tableau corresponding to $\underset{\sim}{y}$. Now by the formula (5.5) for the Pieri product , if $\underset{\sim}{x} \in L_\lambda W$ is decomposable,

$$\psi_\lambda^\mu (v^k \otimes \underset{\sim}{x}) = c\, \underset{\sim}{y} + \underset{\sim}{z}$$

where $c \neq 0$, $\underset{\sim}{y}$ is decomposable with $J(\underset{\sim}{y}) = J(\lambda)$, and $\underset{\sim}{z}$ is a sum of decomposable terms $\Sigma\, \underset{\sim}{z}_j$ with $J(\underset{\sim}{z}_j) > J(\underset{\sim}{y})$ in the lexicographic ordering.

By induction, assume

$$\sum_{J(\lambda) > J_o} \psi_\lambda^\mu (v^k \otimes \cdot)$$

is an injection. Then if

$$\sum_{J(\lambda) \geq J_o} \psi_\lambda^\mu (v^k \otimes \underset{\sim}{x}_\lambda) = 0$$

for $\underset{\sim}{x}_\lambda \in L_\lambda W$, the only decomposable terms $\underset{\sim}{y}$ appearing in this sum with $J(y) = J_o$ are those arising from L_{λ_o} with $J(\lambda_o) = J_o$, and these appear with nonzero multiple. Thus $\underset{\sim}{x}_{\lambda_o} = 0$, and the induction step is completed.

Theorem 8.8

Let $0 \neq v \in V$, $\mu \subset \nu$, $k = |\nu/\mu|$. Then

$$\psi_{\mu}^{\nu}(v^k \otimes \zeta) = 0$$

if and only if

$$\zeta \in \sum_{\lambda \in S^{0}(\mu,\nu)} \operatorname{im} \psi_{\lambda}^{\mu}(v^j \otimes \cdot) \ , \ j = |\lambda/\mu| \ .$$

Proof

Using the direct sum decomposition (3.7), and the previous lemma,

$$\zeta = \sum_{j} \sum_{\lambda \in S_j^1(\mu)} \psi_{\lambda}^{\mu}(v^j \otimes \theta_{\lambda})$$

where $\theta_{\lambda} \in L_{\lambda}W$ is uniquely determined by ζ . Now by commutivity of the hypercomplex

$$\psi_{\mu}^{\nu}(\zeta) = \sum_{j} \sum_{\lambda \in S_j^1(\mu) \cap S_{j+k}^1(\nu)} \psi_{\lambda}^{\nu}(v^{j+k} \otimes \theta_{\lambda}) \ .$$

Again by lemma 8.6, $\psi_{\mu}^{\nu}(\zeta) = 0$ if and only if

$$\theta_{\lambda} = 0 \ \text{ for all } \ \lambda \in S_j^1(\mu) \cap S_{j+k}^1(\nu) \ .$$

Therefore

$$\zeta \in \sum_{\lambda \in S^1(\mu) \cap S^2(\nu)} \operatorname{im} \psi_{\lambda}^{\mu}(v^j \otimes \cdot) \ .$$

Finally, if λ is not μ,ν - maximal, then there exists $\lambda' \in S^1(\mu) \cap S^2(\nu)$ with $\lambda \subset \lambda'$, and

$$\operatorname{im} \psi^{\mu}_{\lambda}(v^j \otimes \cdot) = \operatorname{im} \psi^{\mu}_{\lambda'}(v^{j'} \otimes \cdot)$$

since

$$\psi^{\mu}_{\lambda}(v^j \otimes \eta) = \psi^{\mu}_{\lambda}(v^{j'} \otimes \psi^{\lambda'}_{\lambda}(v^{j-j'} \otimes \eta)) .$$

This completes the proof.

Corollary 8.9 The Schur hypercomplex is exact.

This follows directly from theorem 8.8 and theorem 7.5. (Note that theorem 8.8 is stronger than exactness since ν_1 does not have to equal μ_1 .)

Example 8.9

Let $\mu = (3,2,1)$, $\nu = (3,3,2)$, so $S^0(\mu,\nu) = \{(2,2,1),(3,1,1)\}$. The symbols x,y,z,w,u,v denote linearly independent elements of V , ordered in the manner indicated. We restrict attention to the summand $L^{(1)}_{\mu} W \subset L_{\mu} V$, with $x,y,z,w,u \in W$. Then from the definition of Pieri products and the straightening formulae, (omitting the symmetric product symbol for simplicity)

$$\psi^{\nu}_{\mu}[v^2 \otimes (xyv) \otimes (zw) \otimes u] = \frac{3}{4}(xyv) \otimes (zwv) \otimes (uv) + \frac{1}{4}(xyv) \otimes (zwu) \otimes (vv)$$

$$= \frac{1}{8}(xyz) \otimes (wuv) \otimes (vv) + \frac{1}{8}(xyw) \otimes (zuv) \otimes (vv)$$

$$- \frac{1}{4}(xyu) \otimes (zwv) \otimes (vv) \quad ,$$

$$\psi^{\nu}_{\mu}[v^2 \otimes (xyz) \otimes (wv) \otimes u] = -\frac{1}{2}(xyz) \otimes (wuv) \otimes (vv) \quad ,$$

$$\psi^{\nu}_{\mu}[v^2 \otimes (xyz) \otimes (wu) \otimes v] = (xyz) \otimes (wuv) \otimes (vv) \quad .$$

Therefore $\ker \psi^{\nu}_{\mu}(v^2 \otimes \cdot)$ is spanned by elements of the forms

$$(xyv) \otimes (zw) \otimes u + \frac{1}{2}[(xyz) \otimes (wv) \otimes u + (xyw) \otimes (zv) \otimes u] +$$

$$+ \frac{1}{4}(xyu) \otimes (zw) \otimes v + \frac{1}{8}[(xyz) \otimes (wu) \otimes v + (xyw) \otimes (zu) \otimes v] \quad ,$$

and $$(xyz) \otimes (wv) \otimes u + \frac{1}{2}(xyz) \otimes (wu) \otimes v \quad .$$

But these are equal to

$$\frac{5}{2}\psi^{\mu}_{\lambda}[v \otimes (xy) \otimes (zw) \otimes u] \quad , \quad \lambda = (2,2,1) \quad ,$$

$$\frac{3}{2}\psi^{\mu}_{\lambda'}[v \otimes (xyz) \otimes w \otimes u] \quad , \quad \lambda' = (3,1,1) \quad ,$$

respectively, hence we have

$$\ker \psi^{\nu}_{\mu}(v^2 \otimes \cdot) = \operatorname{im} \psi^{\mu}_{\lambda}(v \otimes \cdot) + \operatorname{im} \psi^{\mu}_{\lambda'}(v \otimes \cdot) \quad .$$

in accordance with the theorem. Note that the sum is not direct, since

$$(xyv) \otimes (zv) \otimes w + \frac{1}{2}(xyv) \otimes (zw) \otimes v -$$

$$- \frac{3}{8}(xyz) \otimes (wv) \otimes v + \frac{3}{8}(xyw) \otimes (zv) \otimes v$$

lies in both. Indeed, the above element is just

$$\psi^{\mu}_{\lambda''}(v^2 \otimes (xy) \otimes z \otimes w\ ) \ , \qquad \lambda'' = (2,1,1) \ ,$$

which clearly factors through both $L_\lambda W$ and $L_{\lambda'} W$.

The Schur hypercomplex sits inside a larger hypercomplex, whose properties will be useful when we come to prove exactness of the differential hypercomplex. Given $\lambda \subset \mu$, $|\mu/\lambda| = k$, define the map

$$\Psi^{\mu}_{\lambda} : \Theta_* \otimes L_\lambda \to \Theta_* \otimes L_\mu$$

as the composition

$$\Theta_* \otimes L_\lambda \xrightarrow{\Delta \otimes 1} \Theta_* \otimes \Theta_* \otimes L_\lambda \xrightarrow{(1 \otimes \tilde{\pi}_k \otimes 1)} \Theta_* \otimes \Theta_k \otimes L_\lambda \xrightarrow{1 \otimes \psi^{\mu}_{\lambda}} \Theta_* \otimes L_\mu \ ,$$

where $\Delta : \Theta_* \to \Theta_* \otimes \Theta_*$ is the diagonal map on the symmetric algebra of V , $\tilde{\pi}_k: \Theta_* \to \Theta_k$ is the (modified) projection $\tilde{\pi}_k(w) = 0$, $w \in \Theta_\ell$, $\ell \neq k$, $\tilde{\pi}_k(w) = k!\, w$, $w \in \Theta_k$ (note the factor $k!$) and $\psi^{\mu}_{\lambda} : \Theta_k \otimes L_\lambda \to L_\mu$ is the Pieri map.

Lemma 8.10 For $\lambda \subset \mu \subset \nu$,

$$\Psi^{\nu}_{\lambda} = \Psi^{\nu}_{\mu} \circ \Psi^{\mu}_{\lambda} \ . \tag{8.10}$$

Proof

Since $\{v^n | v \in V\}$ generates Θ_n , it suffices to check (8.10) on $v^n \otimes \xi$ for $\xi \in L_\lambda$ arbitrary. Now if $|\mu/\lambda| = k$, then

$$\Psi^{\mu}_{\lambda}(v^n \otimes \xi) = \frac{n!}{(n-k)!}\, v^{n-k} \otimes \psi^{\mu}_{\lambda}(v^k \otimes \xi) \ , \tag{8.11}$$

for $n \geq k+\ell$ (and is zero if $n<k+\ell$) . The proof of (8.10) now follows easily from the commutativity of the Schur hypercomplex.

<u>Theorem 8.11</u> The <u>polynomial hypercomplex</u> $\Psi^{\mu}_{\lambda} : \Theta_* \otimes L_\lambda \to \Theta_* \otimes L_\mu$ is exact.

<u>Proof</u>

Commutativity and closure follow directly from (8.10) and the corresponding properties of the Schur hypercomplex. The proof of exactness rests on the following lemma:

<u>Lemma 8.12</u> Suppose $\lambda \subset \mu$, $|\mu/\lambda| = k$, and suppose $\rho \in \mathcal{J}^1_{\ell+k}(\lambda) \cap \mathcal{J}^1_{\ell}(\mu)$. Then the diagram

$$\begin{array}{ccc} \Theta_{k+\ell} \otimes L_\lambda & \xrightarrow{\Psi^{\mu}_{\lambda}} & \Theta_\ell \otimes L_\mu \\ \downarrow{\scriptstyle \psi^{\rho}_{\lambda}} & & \downarrow{\scriptstyle \psi^{\rho}_{\mu}} \\ L_\rho & \xrightarrow{c\,\mathbb{1}} & L_\rho \end{array}$$

commutes. Here $c = (k+\ell)!/\ell!$ (The top map Ψ^{μ}_{λ} is really the restriction of Ψ^{μ}_{λ} to $\Theta_{k+\ell} \otimes L_\lambda$.)

<u>Proof</u>

By Schur's lemma, the bottom map must be a multiple of the identity as all maps commute with the representation of $GL(V)$. To compute c , by (8.11)

$$\psi^{\rho}_{\mu}(\Psi^{\mu}_{\lambda}(v^{k+\ell} \otimes \xi)) = \frac{(k+\ell)!}{\ell!} \; \psi^{\rho}_{\mu}(v^{\ell} \otimes \psi^{\mu}_{\lambda}(v^{k} \otimes \xi))$$

$$= \frac{(k+\ell)!}{\ell!} \; \psi^{\mu}_{\lambda}(v^{k+\ell} \otimes \xi) \; ,$$

proving the lemma.

Clearly, to prove exactness of the polynomial hypercomplex it suffices to look at homogeneous pieces. Let λ,μ,ν be min-max related with $|\mu/\lambda| = k$, $|\nu/\mu| = \ell$. Using the Pieri formula (5.1), we have, for $n \geq 0$,

$$\begin{array}{ccccc} \Theta_{n+k+\ell} \otimes L_\lambda & \xrightarrow{\Psi^\mu_\lambda} & \Theta_{n+k} \otimes L_\mu & \xrightarrow{\Psi^\nu_\mu} & \Theta_n \otimes L_\nu \\ \wr\! \| & & \wr\! \| & & \wr\! \| \\ \bigoplus\limits_{\rho \in \mathcal{T}^1_{n+k+\ell}(\lambda)} L_\rho & \xrightarrow{\Phi^\mu_\lambda} & \bigoplus\limits_{\rho \in \mathcal{T}^1_{n+k}(\mu)} L_\rho & \xrightarrow{\Phi^\nu_\mu} & \bigoplus\limits_{\rho \in \mathcal{T}^1_n(\nu)} L_\rho \end{array} \quad . \tag{8.12}$$

By Schur's lemma, and lemma 8.12, if $\rho \in \mathcal{T}^1_{n+k+\ell}(\lambda) \cap \mathcal{T}^1_{n+k}(\mu)$, then $\Phi^\mu_\lambda | L_\rho : L_\rho \to L_\rho$ is a nonzero multiple of the identity; otherwise $\Phi^\mu_\lambda | L_\rho$ vanishes. But by lemma 2.3,

$$\mathcal{T}^1_{n+k}(\mu) = \mathcal{T}^1_{n+k+\ell}(\lambda) \cup \mathcal{T}^1_n(\nu) ,$$

$$\mathcal{T}^1_{n+k+\ell}(\lambda) \cap \mathcal{T}^1_n(\nu) = \emptyset ,$$

so the lower sequence is trivially exact. For $-k \leq n < 0$ the above still works, only the right-most terms in (8.12) are $\{0\}$. Finally theorem 7.5 completes the proof.

