Infinite–Dimensional Symmetry Groups

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Sur la théorie, si importante sans doute, mais pour nous si obscure, des ≪groupes de Lie infinis≫, nous ne savons rien que ce qui trouve dans les mémoires de Cartan, première exploration à travers une jungle presque impénétrable; mais celle-ci menace de se refermer sur les sentiers déjà tracés, si l'on ne procède bientôt à un indispensable travail de défrichement.

— André Weil, 1947

What's the Deal with Infinite–Dimensional Groups?

• Lie invented Lie groups to study symmetry and solution of differential equations.

♢ In Lie's time, there were no abstract Lie groups. All groups were realized by their action on a space.

♠ Therefore, Lie saw no essential distinction between finite-dimensional and infinite-dimensional group actions.

However, with the advent of abstract Lie groups, the two subjects have gone in radically different directions.

♡ The general theory of finite-dimensional Lie groups has been rigorously formalized and applied.

♣ But there is still no generally accepted abstract object that represents an infinite-dimensional Lie pseudo-group!

Ehresmann's Trinity

1953:

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1953:

- Lie Pseudo-groups

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- Jets

Ehresmann's Trinity

1953:

- Lie Pseudo-groups
- Jets
- Groupoids

Lie Pseudo-groups in Action

- Lie — Medolaghi — Vessiot
- Cartan
- Ehresmann
- Kuranishi, Spencer, Goldschmidt, Guillemin, Sternberg, Kumpera, . . .

- Relativity
- Noether's (Second) Theorem

- Gauge theory and field theories:
  Maxwell, Yang–Mills, conformal, string, . . .
- Fluid mechanics, metereology: Euler, Navier–Stokes, boundary layer, quasi-geostropic, . . .
- Solitons (in $2 + 1$ dimensions):
  K–P, Davey-Stewartson, . . .
- Kac–Moody
- Linear and linearizable PDEs
- *Lie groups*!

Moving Frames

In collaboration with Juha Pohjanpelto and Jeongoo Cheh, I have recently established a moving frame theory for infinite-dimensional Lie pseudo-groups mimicking the earlier equivariant approach for finite-dimensional Lie groups developed with Mark Fels and others.

The finite-dimensional theory and algorithms have had a very wide range of significant applications, including differential geometry, differential equations, calculus of variations, computer vision, Poisson geometry and solitons, numerical methods, relativity, classical invariant theory, . . .

What's New?

In the infinite-dimensional case, the moving frame approach provides new constructive algorithms for:

- Invariant Maurer–Cartan forms
- Structure equations
- Moving frames
- Differential invariants
- Invariant differential operators
- Basis Theorem
- Syzygies and recurrence formulae

- Further applications:

  $\Longrightarrow$ Symmetry groups of differential equations

  $\Longrightarrow$ Vessiot group splitting; explicit solutions

  $\Longrightarrow$ Gauge theories

  $\Longrightarrow$ Calculus of variations

  $\Longrightarrow$ Numerical methods

Symmetry Groups — Review

System of differential equations:

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, 2, \ldots, k$$

By a symmetry, we mean a transformation that maps solutions to solutions.

Lie: To find the symmetry group of the differential equations, work infinitesimally.

The vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

is an infinitesimal symmetry if its flow $\exp(t\,\mathbf{v})$ is a one-parameter symmetry group of the differential equation.

To find the infinitesimal symmetry conditions, we prolong $\mathbf{v}$ to the jet space whose coordinates are the derivatives appearing in the differential equation:

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J=0}^{n} \varphi_\alpha^J \frac{\partial}{\partial u_J^\alpha}$$

where

$$\varphi_\alpha^J = D_J \left(\varphi^\alpha - \sum_{i=1}^{p} u_i^\alpha \, \xi^i \right) + \sum_{i=1}^{p} u_{J,i}^\alpha \, \xi^i$$

$$\equiv \Phi_\alpha^J(x, u^{(n)}; \xi^{(n)}, \varphi^{(n)})$$

Infinitesimal invariance criterion:

$$\mathbf{v}^{(n)}(\Delta_\nu) = 0 \qquad \text{whenever} \qquad \Delta = 0.$$

Infinitesimal determining equations:

$$\mathcal{L}(x, u; \xi^{(n)}, \varphi^{(n)}) = 0$$

The Korteweg–deVries equation

$$\boxed{u_t + u_{xxx} + u u_x = 0}$$

Symmetry generator:

$$\mathbf{v} = \tau(t,x,u)\,\frac{\partial}{\partial t} + \xi(t,x,u)\,\frac{\partial}{\partial x} + \varphi(t,x,u)\,\frac{\partial}{\partial u}$$

Prolongation:

$$\mathbf{v}^{(3)} = \mathbf{v} + \varphi^t\,\frac{\partial}{\partial u_t} + \varphi^x\,\frac{\partial}{\partial u_x} + \;\cdots\; + \varphi^{xxx}\,\frac{\partial}{\partial u_{xxx}}$$

where

$$\varphi^t = \varphi_t + u_t \varphi_u - u_t \tau_t - u_t^2 \tau_u - u_x \xi_t - u_t u_x \xi_u$$

$$\varphi^x = \varphi_x + u_x \varphi_u - u_t \tau_x - u_t u_x \tau_u - u_x \xi_x - u_x^2 \xi_u$$

$$\varphi^{xxx} = \varphi_{xxx} + 3\,u_x \varphi_u + \;\cdots$$

Infinitesimal invariance:

$$\mathbf{v}^{(3)}(u_t + u_{xxx} + u u_x) = \varphi^t + \varphi^{xxx} + u\,\varphi^x + u_x\,\varphi = 0$$

on solutions

Infinitesimal determining equations:

$$\tau_x = \tau_u = \xi_u = \varphi_t = \varphi_x = 0$$

$$\varphi = \xi_t - \tfrac{2}{3}\,u\tau_t \qquad \varphi_u = -\tfrac{2}{3}\,\tau_t = -\,2\,\xi_x$$

$$\tau_{tt} = \tau_{tx} = \tau_{xx} = \;\cdots\; = \varphi_{uu} = 0$$

General solution:

$$\tau = c_1 + 3\,c_4 t, \qquad \xi = c_2 + c_3 t + c_4 x, \qquad \varphi = c_3 - 2c_4 u.$$

Basis for symmetry algebra $\mathfrak{g}_{KdV}$:

$$\mathbf{v}_1 = \partial_t,$$

$$\mathbf{v}_2 = \partial_x,$$

$$\mathbf{v}_3 = t\,\partial_x + \partial_u,$$

$$\mathbf{v}_4 = 3\,t\,\partial_t + x\,\partial_x - 2\,u\,\partial_u.$$

The symmetry group $\mathcal{G}_{KdV}$ is four-dimensional

$$(x, t, u) \longmapsto (\lambda^3\,t + a, \lambda\,x + c\,t + b, \lambda^{-2}\,u + c)$$

$$\mathbf{v}_1 = \partial_t, \qquad \mathbf{v}_2 = \partial_x,$$

$$\mathbf{v}_3 = t\,\partial_x + \partial_u, \qquad \mathbf{v}_4 = 3\,t\,\partial_t + x\,\partial_x - 2\,u\,\partial_u.$$

Commutator table:

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	0	0	0	$\mathbf{v}_1$
$\mathbf{v}_2$	0	0	$\mathbf{v}_1$	$3\,\mathbf{v}_2$
$\mathbf{v}_3$	0	$-\,\mathbf{v}_1$	0	$-2\,\mathbf{v}_3$
$\mathbf{v}_3$	$-\,\mathbf{v}_1$	$-3\,\mathbf{v}_2$	$2\,\mathbf{v}_3$	0

Entries: $[\mathbf{v}_i, \mathbf{v}_j] = \sum_k C_{ij}^k \mathbf{v}_k \qquad C_{ij}^k$ — structure constants of $\mathfrak{g}$

Navier–Stokes Equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu\, \Delta \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0.$$

