Recursive Moving Frames

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1. Introduction.

In the equivariant method of moving frames for finite-dimensional Lie group actions, originally formulated in [**8**], a crucial insight was to delay initiating the analysis until the group action has been prolonged to a sufficiently high order jet space in order that it becomes (locally) free. Once freeness is attained, the specification of a local cross-section to the prolonged group orbits enables one to simultaneously normalize all the group parameters, and thereby produce a moving frame, defined as an equivariant map from (an open subset of) the jet space back to the group. With a moving frame in hand, one immediately produces complete systems of differential invariants, invariant differential operators, invariant differential forms, etc. The equivariant method is endowed with a number of innate advantages over the classical Cartan approach in that, not only is it much simpler to formulate and implement, but, moreover, it can be readily applied to (almost) arbitrary group actions, thereby moving far beyond the special geometries handled by classical moving frames, [**5, 9**]. The most important new contribution is the powerful recurrence formulae, that relate the normalized and differentiated differential invariants and invariant differential forms, producing what Mansfield, [**20**] calls the "symbolic invariant calculus", and forming the foundation of the present work. Subsequently, the equivariant moving frame method was extended to infinite-dimensional Lie pseudo-group actions, [**33, 34**], based on a new, direct approach to the Maurer–Cartan forms and consequential structure theory, [**32, 35**]. The claim that the equivariant method is the "correct" formulation of

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moving frames is borne out by the ever expanding range of new fields of application, including joint invariants and joint differential invariants, [**26**], invariant numerical schemes, [**27**, **13**], object recognition and symmetry detection in image processing, [**4**, **10**], classical invariant theory, [**2**, **24**], invariant variational problems and invariant submanifold flows, [**17**, **29**], Poisson geometry and integrable systems, [**21**], Laplace invariants of differential operators, [**36**], invariants and covariants of Killing tensors arising in general relativity, [**6**, **22**], and invariants of Lie algebras, with applications to the classification of subalgebras and in quantum mechanics, [**3**]. Surveys of recent developments can be found in [**20**, **30**].

While the idea of fully prolonging the group action until the onset of freeness proved to be of crucial importance for developing the general theory and basic algorithms, the direct computation of higher order prolonged group actions, which relies on implicit differentiation, can rapidly overwhelm symbolic software, thus limiting the method's practical scope. In the classical version, e.g., [**5**, **9**, **12**], one instead incrementally normalizes group parameters order by order, producing a succession of what Cartan calls (partial) moving frames at each jet space order. Although much harder to rigorously justify and then develop into a fully general, practical method — in part due to the appearance of a succession of intricate nondegeneracy conditions, that often prove to be irrelevant to the eventual freeness requirement — the recursive approach remains attractive from a computational standpoint, and is thus worth recasting into the more powerful equivariant framework. Indeed, in the original paper [**8**; Section 17], some indications of a recursive equivariant algorithm were presented in the context of a specific example — the equi-affine geometry of plane curves. As noted there, the key complication is that one cannot, in general, partially normalize the implicit differentiation operators and retain their invariance, owing to the appearance of additional connection-like terms in the underlying partially reduced recurrence formulae. This example inspired Kogan, [**14**, **15**, **16**], to develop two recursive/inductive algorithms for equivariant moving frame computations which, however, place restrictions on the allowable group actions. Her recursive algorithm requires the existence of a *slice*, meaning a cross-section that has constant isotropy subgroup at each point, to the prolonged group orbits. Her inductive algorithm relies on a factorization of the full group into a product of subgroups with discrete intersection, and then relates the moving frame and invariants for the full group with those of a subgroup. To date, neither method has been extended to infinite-dimensional Lie pseudo-group actions.

The goal of this paper is to propose an unrestricted, general algorithm for recursively constructing equivariant moving frames — for both finite-dimensional Lie group actions and infinite-dimensional Lie pseudo-groups. The method can also be adapted to provide the explicit relationships to the moving frames and invariants of any of its Lie subgroups. The key insight is to base the computations on the lifted recurrence formulae and the recursively normalized Maurer–Cartan forms, rather than the implicit differentiation operators employed in the standard prolongation approach. As such, the method is somewhat closer in spirit to the differential form-based moving frame method proposed in [**7**], although the insights and results coming from the equivariant approach of [**8**] — particularly the recurrence formulae — are essential to the success of our new recursive approach. We illustrate the method with several reletively simple finite- and infinite-dimensional examples, leaving more substantial applications of these techniques to future works.

In this paper, we assume that the reader is familiar with jet bundles and contact forms, [**23**], groupoids, [**19**], the equivariant moving frame method, [**8, 20, 30**], its extension to Lie pseudo-groups, [**11, 31, 32, 33, 34, 35**], as well as the variational bicomplex, [**1, 37**], and its moving frame invariantization, [**17**].

2. Lie Group Actions.

For simplicity, we begin by developing the finite-dimensional version. Let G be a Lie group acting on a smooth manifold M. The trivial principal bundle $B = G \times M$ carries the structure of a groupoid, [**19, 38**], with *source map* $\sigma(g,z) = z$ and *target map* $Z = \tau(g,z) = g \cdot z$. We will adopt the Cartan convention that employs lower case letters z, x, u, etc., for source coordinates, and capital letters Z, X, U, etc., for the corresponding target coordinates, throughout.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ denote the infinitesimal generators of the action of G on M, which form a basis for its Lie algebra $\mathfrak{g}$. Let $\mu^1, \ldots, \mu^r$ denote the corresponding dual basis of right-invariant Maurer–Cartan one-forms, which we identify with their pull-backs to B via the projection $B \to G$. The Cartesian product structure splits the cotangent bundle $T^*B \simeq T^*G \oplus T^*M$ into a group component, spanned by the Maurer–Cartan forms μ^κ, and a manifold component, spanned by the (pull backs of the) differentials dz^i of local coordinates $z = (z^1, \ldots, z^m)$ on M. This induces a corresponding splitting of the differential d on B into manifold and group components, written $d = d_M + d_G$.

By the *lift* of a function $f\colon M \to \mathbb{R}$ we mean its pull back to B via the target map, so $\lambda(f)(g,z) = \tau^* f(g,z) = f(g \cdot z)$. In particular, the lift of the source coordinates z are the corresponding target coordinates $Z = g \cdot z$, written out as functions of the group parameters and source coordinates:

$$Z^a = \lambda(z^a), \qquad a = 1, \ldots, m. \tag{2.1}$$

More generally, the *lift* $\lambda(\omega)$ of a differential form ω on M is defined as the purely manifold component of its target pull-back to B, written $\lambda(\omega) = \pi_M(\tau^*\omega)$, and obtained formally by setting all group differentials (or, equivalently, Maurer–Cartan forms) in the pull-back to zero. In particular,

$$\lambda(d\omega) = d_M \lambda(\omega). \tag{2.2}$$

The basic formula for the group differential of a lifted function or form provides the key to the all-important *lifted recurrence formulae*. See [**17**; Lemma 5.1] for a proof.