The Schur and polynomial hypercomplexes are special cases of the hypercomplex

$$\Psi^\mu_{\lambda,W} : \Theta_* W \otimes L_\lambda V \to \Theta_* W \otimes L_\mu V$$

defined for any subspace $W \subset V$. The Schur hypercomplex corresponds to the case W being one-dimensional, spanned by $v \in V$. I am sure that all these intermediate hypercomplexes are exact, but I have been unable to generalize either proof to this situation. Even the decomposition of theorem 3.3 is of little direct help, as the skew Schur spaces $L_{\mu/\lambda}Z$ are no longer irreducible under the representations of $GL(Z)$. It would, however, be nice to have a proof of exactness which works for all $W \subset V$.

9. Duality and Interior Products

Since we are working in characteristic zero, if V is a finite dimensional vector space with dual V^*, there is a natural isomorphism $(L_\lambda V)^* \simeq L_\lambda(V^*)$ identifying the Schur spaces of V^* as the duals to the Schur spaces of V. Let $\{e_1,\ldots,e_n\}$, $\{\epsilon_1,\ldots,\epsilon_n\}$ be dual bases of V and V^* respectively. It is an unfortunate fact of life, however, that the corresponding bases $\{e_T\}$ and $\{\epsilon_T\}$, for standard tableaux T of shape λ, do not form dual bases for $L_\lambda V$ and $L_\lambda V^*$ in any natural way.

For instance, in $L_{(2,1)}\mathbb{R}^3$, if one requires

$$\langle (e_1 \odot e_2) \otimes e_3 \,,\, (\epsilon_1 \odot \epsilon_2) \otimes \epsilon_3 \rangle = 1 \,,$$

$$\langle (e_1 \odot e_3) \otimes e_2 \,,\, (\epsilon_1 \odot \epsilon_2) \otimes \epsilon_3 \rangle = 0 \,,$$

then the relation (3.1) requires that

$$\langle (e_2 \odot e_3) \otimes e_1 \,,\, (\epsilon_1 \odot \epsilon_2) \otimes \epsilon_3 \rangle = -1 \,.$$

In other words, the only way to make $\{e_T\}$, $\{\epsilon_T\}$ dual is to make the pairing between $L_\lambda V$ and $L_\lambda V^*$ depend on the order in which the basis elements of V are written.

A more natural way to proceed is to induce the pairing between $L_\lambda V$ and $L_\lambda V^*$ from that on $\otimes_{\tilde\lambda}\wedge_* V$, $\otimes_{\tilde\lambda}\wedge_* V^*$. Namely define

$$\langle \omega, \omega^* \rangle = \frac{1}{\lambda!} \langle \delta_\lambda \omega \,,\, \delta_\lambda \omega^* \rangle \tag{9.1}$$

for $\omega \in \otimes_\lambda \odot_* V$, $\omega^* \in \otimes_\lambda \odot_* V^*$, and δ_λ is the map in the definition of the Schur spaces. It is easy to check that $\langle\ ,\ \rangle$ induces a non-trivial pairing between $L_\lambda V$, $L_\lambda V^*$. The following lemma is a direct consequence of (9.1).

Lemma 9.1 Let T,S be tableaux of shape λ , let $K(T,S)$ denote the set of all triples of permutations (ρ_1,π,ρ_2) acting on the diagram λ with ρ_i preserving rows and π preserving columns, such that $\rho_1\pi\,\rho_2(T) = S$. Then

$$\langle e_T,\epsilon_S\rangle = \frac{1}{\lambda!}\sum_{K(T,S)} \operatorname{sgn}(\pi)\ . \tag{9.2}$$

In particular, $\langle e_T,\epsilon_S\rangle = 0$ if T and S have different contents.

Thus, for example, in $L_{(2,1)}\mathbb{R}^3$,

$$\langle (e_1 \odot e_2) \otimes e_3\ ,\ (\epsilon_1 \odot \epsilon_2) \otimes \epsilon_3\rangle = 1\ ,$$

$$\langle (e_1 \odot e_3) \otimes e_2\ ,\ (\epsilon_1 \odot \epsilon_2) \otimes \epsilon_3\rangle = -\frac{1}{2}\ ,$$

$$\langle e_1^2 \otimes e_2\ ,\ \epsilon_1^2 \otimes \epsilon_2\rangle = 2\ ,$$

etc. More generally, if T is any tableau with content $c_i(T) \le 1$ for each i . (i.e. each symbol appears at most once in T), then

$$\langle e_T\ ,\ \epsilon_T\rangle = 1\ . \tag{9.3}$$

At the other extreme, if T_o is the simple tableau of shape λ , then

$$\langle e_{T_o}\ ,\ \epsilon_{T_o}\rangle = \lambda!\ . \tag{9.4}$$

The Pieri products on $L_\lambda V$ induce dual maps, the interior products, on $L_\lambda V^*$:

<u>Definition 9.2</u> Let $\mu = \lambda + j$. The <u>interior product</u>

$$\lrcorner_j : V \otimes L_\mu V^* \to L_\lambda V^* , \tag{9.5}$$

is defined so that

$$\langle \zeta, v \lrcorner_j \omega \rangle = \langle v *_j \zeta, \omega \rangle \tag{9.6}$$

for all $v \in V$, $\zeta \in L_\lambda V$, $\omega \in L_\mu V^*$.

An explicit formula for the interior product can be given in terms of the partial polarization maps:

<u>Theorem 9.3</u> For $\mu = \lambda + j$, $m(\mu) = m$,

$$v \lrcorner \omega = b^\mu_\lambda \, \pi \otimes \mathbb{1} [v \otimes \chi^\lambda_\mu(\omega)] , \tag{9.7}$$

for $v \in V$, $\omega \in L_\mu V^*$. Here $\pi \otimes \mathbb{1}$: $V \otimes V^* \otimes L_\lambda V^* \to L_\lambda V^*$ is the pairing between V and V^* , and

$$b^\mu_\lambda = \frac{\mu_j!}{L(\tilde{\mu},\tilde{\lambda})} \; \frac{\mu_j + m - j}{\mu_j} \prod_{k<j} \frac{\ell_{jk}(\mu) - 2}{\ell_{jk}(\mu) - 1} . \tag{9.8}$$

<u>Proof</u>

By (9.1), (9.7) is equivalent to the relation

$$\langle v * \xi, \omega \rangle = b^\mu_\lambda \langle v \otimes \xi, \chi^\lambda_\mu(\omega) \rangle \tag{9.9}$$

for all $\xi \in L_\lambda V$, where the right hand $\langle \; , \; \rangle$ is the pairing between $V \otimes L_\lambda V$ and $V^* \otimes L_\lambda V^*$. By functoriality, we need only check (9.9) for the specific choices $v = e_j$, and $\xi = e^\lambda_0$, $\omega = \epsilon^\mu_0$ corresponding to the simple tableaux of shape λ , μ respectively. Now by (6.1)

$$\chi_\mu^\lambda(\epsilon_o^\mu) = \mu_j \, \epsilon_j \otimes \epsilon_o^\lambda + \cdots ,$$

where the omitted terms are all of the form $\epsilon_k \otimes \alpha$ for $k < j$.
From (9.4) ,

$$\langle e_j \otimes e_o^\lambda , \chi_\mu^\lambda(\epsilon_o^\mu)\rangle = \mu_j \cdot \lambda! = \mu! .$$

On the other hand, by the same techniques as used to prove theorem 6.2, it is not difficult to show that

$$\varphi_\lambda^\mu(e_j \otimes e_o^\lambda) = \frac{\mu_j + n - k}{\mu_j} \prod_{k>j} \frac{\ell_{jk}(\lambda) - 1}{\ell_{jk}(\lambda)} \; e_o^\mu . \tag{9.10}$$

Substituting these last two identities into (9.9) and using (8.1), completes the proof of the theorem.

Example 9.4

Let $\lambda = (3,1)$, $\mu = (3,2)$. Then $b_\lambda^\mu = 1$, hence

$$e_1 \lrcorner (\epsilon_1 \odot \epsilon_2 \odot \epsilon_3) \otimes (\epsilon_4 \odot \epsilon_5) = -\frac{1}{2}\,[(\epsilon_2 \odot \epsilon_3 \odot \epsilon_4) \otimes \epsilon_5 + (\epsilon_2 \odot \epsilon_3 \odot \epsilon_5) \otimes \epsilon_4]$$

whereas

$$e_5 \lrcorner (\epsilon_1 \odot \epsilon_2 \odot \epsilon_3) \otimes (\epsilon_4 \odot \epsilon_5) = (\epsilon_1 \odot \epsilon_2 \odot \epsilon_3) \otimes \epsilon_4 .$$

The dual hypercomplex corresponding the Schur hypercomplex is based on these interior products. As a direct result of lemma 7.9 and theorem 8.2 we have

Theorem 9.5 Fix $0 \neq v \in V$. Then the interior products

$$v \lrcorner : L_\mu V^* \to L_\lambda V^*$$

form an exact cohypercomplex, called the Schur cohypercomplex.

There is also a dual cohypercomplex corresponding to the polynomial hypercomplex. Namely, for $\mu \supset \lambda$ let

$$X_\mu^\lambda : \Theta_* V^* \otimes L_\mu V^* \to \Theta_* V^* \otimes L_\lambda V^*$$

denote the dual map to Ψ_λ^μ. In the special case $|\mu/\lambda| = 1$, (9.9) implies the X_μ^λ can be obtained as the composition

$$\Theta_* V^* \otimes L_\mu V^* \xrightarrow{\mathbf{1} \otimes b_\lambda^\mu X_\mu^\lambda} \Theta_* V^* \otimes V^* \otimes L_\lambda V^* \to \Theta_* V^* \otimes L_\lambda V^* ,$$

where the second arrow is the multiplication map $\Theta_* V^* \otimes V^* \to \Theta_* V^*$. We immediately have

Theorem 9.6 The polynomial cohypercomplex X_μ^λ is exact.

The polynomial cohypercomplex can be characterized in a slightly different way. Namely, if we identify $\Theta_* V^*$ with the space $\mathcal{P}$ of polynomial functions on V, then by theorem 9.3,

$$X_\mu^\lambda = v_0 \lrcorner : \mathcal{P} \otimes L_\mu V^* \to \mathcal{P} \otimes L_\lambda V^* , \tag{9.11}$$

where $v_0 \in V^* \otimes V$ is the diagonal element given by

$$v_0 = \Sigma\, \epsilon_i \otimes e_i$$

for dual bases on V, V^*. We leave the verification of (9.11) to the reader.

10. The Littlewood - Richardson Rule

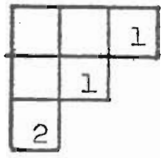
More generally, we may consider the decomposition of the tensor product of two Schur functors into irreducible representations. The basic result was formulated by Littlewood and Richardson, [18], in the context of Schur functions. To state this, we need the concept of a word of Yamanouchi. This is a sequence $y_1, y_2, \dots$ of positive integers subject to the condition that in any subsequence $y_1, y_2, \dots, y_k$ any integer i occurs at least as often as any larger integer $j \geq i$. Thus 1,2,1,3,2,1 is a word of Yamanouchi, whereas 1,2,1,2,2 is not.

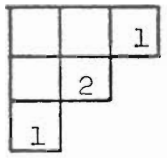
Theorem 10.1 For any vector space V, the tensor product decomposition

$$L_\lambda \otimes L_\mu = \oplus N^\nu_{\lambda\mu} L_\nu \tag{10.1}$$

holds. Here $N^\nu_{\lambda\mu}$ is the number of standard tableau T of shape ν/λ with entries in $\{1,\dots,|\mu|\}$, of content μ and such that the sequence obtained by listing the entries of T row by row, but from right to left in each row, forms a word of Yamanouchi.

For a proof see [2]. Note that $N^\nu_{\lambda\mu} = 0$ unless $\nu \supseteq \lambda, \mu$ and $|\nu| = |\lambda| + |\mu|$. In practice, one obtains $N^\nu_{\lambda\mu}$ by finding all possible ways to fill in $\nu \backslash \lambda$ with μ_1 1's, μ_2 2's, etc. in such a way that the resulting tableau is standard, and listing the rows from right to left in order forms a word of Yamanouchi. Thus, for example, if $\lambda = \mu = (2,1)$ and $\nu = 3,2,1$, then $N^\nu_{\lambda\mu} = 2$ since there are two possible ways to place 2 1's and 1 2 in ν/λ to form a word of Yamanouchi:





See [15], [31], [32] for further examples.

Thus the space of functorial maps $\mathcal{H} = \mathcal{H}(L_\lambda \otimes L_\mu; L_\nu)$ is of dimension $N^{\nu}_{\lambda\mu}$. Each such map φ in $\mathcal{H}$ can be used to define a Pieri - type product

$$\varphi\colon\ L_\lambda \otimes L_\mu \to L_\nu ,$$

but now, unless $N^{\nu}_{\lambda\mu} = 1$, there is much more freedom in the definition.