Symmetry generators:

$$\mathbf{v}_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}(t) \cdot \partial_{\mathbf{x}} + \boldsymbol{\alpha}'(t) \cdot \partial_{\mathbf{u}} - \boldsymbol{\alpha}''(t) \cdot \mathbf{x}\, \partial_p$$

$$\mathbf{v}_0 = \partial_t$$

$$\mathbf{s} = \mathbf{x} \cdot \partial_{\mathbf{x}} + 2\,t\,\partial_t - \mathbf{u} \cdot \partial_{\mathbf{u}} - 2\,p\,\partial_p$$

$$\mathbf{r} = \mathbf{x} \wedge \partial_{\mathbf{x}} + \mathbf{u} \wedge \partial_{\mathbf{u}}$$

$$\mathbf{w}_h = h(t)\,\partial_p$$

Kadomtsev–Petviashvili (KP) Equation

$$(u_t + \tfrac{3}{2}\, u\, u_x + \tfrac{1}{4}\, u_{xxx})_x \pm \tfrac{3}{4}\, u_{yy} = 0$$

Symmetry generators:

$$\begin{aligned}
\mathbf{v}_f &= f(t)\,\partial_t + \tfrac{2}{3}\, y\, f'(t)\,\partial_y + \left(\tfrac{1}{3}\, x\, f'(t) \mp \tfrac{2}{9}\, y^2 f''(t) \right) \partial_x \\
&\qquad + \left(-\tfrac{2}{3}\, u\, f'(t) + \tfrac{2}{9}\, x\, f''(t) \mp \tfrac{4}{27}\, y^2 f'''(t) \right) \partial_u, \\
\mathbf{w}_g &= g(t)\,\partial_y \mp \tfrac{2}{3}\, y\, g'(t)\,\partial_x \mp \tfrac{4}{9}\, y\, g''(t)\,\partial_u, \\
\mathbf{z}_h &= h(t)\,\partial_x + \tfrac{2}{3}\, h'(t)\,\partial_u.
\end{aligned}$$

$\Longrightarrow$ Kac–Moody loop algebra $A_4^{(1)}$

Main Goals

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- Find the structure of its symmetry (pseudo-) group $\mathcal{G}$ directly from the determining equations.
- Find and classify its differential invariants.
- Use symmetry reduction or group splitting to construct explicit solutions.

Pseudo-groups

M — smooth (analytic) manifold

Definition. A pseudo-group is a collection of local diffeomorphisms $\varphi\colon M \to M$ such that

- *Identity:* $\mathbf{1}_M \in \mathcal{G}$,
- *Inverses:* $\varphi^{-1} \in \mathcal{G}$,
- *Restriction:* $U \subset \operatorname{dom} \varphi \implies \varphi \mid U \in \mathcal{G}$,
- *Continuation:* $\operatorname{dom} \varphi = \bigcup U_\kappa$ and $\varphi \mid U_\kappa \in \mathcal{G} \implies \varphi \in \mathcal{G}$,
- *Composition:* $\operatorname{im} \varphi \subset \operatorname{dom} \psi \implies \psi \circ \varphi \in \mathcal{G}$.

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$\implies$ small category with inverses

Lie Pseudo-groups

Definition. A Lie pseudo-group $\mathcal{G}$ is a pseudo-group whose transformations are the solutions to an involutive system of partial differential equations:

$$F(z, \varphi^{(n)}) = 0.$$

called the nonlinear determining equations.

$\Longrightarrow$ *analytic* (*Cartan–Kähler*)

$\star\star$ Key complication: $\nexists$ abstract object $\mathcal{G}$ $\star\star$

A Non-Lie Pseudo-group

Acting on $M = \mathbb{R}^2$:

$$\boxed{X = \varphi(x) \qquad Y = \varphi(y)}$$

where $\varphi \in \mathcal{D}(\mathbb{R})$ is any local diffeomorphism.

♠ Cannot be characterized by a system of partial differential equations

$$\Delta(x, y, X^{(n)}, Y^{(n)}) = 0$$

Theorem. (Johnson, Itskov) Any non-Lie pseudo-group can be completed to a Lie pseudo-group with the same differential invariants.

Completion of previous example:

$$\boxed{X = \varphi(x), \qquad Y = \psi(y)}$$

where $\varphi, \psi \in \mathcal{D}(\mathbb{R})$.

Infinitesimal Generators

$\mathfrak{g}$ — Lie algebra of infinitesimal generators of the pseudo-group $\mathcal{G}$

$z = (x, u)$ — local coordinates on M

Vector field:

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha \frac{\partial}{\partial u^\alpha}$$

Vector field jet:

$$\mathrm{j}_n \mathbf{v} \longmapsto \zeta^{(n)} = (\ \ldots\ \zeta^b_A\ \ldots\ )$$

$$\zeta^b_A = \frac{\partial^{\# A} \zeta^b}{\partial z^A} = \frac{\partial^k \zeta^b}{\partial z^{a_1} \cdots \partial z^{a_k}}$$

The infinitesimal generators of $\mathcal{G}$ are the solutions to the *Infinitesimal (Linearized) Determining Equations*

$$\mathcal{L}(z, \zeta^{(n)}) = 0 \tag{$*$}$$

Remark: If $\mathcal{G}$ is the symmetry group of a system of differential equations $\Delta(x, u^{(n)}) = 0$, then $(*)$ is the (involutive completion of) the usual Lie determining equations for the symmetry group.

The Diffeomorphism Pseudo-group

M — smooth m-dimensional manifold

$\mathcal{D} = \mathcal{D}(M)$ — pseudo-group of all local diffeomorphisms

$$Z = \varphi(z)$$

$$\begin{cases} z = (z^1, \ldots, z^m) \text{ — source coordinates} \\ Z = (Z^1, \ldots, Z^m) \text{ — target coordinates} \end{cases}$$

Jets

Jets are a fancy name for Taylor polynomials/series.

For $0 \leq n \leq \infty$:

Given a smooth map $\varphi\colon M \to M$, written in local coordinates as $Z = \varphi(z)$, let $\mathrm{j}_n\varphi|_z$ denote its n–jet at $z \in M$, i.e., its n^{th} order Taylor polynomial or series based at z.

$\mathrm{J}^n(M, M)$ is the n^{th} order jet bundle, whose points are the jets.

Local coordinates on $\mathrm{J}^n(M, M)$:

$$(z, Z^{(n)}) = (\ \ldots\ z^a\ \ldots\ Z^b_A\ \ldots\ ), \qquad Z^b_A = \frac{\partial^k Z^b}{\partial z^{a_1} \cdots \partial z^{a_k}}$$

Diffeomorphism Jets

The n^{th} order diffeomorphism jet bundle is the subbundle

$$\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M) \subset \mathrm{J}^n(M, M)$$

consisting of n^{th} order jets of local diffeomorphisms $\varphi \colon M \to M$.

The Inverse Function Theorem tells us it is defined by the non-vanishing of the Jacobian determinant:

$$\det(\, Z^a_b \,) = \det(\, \partial Z^a / \partial z^b \,) \neq 0$$

A Lie pseudo-group $\mathcal{G} \subset \mathcal{D}$ defines the subbundle

$$\mathcal{G}^{(n)} = \{\, F(z, Z^{(n)}) = 0 \,\} \subset \mathcal{D}^{(n)}$$

consisting of the jets of pseudo-group diffeomorphisms, and therefore characterized by the pseudo-group's nonlinear determining equations.

$$\mathcal{G}^{(n)} = \{ F(z, Z^{(n)}) = 0 \} \subset \mathcal{D}^{(n)}$$

♡ Local coordinates on $\mathcal{G}^{(n)}$, e.g., the restricted diffeomorphism jet coordinates z^c, Z^a_B, are viewed as the pseudo-group parameters, playing the same role as the local coordinates on a Lie group G.

♠ The pseudo-group jet bundle $\mathcal{G}^{(n)}$ does not form a group, but rather a groupoid.

Groupoid Structure

Double fibration:

$$\begin{tikzcd}
& \mathcal{G}^{(n)} \arrow[dl, "\boldsymbol{\sigma}^{(n)}"'] \arrow[dr, "\boldsymbol{\tau}^{(n)}"] & \\
M & & M
\end{tikzcd}$$

$$\boldsymbol{\sigma}^{(n)}(z, Z^{(n)}) = z \qquad \text{— source map}$$

$$\boldsymbol{\tau}^{(n)}(z, Z^{(n)}) = Z \qquad \text{— target map}$$

You are only allowed to multiply $h^{(n)} \cdot g^{(n)}$ if

$$\boldsymbol{\sigma}^{(n)}(h^{(n)}) = \boldsymbol{\tau}^{(n)}(g^{(n)})$$

★ Composition of Taylor polynomials/series is well-defined only when the source of the second matches the target of the first.