Proposition 2.1. *Let ω be a differential form on M. Then*

$$d_G \lambda(\omega) = \sum_{\kappa=1}^{r} \mu^\kappa \wedge \lambda(\mathbf{v}_\kappa(\omega)), \tag{2.3}$$

and therefore,

$$d\lambda(\omega) = \lambda(d\omega) + \sum_{\kappa=1}^{r} \mu^\kappa \wedge \lambda(\mathbf{v}_\kappa(\omega)). \tag{2.4}$$

Remark: Assuming local effectiveness, if we let ω range over the coordinate functions $z^1, \ldots, z^m$ on M, we can use the system of equations resulting from (2.3) to read off the explicit formulas for the Maurer–Cartan forms. Examples of this procedure appear below.

We are interested in the induced action of G on p-dimensional submanifolds $S \subset M$, where $1 \leq p < m$ is fixed. For $0 \leq n \leq \infty$, let $\mathrm{J}^n = \mathrm{J}^n(M,p)$ denote the (extended) submanifold jet bundle of order n, [**23**]. For $n \geq k$, we let $\widetilde{\pi}^n_k\colon \mathrm{J}^n \to \mathrm{J}^k$ denote the natural projection. Given local coordinates $z = (x,u) = (x^1,\ldots,x^p,u^1,\ldots,u^q)$, with $q = m - p$, we view the x's as independent variables and the u's as dependent variables, with the submanifolds that are transverse to the vertical fibers $\{x = x_0\}$ being locally given as the graphs of functions $u = f(x)$. The induced coordinates on J^n are denoted by $z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )$, where u^α_J denote the derivative coordinates of orders $0 \leq \#J \leq n$. In coordinates,

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^p u^\alpha_{J,i}\, dx^i, \qquad \alpha = 1,\ldots,q, \qquad 0 \leq \#J, \tag{2.5}$$

are the *basic contact one-forms*, spanning the intrinsic *contact* or *vertical subbundle* $\mathcal{C} \subset T^*\mathrm{J}^\infty$. The complementary *horizontal subbundle* $\mathcal{H} \subset T^*\mathrm{J}^\infty$ is spanned by the coordinate one-forms $dx^1,\ldots,dx^p$, and so relies on a choice of independent variables. The splitting $T^*\mathrm{J}^\infty = \mathcal{H} \oplus \mathcal{C}$ induces the variational bicomplex structure on J^∞, [**1, 17**]. The jet differential splits into horizontal and contact components, $d_J = d_H + d_V$, so that, in particular,

$$d_H F = \sum_{i=1}^p (D_i F)\, dx^i, \qquad d_V F = \sum_{\alpha,J} \frac{\partial F}{\partial u^\alpha_J}\, \theta^\alpha_J\,, \tag{2.6}$$

for any differential function $F(x,u^{(n)})$, where $D_1,\ldots,D_p$ are the usual *total derivative operators*.

Given a Lie group action on M, the *lifted horizontal coframe* consists of the horizontal differentials of the target independent variables:

$$\omega^i = d_H X^i = \sum_{j=1}^p (D_j X^i)\, dx^j, \qquad i = 1,\ldots,p. \tag{2.7}$$

As long as the *total Jacobian matrix* $DX = \big(D_j X^i \big)$ is nonsingular[†], these span the space of horizontal forms. The dual *implicit differentiation operators* $\mathcal{D}_1,\ldots,\mathcal{D}_p$ are defined so so that

$$d_H F = \sum_{i=1}^p (D_i F)\, dx^i = \sum_{i=1}^p (\mathcal{D}_i F)\, \omega^i, \tag{2.8}$$

for any differential function $F(x,u^{(n)})$.

There is an induced action of G on J^n, called the n^{th} *prolongation* of G, and written $Z^{(n)} = g^{(n)} \cdot z^{(n)}$ for $z^{(n)} \in \mathrm{J}^n$ and $g \in G$. The prolonged action can be calculated

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[†] At singularities, one needs to introduce an alternative set of independent and dependent variables.

by successively applying the implicit differentiation operators to the target dependent variables U^α, whereby

$$U^\alpha_J = \mathcal{D}_J U^\alpha, \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0. \tag{2.9}$$

Since implicit differentiation involves the entries of the inverse of the total Jacobian matrix, the explicit formulas for the prolonged action (2.9) rapidly lead to the unwieldy expression swell that overwhelms the automated computation of substantial examples.

A Lie group G is said to act *effectively* if the only group element that fixes every point, $g \cdot z = z$, is the identity $g = e$, and *locally effectively* if the set of such group elements forms a discrete normal subgroup. According to [**25**], if G acts locally effectively on all open subsets $W \subset M$, then, for $n \gg 0$ sufficiently large, the action is locally free[†] on a dense open subset of the submanifold jet space J^n. Once the group action becomes (locally) free, one specifies a moving frame by the choice of a local *cross-section* $K^n \subset \mathrm{J}^n$, that is, a submanifold of complementary dimension that intersects the (regular) prolonged group orbits transversally. Typically, but not always, one chooses a coordinate cross-section, fixed by setting an appropriate collection of jet coordinates to adroitly selected constant values. The moving frame is then obtained by solving the corresponding *normalization equations* $Z^{(n)} = g^{(n)} \cdot z^{(n)} \in K^n$ for the group parameters $g = \rho(z^{(n)})$. It is easily seen that ρ is a right-equivariant map, meaning $\rho(g^{(n)} \cdot z^{(n)}) = \rho(z^{(n)}) \cdot g^{-1}$, where defined, for all $g \in G$. In classical geometrical situations, [**5**, **9**], the moving frame can be identified with the left-equivariant counterpart obtained by composing with the group inversion: $\widetilde{\rho}(z^{(n)}) = \rho(z^{(n)})^{-1}$.

In the recursive approach to be developed here, one employs a succession of group parameter normalizations at each order, based on a sequence of cross-sections $K^k \subset \mathrm{J}^k$ for $0 \leq k \leq n$, where n is the order of freeness, to the regular prolonged group orbits at each order, satisfying the compatibility condition $K^{k-1} = \widetilde{\pi}^k_{k-1}(K^k)$ for $1 \leq k \leq n$. The normalization equations $Z^{(k)} = g^{(k)} \cdot z^{(k)} \in K^k$ at order k will be solved for some of the group parameters in terms of the k^{th} order jet coordinates and the remaining group parameters, resulting in a suitably right-equivariant *partial moving frame* of order k. (See below for a more formal definition.) Compatibility implies that one can retain the already established lower order normalizations when proceeding. The key to the efficacy of the algorithm is that one can then compute the resulting partially normalized prolonged group action at the next highest order $k+1$ by making use of the partially normalized lifted recurrence relations at order k along with the explicit formulas for the partially normalized Maurer–Cartan forms. However, before delving further into the theoretical framework, it is best to work through a couple of examples.

Example 2.2. *Equi-affine plane curves*: Let $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$ act on $M = \mathbb{R}^2$ by unimodular affine transformations:

$$X = \alpha\, x + \beta\, u + a, \qquad U = \gamma\, x + \delta\, u + b, \qquad \text{where} \qquad \alpha\,\delta - \beta\,\gamma = 1. \tag{2.10}$$

[†] In all known examples, the prolonged action eventually becomes free on an open subset of J^n. Unfortunately, there is as yet no proof of this observation.