Repeated applications of the Littlewood Richardson rule lead to more general decompositions

$$\otimes_{\underset{\sim}{\lambda}} L_* = L_{\lambda^1} \otimes \ldots \otimes L_{\lambda^m} = \oplus\, N^{\mu}_{\underset{\sim}{\lambda}}\, L_\mu \qquad (10.2)$$

for tensor products of Schur functors, and hence $N^{\mu}_{\underset{\sim}{\lambda}}$ - dimensional spaces $\mathcal{H}(\otimes_{\underset{\sim}{\lambda}} L_*\ ,\ L_\mu)$, each element of which defines a r-fold product between Schur spaces.

Problems arise, however, if one tries to choose such products in some consistent fashion so as to make them (at the very least) associative. Indeed, if $\varphi \in \mathcal{H}(L_\lambda \otimes L_{\lambda'}, L_\mu)$, $\tilde{\varphi} \in \mathcal{H}(L_\mu \otimes L_{\lambda''}\ ,\ L_\nu)$ then $\tilde{\varphi} \circ (\varphi \otimes \mathbb{1}) \in \mathcal{H}(L_\lambda \otimes L_{\lambda'} \otimes L_{\lambda''}\ ,\ L_\nu)$, but unless this latter space has dimension 1 , this in general will not agree (even up to multiple) with a composition $\tilde{\psi} \circ (\mathbb{1} \otimes \psi)$ for $\psi \in \mathcal{H}(L_{\lambda'} \otimes L_{\lambda''}\ ,\ L_{\mu'})$, $\tilde{\psi} \in \mathcal{H}(L_\lambda \otimes L_{\mu'}\ ,\ L_\nu)$. In other words, no amount of juggling coefficients will make the partial product $\tilde{\varphi}(\varphi(\xi \otimes \eta) \otimes \zeta)$ and $\tilde{\psi}(\xi \otimes \psi(\eta \otimes \zeta))$ agree, as the reader can check in simple examples. We thus leave aside problems of defining products between elements of Schur spaces except in one important special case, which we discuss in section 12.

11. Algebraic Differentials

Given a functorial map on a tensor product of Schur spaces, there are induced maps on slightly larger tensor products called the algebraic differentials of the given functorial map. These will be important when we establish a Leibnitz-type rule for differential hyperforms in section 14. In this section we introduce this concept and discuss a few elementary algebraic properties.

Let $\underset{\sim}{\lambda} = (\lambda^1,\ldots,\lambda^m)$ be an ordered k-tuple of shapes, with $\otimes_{\underset{\sim}{\lambda}} L_*$ denoting the corresponding tensor product of Schur spaces, cf. (10.2). Let L_μ occur with positive multiplicity in $\otimes_{\underset{\sim}{\lambda}} L_*$, and suppose

$$\psi : \otimes_{\underset{\sim}{\lambda}} L_* \to L_\mu$$

is a functorial map. For $1 \leq j \leq m$, let

$$\underset{\sim}{\lambda} \oplus j = (\lambda^1,\ldots,\lambda^j, 1, \lambda^{j+1},\ldots,\lambda^m),$$

$$\underset{\sim}{\lambda} + j_k = (\lambda^1,\ldots,\lambda^j + k, \lambda^{j+1},\ldots,\lambda^m),$$

(provided $\lambda^j + k$ is a shape). Further let $\nu \supset \mu$, $|\nu/\mu| = 1$.

Definition 11.1 For ψ,λ,μ,ν as above, the j-th tensor differential of ψ is the functorial map

$$\psi_{\otimes j}: \otimes_{\underset{\sim}{\lambda} \oplus j} L_* \to L_\nu$$

given by the composition

$$\otimes_{\underset{\sim}{\lambda} \oplus j} L_* \to V \otimes \otimes_{\underset{\sim}{\lambda}} L_* \xrightarrow{1 \otimes \psi} V \otimes L_\mu \xrightarrow{*} L_\nu$$

where the last map is the Pieri product ψ_{μ}^{ν} . The j,k-th <u>algebraic differential</u> of ψ is the map

$$\psi_{,j_k} : \otimes_{\lambda + j_k} L_* \to L_\nu$$

given by composing $\psi_{\otimes j}$ with the partial polarization

$$p : L_{\lambda^j + k} \to L_{\lambda^j} \otimes V ,$$

cf. (8.3). If $\lambda^j = (i)$, $k = 1$, we denote $\psi_{,j_1}$ by just $\psi_{,j}$.

<u>Example 11.2</u>

Let $\mu = (2,1)$, $\nu = (3,1)$. Consider the map

$$\psi : V \otimes V \otimes V \to L_\mu$$

given by

$$\psi(x \otimes y \otimes z) = (x \odot y) \otimes z .$$

Then

$$\psi_{,1} : \odot_2 \otimes V \otimes V \to L_\nu$$

is

$$\psi_{,1}((x_1 \odot x_2) \otimes y \otimes z) = \frac{2}{3} (x_1 \odot x_2 \odot y) \otimes z - \frac{1}{3} (x_1 \odot x_2 \odot z) \otimes y ,$$

whereas

$$\psi_{,3} : V \otimes V \otimes \odot_2 \to L_\nu$$

is the different functorial map

$$\psi_{,3}(x \otimes y \otimes (z_1 \odot z_2)) = -\frac{2}{3}(z_1 \odot z_2 \odot x) \otimes y - \frac{2}{3}(z_1 \odot z_2 \odot y) \otimes x .$$

Commutativity of the hypercomplex immediately gives

Lemma 11.3 For ψ as above,

$$\psi_{(\otimes j) \otimes j'} = \psi_{(\otimes j') \otimes j} \equiv \psi_{\otimes(j,j')} \tag{11.1}$$

and

$$\psi_{,j_k,j'_{k'}} = \psi_{,j'_{k'},j_k} \equiv \psi_{,j_k j'_{k'}} . \tag{11.2}$$

whenever both make sense.

Lemma 11.4 For ψ as above,

$$\psi_{\otimes j} = \Sigma\, \psi_{,j_k} , \tag{11.3}$$

the sum being over all k such that $\lambda^j + k$ is a shape.

12. Products and Differentials for Fat Shapes

In this section we restrict attention to the n-subcomplexes of the Schur hypercomplex given by the n-fat shapes $\lambda = n^q r$, $0 \leq r < n$. Throughout this section, all shapes are n-fat unless specifically mentioned otherwise. The first step is to define a product

$$* : L_\lambda \otimes L_\mu \to L_\nu$$

for $|\lambda| + |\mu| = |\nu|$. We first show that there is exactly one such functorial product, up to constant multiple.

Lemma 12.1 Let λ, μ, ν be n-fat shapes with $|\lambda| + |\mu| = |\nu|$. Then L_ν occurs with unit multiplicity in $L_\lambda \otimes L_\mu$.

Proof

Let $\mu = n^q r$, $0 \leq r < n$. In the implementation of the Littlewood-Richardson rule, we must fill ν/λ with n 1's, n 2's , ... , n q's and r q+1's . Since ν/λ has only n columns, and the resulting tableau must be standard, the positions of each of the symbols $1,\ldots,q+1$ is dictated. The resulting sequence is, furthermore, easily seen to be a word of Yamanouchi.

Theorem 12.2 Let $\lambda^1,\ldots,\lambda^k,\mu$ be n-fat shapes with $|\mu| = \Sigma|\lambda^i|$. Then L_μ occurs with unit multiplicity in $\otimes_{\underset{\sim}{\lambda}} L_*$ (cf. (8.2)) if and only if either

$$\text{i)} \quad \Sigma\, r(\lambda^i) \leq n \tag{12.1a}$$

$$\text{or ii)} \quad \Sigma\, r(\lambda^i) \geq (k-1)n\ . \tag{12.1b}$$

Proof

We prove case ii), the proof of case i) being similar, but easier to implement. Assume, by induction, that the tensor product $L_{\lambda^1} \otimes \cdots \otimes L_{\lambda^j}$ decomposes into a direct sum of L_μ's corresponding to shapes of the following three types.

a) μ n-fat. This occurs with multiplicity 1 .

b) μ's with $\mu_1 \geq n+1$

c) μ's which are n-bounded, but have $2 \leq \ell \leq j$ rows with less than n boxes in them, satisfying

$$\mathfrak{m}(\mu)n - |\mu| = j\,n - \sum_{i=1}^{j} r(\lambda_i) \; . \tag{12.2}$$

(This is the number of open boxes needed to complete μ into a rectangle.)

Tensoring with L_λ , where $\lambda = \lambda^{j+1}$, we must investigate the summands of $L_\lambda \otimes L_\mu$ for each of the above types of shapes μ .

For type a), by the previous lemma the n-fat shape ν of size $|\mu| + |\lambda|$ appears with multiplicity 1 . Furthermore, by the Littlewood - Richardson rule, the other constituents of $L_\lambda \otimes L_\mu$ are either of type b), or, if n-bounded, have exactly two rows with fewer than n boxes in them. (cf. the proof of lemma 12.1.) Moreover, the number of open boxes, is

$$\begin{aligned} 2n - r(\mu) - r(\lambda) &= 2n - \Big(\sum_{i=1}^{j} r(\lambda^i) - (j-1)n\Big) - r(\lambda) \\ &= (j+1)n - \sum_{i=1}^{j+1} r(\lambda^i) \; , \end{aligned}$$

so the shape is of type c).

If $\mu_1 \geq n+1$, and L_ν appears in $L_\lambda \otimes L_\mu$, then $\nu_1 \geq \mu_1 \geq n+1$, so only shapes of type b) appear when μ is of type b).

So far we have not used the restriction ii) in the sizes of the λ^i. We need show that if μ is of type c), then all summands in $L_\lambda \otimes L_\mu$, $\lambda = \lambda^{j+1}$, are either of type b) or c). Assume ν is n-bounded, and appears with positive multiplicity in $L_\lambda \otimes L_\mu$. As in the proof of lemma 12.1, the positions of the n 1's,...,n $q = q(\lambda)$'s in ν/μ in the implementation of the Littlewood - Richardson rule are uniquely determined, so we need only discuss how to place the remaining $r = r(\lambda)$ $q+1$'s, keeping in mind that no two can go in any one column. Since by (12.1b)

$$r(\lambda) \geq j\, n - \sum_{i=1}^{j} r(\lambda^i) ,$$

comparison with (12.2) shows that one of the $q+1$'s must appear at the beginning of a new row of ν. Thus there are at most $j+1$ rows with less than n boxes and

$$(j+1)n - \sum_{i=1}^{j+1} r(\lambda^i)$$

open boxes. Finally, since in the last column of μ has at least two open boxes, and ν has one more row than (n^q,μ), then ν must still have at least two rows with less than n boxes. Thus ν is of type c).

Conversely, a closer inspection of the proof will show that if neither i) nor ii) is satisfied, then tensoring L_λ, $\lambda = \lambda^{j+1}$ with μ's of type c) will produce n-fat shapes with positive multiplicity, so the theorem is proven.

We now turn to an explicit formula for the map of lemma 12.1. Given n-fat shapes λ,μ,ν with $|\lambda| + |\mu| = |\nu|$, define the linear map

$$\psi : \otimes_\lambda \Theta_* \otimes \otimes_\mu \Theta_* \to \otimes_\nu \Theta_*$$

as follows: For

$$\xi = \underset{\sim}{x} \otimes y = x_1 \otimes \ldots \otimes x_q \otimes y \in \otimes_\lambda \Theta_* \; , \tag{12.3}$$

$$\eta = \underset{\sim}{x}' \otimes y' = x_1' \otimes \ldots \otimes x_{q'}' \otimes y' \in \otimes_\mu \Theta_* ,$$

where $x_i, x_i' \in \Theta_n$, $y = y_1 \odot \ldots \odot y_r \in \Theta_r$, $y' = y_1' \odot \ldots \odot y_{r'}' \in \Theta_{r'}$, $|\lambda| = qn + r$, $|\mu| = q'n + r'$, we define

$$\psi(\xi \otimes \eta) = \begin{cases} \underset{\sim}{x} \otimes \underset{\sim}{x}' \otimes (y \odot y') \; , & r + r' < n \; , \\ (-1)^r \Sigma \, \underset{\sim}{x} \otimes (y \odot y_I') \otimes \underset{\sim}{x}' \otimes y_J' \; , & r + r' \geq n \end{cases} \tag{12.4}$$

where the sum is over all $I \cup J = \{1,\ldots,r'\}$, $I \cap J = \emptyset$, $\# I = n - r$. From the relations of theorem 3.1 it is easy to verify that these induce nontrivial functorial maps on Schur spaces.

<u>Lemma 12.3</u> The map ψ induces a functorial map (also denoted ψ)

$$\psi : L_\lambda \otimes L_\mu \to L_\nu$$

for n-fat Schur spaces. We thus define a product

$$\xi * \eta = \psi(\xi \otimes \eta) \tag{12.5}$$

for $\xi \in L_\lambda$, $\eta \in L_\mu$.