One-dimensional case: $M = \mathbb{R}$

Source coordinate: x Target coordinate: X

Local coordinates on $\mathcal{D}^{(n)}(\mathbb{R})$

$$g^{(n)} = (x, X, X_x, X_{xx}, X_{xxx}, \ \ldots \ , X_n)$$

Jet:

$$X[\![\, h \,]\!] = X + X_x \, h + \tfrac{1}{2} \, X_{xx} \, h^2 + \tfrac{1}{6} \, X_{xxx} \, h^3 + \ \cdots$$

$\Longrightarrow$ Taylor polynomial/series at a source point x

Groupoid multiplication of diffeomorphism jets:

$$
\begin{aligned}
(X, \mathbf{X}, \mathbf{X}_X, \mathbf{X}_{XX}, \ \ldots\ ) \cdot (x, X, X_x, X_{xx}, \ \ldots\ ) \\
= (x, \mathbf{X}, \mathbf{X}_X\, X_x, \mathbf{X}_X\, X_{xx} + \mathbf{X}_{XX}\, X_x^2, \ \ldots\ )
\end{aligned}
$$

$\Longrightarrow$ Composition of Taylor polynomials/series

The higher order terms are expressed in terms of Bell polynomials according to the general Fàa–di–Bruno formula.

- The groupoid multiplication (or Taylor composition) is only defined when the source coordinate X of the first multiplicand matches the target coordinate X of the second.

Structure of Lie Pseudo-groups

The structure of a finite-dimensional Lie group G is specified by its Maurer–Cartan forms — a basis $\mu^1, \ldots, \mu^r$ for the right-invariant one-forms:

$$d\mu^k = \sum_{i<j} C_{ij}^k \, \mu^i \wedge \mu^j$$

What should be the Maurer–Cartan forms of a Lie pseudo-group?

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Cartan: Use exterior differential systems and prolongation to determine the structure equations.

I propose a direct approach based on the following observation:

The Maurer–Cartan forms for a pseudo-group can be identified with the right-invariant one-forms on the jet groupoid $\mathcal{G}^{(\infty)}$.

The structure equations can be determined immediately from the infinitesimal determining equations.

The Variational Bicomplex

⋆ The differential one-forms on an infinite jet bundle split into two types:

- horizontal forms
- contact forms

Definition. A contact form θ is a differential form that vanishes on all jets: $\theta \mid \mathrm{j}_n \varphi = 0$ for all local diffeomorphisms $\varphi \in \mathcal{D}$.

For the diffeomorphism jet bundle

$$\mathcal{D}^{(\infty)} \subset \mathrm{J}^{\infty}(M, M)$$

Local coordinates:

$$\underbrace{z^1, \ \ldots \ z^m,}_{\text{source}} \quad \underbrace{Z^1, \ \ldots \ Z^m,}_{\text{target}} \quad \underbrace{\ldots \ Z^b_A, \ \ldots}_{\text{jet}}$$

Horizontal forms:

$$dz^1, \ \ldots \ , dz^m$$

Basis contact forms:

$$\Theta^b_A = d_G \, Z^b_A = dZ^b_A - \sum_{a=1}^{m} Z^a_{A,a} \, dz^a$$

One-dimensional case: $M = \mathbb{R}$

Local coordinates on $\mathcal{D}^{(\infty)}(\mathbb{R})$

$$(x, X, X_x, X_{xx}, X_{xxx}, \ \ldots \ , X_n, \ \ldots \ )$$

Horizontal form:

$$dx$$

Contact forms:

$$\Theta = dX - X_x\, dx$$

$$\Theta_x = dX_x - X_{xx}\, dx$$

$$\Theta_{xx} = dX_{xx} - X_{xxx}\, dx$$

$$\vdots$$

- the contact forms vanish when $X = \varphi(x)$

The Variational Bicomplex

$\Longrightarrow$ *Vinogradov, Tsujishita, I. Anderson*

Infinite jet space

$$\mathrm{J}^\infty = \lim_{n \to \infty} \mathrm{J}^n$$

Local coordinates

$$z^{(\infty)} = (x, u^{(\infty)}) = (\; \ldots \; x^i \; \ldots \; u_J^\alpha \; \ldots \;)$$

Horizontal one-forms

$$dx^1, \ldots, dx^p$$

Contact (vertical) one-forms

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^{p} u_{J,i}^\alpha \, dx^i$$

Bigrading of the differential forms on J^∞:

$$\Omega^* = \bigoplus_{r,s} \Omega^{r,s}$$

$$r = \# \quad \text{of} \quad dx^i$$

$$s = \# \quad \text{of} \quad \theta_J^\alpha$$

Vertical and Horizontal Differentials

$$d = d_H + d_V$$

The Variational Bicomplex

$$d_H : \Omega^{r,s} \longrightarrow \Omega^{r+1,s}$$

$$d_V : \Omega^{r,s} \longrightarrow \Omega^{r,s+1}$$

$$d_H F = \sum_{i=1}^{p} (D_i F)\, dx^i \qquad \text{— total differential}$$

$$d_V F = \sum_{\alpha,J} \frac{\partial F}{\partial u_J^\alpha}\, \theta_J^\alpha \qquad \text{— “variation”}$$

$$\pi : \Omega^{p,k} \longrightarrow \mathcal{F}^k = \Omega^{p,k} / d_H [\, \Omega^{p-1,k} \,]$$

— integration by parts

The Variational Bicomplex

$$
\begin{tikzcd}
 & \vdots & \vdots & & \vdots & \vdots & \vdots \\
 & \Omega^{0,3} \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{1,3} \arrow[r, "d_H"] \arrow[u, "d_V"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,3} \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{p,3} \arrow[r, "\pi"] \arrow[u, "d_V"] & \mathcal{F}^3 \arrow[u, "\delta"] \\
 & \Omega^{0,2} \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{1,2} \arrow[r, "d_H"] \arrow[u, "d_V"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,2} \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{p,2} \arrow[r, "\pi"] \arrow[u, "d_V"] & \mathcal{F}^2 \arrow[u, "\delta"] \\
 & \Omega^{0,1} \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{1,1} \arrow[r, "d_H"] \arrow[u, "d_V"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,1} \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{p,1} \arrow[r, "\pi"] \arrow[u, "d_V"] & \mathcal{F}^1 \arrow[u, "\delta"] \\
\mathbb{R} \arrow[r] & \Omega^{0,0} \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{1,0} \arrow[r, "d_H"] \arrow[u, "d_V"] & \cdots \arrow[r, "d_H"] & \Omega^{p-1,0} \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{p,0} \arrow[u, "d_V"] \arrow[ur, "\mathbf{E}"'] &
\end{tikzcd}
$$

• conservation laws Lagrangians PDEs (Euler–Lagrange) Helmholtz conditions

The Simplest Example. $M = \mathbb{R}^2 \quad x, u \in \mathbb{R}$

Horizontal form

$$dx$$

Contact (vertical) forms

$$\theta = du - u_x\, dx$$

$$\theta_x = D_x \theta = du_x - u_{xx}\, dx$$

$$\theta_{xx} = D_x^2 \theta = du_{xx} - u_{xxx}\, dx$$

$$\vdots$$

Differential $\qquad F = F(x, u, u_x, u_{xx}, \ \ldots\ )$

$$dF = \frac{\partial F}{\partial x}\ dx + \frac{\partial F}{\partial u}\ du + \frac{\partial F}{\partial u_x}\ du_x + \frac{\partial F}{\partial u_{xx}}\ du_{xx} + \cdots$$

$$= (D_x F)\, dx \ \ + \ \ \frac{\partial F}{\partial u}\,\theta + \frac{\partial F}{\partial u_x}\,\theta_x + \frac{\partial F}{\partial u_{xx}}\,\theta_{xx} + \cdots$$

$$= \ \ d_H\, F \qquad + \qquad\qquad d_V\, F$$

Total derivative

$$D_x F = \frac{\partial F}{\partial u}\ u_x + \frac{\partial F}{\partial u_x}\ u_{xx} + \frac{\partial F}{\partial u_{xx}}\ u_{xxx} + \cdots$$

Lagrangian form: $\qquad \lambda = L(x, u^{(n)})\, dx \;\in\; \Omega^{1,0}$

Vertical derivative — variation:

$$\begin{aligned} d\lambda = \; d_V \, \lambda = \; d_V \, L \wedge dx \\ = \left(\frac{\partial L}{\partial u}\, \theta + \frac{\partial L}{\partial u_x}\, \theta_x + \frac{\partial L}{\partial u_{xx}}\, \theta_{xx} + \cdots \right) \wedge \, dx \;\in\; \Omega^{1,1} \end{aligned}$$

Integration by parts — compute modulo im d_H :

$$\begin{aligned} d\lambda \;\sim\; \delta\lambda = \left(\frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} + D_x^2 \frac{\partial L}{\partial u_{xx}} - \cdots \right) \theta \wedge \, dx \\ = \mathcal{E}(L)\, \theta \wedge \, dx \end{aligned}$$

$\Longrightarrow$ Euler-Lagrange source form.