The infinitesimal generators are

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_u, \qquad \mathbf{v}_3 = -x\,\partial_x + u\,\partial_u, \qquad \mathbf{v}_4 = u\,\partial_x, \qquad \mathbf{v}_5 = x\,\partial_u. \tag{2.11}$$

Let $\mu^1, \ldots, \mu^5$ be the dual Maurer–Cartan forms. We can recover their formulae by calculating the group differentials of the lifted coordinates X, U, as given in (2.10), and comparing with the expressions resulting from formula (2.3):

$$\begin{aligned} \mu^1 - X\,\mu^3 + U\,\mu^4 &= d_G X = x\,d\alpha + u\,d\beta + da \\ &= \big[\,\delta\,(X-a) - \beta\,(U-b)\,\big]\,d\alpha + \big[-\gamma\,(X-a) + \alpha\,(U-b)\,\big]\,d\beta + da, \\ \mu^2 + U\,\mu^3 + X\,\mu^5 &= d_G U = x\,d\gamma + u\,d\delta + db \\ &= \big[\,\delta\,(X-a) - \beta\,(U-b)\,\big]\,d\gamma + \big[-\gamma\,(X-a) + \alpha\,(U-b)\,\big]\,d\delta + db. \end{aligned} \tag{2.12}$$

Comparing the terms involving the various powers of X and U, we immediately deduce the well-known formulas for the right-invariant Maurer–Cartan forms on SA(2):

$$\begin{gathered} \mu^1 = da + a\,\mu^3 - b\,\mu^4, \qquad \mu^2 = db - a\,\mu^5 - b\,\mu^3, \\ \mu^3 = \gamma\,d\beta - \delta\,d\alpha = \alpha\,d\delta - \beta\,d\gamma, \qquad \mu^4 = \alpha\,d\beta - \beta\,d\alpha, \qquad \mu^5 = \delta\,d\gamma - \gamma\,d\delta. \end{gathered} \tag{2.13}$$

To construct an equivariant moving frame, we must prolong the action to a sufficiently high order curve jet space $\mathrm{J}^n = \mathrm{J}^n(\mathbb{R}^2, 1)$ in order that the action becomes (locally) free. In keeping with Cartan's convention, we will use $x, u, u_x, u_{xx}, u_{xxx}, \ldots$, to denote the (source) jet coordinates, and $X, U, U_X, U_{XX}, U_{XXX}, \ldots$, to denote the corresponding target jet coordinates. The latter can be obtained by implicit differentiation, but the higher order formulas become more and more unwieldy. While the direct equivariant method would require their determination ab initio, the recursive method, as we will see, completely avoids this calculation.

To streamline the presentation, we will ignore contact forms from here on, since these do not play a role in determining the moving frame and the differential invariants. (Contact forms do, however, play a key role in the analysis of invariant variational problems and invariant flows, [**29**]. Our method will also produce them with a bit of extra work.) We will use $\equiv$ to indicate equality modulo contact forms, so that $\omega \equiv \zeta$ when $\omega - \zeta$ is a contact form. In particular, the (jet) differential of a differential function $F\colon \mathrm{J}^n \to \mathbb{R}$ is equivalent, modulo contact forms, to its horizontal component:

$$dF \equiv d_H F = D_x F \; dx,$$

where D_x is the usual total derivative. Also, keep in mind that the lift operation takes contact forms to contact forms.

We begin the recursive algorithm by analyzing the differentials of the target coordinates. According to the lifted recurrence formula (2.4),

$$dX = d\,\lambda(x) = \lambda(dx) + \sum_{\nu=1}^{5} \lambda[\mathbf{v}_\nu(x)]\,\mu^\nu = \omega + \mu^1 - X\,\mu^3 + U\,\mu^4, \tag{2.14}$$

where, again modulo contact forms,

$$\lambda(dx) \equiv \omega = d_H X = (\alpha + \beta\,u_x)\,dx \tag{2.15}$$

is the basic lifted horizontal form. Similarly,

$$\begin{aligned} dU = d\,\lambda(u) = \lambda(du) + \sum_{\nu=1}^{5} \lambda[\mathbf{v}_\nu(u)]\,\mu^\nu \\ \equiv \lambda(u_x\,dx) + \mu^2 + U\,\mu^3 + X\,\mu^5 = U_X\,\omega + \mu^2 + U\,\mu^3 + X\,\mu^5, \end{aligned} \tag{2.16}$$

has horizontal component

$$U_X\,\omega = d_H U = (\gamma + \delta\,u_x)\,dx, \tag{2.17}$$

which, in view of (2.15), produces the formula

$$U_X = \frac{\gamma + \delta\,u_x}{\alpha + \beta\,u_x} \tag{2.18}$$

for the first order prolonged action.

The higher order lifted recurrence formulas are obtained using (2.4), where the group now acts on the submanifold jet space J^n, and so we must prolong the infinitesimal generators (2.11) using the standard formula, [**23**]:

$$\begin{aligned} &\mathbf{v}_1 = \partial_x, \\ &\mathbf{v}_2 = \partial_u, \\ &\mathbf{v}_3 = -x\,\partial_x + u\,\partial_u + 2u_x\,\partial_{u_x} + 3u_{xx}\,\partial_{u_{xx}} + 4u_{xxx}\,\partial_{u_{xxx}} + 5u_{xxxx}\,\partial_{u_{xxxx}} + \cdots, \\ &\mathbf{v}_4 = u\,\partial_x - u_x^2\,\partial_{u_x} - 3u_xu_{xx}\,\partial_{u_{xx}} - (4u_xu_{xxx} + 3u_{xx}^2)\,\partial_{u_{xxx}} \\ &\qquad\qquad - (5u_xu_{xxxx} + 10u_{xx}u_{xxx})\,\partial_{u_{xxxx}} + \cdots, \\ &\mathbf{v}_5 = x\,\partial_u + \partial_{u_x}. \end{aligned} \tag{2.19}$$

Consequently, the lifted recurrence formulae, up to order 4, are

$$\begin{aligned} &dX \equiv \omega + \mu^1 - X\,\mu^3 + U\,\mu^4, \\ &dU \equiv U_X\,\omega + \mu^2 + U\,\mu^3 + X\,\mu^5, \\ &dU_X \equiv U_{XX}\,\omega + 2U_X\,\mu^3 - U_X^2\,\mu^4 + \mu^5, \\ &dU_{XX} \equiv U_{XXX}\,\omega + 3U_{XX}\,\mu^3 - 3U_X\,U_{XX}\,\mu^4, \\ &dU_{XXX} \equiv U_{XXXX}\,\omega + 4U_{XX}\,\mu^3 - (4U_X\,U_{XXX} + 3U_{XX}^2)\,\mu^4, \\ &dU_{XXXX} \equiv U_{XXXXX}\,\omega + 5U_{XX}\,\mu^3 - (5U_X\,U_{XXXX} + 10U_{XX}\,U_{XXX})\,\mu^4, \end{aligned} \tag{2.20}$$

We will retain the same Cartan-style notation for the recursively normalized functions and forms. Eventually, once all the group parameters have been normalized, these will reduce to the invariantized functions and forms, which are interrelated by the fully normalized recurrence formulae, obtained by recursively reducing (2.20). The key feature of the recursive approach is that there is no need to a priori compute explicit formulas for the higher order lifted quantities, that is, the formulae for the prolonged group action; their reduced expressions will appear, in much simpler fashion, at the appropriate stage of the procedure.