This product agrees, up to multiple, with the product introduced in [23] for the hyperforms defined there. In particular, it shares the basic properties of (anti-) commutativity and associativity under appropriate restrictions:

Lemma 12.4 Let $|\lambda| = qn + r$, $|\mu| = q'n + r'$. Then

$$\xi * \eta = (-1)^{s}\, \eta * \xi \qquad \xi \in L_{\lambda},\ \eta \in L_{\mu}, \tag{12.6}$$

where

$$s = \begin{cases} nqq' & r + r' < n, \\ n(q+1)(q'+1) & r + r' \geq n. \end{cases}$$

Proof

By functoriality it suffices to check (12.6) for some particular ξ, η which we take to be of the form (12.3) with $y = v^{r}$, $y' = w^{r'}$ for $v, w \in V$. We concentrate on the case $r + r' \geq n$, leaving the other to the reader. Now

$$\xi * \eta = (-1)^{r}\binom{r'}{n-r}\, \underset{\sim}{x} \otimes (v^{r} \odot w^{n-r}) \otimes \underset{\sim}{x}' \otimes w^{n-r-r'} \tag{12.7}$$

by the definition (12.4). From the Young symmetry (3.1) we have

$$(j+1)(x^{i} \odot y^{j}) \otimes (x^{k} \odot y^{\ell}) + i(x^{i-1} \odot y^{j+1}) \otimes (x^{k+1} \odot y^{\ell-1}) = 0 \tag{12.8}$$

for any $i + j \geq k + \ell$. From this it is easy to prove that

$$\xi * \eta = (-1)^{r'-n}\binom{r}{n-r'}\underset{\sim}{x} \otimes (v^{n-r'} \odot w^{r'}) \otimes \underset{\sim}{x}' \otimes v^{r+r'-n},$$

from which (12.6) immediately follows.

<u>Lemma 12.5</u> Let $|\lambda| = qn + r$, $|\mu| = q'n + r'$, $|\nu| = q''n + r''$, $|\rho| = |\lambda| + |\mu| + |\nu|$. Then the $*$ product for L_λ, L_μ, L_ν is associative:

$$\xi * (\eta * \zeta) = (\xi * \eta) * \zeta \tag{12.9}$$

provided either

i) $r + r' + r'' < n$

or ii) $r + r' + r'' \geq 2n$.

<u>Proof</u>

Again, we restrict attention to case ii). Note first that by theorem 12.2 there is precisely one functorial map $\psi : L_\lambda \otimes L_\mu \otimes L_\nu \to L_\rho$, up to multiple, so it suffices to check (12.8) for the specific choices

$$\xi = \underset{\sim}{x} \otimes u^r , \; \eta = \underset{\sim}{y} \otimes v^{r'} , \; \zeta = \underset{\sim}{z} \otimes w^{r''} .$$

Then by (12.7)

$$(\xi * \eta) * \zeta = (-1)^{r'-n} \binom{r'}{n-r} \binom{r''}{2n-r-r'} \underset{\sim}{x} \otimes (u^r v^{n-r}) \otimes \underset{\sim}{y} \otimes (v^{r+r'-n} w^{2n-r-r'}) \otimes \underset{\sim}{z} \otimes w^{r+r'+r''-2n}$$

whereas

$$\xi * (\eta * \zeta) = (-1)^{r+r'} \binom{r''}{n-r'} \binom{r'+r''-n}{n-r} \underset{\sim}{x} \otimes (u^r w^{n-r}) \otimes \underset{\sim}{y} \otimes (v^{r'} w^{n-r'}) \otimes \underset{\sim}{z} \otimes w^{r+r'+r''-2n}$$

Some elementary manipulations using (12.8) proves that these two expressions are indeed equal in L_ρ .

Example 12.7

Let $n=3$. Consider the shapes $\lambda=\mu=\nu=(2)$. If $\xi=u^2=u\odot u$, $\eta = v^2$, then

$$\xi * \eta = 2(u^2v) \otimes v$$

in $L_{(3,1)}$. On the other hand

$$\eta * \xi = 2(v^2u) \otimes u ,$$

but by (3.1)

$$2(u^2v) \otimes v + 2(v^2u) \otimes u = 0 ,$$

so

$$\xi * \eta = -\eta * \xi ,$$

which agrees with (12.6) .

Furthermore, in $L_{(3,3)}$, if $\zeta = w^2$

$$(\xi * \eta) * \zeta = -2(u^2v) \otimes v^2w .$$

However, by (3.1) again

$$\begin{aligned}(u^2v) \otimes (v\ w^2) &= -(uv^2) \otimes (u\ w^2)\\ &= +(w\ v^2) \otimes (u^2w)\\ &= -(u^2w) \otimes (w\ v^2),\end{aligned}$$

so the product is associative. Note that in this case $\xi * \eta * \zeta$

is alternating in $\xi,\eta,\zeta \in \Theta_2$, hence defines a map

$$\Lambda_3\Theta_2 \to L_{(3,3)} .$$

Next we compute the differential of our product, but, as the general computation is rather complicated, we do only one special case here.

<u>Lemma 12.8</u> Suppose $|\mu| = |\lambda| + k$, $r = r(\lambda)$, $k \leq n$. Define

$$\psi(x \otimes \xi) = x * \xi$$

for $x \in \Theta_k$, $\xi \in L_\lambda$, so $x * \xi \in L_\mu$. Then, if $|\nu| = |\mu| + 1$, the differential $\psi_{,1} : \Theta_{k+1} \otimes L_\lambda \to L_\nu$ is given by

$$\psi_{,1}(y \otimes \xi) = \begin{cases} \frac{1}{k+1}\, y * \xi , & k + r < n , \\ \frac{k+r-n+1}{k+1}\, y * \xi & k + r \geq n , \ k < n , \\ 0 & k = n ; \end{cases} \qquad (12.10)$$

for $y \in \Theta_{k+1}$, $\xi \in L_\lambda$.

<u>Proof</u>

The case $k = n$ is trivial. Here we only do the case $k + r \geq n$, $k < n$, leaving the other, easier case to the reader. Recall that $\psi_{,1}$ is the composition of maps

$$\Theta_{k+1} \otimes L_\lambda \xrightarrow{p \otimes 1} V \otimes \Theta_k \otimes L_\lambda \xrightarrow{1 \otimes \psi} V \otimes L_\mu \xrightarrow{*} L_\nu ,$$

where p is the partial polarization (8.3). By lemma 12.1, $\psi_{,1}(y \otimes \xi)$ must be a multiple of $y * \xi$, so we need only evaluate the differential on the specific elements $y = v^{k+1}$, $\xi = \underset{\sim}{x} \otimes w^r$ for $v,w \in V$. By (12.7),

$$(\mathbb{1}\otimes\psi)\circ(p\otimes\mathbb{1})(y\otimes\xi) = v\otimes\psi(v^k\otimes(\underset{\sim}{x}\otimes w^r))$$

$$= (-1)^k\binom{r}{n-k} v\otimes(v^k\odot w^{n-k})\otimes\underset{\sim}{x}\otimes w^{k+r-n},$$

hence, using (12.8),

$$\psi_{,1}(y\otimes\xi) = (-1)^k\binom{r}{n-k}(v^k\odot w^{n-k})\otimes\underset{\sim}{x}\otimes(v\odot w^{k+r-n})$$

$$= (-1)^{k+1}\frac{n-k}{k+1}\binom{r}{n-k}(v^{k+1}\odot w^{n-k-1})\otimes\underset{\sim}{x}\otimes w^{k+r+1-n}.$$

Comparison with $y*\xi$ completes the proof.

In the more general case $\psi(\xi\otimes\eta) = \xi*\eta$ for $\xi\in L_\lambda$, $\eta\in L_\mu$, we have $\psi_{,1}$ and $\psi_{,2}$ both multiples of the product on the appropriate n-fat Schur spaces, but we leave it to the reader to ascertain what these multiples are!

Let α be an n-bounded diagram, and $\psi_\alpha: \otimes_\alpha\Theta_* \to L_\mu$ a functorial map, with $|\mu| = |\alpha|$, μ n-fat. According to theorem 12.2, ψ_α is uniquely determined up to constant multiple provided either a) $|\alpha| \leq n$ or b) $|\alpha| \geq (k-1)n$, where $k = m(\alpha)$ is the number of rows in α. Moreover, since the $*$ products are associative, we can prescribe

$$\psi_\alpha(x^1\otimes\ldots\otimes x^k) = x^1*\ldots*x^k, \quad x^j\in\Theta_{\alpha_j}, \tag{12.11}$$

where the $*$ products are taken in any convenient order provided either a) or b) holds. We now compute differentials of ψ_α:

Theorem 12.9 Let α be n-bounded, and ψ_α given by (12.11). Then for $1 \leq j \leq k = m(\alpha)$ the j-th algebraic differential

$$\psi_{\alpha,j}: \otimes_{\alpha+j} \Theta_* \to L_\nu \,,$$

$|\nu| = |\mu| + 1$, ν n-fat, is given by

$$\psi_{\alpha,j} = \begin{cases} \psi_{\alpha+j} \,, & |\alpha| + 1 \leq n \,, \\ \frac{r+1}{\alpha_j+1} \psi_{\alpha+j} \,, & |\alpha| \geq (k-1)n \,,\ \alpha_j < n \,, \\ 0 & \alpha_j = n \,. \end{cases}$$

where $r = r(\mu) = |\alpha| - (k-1)n$ in the second case.

The proof is immediate from lemma 12.8 and the commutation formula (12.6). The details are left to the reader.

Corollary 12.10 Let $\lambda = n^{k-1}$ $\mu = n^k$. Let $n = sk + t$, $s \geq 0$, $0 \leq t < k$, and let $\alpha = (n-s-1)^t (n-s)^{k-t}$, so $|\alpha| = |\lambda|$, $\beta = (s+1)^t s^{k-t}$. Then the n-fold iterated differential $\psi_{\alpha,\beta} : \otimes_\mu \Theta_* \to L_\mu$ is given by

$$\psi_{\alpha,\beta} = \frac{((s-1)!)^t (s!)^{k-t}}{(n!)^{k-1}} \psi_\mu \qquad (12.12)$$

(Note by lemma 11.3 it doesn't matter what order the differentials are taken.)

Example 12.11 Consider the case $n=2$, $k=2$, so $\mu = (2,2)$, $\alpha = (1,1)$

$$\psi_\alpha: V \otimes V \to L_{(2)}$$

is given by

$$\phi_\alpha(v \otimes w) = -v \odot w\ , \quad v,w \in V\ ,$$

cf. (12.4). Then (12.12) shows that if $\beta = (1,1)$,

$$\psi_{\alpha,\beta}\colon\ \odot_2 \otimes \odot_2 \to L_{(2,2)}$$

is given by

$$\psi_{\alpha,\beta} = \frac{1}{2}\psi_\mu\ ,$$

where

$$\psi_\mu((v \odot v') \otimes (w \odot w')) = (v \odot v') \otimes (w \odot w')$$

considered as an element of $L_{(2,2)}$, as the reader can easily verify.

13. The Differential Hypercomplex

In this section the theory of higher order differential forms (hyperforms) will be developed over Euclidean space. Let $M = \mathbb{R}^p$ with coordinates $x = (x^1,\ldots,x^p)$. The cotangent bundle T^*M is spanned by the basis one forms $dx^1,\ldots,dx^p$, so for each $x \in M$, $T^*M|_x$ is a vector space of dimension p .

Define the hyperform bundle Ξ_λ so

$$\Xi_\lambda|_x = L_\lambda(T^*M|_x)$$

for each $x \in M$. A <u>differential λ-hyperform</u> ω is a section of Ξ_λ . Thus, using the basis theorem 3.2, a λ-hyperform is of the form

$$\omega(x) = \Sigma f_T(x)dx_T , \tag{13.1}$$

the sum being over all standard tableaux T of shape λ with values in $\{1,\ldots,p\}$, and dx_T is the corresponding basis element of Ξ_λ . The coefficient functions $f_T(x)$ will be in some function space $\mathfrak{J}$, e.g. $C^\infty(M)$, and we write $\Gamma(\Xi_\lambda,\mathfrak{J}) \sim \mathfrak{J} \otimes \Xi_\lambda$ for the space of all such $\mathfrak{J}$-valued sections. Most of the time $\mathfrak{J} = C^\infty(M)$, and we denote $\Gamma(\Xi_\lambda,C^\infty)$, the space of smooth sections of Ξ_λ , by $\Gamma\,\Xi_\lambda$. Note that all our algebraic operations on Schur spaces - Pieri products, polarizations, etc. - carry over point wise to the hyperform bundles Ξ_λ . In addition, the notion of exterior derivative is of great interest.

<u>Definition 13.1</u> Let ω be a λ-hyperform given by (13.1). Then given any $\mu \supset \lambda$, $|\mu| = |\lambda| + 1$, the λ,μ <u>exterior derivative</u> $d\omega = d_\lambda^\mu \omega$ is the μ-hyperform.

$$d\omega = \sum_T df_T * dx_T , \tag{13.2}$$

where $*$ is the Pieri product from $T^* \otimes \Xi_\lambda$ to Ξ_μ, and $df_T = \Sigma\, \partial f_T/\partial x^j . dx^j$ is the ordinary differential of the function f_T.

<u>Example 13.2</u> If $\lambda = 1^k$, $\mu = 1^{k+1}$, so $\Xi_\lambda \simeq \Lambda_k T^*M$, $\Xi_\mu \simeq \Lambda_{k+1} T^*M$, then the above exterior derivative agrees with the ordinary exterior derivative of differential forms, since the Pieri product agrees with the wedge product in this case.

<u>Theorem 13.3</u> The exterior derivatives

$$d_\lambda^\mu : \Gamma \Xi_\lambda \to \Gamma \Xi_\mu$$

form a hypercomplex called the (smooth) <u>differential hypercomplex</u> of Euclidean space.