Maurer–Cartan Forms

The Maurer–Cartan forms for the diffeomorphism pseudo-group are the right-invariant one-forms on the diffeomorphism jet groupoid $\mathcal{D}^{(\infty)}$.

Key observation:

The target coordinate functions Z^a are right-invariant.

Thus, when we decompose

$$dZ^a = \underbrace{\sigma^a}_{\text{horizontal}} + \underbrace{\mu^a}_{\text{contact}}$$

the two constituents are also right-invariant.

Invariant horizontal forms:

$$\sigma^a = d_M\, Z^a = \sum_{b=1}^{m} Z^a_b\, dz^b$$

Invariant total differentiation (dual operators):

$$\mathbb{D}_{Z^a} = \sum_{b=1}^{m} (\, Z^a_b\,)^{-1}\, \mathbb{D}_{z^b}$$

Invariant contact forms:

$$\mu^b = d_G\, Z^b = \Theta^b = dZ^b - \sum_{a=1}^{m} Z^b_a\, dz^a$$

$$\mu^b_A = \mathbb{D}^A_Z \mu^b = \mathbb{D}_{Z^{a_1}} \cdots \mathbb{D}_{Z^{a_n}} \Theta^b$$

$$b = 1, \ldots, m, \;\; \#A \geq 0$$

One-dimensional case: $M = \mathbb{R}$

Contact forms:

$$\Theta = dX - X_x\, dx$$

$$\Theta_x = \mathbb{D}_x \Theta = dX_x - X_{xx}\, dx$$

$$\Theta_{xx} = \mathbb{D}_x^2 \Theta = dX_{xx} - X_{xxx}\, dx$$

Right-invariant horizontal form:

$$\sigma = d_M\, X = X_x\, dx$$

Invariant differentiation:

$$\mathbb{D}_X = \frac{1}{X_x}\, \mathbb{D}_x$$

Invariant contact forms:

$$\mu = \Theta = dX - X_x \, dx$$

$$\mu_X = \mathbb{D}_X \mu = \frac{\Theta_x}{X_x} = \frac{dX_x - X_{xx} \, dx}{X_x}$$

$$\mu_{XX} = \mathbb{D}_X^2 \mu = \frac{X_x \, \Theta_{xx} - X_{xx} \, \Theta_x}{X_x^3}$$

$$= \frac{X_x \, dX_{xx} - X_{xx} \; dX_x + (X_{xx}^2 - X_x X_{xxx}) \, dx}{X_x^3}$$

$$\vdots$$

$$\mu_n = \mathbb{D}_X^n \mu$$

The Structure Equations for the Diffeomorphism Pseudo–group

Maurer–Cartan series:

$$\mu^b[\![\,H\,]\!] = \sum_A \frac{1}{A!}\,\mu^b_A\,H^A$$

$$H = (H^1,\ldots,H^m) \quad \text{— formal parameters}$$

$$\boxed{\begin{aligned} d\mu[\![\,H\,]\!] &= \nabla\mu[\![\,H\,]\!] \wedge (\,\mu[\![\,H\,]\!] - dZ\,) \\ d\sigma = -\,d\mu[\![\,0\,]\!] &= \nabla\mu[\![\,0\,]\!] \wedge \sigma \end{aligned}}$$

One-dimensional case: $M = \mathbb{R}$

Structure equations:

$$\boxed{d\sigma = \mu_X \wedge \sigma \qquad d\mu[\![\, H \,]\!] = \frac{d\mu}{dH}[\![\, H \,]\!] \wedge (\mu[\![\, H \,]\!] - dZ)}$$

where

$$\sigma = X_x\, dx = dX - \mu$$

$$\mu[\![\, H \,]\!] = \mu + \mu_X\, H + \tfrac{1}{2}\, \mu_{XX}\, H^2 + \cdots$$

$$\mu[\![\, H \,]\!] - dZ = -\,\sigma + \mu_X\, H + \tfrac{1}{2}\, \mu_{XX}\, H^2 + \;\cdots$$

$$\frac{d\mu[\![\, H \,]\!]}{dH} = \mu_X + \mu_{XX}\, H + \tfrac{1}{2}\, \mu_{XXX}\, H^2 + \;\cdots$$

In components:

$$
\begin{aligned}
d\sigma &= \mu_1 \wedge \sigma \\
d\mu_n &= -\,\mu_{n+1} \wedge \sigma \ + \sum_{i=0}^{n-1} \binom{n}{i}\, \mu_{i+1} \wedge \mu_{n-i} \\
&= \sigma \wedge \mu_{n+1} - \sum_{j=1}^{\left[\frac{n+1}{2}\right]} \frac{n-2j+1}{n+1}\binom{n+1}{j}\, \mu_j \wedge \mu_{n+1-j}.
\end{aligned}
$$

$\implies$ Cartan

The Maurer–Cartan Forms for a Lie Pseudo-group

The Maurer–Cartan forms for $\mathcal{G}$ are obtained by restricting the diffeomorphism Maurer–Cartan forms σ^a, μ^b_A to $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$.

$\star$ The resulting one-forms are no longer linearly independent.

Theorem. The Maurer–Cartan forms on $\mathcal{G}^{(\infty)}$ satisfy the invariant infinitesimal determining equations

$$\mathcal{L}(\; \ldots \; Z^a \; \ldots \; \mu^b_A \; \ldots \;) = 0 \qquad (\star\star)$$

obtained from the infinitesimal determining equations

$$\mathcal{L}(\; \ldots \; z^a \; \ldots \; \zeta^b_A \; \ldots \;) = 0 \qquad (\star)$$

by replacing

- source variables z^a by target variables Z^a
- derivatives of vector field coefficients ζ^b_A by
  right-invariant Maurer–Cartan forms μ^b_A

The Structure Equations for a Lie Pseudo-group

Theorem. The structure equations for the pseudo-group $\mathcal{G}$ are obtained by restricting the universal diffeomorphism structure equations

$$d\mu[\![\, H \,]\!] = \nabla\mu[\![\, H \,]\!] \wedge (\, \mu[\![\, H \,]\!] - dZ \,)$$

to the solution space of the linearized involutive system

$$\mathcal{L}(\, \ldots \; Z^a, \; \ldots \; \mu^b_A, \; \ldots \,) = 0.$$

The Korteweg–deVries Equation

$$\boxed{u_t + u_{xxx} + u u_x = 0}$$

Diffeomorphism Maurer–Cartan forms:

$$\mu^t,\ \mu^x,\ \mu^u,\ \mu^t_T,\ \mu^t_X,\ \mu^t_U,\ \mu^x_T,\ \ldots,\ \mu^u_U,\ \mu^t_{TT},\ \mu^T_{TX},\ \ldots$$

Maurer–Cartan determining equations:

$$\mu^t_X = \mu^t_U = \mu^x_U = \mu^u_T = \mu^u_X = 0,$$

$$\mu^u = \mu^x_T - \tfrac{2}{3}\,U\mu^t_T, \qquad \mu^u_U = -\tfrac{2}{3}\,\mu^t_T = -\,2\,\mu^x_X,$$

$$\mu^t_{TT} = \mu^t_{TX} = \mu^t_{XX} = \;\cdots\; = \mu^u_{UU} = \;\ldots\; = 0.$$

Basis ($\dim \mathcal{G}_{KdV} = 4$):

$$\mu^1 = \mu^t, \qquad \mu^2 = \mu^x, \qquad \mu^3 = \mu^u, \qquad \mu^4 = \mu^t_T.$$

Structure equations:

$$
\begin{aligned}
d\mu^1 &= -\,\mu^1 \wedge \mu^4,\\
d\mu^2 &= -\mu^1 \wedge \mu^3 - \tfrac{2}{3}\,U\,\mu^1 \wedge \mu^4 - \tfrac{1}{3}\,\mu^2 \wedge \mu^4,\\
d\mu^3 &= \tfrac{2}{3}\,\mu^3 \wedge \mu^4,\\
d\mu^4 &= 0.
\end{aligned}
$$

$$\boxed{d\mu^i = C^i_{\;jk}\,\mu^j \wedge \mu^k}$$

♠ The structure equations are on the principal bundle $\mathcal{G}^{(\infty)}$; if G is a finite-dimensional Lie group, then $\mathcal{G}^{(\infty)} \simeq M \times G$, and the usual Lie group structure equations are found by restriction to the target fibers $\{\, Z = c \,\} \simeq G$.