We will work with the normalizations that produce the standard cross-section leading to the classical moving frame, [**8**, **9**]. At order 0, since SA(2) acts transitively on M, a cross-section consists of a single point, and we set $K^0 = \{x = u = 0\}$. The corresponding order 0 normalization equations $Z = g \cdot z \in K^0$ are obtained by setting $X = U = 0$, producing the formulae

$$a = -\alpha\, x - \beta\, u, \qquad b = -\gamma\, x - \delta\, u, \tag{2.21}$$

for the first two group parameters. Since we've normalized X and U to be constant, their differentials now vanish: $dX = dU = 0$; substituting into the first two lifted recurrence formulae (2.20) produces the corresponding formulae for the partially normalized Maurer–Cartan forms

$$\mu^1 = -\omega, \qquad \mu^2 = -U_X\, \omega - \vartheta \equiv -U_X\, \omega, \tag{2.22}$$

which can also be checked directly. The other 3 Maurer–Cartan forms are not affected by (2.21).

We now proceed to the order 1 moving frame, based on the compatible cross-section $K^1 = \{x = u = u_x = 0\}$. The corresponding first order normalization requires

$$0 = U_X = \frac{\gamma + \delta\, u_x}{\alpha + \beta\, u_x}, \qquad \text{whence} \qquad \gamma = -\delta\, u_x \quad \text{and} \quad \alpha = \frac{1}{\delta} - \beta\, u_x, \tag{2.23}$$

the latter equation coming from the unimodularity constraint $\alpha\,\delta - \beta\,\gamma = 1$. Consequently, the lifted horizontal form (2.15) reduces to

$$\omega = \frac{dx}{\delta} + \beta\,\theta \equiv \frac{dx}{\delta}\,. \tag{2.24}$$

Substituting (2.23, 24) into (2.13) produces the partially normalized Maurer–Cartan forms

$$\begin{aligned}
\mu^3 &= \beta\,\delta\, du_x + \frac{d\delta}{\delta} \equiv \beta\,\delta\, u_{xx}\, dx + \frac{d\delta}{\delta} \equiv \beta\,\delta^2\, u_{xx}\,\omega + \frac{d\delta}{\delta}\,, \\
\mu^4 &= \beta^2\, du_x + \frac{d\beta}{\delta} + \frac{\beta\, d\delta}{\delta^2} \equiv \beta^2\, u_{xx}\, dx + \frac{d\beta}{\delta} + \frac{\beta\, d\delta}{\delta^2} \equiv \beta^2\,\delta\, u_{xx}\omega + \frac{d\beta}{\delta} + \frac{\beta\, d\delta}{\delta^2}\,, \\
\mu^5 &= -\delta^2\, du_x \equiv -\delta^2\, u_{xx}\, dx \equiv -\delta^3\, u_{xx}\,\omega.
\end{aligned} \tag{2.25}$$

On the other hand, our normalization of $U_X = 0$ implies $dU_X = 0$, and so the first order recurrence formula in (2.20) reduces to

$$\mu^5 = -U_{XX}\,\omega - \vartheta_1 \equiv -U_{XX}\,\omega.$$

Comparison with the last equation in (2.25) leads to the formula for the partially normalized second order jet coordinate:

$$U_{XX} = \delta^3\, u_{xx}. \tag{2.26}$$

The advantage of the recursive approach is that we did *not* need to compute the original, more complicated second order prolonged action in order to arrive at this expression.

The order 2 cross-section sets $u_{xx} = 1$; solving the consequent normalization equation $U_{XX} = 1$ using the partially normalized formula (2.26) produces

$$\delta = \frac{1}{\sqrt[3]{u_{xx}}}\,, \qquad \text{and thus, from (2.24),} \qquad \omega = \sqrt[3]{u_{xx}}\; dx, \tag{2.27}$$

which recovers the usual equi-affine arc length element. Substituting the formula for δ into (2.25) produces

$$\mu^3 \equiv -\left(\frac{u_{xxx}}{3\,u_{xx}^{4/3}} - \beta\,u_{xx}^{1/3}\right)\omega, \qquad \mu^4 \equiv \left(\beta^2\,u_{xx}^{2/3} - \frac{\beta\,u_{xxx}}{3\,u_{xx}}\right)\omega + u_{xx}^{1/3}\,d\beta. \tag{2.28}$$

Comparison of the former with the recurrence formula for $0 = dU_{XX}$ in (2.20) produces

$$\mu^3 \equiv -\tfrac{1}{3}\,U_{XXX}\,\omega, \qquad \text{and thus} \qquad U_{XXX} = \frac{u_{xxx}}{u_{xx}^{4/3}} - 3\,\beta\,u_{xx}^{1/3}.$$

Again, no initial prolongation was needed to arrive at this formula for the partially reduced third order lifted jet coordinate.

The final order 3 normalization sets

$$U_{XXX} = 0, \qquad \text{whence} \qquad \beta = \frac{u_{xxx}}{3\,u_{xx}^{5/3}}\,. \tag{2.29}$$

At this stage, collecting (2.23, 27, 29), we have

$$\alpha = \sqrt[3]{u_{xx}} - \frac{u_x u_{xxx}}{3\,u_{xx}^{5/3}}\,, \qquad \beta = \frac{u_{xxx}}{3\,u_{xx}^{5/3}}\,, \qquad \gamma = -\frac{u_x}{\sqrt[3]{u_{xx}}}\,, \qquad \delta = \frac{1}{\sqrt[3]{u_{xx}}}\,, \tag{2.30}$$

which, when combined with (2.21), forms the right-equivariant moving frame. (The classical moving frame, [**9**], is left-equivariant and obtained by inverting the equi-affine group element corresponding to (2.21, 30).) We substitute the moving frame formulas for β and δ into the partially reduced expression (2.28) for μ^4, and then compare with the recurrence formula (2.20) for $0 = dU_{XXX}$, to deduce $\mu^4 \equiv \frac{1}{3}\,\kappa\,\omega$, where

$$\kappa = U_{XXXX} = D_x\left(\frac{u_{xxx}}{u_{xx}^{5/3}}\right) = \frac{u_{xxxx}}{u_{xx}^{5/3}} - \frac{5\,u_{xxx}^2}{3\,u_{xx}^{8/3}} \tag{2.31}$$

is the fundamental differential invariant — the equi-affine curvature. We now have the complete system of invariantized Maurer–Cartan forms:

$$\mu^1 = -\,\omega, \quad \mu^2 = -\,\vartheta, \quad \mu^3 = -\tfrac{1}{3}\,\vartheta_2, \quad \mu^4 = \tfrac{1}{3}\,\kappa\,\omega + \tfrac{1}{3}\,\vartheta_3 - \tfrac{4}{3}\,\vartheta_2, \quad \mu^5 = -\,\omega - \vartheta_1, \tag{2.32}$$

from which the higher order recurrence formula for the differential invariants and invariant differential forms follow as in the standard treatment, [**8**]. The explicit formulae for the higher order contact forms ϑ_j can also be found via recurrence, as in [**17**].

Example 2.3. Consider the following intransitive action of the abelian Lie group $G = \mathbb{R}^3$ on $M = \mathbb{R}^2$:

$$X = x, \qquad U = u + a + b\,x + c\,x^2. \tag{2.33}$$

Although almost completely trivial, this provides an example of a group action that does not admit a slice (either at order 0 or, as we will see, order 1), and hence serves to illustrate our contention that the recursive algorithm will proceed even when there is no slice at hand.