<u>Proof</u>

To prove commutativity, suppose $\mu = \lambda + j$, $\mu' = \lambda + k$, $\nu = \mu + k = \mu' + j$. For $\omega = f(x)dx_T \in \Xi_\lambda$,

$$d_\mu^\nu(d_\lambda^\mu \omega) = d_\mu^\nu(\sum_\ell \partial_\ell f\ dx^\ell *_j dx_T)$$

$$= \sum_{\ell,m} \partial_m \partial_\ell f\ dx^m *_k (dx^\ell *_j dx_T) ,$$

where $\partial_\ell = \partial/\partial x^\ell$. Equality of mixed partials and corollary 8.5 complete the proof of commutativity:

$$d_\mu^\nu(d_\lambda^\mu \omega) = d_{\mu'}^\nu(d_\lambda^{\mu'} \omega).$$

The proof of closure follows similarly, using lemma 8.6.

We defer the discussion of exactness to section 15. The fact that the exterior derivatives mutually commute allows us to unambiguously define iterated derivatives

$$d_\lambda^\nu : \Gamma\Xi_\lambda \to \Gamma\Xi_\nu$$

whenever $\lambda \subset \nu$. These vanish if ν/λ has a column with two or more boxes in it.

Example 13.4

Let $\lambda = (k)$ be a single row, so $\Xi_\lambda \simeq \odot_k T^*M$. Then the k-fold derivative

$$d^k f = d_o^\lambda f ,$$

where $f(x)$ is a smooth function, is just the k-th order homogeneous Taylor polynomial of f :

$$d^k f = \Sigma \binom{k}{J} \partial_J f \, dx^J ,$$

summed over all $J = (j_1, \ldots j_p)$ with $|J| = k$, and

$$\binom{k}{J} = \frac{k!}{J!} , \quad \partial_J = \partial_1^{j_1} \ldots \partial_p^{j_p} , \; dx^J = (dx^1)^{j_1} \odot \ldots \odot (dx^p)^{j_p} .$$

Note that since $d^k f$ involves only the k-th order derivatives of f, it cannot be invariant under changes of coordinates on M. This problem and its resolution, using the theory of prolongation of the general linear group $GL^{(k)}(p)$, [16], will be dealt with in the sequel to this paper.

Example 13.5

Let $M = \mathbb{R}^2$ with coordinates x,y . Consider the "smallest" bundles over M corresponding to shapes

$$\lambda = (1) \ , \ \mu = (1,1) \ , \ \mu' = (2) \ , \ \nu = (2,1) \ .$$

The Young symmetry properties and the formulae for the Pieri product (see example 5.3) give the following formulae for the various differentials:

a) d^{μ}_{λ}: $f\ dx + g\ dy \to (f_y - g_x)\, dx \otimes dy$

b) $d^{\mu'}_{\lambda}$: $f\ dx + g\ dy \to f_x dx^2 + (f_y + g_x)\, dxdy + g_y dy^2$

c) d^{ν}_{μ} : $\alpha\ dx \otimes dy \to \frac{1}{2}\alpha_x dx^2 \otimes dy + \alpha_y dxdy \otimes dy$

d) $d^{\nu}_{\mu'}$: $a\ dx^2 + b\ dxdy + c\ dy^2 \to (a_y - \frac{1}{2}\ b_x)\, dx^2 \otimes dy + (b_y - 2c_x)\, dxdy \otimes dx$.

It is easy to check commutativity:

$$d^{\nu}_{\lambda} = d^{\nu}_{\mu} d^{\mu}_{\lambda} = d^{\nu}_{\mu'} d^{\mu'}_{\lambda} : f\ dx + g\ dy \to \frac{1}{2}\ (f_{xy} - g_{xx})\, dx^2 \otimes dy + (f_{yy} - g_{xy})\, dxdy \otimes dy \ .$$

Furthermore, if $\varphi(x,y)$ is any smooth function, then

$$d^{\nu}_{o}\ \varphi = 0$$

since $d\varphi = d^{\lambda}_{o}\ \varphi = \varphi_x dx + \varphi_y dy$, proving closure at ν . Conversely, if $d^{\nu}_{\lambda}(f\ dx + g\ dy) = 0$, then $f_y = g_x + c$ for some constant c , hence

$$f\ dx + g\ dy = d\varphi + c\ xdy \ .$$

The reason "exactness" fails here is that $0,\lambda,\nu$ are <u>not</u> min-max related. In general, adding boxes onto the first row always causes probelms with "exactness" in the differential hypercomplex. (This already crops up in the deRham complex when $df = 0$ implies f is constant, not necessarily zero.)

14. Leibnitz' Rule

As remarked in the previous section, all the algebraic operations on Schur spaces carry over to corresponding operations on hyperforms. Thus, if $\underset{\sim}{\lambda} = (\lambda^1,\ldots,\lambda^k)$ are shapes and

$$\psi: \otimes_{\underset{\sim}{\lambda}} L_* \to L_\mu$$

is functorial, then there is a functorial map

$$\psi: \otimes_{\underset{\sim}{\lambda}} \Xi_* = \Xi_{\lambda^1} \otimes \ldots \otimes \Xi_{\lambda^k} \to \Xi_\mu$$

obtained by applying ψ pointwise over M. Each such map defines a "product" between hyperforms. In this section we discuss the "Leibnitz rule" for the differential of such a product.

First we introduce the notion of a tensor differential of a hyperform. Given a shape λ, and a λ-hyperform ω as in (13.1), define

$$d_\otimes \omega(x) = \Sigma\, dx_T \otimes d\, f_T , \tag{14.1}$$

which as a section of the bundle $\Xi_\lambda \otimes T^*$. By the Pieri formula,

$$\Xi_\lambda \otimes T^* \simeq \oplus\ \Xi_\mu ,$$

the sum being over all $\mu \supset \lambda$ with $|\mu/\lambda| = 1$; hence, summing over the same shapes,

$$d_\otimes \omega = \Sigma\, d_\lambda^\mu\, \omega . \tag{14.2}$$

Note in particular, if $\omega = d^k f \in \Gamma\, \Theta_k$ for some smooth function $f(x)$, then

$$d_\otimes \omega = d^{k+1} f(x) \tag{14.3}$$

since the only other summand in (14.2) vanishes by closure of the differential hypercomplex.

We now state our version of the Leibnitz rule for hyperforms.

Theorem 14.1 Let $\psi : \otimes_{\underset{\sim}{\lambda}} L_* \to L_\mu$ be functorial. Let $\nu \supset \mu$, $|\nu/\mu| = 1$. Let $\omega_j \in \Gamma\, \Xi_{\lambda^j}$ be hyperforms. Then

$$d_\mu^\nu[\psi(\omega_1 \otimes \ldots \otimes \omega_k)] = \sum_{j=1}^{k} \psi_{\otimes j}(\omega_1 \otimes \ldots \otimes d_\otimes \omega_j \otimes \ldots \otimes \omega_k) , \quad (14.4)$$

where $\psi_{\otimes j}$ is the j-th tensor differential of ψ.

The proof is almost immediate from the definition of the tensor differential $\psi_{\otimes j}$ in section 11, and is left to the reader. In particular, if

$$\psi : \otimes_\alpha \Theta_* \to L_\mu$$

for $\alpha = (\alpha_1, \ldots, \alpha_k)$, is tensorial, then

$$d[\psi(d^{\alpha_1} f^1 \otimes \ldots \otimes d^{\alpha_k} f^k)] = \sum_{j=1}^{k} \psi_{,j}(d^{\alpha_1} f^1 \otimes \ldots \otimes d^{\alpha_j+1} f^j \otimes \ldots \otimes d^{\alpha_k} f^k) , \quad (14.5)$$

for smooth functions $f^1, \ldots, f^k$. This follows directly from the definition 11.1 of $\psi_{,j}$ and (14.3).

15. Hyperjacobian Identities

We now again restrict attention to the differential subcomplexes given by the fat shapes. As in section 11, all shapes in this section will be n-fat unless explicitly noted, where n is some fixed positive integer.

Given a C^∞ function $u(x)$, $d^j u(x) \in \odot_j T^*$ is the j-th differential. Let $\mu = n^k$ be rectangular. The hyperform

$$\omega = d^n u^1 * \ldots * d^n u^k , \tag{15.1}$$

where $u^1, \ldots, u^k \in C^\infty$, $*$ denotes the product (12.4), which takes the particularly simple form

$$x * \xi = \xi \otimes x \in \Xi_{n^{j+1}}$$

for $x \in \odot_n$, $\xi \in \Xi_{n^j}$, is of great interest.

Definition 15.1 Let T be a standard tableau of shape λ. The n-th order hyperjacobian

$$\frac{\partial^n (u^1, \ldots, u^k)}{\partial x_T} = J^k_T(u) \tag{15.2}$$

is defined as $\hat{c}(T)!/k!$ times the coefficient of dx_T in (15.1). Here $\hat{c}(T)! = \hat{c}_1! \ldots \hat{c}_p!$, where the tableau T has $\hat{c}_1$ identical columns, followed by $\hat{c}_2$ identical columns etc. For example, if

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 3 & 3 & 3 \\ \hline \end{array} , \qquad \lambda = 5^2 ,$$

then

$$\hat{c}(T)! = 2!\ 3!\ 1! = 12 .$$

Example 15.2 Let $p = 2$, $n = 2$, so

$$d^2u = u_{xx}dx^2 + 2u_{xy}dx \odot dy + u_{yy}dy^2 .$$

Then for $\lambda = (2,2)$,

$$d^2u * d^2v = (u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx})dx^2 \otimes dy^2 , \qquad (15.3)$$

since

$$dx^2 \otimes dy^2 + 2(dxdy) \otimes (dxdy) = 0 . \qquad (15.4)$$

Thus

$$\frac{\partial^2(u,v)}{\partial(x,y)^2} = u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx} ,$$

which is the second order hyperjacobian introduced in [23].

In general, if T has entries m^i_j , then we will write

$$\partial x_T = \partial(x_{m^1_1},\ldots,x_{m^k_1})\partial(x_{m^1_2},\ldots,x_{m^k_2})\cdots$$

in (15.2). The fact that the above definition of hyperjacobians agrees with the constructive definition given in [23] depends on the row expansion formulae for hyperjacobians, and proceeds along the same lines as the proof of Lemma 4.14 of that paper. We will not repeat the arguments here.

The basic fact, proved in [23], is that n-hyperjacobians can be written as n-th order divergences, and that essentially these are all the possible n-th order divergences depending exclusively on n-th order partial derivatives of the relevant functions. The primitive version of differential hyperforms developed in [23] was used to construct these hyperform identities. As these are equivalent to the present Schur functor techniques for fat forms, the identities can be equivalently constructed using hyperform theory. The first step is to express (15.1) as the n-th order differential of some other hyperform.

Theorem 15.3. Let $\lambda = n^{k-1}$, $\mu = n^k$, and suppose $n = sk + t$, $s \geq 0$, $0 \leq t < k$. Let d^n denote the differential $\Xi_\lambda \to \Xi_\mu$. Set $\zeta = d^{n-s-1}u^1 * \ldots * d^{n-s-1}u^t * d^{n-s}u^{t+1} * \ldots * d^{n-s}u^k$. Then

$$d^n\zeta = \frac{((s-1)!)^t(s!)^{k-t}}{(n!)^{k-1}}\,\omega\,, \tag{15.5}$$

where $\omega = d^n u^1 * \ldots * d^n u^k$.

Proof

This follows immediately from (14.3), theorem 14.1 and corollary 12.10.

The coefficient of dx_T, for T a standard tableau of shape μ, in (15.5) will yield the hyperjacobian identity for $J_T^k(u)$. We illustrate this with a couple of examples, see [23; section 4] for more examples.

Example 15.4 Let $p = n = k = 2$, and consider

$$\omega = d^2u * d^2v$$

as given in (15.3). According to (15.5),

$$\omega = \frac{1}{2}\; d^2(du * dv).$$

Now

$$du * dv = - du \odot dv$$
$$= -(u_x v_x dx^2 + (u_x v_y + u_y v_x) dxdy + u_y v_y dy^2)$$

so, in $L_{(2,2)}$, using (15.4),

$$\omega = - \frac{1}{2} [d^2(u_x v_x) * dx^2 + d^2(u_x v_y + u_y v_x) * dxdy + d^2(u_y v_y) * dy^2]$$
$$= [-D_y^2(u_x v_x) + D_x D_y(u_x v_y + u_y v_x) - D_x^2(u_y v_y)] dx^2 \otimes dy^2 .$$

Therefore,

$$\frac{\partial^2(u,v)}{\partial(x,y)^2} = -D_x^2(u_y v_y) + D_x D_y(u_x v_y + u_y v_x) - D_y^2(u_x v_x) ,$$

which, in the case $u = v$, reproduces (1.1) .