Lie–Kumpera Example

$$\boxed{X = f(x) \qquad\qquad U = \frac{u}{f'(x)}}$$

Linearized determining system

$$\xi_x = -\frac{\varphi}{u} \qquad\qquad \xi_u = 0 \qquad\qquad \varphi_u = \frac{\varphi}{u}$$

Maurer–Cartan forms:

$$\sigma = \frac{u}{U}\,dx = f_x\,dx, \qquad \tau = U_x\,dx + \frac{U}{u}\,du = \frac{-u\,f_{xx}\,dx + f_x\,du}{f_x^{\;2}}$$

$$\mu = dX - \frac{U}{u}\,dx = df - f_x\,dx, \qquad \nu = dU - U_x\,dx - \frac{U}{u}\,du = -\frac{u}{f_x^{\;2}}\,(\,df_x - f_{xx}\,dx\,)$$

$$\mu_X = \frac{du}{u} - \frac{dU - U_x\,dx}{U} = \frac{df_x - f_{xx}\,dx}{f_x}, \qquad \mu_U = 0$$

$$\nu_X = \frac{U}{u}\,(dU_x - U_{xx}\,dx) - \frac{U_x}{u}\,(dU - U_x\,dx)$$

$$= -\frac{u}{f_x^{\;3}}(\,df_{xx} - f_{xxx}\,dx\,) + \frac{u\,f_{xx}}{f_x^{\;4}}(\,df_x - f_{xx}\,dx\,)$$

$$\nu_U = -\frac{du}{u} + \frac{dU - U_x\,dx}{U} = -\frac{df_x - f_{xx}\,dx}{f_x}$$

Right-invariant linearized system:

$$\mu_X = -\frac{\nu}{U} \qquad \mu_U = 0 \qquad \nu_U = \frac{\nu}{U}$$

First order structure equations:

$$d\mu = -\,d\sigma = \frac{\nu \wedge \sigma}{U}, \qquad d\nu = -\,\nu_X \wedge \sigma \; - \; \frac{\nu \wedge \tau}{U}$$

$$d\nu_X = -\,\nu_{XX} \wedge \sigma \; - \; \frac{\nu_X \wedge (\tau + 2\,\nu)}{U}$$

Action of Pseudo-groups on Submanifolds a.k.a. Solutions of Differential Equations

$\mathcal{G}$ — Lie pseudo-group acting on p-dimensional submanifolds:

$$N = \{ u = f(x) \} \subset M$$

For example, $\mathcal{G}$ may be the symmetry group of a system of differential equations

$$\Delta(x, u^{(n)}) = 0$$

and the submanifolds the graphs of solutions $u = f(x)$.

Prolongation

$\mathrm{J}^n = \mathrm{J}^n(M,p) \quad — \quad n^{\text{th}}$ order submanifold jet bundle

Local coordinates :

$$z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )$$

Prolonged action of $\mathcal{G}^{(n)}$ on submanifolds:

$$(x, u^{(n)}) \quad \longmapsto \quad (X, \widehat{U}^{(n)})$$

Coordinate formulae:

$$\widehat{U}^\alpha_J = F^\alpha_J(x, u^{(n)}, g^{(n)})$$

$\Longrightarrow$ Implicit differentiation.

Differential Invariants

A differential invariant is an invariant function $I : \mathrm{J}^n \to \mathbb{R}$ for the prolonged pseudo-group action

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$$\Longrightarrow \quad \text{curvature, torsion, } \ldots$$

Invariant differential operators:

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

$$\Longrightarrow \quad \text{arc length derivative}$$

$\mathbb{I}(\mathcal{G})$ — the algebra of differential invariants

The Basis Theorem

Theorem. The differential invariant algebra $\mathbb{I}(\mathcal{G})$ is locally generated by a finite number of differential invariants

$$I_1, \; \ldots \; , I_\ell$$

and $p = \dim S$ invariant differential operators

$$\mathcal{D}_1, \; \ldots \; , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa .$$

$\Longrightarrow$ Lie groups: *Lie, Ovsiannikov*

$\Longrightarrow$ Lie pseudo-groups: *Tresse, Kumpera, Pohjanpelto–O*

Key Issues

- Minimal basis of generating invariants: $I_1, \ldots, I_\ell$
- Commutation formulae for

  the invariant differential operators:

$$[\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y_{jk}^{i}\, \mathcal{D}_i$$

$\Longrightarrow$ Non-commutative differential algebra

- Syzygies (functional relations) among

  the differentiated invariants:

$$\Phi(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ ) \equiv 0$$

$\Longrightarrow$ Codazzi relations

Examples of Differential Invariants

Euclidean Group on $\mathbb{R}^3$

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$$\Longrightarrow \quad \text{group of rigid motions}$$

$$z \quad \longmapsto \quad R\,z + b \qquad\qquad R \in \mathrm{SO}(3)$$

- Induced action on curves and surfaces.

Euclidean Curves $\quad C \subset \mathbb{R}^3$

- κ — curvature: order $= 2$
- τ — torsion: order $= 3$

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- κ — curvature: order $= 2$
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Theorem. Every Euclidean differential invariant of a space curve $C \subset \mathbb{R}^3$ can be written

$$I = H(\kappa, \tau, \kappa_s, \tau_s, \kappa_{ss}, \ \ldots\ )$$

Euclidean Curves $\quad C \subset \mathbb{R}^3$

- κ — curvature: order $= 2$
- τ — torsion: order $= 3$
- $\kappa_s, \tau_s, \kappa_{ss}, \ldots$ — derivatives w.r.t. arc length ds

Theorem. Every Euclidean differential invariant of a space curve $C \subset \mathbb{R}^3$ can be written

$$I = F(\kappa, \tau, \kappa_s, \tau_s, \kappa_{ss}, \ \ldots\ )$$

Thus, κ and τ *generate* the differential invariants of space curves under the Euclidean group.

Euclidean Surfaces $S \subset \mathbb{R}^3$

- $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ — mean curvature: order $= 2$
- $K = \kappa_1 \kappa_2$ — Gauss curvature: order $= 2$

Euclidean Surfaces $\quad S \subset \mathbb{R}^3$

- $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ — mean curvature: order $= 2$
- $K = \kappa_1 \, \kappa_2$ — Gauss curvature: order $= 2$
- $\mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ldots$ — derivatives with respect to the equivariant Frenet frame on S

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Theorem. Every Euclidean differential invariant of a non-umbilic surface $S \subset \mathbb{R}^3$ can be written

$$I = F(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots\ )$$

Euclidean Surfaces $\quad S \subset \mathbb{R}^3$

- $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ — mean curvature: order $= 2$
- $K = \kappa_1 \kappa_2$ — Gauss curvature: order $= 2$
- $\mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ldots$ — derivatives with respect to the equivariant Frenet frame on S

Theorem. Every Euclidean differential invariant of a non-umbilic surface $S \subset \mathbb{R}^3$ can be written

$$I = F(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots\ )$$

Thus, H, K *generate* the differential invariants of (generic) Euclidean surfaces.

Euclidean Surfaces

Theorem. The algebra of Euclidean differential invariants for a non-degenerate surface is generated by the mean curvature through invariant differentiation.

Euclidean Surfaces

Theorem. The algebra of Euclidean differential invariants for a non-degenerate surface is generated by the mean curvature through invariant differentiation.

$$K = \Phi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \ \ldots\ )$$

Applications of Differential Invariants

Every (regular) $\mathcal{G}$-invariant system of differential equations can be expressed in terms of the differential invariants:

$$F(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ ) = 0$$

Every $\mathcal{G}$-invariant variational problem can be expressed in terms of the differential invariants and an invariant volume form:

$$\mathcal{I}[u] = \int L(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ )\,\Omega$$

Question: How to go directly from the differential invariant form of the variational problem to the differential invariant form of the Euler–Lagrange equations? (See Kogan–O.)