The prolonged infinitesimal generators are

$$\mathbf{v}_1 = \partial_u, \qquad \mathbf{v}_2 = x\,\partial_u + \partial_{u_x}, \qquad \mathbf{v}_3 = x^2\,\partial_u + 2\,x\,\partial_{u_x} + 2\,\partial_{u_{xx}}.$$

Ignoring contact components, the order 0 lifted recurrence formulae are

$$dx = dX = \omega,$$
$$(u_x + b + 2\,c x)\, dx + da + x\, db + x^2\, dc \equiv dU \equiv U_X\, \omega + \mu^1 + X\, \mu^2 + X^2\, \mu^3.$$

Thus, the basic lifted horizontal form is $\omega = dx$. The first prolongation sets $U_X = u_x + b + 2\,c x$, while the Maurer–Cartan forms are

$$\mu^1 = da, \qquad \mu^2 = db, \qquad \mu^3 = dc.$$

Let us start with the order 0 cross-section $K^0 = \{ u = 0 \}$, leading to the normalization equation $U = 0$, with solution $a = -u - bx - cx^2$. Observe that the isotropy subgroup of a point $(x, 0) \in K^0$ is determined by the condition $a = -bx - cx^2$ which, in that it explicitly involves the invariant $X = x$,demonstrates that K^0 is not a slice. Indeed, it is not hard to see that there are no slices. Continuing, we set $U_X = 0$, which implies $b = -u_x - 2\,c x$. Again the isotropy subgroup is x-dependent, and so we still do not have a slice at order 1. We substitute the formula for b into

$$\mu^2 \equiv -(u_{xx} + 2\,c)\, dx - 2\,x\, dc.$$

Comparison with the first order lifted recurrence formula

$$0 = dU_X \equiv U_{XX}\, \omega + \mu^2 + 2\,X\, \mu^3$$

leads to the reduced formula for the second prolongation: $U_{XX} = u_{xx} + 2c$. (For such a simple action, this is, of course, easy to deduce directly.) Finally, we normalize $U_{XX} = 0$ to produce the right-equivariant moving frame

$$a = -\tfrac{1}{2}\, x^2 u_{xx} + x\, u_x - u, \qquad b = x\, u_{xx} - u_x, \qquad c = -\tfrac{1}{2}\, u_{xx}.$$

The basic differential invariant $U_{XXX} = \iota(u_{xxx}) = u_{xxx}$ can be deduced by first computing $\mu^3 \equiv -\frac{1}{2}\, u_{xxx}\, dx$ and then substituting into the second order lifted recurrence formula

$$0 = dU_{XX} = U_{XXX}\, \omega + 2\, \mu^3.$$

Thus, x, u_{xxx} serve to generate the algebra of differential invariants through invariant differentiation with respect to $\mathcal{D} = D_x$.

Let us now describe the geometric framework underlying the recursive algorithm. The starting point is the trivial principal bundle $B = B^{(0)} = G \times M$. Let $B^{(n)} = B_0^{(n)} = (\widetilde{\pi}_0^n)^* B = G \times \mathrm{J}^n$ be the corresponding principal bundle over the n^{th} order jet space. Note that $B^{(n)}$ is also a groupoid, with source map $\sigma^{(n)}(g, z^{(n)}) = z^{(n)}$ and target map $Z^{(n)} = \tau^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}$ determined by the prolonged group action. There is a natural action of G on $B^{(n)}$ given by

$$R_h(g, z^{(n)}) = (g \cdot h^{-1}, h^{(n)} \cdot z^{(n)}), \tag{2.34}$$

which, in view of the G component, we refer to as *right multiplication*. Alternatively, we can realize this by right groupoid multiplication:

$$R_h(g, z^{(n)}) = (g, z^{(n)}) \cdot (h^{-1}, h^{(n)} \cdot z^{(n)}), \quad \text{where} \quad (h^{-1}, h^{(n)} \cdot z^{(n)}) = R_h(e, z^{(n)}). \tag{2.35}$$

The groupoid product is well-defined since $\sigma^{(n)}(g, z^{(n)}) = z^{(n)} = \tau^{(n)}(h^{-1}, h^{(n)} \cdot z^{(n)})$. Moreover,

$$\tau^{(n)}\left[\, R_h(g, z^{(n)}) \,\right] = \tau^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}, \tag{2.36}$$

and thus the components of the target map are invariant under right multiplication — indeed, these are the *lifted invariants* defined in [**8**].

A right-equivariant moving frame can be viewed as a (locally defined) right-invariant section $\widehat{\rho}^{(n)} \colon \mathrm{J}^n \to B^{(n)}$. Indeed, writing

$$\widehat{\rho}^{(n)}(z^{(n)}) = (\rho^{(n)}(z^{(n)}), z^{(n)}),$$

right-invariance requires $R_h \widehat{\rho}^{(n)}(z^{(n)}) = \widehat{\rho}^{(n)}(z^{(n)})$ for all $h \in G$, which, by (2.34), implies

$$\rho^{(n)}(h^{(n)} \cdot z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot h^{-1},$$

which is precisely the right-equivariance condition on the moving frame map $\rho^{(n)}$, cf. [**8**]. With this in mind, let us now formalize the notion of a partial moving frame.

Definition 2.4. A *partial moving frame* of order k is a right-invariant (local) subbundle $\widehat{B}^{(k)} \subset B^{(k)}$, meaning that $R_h(\widehat{B}^{(k)}) \subset \widehat{B}^{(k)}$ for all $h \in G$.

In the case of a moving frame, the right-invariant subbundle is the image of the moving frame section: $\widehat{B}^{(n)} = \widehat{\rho}^{(n)}(V^n)$ where $V^n = \operatorname{dom} \widehat{\rho}^{(n)} \subset \mathrm{J}^n$. Clearly, if $\widehat{B}^{(k)} \subset B^{(k)}$ is right-invariant, so is its pull-back $(\widetilde{\pi}^l_k)^* \widehat{B}^{(k)} \subset B^{(l)}$ for any $l \geq k$.

Given a right-invariant subbundle $\widehat{B}^{(k)} \subset B^{(k)}$ (e.g., $B^{(k)}$ itself) consider the target map $\tau^{(k)} \colon \widehat{B}^{(k)} \to \mathrm{J}^k$. If $K^k \subset \mathrm{J}^k$ is *any* subset, then, as an immediate consequence of (2.36), $(\tau^{(k)})^{-1} K^k \subset \widehat{B}^{(k)}$ is a right-invariant subset. In order that it also be a subbundle, we must impose a suitable transversality condition. For each $z^{(k)} \in \mathrm{J}^k$, we let $\mathfrak{g}^{(k)}|_{z^{(k)}} \subset T\mathrm{J}^k|_{z^{(k)}}$ denote the subspace spanned by the prolonged infinitesimal generators, or, equivalently, the tangent to the prolonged group orbit passing through $z^{(k)}$. The following construction of a partial moving frame is an immediate consequence of the Implicit Function Theorem.