Example 15.5 Let $p = k = n = 3$. We construct the identity for the third order hyperjacobian

$$\frac{\partial^3(u,v,w)}{\partial(x,y,z)^3} = u_{xxx} v_{yyy} w_{zzz} - 3u_{xxx} v_{yyz} w_{yzz} + 6u_{xxy} v_{xyz} w_{yzz}$$
$$+ 3u_{xxy} v_{yyz} w_{xzz} \pm \cdots$$

where the omitted terms are obtained by permuting x,y,z and u,v,w in all possible ways in the displayed terms, keeping track of the signs of the permutation. Thus, for instance, $u_{xyz} v_{xxz} w_{yyz}$ appears with coefficient -6 . This differential polynomial is just the coefficient of $dx^3 \otimes dy^3 \otimes dz^3$ in the hyperform

$$\omega = d^3u * d^3v * d^3w ,$$

cf. (15.2). According to (15.5),

$$\omega = \frac{1}{36} d^3(d^2u * d^2v * d^2w) = \frac{1}{36} d^3\zeta \ .$$

Now according to example 12.7, ζ actually depends on $d^2u \wedge d^2v \wedge d^2\omega$, so its coefficients are determinants in the second order derivatives of u,v,w. Thus, for instance, the coefficient of $dx^2dy \otimes dydz^2$ is

$$\frac{\partial(u,v,w)}{\partial(xx,yy,zz)} + 4\frac{\partial(u,v,w)}{\partial(xy,xz,yz)} ,$$

where the first summand stands for

$$\det \begin{vmatrix} u_{xx} & u_{yy} & u_{zz} \\ v_{xx} & v_{yy} & v_{zz} \\ w_{xx} & w_{yy} & w_{zz} \end{vmatrix} .$$

It is now a straightforward task to construct the rather pretty identity

$$\frac{\partial^3(u,v,w)}{\partial(x,y,z)^3} = - D_z^3 \frac{\partial(u,v,w)}{\partial(xx,xy,yy)} + D_z^2D_y[2\frac{\partial(u,v,w)}{\partial(xx,xy,yz)} - \frac{\partial(u,v,w)}{\partial(xx,yy,xz)}] +$$

$$+ D_xD_yD_z[\frac{\partial(u,v,w)}{\partial(xx,yy,zz)} + 4\frac{\partial(u,v,w)}{\partial(xy,xz,yz)}] + \cdots ,$$

where the omitted terms are obtained by cyclically permuting x,y,z in the displayed terms. This is a good illustration of the power of our hyperform techniques for deriving otherwise unmanageable divergence identities.

16. Exactness and Integrability

Finally we turn to the question of exactness of the differential hypercomplex. The goal of this section is to prove

Theorem 16.1 The smooth differential hypercomplex $d^{\mu}_{\lambda}: \Gamma\,\Xi_{\lambda} \to \Gamma\,\Xi_{\mu}$ is exact.

Rather than try to prove this directly, the most straight forward method is to use the Fourier transform to change the differential hypercomplex into the algebraic Schur hypercomplex, and then use our previous exactness results. However, the implementation of this program requires the sophisticated results of Ehrenpreis and Malgrange on the Fourier transform of distributions and the solution of the "division problem" for systems of linear, constant coefficient partial differential equations. The proof will therefore be deferred until the end of this section, after we consider some applications of the result.

First we write out the statement of the theorem in greater detail: For every $\underset{\sim}{\lambda}, \mu, \underset{\sim}{\nu}$ min-max related, the hyperform equation

$$\sum_{\lambda \in \underset{\sim}{\lambda}} d^{\mu}_{\lambda}\, \xi_{\lambda} = \eta_{\mu} \,, \tag{16.1}$$

where $\eta_{\mu} \in \Gamma\,\Xi_{\mu}$, has a smooth solution $\xi_{\lambda} \in \Gamma\,\Xi_{\lambda}$, $\lambda \in \underset{\sim}{\lambda}$, if and only if

$$d^{\nu}_{\mu}\, \eta_{\mu} = 0 \quad \text{for all} \quad \nu \in \underset{\sim}{\nu} \,. \tag{16.2}$$

These can be viewed in two ways; either (16.1) forms the general solution for the homogeneous system of partial differential equations (16.2), or (16.2) are the integrability conditions for the solution of the non-homogeneous system of partial differential equations (16.1). In the second interpretation, (16.1) form a higher order type of Pfaffian system, [12], [25], (a "hyper-Pfaffian" system!). I have tried without success to find any reference to such higher order Pfaff systems in the literature.

Depending on the relative weights of $\underset{\sim}{\lambda}, \mu, \underset{\sim}{\nu}$, there are a number of possibilities on the relative orders of (15.1,2). In the simplest case, $\underset{\sim}{\lambda} = \{\lambda\}$, $\underset{\sim}{\nu} = \{\nu\}$ each consist of a single shape. If $|\mu/\lambda| = k$, $|\nu/\mu| = \ell$, then (16.1) is a k-th order system of differential equations, with integrability conditions (16.2) involving ℓ-th order derivatives of the coefficients of η_μ. It is possible that $\underset{\sim}{\lambda} = \{\lambda\}$, but $\underset{\sim}{\nu} = \{\nu_1, \ldots, \nu_m\}$, so that the integrability conditions for the $k = |\mu/\lambda|$ - th order system (16.1) involve derivatives of orders $\ell_j = |\nu_j/\mu|$, where the ℓ_j may very well be different. Similarly $\underset{\sim}{\nu}$ might consist of a single shape, but $\underset{\sim}{\lambda}$ has several shapes of different sizes, so a system involving various derivatives has integrability conditions involving only ℓ'th order derivatives. We illustrate these possibilities with a few "easy" examples.

Example 16.2

Consider the part of the differential hypercomplex given in figure 3, where the d^μ_λ's have been relabelled as d_i, $i = 0, \ldots, 10$ for simplicity. Suppose $M = \mathbb{R}^3$, with coordinates $(x,y,z) = (x^1, x^2, x^3)$.

Here we explicitly display the systems (16.1) together with their integrability criteria (16.2) for various choices of $\underset{\sim}{\lambda}, \mu, \underset{\sim}{\nu}$.

We denote general sections of the relevant hyperform bundles as follows

$$\xi = \xi^1 dx + \xi^2 dy + \xi^3 dz \in \Gamma\ \Xi_1 = \Gamma\ T^* ,$$

$$\eta = \eta^1 dx^2 + \eta^2 dxdy + \eta^3 dxdz + \eta^4 dy^2 + \eta^5 dydz + \eta^6 dz^2 \in \Gamma\ \Xi_2 = \Gamma\ \odot_2 T^*$$

$$\zeta = \zeta^1 dx^3 + \zeta^2 dx^2 dy + \zeta^3 dx^2 dz + \zeta^4 dxdy^2 + \zeta^5 dxdydz + \zeta^6 dxdz^2 + \zeta^7 dy^3 + \zeta^8 dy^2 dz + \zeta^9 dydz^2 + \zeta^{10} dz^3 \in \Gamma\ \Xi_3 = \Gamma\ \odot_3 T^*$$

$$\alpha = \alpha^1 dx \otimes dy + \alpha^2 dx \otimes dz + \alpha^3 dy \otimes dz \in \Gamma\ \Xi_{(1,1)} = \Gamma \wedge_2 T^* ,$$

$$\beta = \beta^1 dx^2 \otimes dy + \beta^2 dx^2 \otimes dz + \beta^3 dxdy \otimes dy + \beta^4 dxdy \otimes dz + \beta^5 dxdz \otimes dy + \beta^6 dxdz \otimes dz + \beta^7 dy^2 \otimes dz + \beta^8 dydz \otimes dz \in \Gamma \Xi_{(2,1)}$$

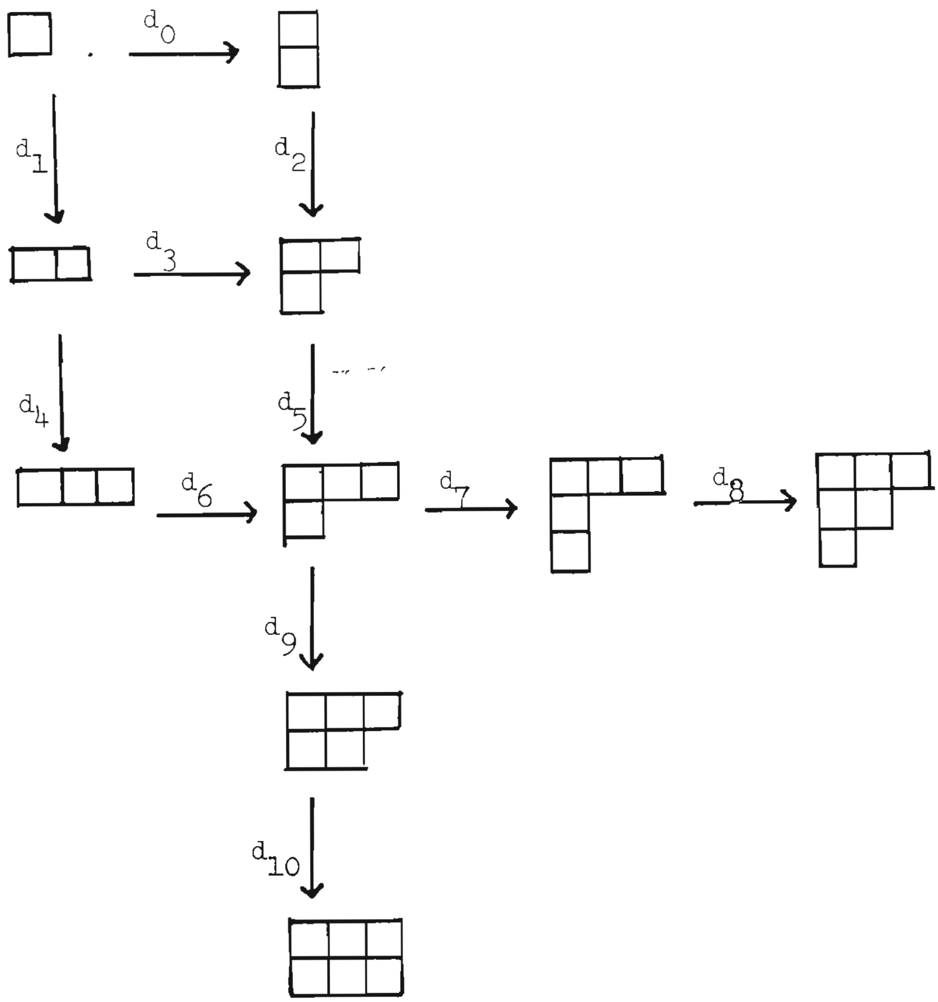


Figure 3: Diagram for example 16.2. (The Ξ have been suppressed.)

$$\gamma = \gamma^1 dx^3 \otimes dy + \gamma^2 dx^3 \otimes dz + \gamma^3 dx^2 dy \otimes dy + \gamma^4 dx^2 dy \otimes dz +$$
$$+ \gamma^5 dx^2 dz \otimes dy + \gamma^6 dx^2 dz \otimes dz + \gamma^7 dxdy^2 \otimes dy + \gamma^8 dxdy^2 \otimes dz +$$
$$+ \gamma^9 dxdydz \otimes dy + \gamma^{10} dxdydz \otimes dz + \gamma^{11} dxdz^2 \otimes dy + \gamma^{12} dxdz^2 \otimes dz +$$
$$+ \gamma^{13} dy^3 \otimes dz + \gamma^{14} dy^2 dz \otimes dz + \gamma^{15} dydz^2 \otimes dz \in \Gamma\,\Xi_{(3,1)},$$

where the coefficients ξ^1, etc. are all functions of x,y,z.

First consider the case $\lambda = (2,1)$, $\mu = (3,1)$, $\nu = (3,3)$. Then

$$\gamma = d_5\beta$$

is the first order system

$$\gamma^1 = \frac{2}{3}\beta^1_x \qquad \gamma^2 = \frac{2}{3}\beta^2_x, \quad \gamma^3 = \beta^1_y + \frac{1}{2}\beta^3_x,$$

$$\gamma^4 = \frac{1}{4}\beta^1_z + \frac{3}{4}\beta^2_y + \frac{5}{8}\beta^4_x - \frac{1}{8}\beta^5_x, \quad \gamma^5 = \frac{3}{4}\beta^1_z + \frac{1}{4}\beta^2_y - \frac{1}{8}\beta^4_x + \frac{5}{8}\beta^5_x,$$

$$\gamma^6 = \beta^2_z + \frac{1}{2}\beta^6_x, \quad \gamma^7 = \beta^3_y, \quad \gamma^8 = \frac{1}{4}\beta^3_z + \frac{3}{4}\beta^4_y + \frac{1}{2}\beta^7_x,$$

$$\gamma^9 = \frac{3}{4}\beta^3_z + \frac{1}{4}\beta^4_y + \beta^5_y - \frac{1}{2}\beta^7_x, \quad \gamma^{10} = \beta^4_z + \frac{1}{4}\beta^5_z + \frac{3}{4}\beta^6_y + \frac{1}{2}\beta^8_x,$$

$$\gamma^{11} = \frac{3}{4}\beta^4_z + \frac{1}{4}\beta^6_y - \frac{1}{2}\beta^8_x, \quad \gamma^{12} = \beta^6_z, \quad \gamma^{13} = \frac{2}{3}\beta^7_y,$$

$$\gamma^{14} = \beta^7_z + \frac{1}{2}\beta^8_y, \quad \gamma^{15} = \beta^8_z.$$