- Characterization of moduli spaces
- Integration of invariant ordinary differential equations.
- Symmetry reduction and group splitting (Vessiot's method) for finding explicit solutions to partial differential equations.
- Equivalence and symmetry of solutions/submanifolds — differential invariant signatures.
  Image processing.
- Design of symmetry-preserving numerical algorithms.

Computing Differential Invariants

♠ The infinitesimal method:

$$\mathbf{v}(I) = 0 \qquad \text{for every infinitesimal generator} \qquad \mathbf{v} \in \mathfrak{g}$$

$\Longrightarrow$ Requires solving differential equations.

♡ Moving frames. (Cartan; PJO–Fels–Pohjanpelto– $\cdots$)

- Completely algebraic.
- Can be adapted to arbitrary group and pseudo-group actions.
- Describes the complete structure of the differential invariant algebra $\mathbb{I}(\mathcal{G})$ — using only linear algebra & differentiation!
- Prescribes differential invariant signatures for equivalence and symmetry detection.

Moving Frames for Pseudo–Groups

In the finite-dimensional Lie group case, a moving frame is defined as an equivariant map

$$\rho^{(n)} : \mathrm{J}^n \longrightarrow G$$

$\Longrightarrow$ All classical moving frames can be thus interpreted.

However, we do not have an appropriate abstract object to represent our pseudo-group $\mathcal{G}$.

Consequently, the moving frame will be an equivariant section

$$\rho^{(n)} : \mathrm{J}^n \longrightarrow \mathcal{H}^{(n)}$$

of the pulled-back pseudo-group jet groupoid:

$$\begin{tikzcd}
\mathcal{G}^{(n)} \arrow[d] & \mathcal{H}^{(n)} \arrow[d] \\
M & \mathrm{J}^n . \arrow[l]
\end{tikzcd}$$

Moving Frames for Pseudo–Groups

Definition. A (right) *moving frame* of *order* n is a right-equivariant section $\rho^{(n)} : V^n \to \mathcal{H}^{(n)}$ defined on an open subset $V^n \subset \mathrm{J}^n$.

$\Longrightarrow$ Groupoid action.

Proposition. A moving frame of order n exists if and only if $\mathcal{G}^{(n)}$ acts *freely* and regularly.

Freeness

♣ For Lie group actions, freeness means no isotropy. For infinite-dimensional pseudo-groups, this definition cannot work, and one must restrict to the transformation jets of order n, using the n^{th} order isotropy subgroup:

$$\mathcal{G}^{(n)}_{z^{(n)}} = \left\{ \, g^{(n)} \in \mathcal{G}^{(n)}_{z} \;\middle|\; g^{(n)} \cdot z^{(n)} = z^{(n)} \, \right\}$$

Definition. At a jet $z^{(n)} \in \mathrm{J}^n$, the pseudo-group $\mathcal{G}$ acts

- freely if $\mathcal{G}^{(n)}_{z^{(n)}} = \{ \, \mathbf{1}^{(n)}_{z} \, \}$
- locally freely if
  - $\mathcal{G}^{(n)}_{z^{(n)}}$ is a discrete subgroup of $\mathcal{G}^{(n)}_{z}$
  - the orbits have $\dim = r_n = \dim \mathcal{G}^{(n)}_{z}$

Freeness Theorem

Theorem. If $n \geq 1$ and $\mathcal{G}^{(n)}$ acts locally freely at $z^{(n)} \in \mathrm{J}^n$, then it acts locally freely at any $z^{(k)} \in \mathrm{J}^k$ with $\widetilde{\pi}^k_n(z^{(k)}) = z^{(n)}$ for all $k > n$.

The Normalization Algorithm

To construct a moving frame :

I. Compute the prolonged pseudo-group action

$$u_K^\alpha \quad \longmapsto \quad U_K^\alpha = F_K^\alpha(x, u^{(n)}, g^{(n)})$$

by implicit differentiation.

II. Choose a cross-section to the pseudo-group orbits:

$$u_{J_\kappa}^{\alpha_\kappa} = c_\kappa, \qquad \kappa = 1, \ldots, r_n = \text{fiber dim } \mathcal{G}^{(n)}$$

III. Solve the normalization equations

$$U^{\alpha_\kappa}_{J_\kappa} = F^{\alpha_\kappa}_{J_\kappa}(x, u^{(n)}, g^{(n)}) = c_\kappa$$

for the n^{th} order pseudo-group parameters

$$g^{(n)} = \rho^{(n)}(x, u^{(n)})$$

IV. Substitute the moving frame formulas into the unnormalized jet coordinates $u^\alpha_K = F^\alpha_K(x, u^{(n)}, g^{(n)})$. The resulting functions form a complete system of n^{th} order differential invariants

$$I^\alpha_K(x, u^{(n)}) = F^\alpha_K(x, u^{(n)}, \rho^{(n)}(x, u^{(n)}))$$

Invariantization

A moving frame induces an invariantization process, denoted ι, that projects functions to invariants, differential operators to invariant differential operators; differential forms to invariant differential forms, etc.

Geometrically, the invariantization of an object is the unique invariant version that has the same cross-section values.

Algebraically, invariantization amounts to replacing the group parameters in the transformed object by their moving frame formulas.

Invariantization

In particular, invariantization of the jet coordinates leads to a complete system of functionally independent differential invariants:

$$\iota(x^i) = H^i \qquad \iota(u^\alpha_J) = I^\alpha_J$$

- Phantom differential invariants: $\quad I^{\alpha_\kappa}_{J_\kappa} = c_\kappa$
- The non-constant invariants form a functionally independent generating set for the differential invariant algebra $\mathcal{I}(\mathcal{G})$

- Replacement Theorem

$$I(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ ) = \iota(\, I(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )\,)$$

$$= I(\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ )$$

$\diamondsuit$ Differential forms $\Longrightarrow$ invariant differential forms

$$\iota(\,dx^i) = \omega^i \qquad\qquad i = 1, \ldots, p$$

$\diamondsuit$ Differential operators $\Longrightarrow$

invariant differential operators

$$\iota\,(\,\mathrm{D}_{x^i}\,) = \mathcal{D}_i \qquad\qquad i = 1, \ldots, p$$

Recurrence Formulae

$\star\star$ Invariantization and differentiation do not commute $\star\star$

The *recurrence formulae* connect the differentiated invariants with their invariantized counterparts:

$$\boxed{\boxed{\mathcal{D}_i I_J^\alpha = I_{J,i}^\alpha + M_{J,i}^\alpha}}$$

$\Longrightarrow$ $M_{J,i}^\alpha$ — correction terms

♡ Once established, the recurrence formulae completely prescribe the structure of the differential invariant algebra $\mathbb{I}(\mathcal{G})$ — thanks to the functional independence of the non-phantom normalized differential invariants.

★ ★ The recurrence formulae can be explicitly determined using only the infinitesimal generators and linear differential algebra!

Korteweg–deVries Equation

Prolonged Symmetry Group Action:

$$T = e^{3\lambda_4}(t + \lambda_1)$$

$$X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2)$$

$$U = e^{-2\lambda_4}(u + \lambda_3)$$

$$U_T = e^{-5\lambda_4}(u_t - \lambda_3 u_x)$$

$$U_X = e^{-3\lambda_4} u_x$$

$$U_{TT} = e^{-8\lambda_4}(u_{tt} - 2\lambda_3 u_{tx} + {\lambda_3}^2 u_{xx})$$

$$U_{TX} = D_X D_T U = e^{-6\lambda_4}(u_{tx} - \lambda_3 u_{xx})$$

$$U_{XX} = e^{-4\lambda_4} u_{xx}$$

$$\vdots$$

Cross Section:

$$T = e^{3\lambda_4}(t + \lambda_1) = 0$$

$$X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1 \lambda_3 + \lambda_2) = 0$$

$$U = e^{-2\lambda_4}(u + \lambda_3) = 0$$

$$U_T = e^{-5\lambda_4}(u_t - \lambda_3 u_x) = 1$$

Moving Frame:

$$\lambda_1 = -t, \qquad \lambda_2 = -x, \qquad \lambda_3 = -u, \qquad \lambda_4 = \tfrac{1}{5}\log(u_t + u u_x)$$

Moving Frame:

$$\lambda_1 = -t, \qquad \lambda_2 = -x, \qquad \lambda_3 = -u, \qquad \lambda_4 = \frac{1}{5}\log(u_t + uu_x)$$