Proposition 2.5. *If $K^k \subset \mathrm{J}^k$ is a cross-section to the prolonged group orbits, or, more generally, satisfies $TK^k|_{z^{(k)}} + \mathfrak{g}^{(k)}|_{z^{(k)}} = T\mathrm{J}^n|_{z^{(k)}}$, then $\widehat{B}^{(k)} = (\tau^{(n)})^{-1} K^k$ defines a partial moving frame of order k.*

In particular, if the action is free on an open subset of J^n, then the partial moving frame $\widehat{B}^{(n)} = (\tau^{(n)})^{-1} K^n \subset B^{(n)}$ associated with a local cross-section $K^n \subset \mathrm{J}^n$ coincides the image of an equivariant moving frame section, reproducing the construction originally proposed in [**8**].

The recursive procedure can now be formalized as follows. To keep matters simple, we will only construct moving frames of minimal order; [**28**]. (This restriction can be relaxed by not performing all possible normalizations at low orders; but this variant will not be developed here.) The order 0 lifted recurrence formulae (2.4) can be used to compute the Maurer–Cartan forms $\mu^1, \ldots, \mu^r$, the lifted horizontal forms $(\omega^1, \ldots, \omega^p)$, and hence the dual implicit differentiation operators $(\mathcal{D}_1, \ldots, \mathcal{D}_p)$. (These formulas can also be used to compute the lifted order 0 contact forms, although in this paper these will be suppressed.) One can also read off the formulae for the first prolongation of G on J^1, although these are also not required until after the initial choice of cross-section and resulting order 0 normalization of group parameters has been performed.

Let $K^0 \subset M$ be a cross-section to the group orbits. (If G acts transitively, K^0 is a single point.) Set $\widehat{B}^{(0)} = \tau^{-1}(K^0) \subset B$ so that $\widehat{B}^{(0)}$ defines a partial moving frame of order 0 according to Proposition 2.5. If K^0 has codimension k_0, then, by transversality and the Implicit Function Theorem, the subbundle $\widehat{B}^{(0)}$ will also have codimension k_0. In other words, the normalization equations $\tau(g,z) \in K^0$ will result in expressions for k_0 of the group parameters in terms of the coordinates $z=(x,u)$ on M and the remaining $r-k_0$ unnormalized parameters. We then substitute these expressions into the Maurer–Cartan forms and the lifted horizontal forms. Using the results in the lifted recurrence formulae produces the formulae for the partial normalization of the first prolongation of G.

The recursive step proceeds in a similar fashion. At each order $1 \le k \le n$, where n is the order of (local) freeness of the prolonged group action, we start with a partial moving frame $\widehat{B}^{(k-1)} \subset B^{(k-1)}$ of order $k-1$. We pull the subbundle back to J^k via the jet space projection: $\widetilde{B}^{(k)} = (\widetilde{\pi}^k_{k-1})^* \widehat{B}^{(k-1)} \subset B^{(k)}$. Choose a cross-section $K^k \subset \mathrm{J}^k$ to the prolonged group orbits satisfying $\widetilde{\pi}^k_{k-1}(K^k) \subset K^{k-1}$. Using the restricted target map $\tau^{(k)}\colon \widetilde{B}^{(k)} \to \mathrm{J}^k$, define $\widehat{B}^{(k)} = (\tau^{(k)})^{-1} K^k \subset \widetilde{B}^{(k)}$. The remaining steps consist of normalizing the group parameters using the cross-section conditions $\tau^{(k)}(g, z^{(k)}) \in K^k$ with the group parameters restricted to $(g, z^{(k)}) \in \widetilde{B}^{(k)}$, having been partially normalized at order $k-1$, then determining the partially normalized Maurer–Cartan forms and lifted horizontal forms, and then finally using the resulting partially normalized lifted recurrence formulae to compute the prolonged action at order $k+1$.

Example 2.6. Let us illustrate the construction in the context of the equi-affine curve Example 2.2. The initial bundle has coordinates

$$B = G \times M \to M : \qquad (g;z) = (\alpha,\beta,\gamma,\delta,a,b\,;x,u), \qquad \text{where} \qquad \alpha\,\delta - \beta\gamma = 1.$$

The target map is

$$\tau(g;z) = Z = (\alpha\,x + \beta\,u + a, \gamma\,x + \delta\,u + b).$$

The order 0 normalizations (2.21) produce the subbundle coordinatized by

$$B \supset \widehat{B}^{(0)} \to M : \qquad (\alpha,\beta,\gamma,\delta,-\alpha\,x - \beta\,u, -\gamma\,x - \delta\,u; x,u).$$

Observe that the restricted target map satisfies

$$\tau(\alpha,\beta,\gamma,\delta,-\alpha\,x - \beta\,u, -\gamma\,x - \delta\,u; x,u) = (0,0),$$

and so, in accordance with its construction, mapping the entire subbundle to the cross-section $K^0 = \{\,(0,0)\,\}$. Moreover, $\widehat{B}^{(0)}$ is easily seen to be right-invariant: $R_h(\widehat{B}^{(k)}) \subset \widehat{B}^{(k)}$ for all $h \in \mathrm{SA}(2)$, and hence forms a partial moving frame of order 0. We then lift the bundle $\widehat{B}^{(0)}$ to the first jet space, producing

$$B^{(1)} \supset \widetilde{B}^{(1)} \to \mathrm{J}^1 : \qquad (\alpha,\beta,\gamma,\delta,-\alpha\,x - \beta\,u, -\gamma\,x - \delta\,u; x,u,u_x),$$

with restricted target map provided by (2.18):

$$\tau(\alpha,\beta,\gamma,\delta,-\alpha x-\beta u,-\gamma x-\delta u;x,u,u_x)=\left(0,0,\frac{\gamma+\delta u_x}{\alpha+\beta u_x}\right).$$

The first order normalization (2.23) produces the partial moving frame of order 1:

$$\widetilde{B}^{(1)}\supset\widehat{B}^{(1)}\to \mathrm{J}^1: \quad \left(\delta^{-1}-\beta u_x,\beta,-\delta u_x,\delta,-\delta^{-1}x+\beta(x u_x-u),\delta(x u_x-u);x,u,u_x\right).$$

We then lift $\widehat{B}^{(1)}$ to J^2 and apply the normalization (2.27), producing the order 2 partial moving frame $\widehat{B}^{(2)}\to V^2$, where $V^2=\{u_{xx}\neq 0\}\subset \mathrm{J}^2$:

$$\left(u_{xx}^{1/3}-\beta u_x,\beta,-u_x u_{xx}^{-1/3},u_{xx}^{-1/3},-x u_{xx}^{1/3}+\beta(x u_x-u),(x u_x-u)u_{xx}^{-1/3};x,u,u_x,u_{xx}\right),$$

which all goes to the cross-section $K^2=\{(0,0,0,1)\}$ under the target map. Finally, the complete moving frame, of order 3, is obtained by lifting $\widehat{B}^{(2)}$ to J^3 and then applying the normalization (2.29), producing $\widehat{B}^{(3)}\to \mathrm{J}^3$:

$$\begin{aligned}&\big(u_{xx}^{1/3}-\tfrac{1}{3}u_x u_{xx}^{-5/3}u_{xxx},\tfrac{1}{3}u_{xx}^{-5/3}u_{xxx},-u_x u_{xx}^{-1/3},u_{xx}^{-1/3},\\ &\qquad -x u_{xx}^{1/3}+\tfrac{1}{3}(x u_x-u)u_{xx}^{-5/3}u_{xxx},(x u_x-u)u_{xx}^{-1/3};x,u,u_x,u_{xx},u_{xxx}\big),\end{aligned}$$

which is the image of a right-equivariant section $\widehat{\rho}^{(3)}\colon V^3=\{u_{xx}\neq 0\}\subset \mathrm{J}^3\to B^{(3)}$.