These have the second order integrability criteria

$$d_{10}d_9(\gamma) = 0,$$

or, in components,

$$\gamma^1_{yy} - \frac{2}{3}\gamma^3_{xy} + \frac{1}{3}\gamma^7_{yy} = 0 \qquad dx^3 \otimes dy^3$$

$$2\gamma^1_{yz} + \gamma^2_{yy} - \frac{2}{3}\gamma^3_{xz} - \frac{2}{3}\gamma^4_{xy} - \frac{2}{3}\gamma^5_{xy} + \frac{1}{3}\gamma^8_{xx} + \frac{1}{3}\gamma^9_{xx} = 0 , \qquad dx^3 \otimes dy^2 dz$$

$$\gamma^1_{zz} - 2\gamma^2_{yz} - \frac{2}{3}\gamma^4_{xz} - \frac{2}{3}\gamma^5_{xz} - \frac{2}{3}\gamma^6_{xy} - \frac{1}{3}\gamma^{10}_{xx} + \frac{1}{3}\gamma^{11}_{xx} = 0 , \qquad dx^3 \otimes dy dz^2$$

$$\gamma^2_{zz} - \frac{2}{3}\gamma^6_{xz} + \frac{1}{3}\gamma^{12}_{xx} = 0 , \qquad dx^3 \otimes dz^3$$

$$2\gamma^3_{yz} + \gamma^4_{yy} - \frac{1}{3}\gamma^5_{yy} - 2\gamma^7_{xz} - 2\gamma^8_{xy} + \frac{2}{3}\gamma^9_{xy} + 3\gamma^{13}_{xx} = 0 \qquad dx^2 dy \otimes dy^2 dz$$

$$\gamma^3_{zz} + 2\gamma^4_{yz} - 2\gamma^5_{yz} - \gamma^6_{yy} - 2\gamma^8_{xz} + 2\gamma^{11}_{xy} + \gamma^{14}_{xx} = 0 , \qquad dx^2 dy \otimes dy dz^2$$

$$\gamma^4_{zz} - \frac{1}{3}\gamma^5_{zz} - \frac{2}{3}\gamma^6_{yz} - \frac{2}{3}\gamma^{10}_{xz} + \frac{2}{3}\gamma^{11}_{xz} + \frac{2}{3}\gamma^{12}_{xy} + \frac{1}{3}\gamma^{15}_{xx} = 0 \qquad dx^2 dy \otimes dz^3$$

$$\gamma^7_{zz} + 2\gamma^8_{yz} - 2\gamma^9_{yz} - \gamma^{10}_{yy} + 3\gamma^{11}_{yy} - 6\gamma^{13}_{xz} + 2\gamma^{14}_{xy} = 0 , \qquad dx dy^2 \otimes dy dz^2$$

$$\gamma^8_{zz} - \frac{1}{3}\gamma^9_{zz} - \frac{2}{3}\gamma^{10}_{yz} + \frac{2}{3}\gamma^{11}_{yz} + \frac{1}{3}\gamma^{12}_{yy} - \frac{2}{3}\gamma^{14}_{xz} + \frac{1}{3}\gamma^{15}_{yy} + 0 , \qquad dx dy^2 \otimes dz^3$$

$$\gamma^{13}_{zz} - \frac{2}{3}\gamma^{14}_{yz} + \frac{1}{3}\gamma^{15}_{yy} = 0 . \qquad dy^3 \otimes dz^3$$

where the second column indicates of which basis element of $\Xi_{(3,3)}$ the corresponding equation is a coefficient.

An example of a second order system with second order integrability conditions is given by $\lambda = (1)$, $\mu = (3)$, $\nu = (3,2)$, which, for simplicity, we illustrate on $\mathbb{R}^2$, so all dependence on z and dz is omitted. Then

$$\zeta = d_4 d_1(\xi)$$

reads

$$\zeta^1 = \xi^1_{xx} ,$$
$$\zeta^2 = 2\xi^1_{xy} + \xi^2_{xx} ,$$
$$\zeta^4 = \xi^1_{yy} + 2\xi^2_{xy} , \qquad (16.4)$$
$$\zeta^7 = \xi^2_{yy} .$$

The integrability criteria are

$$d_9 d_6(\zeta) = 0 ,$$

or,

$$\zeta^1_{yy} - \frac{2}{3}\zeta^2_{xy} + \frac{1}{3}\zeta^4_{xx} = 0 , \qquad dx^3 \otimes dy^2$$
$$\zeta^2_{yy} - 2\zeta^4_{xy} + 3\zeta^7_{xx} = 0 , \qquad dx^2 dy \otimes dy^2 \qquad (16.5)$$

In other words, exactness of the hypercomplex says that (16.4) have a solution ξ^i if and only if (16.5) are satisfied. The three-dimensional case is similar, but more complicated.

For an example of a second order system with both first and second order integrability conditions, consider

$$\lambda = (2) \quad , \quad \mu = (3,1) \quad , \quad \underset{\sim}{\nu} = \{(3,3),(3,1,1)\} .$$

Then

$$\gamma = d_6 d_4(\eta) = d_5 d_3(\eta)$$

is the system

$$\gamma^1 = \frac{2}{3}\eta^1_{xy} - \frac{1}{3}\eta^2_{xx} , \qquad \gamma^2 = \frac{2}{3}\eta^1_{xz} - \frac{1}{3}\eta^3_{xx}$$

$$\gamma^3 = \eta^1_{yy} - \eta^4_{xx} , \qquad \gamma^4 = \eta^1_{yz} + \frac{1}{2}\eta^2_{xz} - \frac{1}{2}\eta^3_{xy} - \frac{1}{2}\eta^5_{xx} ,$$

$$\gamma^5 = \eta^1_{yz} - \frac{1}{2}\eta^2_{xz} + \frac{1}{2}\eta^3_{xy} - \frac{1}{2}\eta^5_{xx} , \qquad \gamma^6 = \eta^1_{zz} - \eta^6_{xx} ,$$

$$\gamma^7 = \eta^2_{yy} - 2\eta^4_{xy} , \qquad \gamma^8 = \eta^2_{yz} - \eta^5_{xy} ,$$

$$\gamma^9 = \eta^2_{yz} + \eta^3_{yy} - 2\eta^4_{xz} - \eta^5_{xy} , \qquad \gamma^{10} = \eta^2_{zz} + \eta^3_{yz} - \eta^5_{xz} - 2\eta^6_{xy} ,$$

$$\gamma^{11} = \eta^3_{yz} - \eta^5_{xy} , \qquad \gamma^{12} = \eta^3_{zz} - 2\eta^6_{xz} ,$$

$$\gamma^{13} = -\frac{2}{3}\eta^4_{yz} - \frac{1}{3}\eta^5_{yy} , \qquad \gamma^{14} = \eta^4_{zz} - \eta^6_{yy} ,$$

$$\gamma^{15} = \eta^5_{zz} - 2\eta^4_{yz} .$$

This is solvable if and only if

$$d_{10}d_9\,\gamma = 0 = d_7\gamma .$$

The first of these is the second order system (16.3); the second, $d_7\gamma = 0$, is the first order system

$$\gamma^1_z - \gamma^2_y + \frac{1}{3}\gamma^4_x - \frac{1}{3}\gamma^5_y = 0 , \qquad dx^3 \otimes dy \otimes dz ,$$

$$\gamma^2_z - \gamma^3_y + \gamma^8_x - \frac{1}{2}\gamma^9_x = 0 , \qquad dx^2 dy \otimes dy \otimes dz ,$$

$$\gamma^5_z - \gamma^6_y + \frac{1}{2}\gamma^{10}_x - \gamma^{11}_x = 0 , \qquad dx^2 dz \otimes dy \otimes dz ,$$

$$\gamma^7_z - \gamma^8_y + 3\gamma^{13}_x = 0 , \qquad dx dy^2 \otimes dy \otimes dz ,$$

$$\gamma^9_z - \gamma^{10}_y + 2\gamma^{14}_x = 0 , \qquad dx dy dz \otimes dy \otimes dz ,$$

$$\gamma^{11}_z - \gamma^{12}_y + \gamma^{15}_x = 0 , \qquad dx dz^2 \otimes dy \otimes dz .$$

Both the first and second order integrability criteria must be satisfied for (16.6) to have a solution.

Finally, consider $\underset{\sim}{\lambda} = \{(1,1),(3)\}$, $\mu = (3,1)$, $\nu = (3,2,1)$.

Exactness says

$$\gamma = d_5 d_2(\alpha) + d_6(\zeta) \tag{16.7}$$

if and only if

$$d_8 d_7(\gamma) = 0 \ . \tag{16.8}$$

Here (16.7) is the mixed system

$$\gamma^1 = \frac{1}{3}\alpha^1_{xx} + \zeta^1_y - \frac{1}{3}\zeta^2_x \ ,$$

$$\gamma^2 = \frac{1}{3}\alpha^2_{xx} + \zeta^1_z - \frac{1}{3}\zeta^3_x \ ,$$

$$\gamma^3 = \frac{1}{4}\alpha^1_{xy} + \zeta^2_y - \zeta^4_x$$

$$\gamma^4 = \frac{1}{4}\alpha^1_{xz} + \frac{3}{4}\alpha_{xy} + \frac{1}{4}\alpha^3_{xx} + \zeta^2_z - \frac{1}{2}\zeta^5_x \ ,$$

$$\gamma^5 = \frac{3}{4}\alpha^1_{xz} + \frac{1}{4}\alpha^2_{xy} - \frac{1}{4}\alpha^3_{xx} + \zeta^3_y - \frac{1}{2}\zeta^5_x \ ,$$

$$\gamma^6 = \alpha^2_{xz} + \zeta^3_z - \zeta^6_x \ ,$$

$$\gamma^7 = \alpha^1_{yy} + \zeta^4_y - 3\zeta^7_x \ ,$$

$$\gamma^8 = \frac{1}{2}\alpha^1_{yz} + \frac{1}{2}\alpha^2_{yy} + \frac{1}{2}\alpha^3_{xy} + \zeta^4_z - \zeta^8_x \ ,$$

$$\gamma^9 = \frac{3}{2}\alpha^1_{yz} + \frac{1}{2}\alpha^2_{yy} - \frac{1}{2}\alpha^3_{xx} + \zeta^5_y - 2\zeta^8_x \ ,$$

$$\gamma^{10} = \frac{1}{2}\alpha^1_{zz} + \frac{3}{2}\alpha^2_{yz} + \frac{3}{4}\alpha^3_{xz} + \zeta^5_z - 2\zeta^9_x \ ,$$

$$\gamma^{11} = \frac{1}{2}\alpha^1_{zz} + \frac{1}{2}\alpha^2_{yz} - \frac{3}{4}\alpha^3_{xz} + \zeta^6_y - \zeta^9_x \ ,$$

$$\gamma^{12} = \alpha^2_{zz} + \zeta^6_z - 3\zeta^{10}_x \ ,$$

$$\gamma^{13} = \frac{1}{3}\alpha^3_{yy} + \zeta^7_z - \frac{1}{3}\zeta^8_y \ ,$$

$$\gamma^{14} = \alpha^3_{yz} + \zeta^8_z - \zeta^9_y \ ,$$

$$\gamma^{15} = \alpha^3_{zz} + \zeta^9_z - 3\,\zeta^{10}_y \ ,$$

whereas the integrability conditions (16.8) are purely second order:

$$\frac{1}{2}\gamma^1_{yz} - \frac{1}{2}\gamma^2_{yy} - \frac{1}{6}\gamma^3_{xz} + \frac{1}{3}\gamma^4_{xy} - \frac{1}{6}\gamma^5_{xy} - \frac{1}{6}\gamma^8_{xx} + \frac{1}{12}\gamma^9_{xx} = 0 \ , \qquad dx^3 \otimes dy^2 \otimes dz \ ,$$

$$\gamma^1_{zz} - \gamma^2_{yz} + \frac{1}{3}\gamma^4_{xz} - \frac{2}{3}\gamma^5_{xz} + \frac{1}{3}\gamma^6_{xy} - \frac{1}{6}\gamma^{10}_{xx} + \frac{1}{3}\gamma^{11}_{xx} = 0 \ , \qquad dx^3 \otimes dydz \otimes dz$$

$$\frac{1}{2}\gamma^2_{yz} - \frac{1}{2}\gamma^3_{yy} - \gamma^7_{xz} + \frac{3}{2}\gamma^8_{xy} - \frac{1}{4}\gamma^9_{xy} - 3\,\gamma^{13}_{xx} = 0 \ , \qquad dx^2dy \otimes dy^2 \otimes dz$$

$$\gamma^2_{zz} - \gamma^3_{yz} + \gamma^7_{xz} - \gamma^8_{xz} + \frac{1}{2}\gamma^{10}_{xy} - \gamma^{14}_{xx} = 0 \ , \qquad dx^2dy \otimes dydz \otimes dz$$

$$\frac{1}{2}\gamma^5_{yz} - \frac{1}{2}\gamma^6_{yy} - \frac{1}{4}\gamma^9_{xz} + \frac{1}{2}\gamma^{10}_{xy} - \frac{1}{2}\gamma^{11}_{xy} - \frac{1}{2}\gamma^{14}_{xx} = 0 \ , \qquad dx^2dz \otimes dy^2 \otimes dz \ ,$$

$$\gamma^5_{zz} - \gamma^6_{yz} + \frac{1}{2}\gamma^{10}_{xz} - 2\,\gamma^{11}_{xz} + \gamma^{12}_{xy} - \gamma^{15}_{xx} = 0 \ , \qquad dx^2dz \otimes dydz \otimes dz \ ,$$

$$\frac{1}{2}\gamma^7_{zz} - \frac{1}{2}\gamma^8_{yz} - \frac{1}{2}\gamma^9_{yz} + \frac{1}{2}\gamma^{10}_{yy} + \frac{3}{2}\gamma^{13}_{xz} - \gamma^{14}_{xy} = 0 \ , \qquad dxdy^2 \otimes dydz \otimes dz \ ,$$

$$\gamma^9_{zz} - \gamma^{10}_{yz} - 2\,\gamma^{11}_{yz} + 2\,\gamma^{12}_{yy} + \gamma^{14}_{xz} - 2\,\gamma^{15}_{xy} = 0 \ , \qquad dxdydz \otimes dydz \otimes dz$$

Other examples, of greater complexity, can of course be constructed at will, but the explicit expressions rapidly get out of hand, even in low dimensional spaces. The above examples should give the reader a taste of the variety of systems of partial differential equations together with explicit integrability conditions which may be constructed. It would be interesting to check these with the integrability conditions of Goldschmidt, [14], but I have not attempted to do this.