Invariantization:

$$\iota(u_K) = U_K \,|_{\lambda_1=-t,\lambda_2=-x,\lambda_3=-u,\lambda_4=\log(u_t+uu_x)/5}$$

Phantom Invariants:

$$H^1 = \iota(t) = 0$$

$$H^2 = \iota(x) = 0$$

$$I_{00} = \iota(u) = 0$$

$$I_{10} = \iota(u_t) = 1$$

Normalized differential invariants:

$$I_{01} = \iota(u_x) = \frac{u_x}{(u_t + uu_x)^{3/5}}$$

$$I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2uu_{tx} + u^2 u_{xx}}{(u_t + uu_x)^{8/5}}$$

$$I_{11} = \iota(u_{tx}) = \frac{u_{tx} + uu_{xx}}{(u_t + uu_x)^{6/5}}$$

$$I_{02} = \iota(u_{xx}) = \frac{u_{xx}}{(u_t + uu_x)^{4/5}}$$

$$I_{03} = \iota(u_{xxx}) = \frac{u_{xxx}}{u_t + uu_x}$$

$$\vdots$$

Invariantization:

$$
\begin{aligned}
\iota(\, F(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ \ldots\ )\,) \\
&= F(\iota(t), \iota(x), \iota(u), \iota(u_t), \iota(u_x), \iota(u_{tt}), \iota(u_{tx}), \iota(u_{xx}), \ \ldots\ ) \\
&= F(H^1, H^2, I_{00}, I_{10}, I_{01}, I_{20}, I_{11}, I_{02}, \ \ldots\ ) \\
&= F(0, 0, 0, 1, I_{01}, I_{20}, I_{11}, I_{02}, \ \ldots\ )
\end{aligned}
$$

Replacement Theorem:

$$
0 = \iota(u_t + u\, u_x + u_{xxx}) = 1 + I_{03} = \frac{u_t + u u_x + u_{xxx}}{u_t + u u_x}.
$$

Invariant horizontal one-forms:

$$
\omega^1 = \iota(dt) = (u_t + u u_x)^{3/5}\, dt,
$$

$$
\omega^2 = \iota(dx) = -u(u_t + u u_x)^{1/5}\, dt + (u_t + u u_x)^{1/5}\, dx.
$$

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_t) = (u_t + uu_x)^{-3/5} D_t + u(u_t + uu_x)^{-3/5} D_x,$$

$$\mathcal{D}_2 = \iota(D_x) = (u_t + uu_x)^{-1/5} D_x.$$

Commutation formula:

$$[\,\mathcal{D}_1, \mathcal{D}_2\,] = I_{01}\,\mathcal{D}_1$$

Recurrence formulae:

$$\mathcal{D}_1 I_{01} = I_{11} - \tfrac{3}{5} I_{01}^2 - \tfrac{3}{5} I_{01} I_{20}, \qquad \mathcal{D}_2 I_{01} = I_{02} - \tfrac{3}{5}\, I_{01}^3 - \tfrac{3}{5}\, I_{01} I_{11},$$

$$\mathcal{D}_1 I_{20} = I_{30} + 2I_{11} - \tfrac{8}{5} I_{01} I_{20} - \tfrac{8}{5} I_{20}^2, \qquad \mathcal{D}_2 I_{20} = I_{21} + 2I_{01} I_{11} - \tfrac{8}{5} I_{01}^2 I_{20} - \tfrac{8}{5} I_{11} I_{20},$$

$$\mathcal{D}_1 I_{11} = I_{21} + I_{02} - \tfrac{6}{5} I_{01} I_{11} - \tfrac{6}{5} I_{11} I_{20}, \qquad \mathcal{D}_2 I_{11} = I_{12} + I_{01} I_{02} - \tfrac{6}{5} I_{01}^2 I_{11} - \tfrac{6}{5} I_{11}^2,$$

$$\mathcal{D}_1 I_{02} = I_{12} - \tfrac{4}{5} I_{01} I_{02} - \tfrac{4}{5} I_{02} I_{20}, \qquad \mathcal{D}_2 I_{02} = I_{03} - \tfrac{4}{5} I_{01}^2 I_{02} - \tfrac{4}{5} I_{02} I_{11},$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

Generating differential invariants:

$$I_{01} = \iota(u_x) = \frac{u_x}{(u_t + u u_x)^{3/5}}, \qquad I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2 u u_{tx} + u^2 u_{xx}}{(u_t + u u_x)^{8/5}}.$$

Fundamental syzygy:

$$\mathcal{D}_1^2 I_{01} + \tfrac{3}{5}\, I_{01} \mathcal{D}_1 I_{20} - \mathcal{D}_2 I_{20} + \left(\tfrac{1}{5}\, I_{20} + \tfrac{19}{5}\, I_{01} \right) \mathcal{D}_1 I_{01}$$
$$- \mathcal{D}_2 I_{01} - \tfrac{6}{25}\, I_{01} I_{20}^2 - \tfrac{7}{25}\, I_{01}^2 I_{20} + \tfrac{24}{25}\, I_{01}^3 = 0.$$

Lie–Tresse–Kumpera Example

$$\boxed{X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}}$$

Horizontal coframe

$$d_H X = f_x \, dx, \qquad d_H Y = dy,$$

Implicit differentiations

$$\mathrm{D}_X = \frac{1}{f_x} \mathrm{D}_x, \qquad \mathrm{D}_Y = \mathrm{D}_y.$$

Prolonged pseudo-group transformations on surfaces $S \subset \mathbb{R}^3$

$$X = f \qquad Y = y \qquad U = \frac{u}{f_x}$$

$$U_X = \frac{u_x}{f_x^{\,2}} - \frac{u\, f_{xx}}{f_x^{\,3}} \qquad U_Y = \frac{u_y}{f_x}$$

$$U_{XX} = \frac{u_{xx}}{f_x^{\,3}} - \frac{3u_x\, f_{xx}}{f_x^{\,4}} - \frac{u\, f_{xxx}}{f_x^{\,4}} + \frac{3u\, f_{xx}^{\,2}}{f_x^{\,5}}$$

$$U_{XY} = \frac{u_{xy}}{f_x^{\,2}} - \frac{u_y\, f_{xx}}{f_x^{\,3}} \qquad U_{YY} = \frac{u_{yy}}{f_x}$$

$\Longrightarrow$ action is free at every order.

Coordinate cross-section

$$X = f = 0, \quad U = \frac{u}{f_x} = 1, \quad U_X = \frac{u_x}{f_x^{\,2}} - \frac{u\, f_{xx}}{f_x^{\,3}} = 0, \quad U_{XX} = \cdots = 0.$$

Moving frame

$$f = 0, \qquad f_x = u, \qquad f_{xx} = u_x, \qquad f_{xxx} = u_{xx}.$$

Differential invariants

$$U_Y \longmapsto J = \frac{u_y}{u}$$

$$U_{XY} \longmapsto J_1 = \frac{u u_{xy} - u_x u_y}{u^3} \qquad U_{YY} \longmapsto J_2 = \frac{u_{yy}}{u}$$

Invariant horizontal forms

$$d_H X = f_x \, dx \longmapsto u \, dx, \qquad d_H Y = dy \longmapsto dy,$$

Invariant differentiations

$$\mathcal{D}_1 = \frac{1}{u} \, \mathrm{D}_x \qquad \mathcal{D}_2 = \mathrm{D}_y$$

Higher order differential invariants: $\quad \mathcal{D}_1^m \, \mathcal{D}_2^n \, J$

$$J_{,1} = \mathcal{D}_1 J = \frac{u u_{xy} - u_x u_y}{u^3} = J_1,$$

$$J_{,2} = \mathcal{D}_2 J = \frac{u u_{yy} - u_y^2}{u^2} = J_2 - J^2.$$

Recurrence formulae:

$$\begin{aligned} \mathcal{D}_1 J &= J_1, & \qquad \mathcal{D}_2 J &= J_2 - J^2, \\ \mathcal{D}_1 J_1 &= J_3, & \qquad \mathcal{D}_2 J_1 &= J_4 - 3\, J\, J_1, \\ \mathcal{D}_1 J_2 &= J_4, & \qquad \mathcal{D}_2 J_2 &= J_5 - J\, J_2, \end{aligned}$$

The Master Recurrence Formula

$$\boxed{\; d_H\, I^\alpha_J \;=\; \sum_{i=1}^{p} (\, \mathcal{D}_i I^\alpha_J \,)\, \omega^i \;=\; \sum_{i=1}^{p} I^\alpha_{J,i}\, \omega^i + \widehat{\psi}\,{}^\alpha_J \;}$$

where

$$\widehat{\psi}\,{}^\alpha_J = \iota(\, \widehat{\varphi}\,{}^\alpha_J \,) = \Phi^\alpha_J(\; \ldots \;\; H^i \;\; \ldots \;\; I^\alpha_J \;\; \ldots \;\; ; \;\; \ldots \;\; \gamma^b_A \;\; \ldots \;)$$

are the invariantized prolonged vector field coefficients, which are particular linear combinations of

$\gamma^b_A \;=\; \iota(\zeta^b_A)$ — invariantized Maurer–Cartan forms prescribed by the invariantized prolongation map.