3. The Inductive Method.

Let us now discuss an implementation of the recursive procedure that takes into account the existence of a smaller subgroup $H\subset G$ for which we have already constructed a moving frame and consequent differential invariants and invariant differential forms. The goal is to use this information to both streamline the construction of a moving frame for the larger group G, and also to express G invariant quantities in terms of their H counterparts. Unlike Kogan's inductive approach, [**14, 15, 16**], which requires the existence of a factorization $G=N\cdot H$ into a product of Lie subgroups $N,H\subset G$, with $N\cap H$ discrete, there will be no restrictions on the subgroup $H\subset G$ for the recursive method to succeed. Our general inductive method will be modeled on the recursive algorithm, and thus relies on the lifted recurrence relations in an essential manner.

Rather than describing the general theory, it is easiest to explain how the method proceeds in the context of the preceding Example 2.2, which also appears in [**16**]. Although this is a case in which the group factors, we never require the existence of the complementary subgroup in order to complete the calculations. Also, while we are treating the exact same example, the calculations performed here can be done independently of those appearing above.

Example 3.1. We return to the action (2.10) of the equi-affine group $G=\mathrm{SA}(2)$ on plane curves, and take the Euclidean subgroup $H=\mathrm{SE}(2)\subset\mathrm{SA}(2)$ containing the orientation-preserving rigid motions. For simplicity, we will continue to ignore contact forms, although it would not take much more effort to include them in the computations.

We begin by constructing a Euclidean moving frame using the recursive approach based on the standard cross-section

$$S = \{ x = u = u_x = 0 \} \subset \mathrm{J}^1, \tag{3.1}$$

thereby recovering the well-known result [**8, 30**]. (If you already know the Euclidean moving frame, you can skip this initial computation.) We start with the Euclidean action

$$X = x \cos\phi - u \sin\phi + a, \qquad U = x \sin\phi + u \cos\phi + b, \tag{3.2}$$

and corresponding prolonged infinitesimal generators

$$\begin{gathered} \mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_u, \\ \mathbf{v}_3 = -\,u\,\partial_x + x\,\partial_u + (1+u_x^2)\,\partial_{u_x} + 3\,u_x u_{xx}\,\partial_{u_{xx}} + (4\,u_x u_{xxx} + 3\,u_{xx}^2)\,\partial_{u_x} + \ \cdots\ . \end{gathered} \tag{3.3}$$

Computing

$$\begin{aligned} dX &\equiv (\cos\phi - u_x \sin\phi)\,dx - (x\sin\phi + u\cos\phi)\,d\phi + da \\ &= (\cos\phi - u_x \sin\phi)\,dx - U\,d\phi + da + b\,d\phi, \\ dU &\equiv (\sin\phi + u_x \cos\phi)\,dx + (x\cos\phi - u\sin\phi)\,d\phi + db \\ &= (\sin\phi + u_x \cos\phi)\,dx + X\,d\phi + db - a\,d\phi, \end{aligned}$$

and comparing with the order 0 lifted recurrence formulae

$$dX \equiv \omega + \mu^1 - U\,\mu^3, \qquad dU \equiv U_X\,\omega + \mu^2 + X\,\mu^3, \tag{3.4}$$

we deduce the formulas for the lifted horizontal one-form and first order prolonged action:

$$\omega = (\cos\phi - u_x \sin\phi)\,dx, \qquad U_X = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi}\,, \tag{3.5}$$

while the Maurer–Cartan forms are given by

$$\mu^1 = da + b\,d\phi, \qquad \mu^2 = db - a\,d\phi, \qquad \mu^3 = d\phi. \tag{3.6}$$

The order 0 normalizations $X = U = 0$ imply that

$$a = -\,x\cos\phi + u\sin\phi, \qquad b = -\,x\sin\phi - u\cos\phi, \tag{3.7}$$

while, using either (3.4) or (3.6), the partially normalized Maurer–Cartan forms become

$$\mu^1 \equiv -\,\omega, \qquad \mu^2 \equiv -\,U_X\,\omega. \tag{3.8}$$

Next, we use the order 1 normalization†

$$U_X = 0, \qquad \text{whence} \qquad \phi = -\tan^{-1} u_x, \tag{3.9}$$

† As in most treatments — an exception being [**26**] — we ignore a sign ambiguity resulting from the fact that the action of SE(2) on J^1 is only locally free.

to produce the (locally) right-equivariant moving frame (3.7, 9). (The classical left-equivariant moving frame is obtained by inversion.) Substituting (3.9) back into (3.5) produces the contact-invariant Euclidean arc length element

$$\omega \equiv \sqrt{1+u_x^2}\, dx. \tag{3.10}$$

On the other hand, using the formula for $\mu^3 = d\phi$ along with the first order recurrence formula

$$dU_X \equiv U_{XX}\,\omega + (1+U_X^2)\,\mu^3, \tag{3.11}$$

we find that

$$-\frac{u_{xx}\,dx}{1+u_x^2} \equiv -\,d\tan^{-1} u_x = d\phi = -\,U_{XX}\,\omega, \tag{3.12}$$

from which we conclude that the lifted second order jet coordinate has been reduced to the Euclidean curvature differential invariant:

$$U_{XX} = \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}\,.$$

The fully normalized recurrence formulae can then be used to determine the higher order differential invariants; for instance,

$$\begin{aligned} dU_{XX} &\equiv U_{XXX}\,\omega + 3\,U_X U_{XX}\,\mu^3 = U_{XXX}\,\omega,\\ dU_{XXX} &\equiv U_{XXXX}\,\omega + (4\,U_X U_{XX} + 3\,U_{XX}^2)\,\mu^3 = (U_{XXXX} - 3\,U_{XX}^3)\,\omega, \end{aligned} \tag{3.13}$$

imply that

$$U_{XXX} = \mathcal{D}_s U_{XX} = \kappa_s, \qquad U_{XXXX} = \mathcal{D}_s U_{XXX} + 3\,U_{XX}^3 = \kappa_{ss} + 3\,\kappa^3, \tag{3.14}$$

where the s subscript indicates invariant differentiation with respect to the arc length element (3.10).