Before tackling the general case of smooth hyperforms, we first treat the question of exactness of differential forms with polynomial coefficients. Let $\mathcal{P}$ denote the space of all polynomial functions on $M = \mathbb{R}^p$.

Theorem 16.3 The polynomial hypercomplex

$$d^{\mu}_{\lambda}: \Gamma(\Xi_{\lambda}, \mathcal{P}) \to \Gamma(\Xi_{\mu}, \mathcal{P})$$

is exact.

Proof

It suffices to note that this polynomial hypercomplex is isomorphic to the polynomial hypercomplex discussed at the end of section 8, for then theorem 16.3 is just a restatement of theorem 8.11. To this end, we have for $M = \mathbb{R}^p$,

$$\mathcal{P} \simeq \odot^* M \simeq \odot_* M^* ,$$

cf. [11]. Since M is a vector space, we can identify $M^* \simeq T^*M|_x$ for each $x \in M$, and hence

$$\Gamma(\Xi_{\lambda}, \mathcal{P}) \simeq \mathcal{P} \otimes \Xi_{\lambda} \simeq \odot_* M^* \otimes L_{\lambda} M^* .$$

It is easy to check that this is the required isomorphism between the two polynomial hypercomplexes.

For the proof of theorem 15.1, we begin with a brief review of the relevant distribution theory, using Ehrenpreis, [9], [10], Malgrange, [19], and Gel'fand and Shilov, [13], for basic references. The notation is that of Ehrenpreis. Let $\mathcal{D} = C_0^\infty(M)$ denote the space of test functions, and $\mathcal{D}'$ the dual space of distributions. Let $\mathcal{E} = C^\infty(M)$, which can be viewed as a subspace of $\mathcal{D}'$. The <u>Fourier transform</u> of a test function $\varphi \in \mathcal{D}$ is the entire function

$$\hat{\varphi}(z) = \mathfrak{F}[\varphi(x)] = \int_M \varphi(x) e^{\sqrt{-1}\, z \cdot x}\, dx \ ,$$

where $z \in C^p$ is complex. The Fourier transform is a topological isomorphism

$$\mathfrak{F} : \mathcal{D} \to \underset{\sim}{D} \ ,$$

where $\underset{\sim}{D}$ denotes the space of entire functions of exponential type which are rapidly decreasing on $\mathbb{R}^p \subset C^p$. The adjoint $\underset{\sim}{F}$ of $\mathfrak{F}^{-1}$ defines a topological isomorphism

$$\underset{\sim}{F} : \mathcal{D}' \to \underset{\sim}{D}' \ ,$$

where $\underset{\sim}{D}'$ is the dual of $\underset{\sim}{D}$ with topology of uniform convergence on bounded subsets. This restricts to a topological isomorphism

$$\underset{\sim}{\mathfrak{F}} : \mathcal{D}'_F \to \underset{\sim}{D}'_F \ ,$$

in the subspace $\mathcal{D}'_F \subset \mathcal{D}'$ of distributions of finite order, which we also call the Fourier transform. Note that $\mathcal{E} \subset \mathcal{D}'_F$, so $\underset{\sim}{\mathfrak{F}}$ further restricts

$$\underset{\sim}{\mathfrak{F}} : \mathcal{E} \to \underset{\sim}{E}' ,$$

where the dual space $\underset{\sim}{E}'$ is rather complicated to describe explicitly, but need not concern us here. Similar transforms exist on the Cartesian product spaces $\mathcal{D}^m = \mathcal{D}\times\ldots\times\mathcal{D}$, etc.

If P denotes a linear, constant coefficient partial differential operator, then its Fourier transform

$$Q = \mathfrak{F}(P): \underset{\sim}{D} \to \underset{\sim}{D}$$

is the operation of multiplication by an appropriate polynomial. By duality, this extends to

$$Q = \underset{\sim}{\mathfrak{F}}(P) \; : \underset{\sim}{D}'_F \to \underset{\sim}{D}'_F ,$$

where for $S \in \mathcal{D}'_F$,

$$PS(\varphi) = S(P'\varphi) \; , \; \varphi \in \mathcal{D} ,$$

where P' denotes the adjoint of P . Similarly, if

$$\underset{\sim}{P} : \mathcal{D}^m \to \mathcal{D}^n$$

is an $n \times m$ matrix of linear, constant coefficient partial differential operators, the Fourier transform

$$\underset{\sim}{Q} = \underset{\sim}{\mathfrak{F}}(\underset{\sim}{P}) \; : \underset{\sim}{D}'_F \to \underset{\sim}{D}'_F$$

is given by multiplication by a matrix of polynomials.

The main result required here is on the solvability of systems of linear differential equations in the space $\mathcal{E}$.

Theorem 16.4 The system of constant coefficient partial differential equations

$$\underset{\sim}{P}\,\underset{\sim}{f} = \underset{\sim}{g} \,, \tag{16.9}$$

for $\underset{\sim}{g} \in \mathcal{E}^n$ has a solution $\underset{\sim}{f} \in \mathcal{E}^m$ if and only if

$$\int \underset{\sim}{g}(x) \cdot \underset{\sim}{h}(x)\,dx = 0 \tag{16.10}$$

for all $\underset{\sim}{h} \in \mathcal{D}^m$ with

$$\underset{\sim}{P}'\,\underset{\sim}{h} = 0 \,.$$

This result can easily be inferred from the corresponding result for square matrices $\underset{\sim}{P}$ proved in [9; theorem 7].

By use of theorem 7.5, we first reduce the general problem of exactness of the differential hypercomplex to the following problem: Given $\mu \subset \nu$, $|\nu/\mu| = 1$, let $\{\lambda\} = \mathcal{S}^0(\mu,\nu)$. Let $\omega \in \Gamma\,\Xi_\mu$. If

$$d_\mu^\nu \omega = 0, \tag{16.11}$$

then there exists $\omega \in \Gamma\,\Xi_\lambda$ with

$$d_\lambda^\mu \zeta = \omega \,. \tag{16.12}$$

Now (16.12), when written out in full detail, forms a large system of constant-coefficient partial differential equations of the form (16.9), where $\underset{\sim}{f}$ stands for the coefficient functions of ζ and $\underset{\sim}{g}$ the coefficients of ω, and m and n the dimensions of $L_\lambda M$, $L_\mu M$ respectively, cf. theorem 2.4. These will be solvable provided the integrability criteria (16.10) is satisfied, and this must be shown to follow from (16.11). We thus have to investigate the "adjoint" of d^μ_λ.

Using the underlying metric on M, there is an induced identification between Ξ_λ and its dual obtained from the identification of T^* with T. The <u>codifferential</u> $\delta^\lambda_\mu : \Gamma\, \Xi_\mu \to \Gamma\, \Xi_\lambda$ is defined for $|\mu/\lambda| = 1$ by the formula

$$\delta^\lambda_\mu(\Sigma\, f_T(x)\, dx_T) = \Sigma\, df_T \lrcorner\, dx_T \ , \tag{16.13}$$

where $\lrcorner : T^* \otimes \Xi_\mu \to \Xi_\lambda$ is the interior product (9.1) using the identification of T^* and T. (Compare the definition of the ordinary co-differential used in the construction of the Laplace - Beltrami operator on differential forms over a Riemannian manifold, cf. [28, page 221].)

<u>Lemma 16.5</u> The codifferential $\delta^\lambda_\mu : \Gamma\, \Xi_\mu \to \Gamma\, \Xi_\lambda$ forms a cohypercomplex.

The proof follows the same lines as theorem 13.3. Another easy computation, using (9.2), (13.2) and (16.13) shows that the codifferential δ^λ_μ is (up to sign) the adjoint of the exterior derivative d^μ_λ.

<u>Lemma 16.6</u> Let $|\mu/\lambda| = k$. Then for $\zeta \in \Gamma(\Xi_\lambda, \mathcal{E})$, $\omega \in \Gamma(\Xi_\mu, \mathcal{D})$,

$$\int_M \langle d_\lambda^\mu \zeta,\omega\rangle dx = (-1)^k \int_M \langle \zeta,\delta_\mu^\lambda \omega\rangle dx . \qquad (16.14)$$

This lemma combined with theorem 15.4 reduces the integrability of (16.12) to the problem of whether

$$\int_M \langle \omega,\theta\rangle dx = 0 \qquad (16.15)$$

for all $\theta \in \Gamma(\Xi_\mu,\mathcal{D})$ with

$$\delta_\mu^\lambda \theta = 0 . \qquad (16.16)$$

At this point we change to Fourier transform space. Given a hyperform

$$\zeta = \Sigma f_T(x) dx_T \in \Gamma(\Xi_\lambda,\mathcal{D}_F')$$

with distributional coefficients, define its Fourier transform

$$\hat{\zeta} = \mathfrak{F}(\zeta) = \Sigma \hat{f}_T(z) dz_T \in \Gamma(\Xi_\lambda,\underset{\sim}{D}_F') ,$$

which is a (complex) hyperform with coefficients in $\underset{\sim}{D}_F'$.

<u>Lemma 16.7</u> Let $\lambda \subset \mu$, $|\mu/\lambda| = 1$. For $\zeta \in \Gamma(\Xi_\lambda,\mathcal{D}_F')$,

$$\mathfrak{F}(d_\lambda^\mu \zeta) = v(z) * \mathfrak{F}(\zeta) , \qquad (16.17)$$

where $*$ is the Pieri product from Ξ_λ to Ξ_μ , and

$$v(z) = \sqrt{-1} \sum_{j=1}^{p} z^j dz^j .$$

Similarly, for $\omega \in \Gamma(\Xi_\mu,\mathcal{D}_F')$,

$$\mathfrak{F}[\delta_{\mu}^{\lambda}\omega] = v(z) \lrcorner \mathfrak{F}(\omega) \ , \tag{16.18}$$

where $\lrcorner$ is the interior product (10.1) from Ξ_{μ} to Ξ_{λ} induced by the metric on M .

Thus, by the Plancheral Formula, we need only check that

$$\int_{\mathbb{R}^p} \langle \omega(z), \theta(z) \rangle dz = 0 \tag{16.19}$$

for all $\theta \in \Gamma(\Xi_{\mu}, D)$ satisfying

$$(v(z) \lrcorner)^k \, \theta(z) = 0 \ , \text{ in } \ \Xi_{\lambda} \tag{16.20}$$

for all z . (Here $k = |\mu/\lambda|$.) .

Now if $z \neq 0$, then (16.20) is true if and only if

$$\hat{\theta}(z) = v(z) \lrcorner \hat{\xi}(z) \tag{16.21}$$

pointwise. To see that (16.21) holds when $z = 0$, where $v(z)$ vanishes, we need to check that $\hat{\theta}(0) = 0$. Write

$$\hat{\theta}(z) = \hat{\theta}(0) + \tilde{\theta}(z) \ ,$$

where $\tilde{\theta}(0) = 0$. Substituting into (16.19), the only terms of degree k in z are

$$(v(z) \lrcorner)^k \, \hat{\theta}(0) = 0 \ ,$$

the other terms vanishing to order $(k+1)$ or higher at $z = 0$. But the exactness of the polynomial cohypercomplex, which by (9.9) can be identified with the cohypercomplex

$$v(z) \lrcorner : \mathcal{P} \otimes \Xi_{\mu} \to \mathcal{P} \otimes \Xi_{\lambda} \ , \ |\mu/\lambda| = 1 \ ,$$

(see also theorem 16.3), implies that $\hat{\theta}(0) = 0$ (of course, assuming $\mu \neq 0$) , since $\operatorname{im}(v(z) \lrcorner) \cap \mathbb{R} \otimes \Xi_{\lambda} = \{0\}$.

(Actually, it can be proven that $\hat{\xi}(z)$ in (16.21) can be chosen entire in z , but this is considerably more difficult to do.)

Since (16.21) holds for all z , we have

$$\begin{aligned} \langle \hat{\omega}(z), \hat{\theta}(z) \rangle &= \langle \hat{\omega}(z), v(z) \lrcorner \hat{\xi}(z) \rangle \\ &= \langle v(z) * \hat{\omega}(z), \hat{\xi}(z) \rangle \\ &= 0 \end{aligned}$$

pointwise for all z . Therefore (16.19) holds and the proof of exactness is complete.

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