- The invariantized Maurer–Cartan forms are subject to the *invariantized determining equations*:

$$\mathcal{L}(H^1, \ldots, H^p, I^1, \ldots, I^q, \;\ldots\;, \gamma^b_A, \;\ldots\;) = 0$$

$$\boxed{\quad d_H\, I^\alpha_J \;=\; \sum_{i=1}^{p} I^\alpha_{J,i}\,\omega^i + \widehat{\psi}\,^\alpha_J(\;\ldots\;\gamma^b_A\;\ldots\;)\quad}$$

Step 1: Solve the phantom recurrence formulas

$$0 = d_H\, I^\alpha_J \;=\; \sum_{i=1}^{p} I^\alpha_{J,i}\,\omega^i + \widehat{\psi}\,^\alpha_J(\;\ldots\;\gamma^b_A\;\ldots\;)$$

for the invariantized Maurer–Cartan forms:

$$\gamma^b_A = \sum_{i=1}^{p} J^b_{A,i}\,\omega^i \tag{$*$}$$

Step 2: Substitute $(*)$ into the non-phantom recurrence formulae to obtain the explicit correction terms.

- $\diamondsuit$ Only uses linear differential algebra based on the specification of cross-section.

- $\heartsuit$ Does not require explicit formulas for the moving frame, the differential invariants, the invariant differential operators, or even the Maurer–Cartan forms!

The Korteweg–deVries Equation (continued)

Recurrence formula:

$$dI_{jk} = I_{j+1,k}\omega^1 + I_{j,k+1}\omega^2 + \iota(\varphi^{jk})$$

Invariantized Maurer–Cartan forms:

$$\iota(\tau) = \lambda, \qquad \iota(\xi) = \mu, \qquad \iota(\varphi) = \psi = \nu, \qquad \iota(\tau_t) = \psi^t = \lambda_t, \qquad \ldots$$

Invariantized determining equations:

$$\lambda_x = \lambda_u = \mu_u = \nu_t = \nu_x = 0$$

$$\nu = \mu_t \qquad \nu_u = -\,2\,\mu_x = -\,\tfrac{2}{3}\,\lambda_t$$

$$\lambda_{tt} = \lambda_{tx} = \lambda_{xx} = \ \cdots\ = \nu_{uu} = \ \cdots\ = 0$$

Invariantizations of prolonged vector field coefficients:

$$\iota(\tau) = \lambda, \qquad \iota(\xi) = \mu, \qquad \iota(\varphi) = \nu, \qquad \iota(\varphi^t) = -I_{01}\nu - \tfrac{5}{3}\,\lambda_t,$$

$$\iota(\varphi^x) = -I_{01}\lambda_t, \qquad \iota(\varphi^{tt}) = -2I_{11}\nu - \tfrac{8}{3}I_{20}\lambda_t, \qquad \ldots$$

Phantom recurrence formulae:

$$
\begin{aligned}
0 &= d_H\, H^1 = \omega^1 + \lambda, \\
0 &= d_H\, H^2 = \omega^2 + \mu, \\
0 &= d_H\, I_{00} = I_{10}\omega^1 + I_{01}\omega^2 + \psi = \omega^1 + I_{01}\omega^2 + \nu, \\
0 &= d_H\, I_{10} = I_{20}\omega^1 + I_{11}\omega^2 + \psi^t = I_{20}\omega^1 + I_{11}\omega^2 - I_{01}\nu - \tfrac{5}{3}\,\lambda_t,
\end{aligned}
$$

$$
\Longrightarrow \quad \text{Solve for} \qquad \lambda = -\omega^1, \qquad \mu = -\omega^2, \qquad \nu = -\omega^1 - I_{01}\omega^2,
$$

$$
\lambda_t = \tfrac{3}{5}\,(I_{20} + I_{01})\omega^1 + \tfrac{3}{5}\,(I_{11} + I_{01}^2)\omega^2.
$$

Non-phantom recurrence formulae:

$$
\begin{aligned}
d_H\, I_{01} &= I_{11}\omega^1 + I_{02}\omega^2 - I_{01}\lambda_t, \\
d_H\, I_{20} &= I_{30}\omega^1 + I_{21}\omega^2 - 2I_{11}\nu - \tfrac{8}{3}I_{20}\lambda_t, \\
d_H\, I_{11} &= I_{21}\omega^1 + I_{12}\omega^2 - I_{02}\nu - 2I_{11}\lambda_t, \\
d_H\, I_{02} &= I_{12}\omega^1 + I_{03}\omega^2 - \tfrac{4}{3}I_{02}\lambda_t, \\
&\ \ \vdots
\end{aligned}
$$

$$\begin{aligned}
\mathcal{D}_1 I_{01} &= I_{11} - \tfrac{3}{5} I_{01}^2 - \tfrac{3}{5} I_{01} I_{20}, & \mathcal{D}_2 I_{01} &= I_{02} - \tfrac{3}{5}\, I_{01}^3 - \tfrac{3}{5}\, I_{01} I_{11}, \\
\mathcal{D}_1 I_{20} &= I_{30} + 2 I_{11} - \tfrac{8}{5} I_{01} I_{20} - \tfrac{8}{5} I_{20}^2, & \mathcal{D}_2 I_{20} &= I_{21} + 2 I_{01} I_{11} - \tfrac{8}{5} I_{01}^2 I_{20} - \tfrac{8}{5} I_{11} I_{20}, \\
\mathcal{D}_1 I_{11} &= I_{21} + I_{02} - \tfrac{6}{5} I_{01} I_{11} - \tfrac{6}{5} I_{11} I_{20}, & \mathcal{D}_2 I_{11} &= I_{12} + I_{01} I_{02} - \tfrac{6}{5} I_{01}^2 I_{11} - \tfrac{6}{5} I_{11}^2, \\
\mathcal{D}_1 I_{02} &= I_{12} - \tfrac{4}{5} I_{01} I_{02} - \tfrac{4}{5} I_{02} I_{20}, & \mathcal{D}_2 I_{02} &= I_{03} - \tfrac{4}{5} I_{01}^2 I_{02} - \tfrac{4}{5} I_{02} I_{11}, \\
&\vdots & &\vdots
\end{aligned}$$

Lie–Tresse–Kumpera Example (continued)

$$\boxed{X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}}$$

Phantom recurrence formulae:

$$0 = dH = \varpi^1 + \gamma, \qquad 0 = dI_{10} = J_1\,\varpi^2 + \vartheta_1 - \gamma_2,$$

$$0 = dI_{00} = J\,\varpi^2 + \vartheta - \gamma_1, \qquad 0 = dI_{20} = J_3\,\varpi^2 + \vartheta_3 - \gamma_3,$$

Solve for pulled-back Maurer–Cartan forms:

$$\gamma = -\,\varpi^1, \qquad \gamma_2 = J_1\,\varpi^2 + \vartheta_1,$$

$$\gamma_1 = J\,\varpi^2 + \vartheta, \qquad \gamma_3 = J_3\,\varpi^2 + \vartheta_3,$$

Recurrence formulae: $\qquad dy = \varpi^2$

$$dJ = J_1\,\varpi^1 + (J_2 - J^2)\,\varpi^2 + \vartheta_2 - J\,\vartheta,$$

$$dJ_1 = J_3\,\varpi^1 + (J_4 - 3\,J\,J_1)\,\varpi^2 + \vartheta_4 - J\,\vartheta_1 - J_1\,\vartheta,$$

$$dJ_2 = J_4\,\varpi^1 + (J_5 - J\,J_2)\,\varpi^2 + \vartheta_5 - J_2\,\vartheta,$$