Now, with a Euclidean moving frame in hand, let's implement the recursive construction of the SA(2) moving frame, but base our calculations on the SE(2) lifted coordinates (3.2, 5), which we continue to denote by X, U, U_X, etc. We will use bars to distinguish the SA(2) lifted jet coordinates: $\overline{X}, \overline{U}, \overline{U}_{\overline{X}}$, etc. Thus, on $M = \mathbb{R}^2$, the equi-affine action has the adapted form

$$\overline{X} = \alpha\, X + \beta\, U + a, \qquad \overline{U} = \gamma\, X + \delta\, U + b, \qquad \alpha\,\delta - \beta\,\gamma = 1, \tag{3.15}$$

where X, U are given in (3.2). As we will see, the overspecification of group parameters will be naturally dealt with during the course of the computation. Using the Euclidean recurrence formulae (3.4) to evaluate dX and dU, we find, modulo contact forms,

$$\begin{aligned} d\overline{X} &= \alpha\, dX + \beta\, dU + X\, d\alpha + U\, d\beta + da\\ &\equiv (\alpha + \beta\, U_X)\,\omega + \alpha\,\mu^1 + \beta\,\mu^2 + da + X\,(d\alpha + \beta\,\mu^3) + U\,(d\beta - \alpha\,\mu^3)\\ &= (\alpha + \beta\, U_X)\,\omega + \alpha\,\mu^1 + \beta\,\mu^2 + da +\\ &\qquad + [\,\delta(\overline{X} - a) - \beta(\overline{U} - b)\,]\,(d\alpha + \beta\,\mu^3) + [\,-\gamma(\overline{X} - a) + \alpha(\overline{U} - b)\,]\,(d\beta - \alpha\,\mu^3), \end{aligned}$$

$$
\begin{aligned}
d\overline{U} &= \gamma\, dX + \delta\, dU + X\, d\gamma + U\, d\delta + db \\
&\equiv (\gamma + \delta\, U_X)\,\omega + \gamma\,\mu^1 + \delta\,\mu^2 + db + X\,(d\gamma + \delta\,\mu^3) + U\,(d\delta - \gamma\,\mu^3) \\
&= (\gamma + \delta\, U_X)\,\omega + \gamma\,\mu^1 + \delta\,\mu^2 + db + \\
&\qquad + [\,\delta(\overline{X} - a) - \beta(\overline{U} - b)\,]\,(d\gamma + \delta\,\mu^3) + [\,-\gamma(\overline{X} - a) + \alpha(\overline{U} - b)\,]\,(d\delta - \gamma\,\mu^3).
\end{aligned}
$$

On the other hand, the first two equi-affine recurrence relations in (2.20) are

$$
d\overline{X} \equiv \overline{\omega} + \overline{\mu}^{\,1} - \overline{X}\,\overline{\mu}^{\,3} + \overline{U}\,\overline{\mu}^{\,4}, \qquad d\overline{U} \equiv \overline{U}_{\overline{X}}\,\overline{\omega} + \overline{\mu}^{\,2} + \overline{X}\,\overline{\mu}^{\,5} + \overline{U}\,\overline{\mu}^{\,3}, \tag{3.16}
$$

where the SA(2) Maurer–Cartan forms $\overline{\mu}^{\,\alpha}$ are now indicated with bars in order to distinguish them from the SE(2) Maurer–Cartan forms μ^β. Comparing the preceding two pairs of formulas, the horizontal components imply that

$$
\overline{\omega} = (\alpha + \beta\, U_X)\,\omega, \qquad \overline{U}_{\overline{X}} = \frac{\gamma + \delta\, U_X}{\alpha + \beta\, U_X}\,, \tag{3.17}
$$

where U_X is as in (3.5), while the Maurer–Cartan forms are related by

$$
\begin{aligned}
\overline{\mu}^{\,1} &= da + \alpha\,\mu^1 + \beta\,\mu^2 + (\beta\, b - \delta\, a)\,(d\alpha + \beta\,\mu^3) + (\gamma\, a - \alpha\, b)\,(d\beta - \alpha\,\mu^3), \\
\overline{\mu}^{\,2} &= db + \gamma\,\mu^1 + \beta\,\mu^2 + (\beta\, b - \delta\, a)\,(d\gamma + \delta\,\mu^3) + (\gamma\, a - \alpha\, b)\,(d\delta - \gamma\,\mu^3), \\
\overline{\mu}^{\,3} &= \gamma(d\beta - \alpha\,\mu^3) - \delta(d\alpha + \beta\,\mu^3) = \alpha(d\delta - \gamma\,\mu^3) - \beta(d\gamma + \delta\,\mu^3), \\
\overline{\mu}^{\,4} &= \alpha(d\beta - \alpha\,\mu^3) - \beta(d\alpha + \beta\,\mu^3), \\
\overline{\mu}^{\,5} &= \delta(d\gamma + \delta\,\mu^3) - \gamma(d\delta - \gamma\,\mu^3).
\end{aligned} \tag{3.18}
$$

For the order 0 normalizations, we already employed $X = U = 0$ for the Euclidean action, and we adopt the same cross-section $\overline{X} = \overline{U} = 0$ for the equi-affine action. (Note: in general, it is not necessary that the two cross-sections be identical for the inductive algorithm to proceed.) With this choice, (3.15) implies $a = b = 0$, which thus simplifies

$$
\overline{\mu}^{\,1} = \alpha\,\mu^1 + \beta\,\mu^2, \qquad \overline{\mu}^{\,2} = \gamma\,\mu^1 + \beta\,\mu^2.
$$

Substituting into the order 0 recurrence formulae (3.16), and recalling (3.8, 17),

$$
\overline{\mu}^{\,1} \equiv -\,\overline{\omega} = -\,(\alpha + \beta\, U_X)\,\omega, \qquad \overline{\mu}^{\,2} \equiv -\,\overline{U}_{\overline{X}}\,\overline{\omega} = -\,(\gamma + \delta\, U_X)\,\omega.
$$

The order 1 Euclidean normalization sets $U_X = 0$, and, correspondingly, we adopt the equi-affine normalization $\overline{U}_{\overline{X}} = 0$. Using (3.17), this implies

$$
\gamma = 0, \qquad \text{and hence, by unimodularity} \qquad \alpha = \frac{1}{\delta}\,. \tag{3.19}
$$

At this stage, we have deduced that the isotropy subgroup of any jet on the Euclidean cross-section (3.1) is

$$
N = \left\{ \begin{pmatrix} 1/\delta & \beta \\ 0 & \delta \end{pmatrix} \;\middle|\; \delta \neq 0 \right\} \subset \mathrm{SA}(2). \tag{3.20}
$$

Since N is independent of the cross-section coordinates (equivalently, the subgroup's differential invariants) the cross-section is a slice. (Although, in this case, this is completely trivial, since it consists of a single point.) Moreover, the equiaffine group factors, $\mathrm{SA}(2) = N \cdot \mathrm{SE}(2)$ with $N \cap \mathrm{SE}(2) = \{\pm I\}$ discrete, although these properties are not required for the recursive algorithm to succeed.

Remark: If we know the relevant subgroup N in advance, then we can restrict our initial group action (3.15) to that of N alone, thereby simplifying the preceding calculation. Or, alternatively, the presented algorithm can serve as a means of determining the corresponding isotropy subgroup(s).

Substituting the normalizations (3.19) into (3.18) yields

$$\overline{\mu}^3 = \frac{d\delta}{\delta} - \beta\,\delta\,\mu^3, \qquad \overline{\mu}^4 = \frac{d\beta}{\delta} + \frac{\beta\,d\delta}{\delta^2} - \left(\beta^2 + \frac{1}{\delta^2}\right)\mu^3, \qquad \overline{\mu}^5 = \delta^2\,\mu^3, \tag{3.21}$$

while

$$\overline{\omega} = \frac{1}{\delta}\,\omega, \qquad \overline{\mu}^1 \equiv -\,\overline{\omega}, \qquad \overline{\mu}^2 \equiv 0, \tag{3.22}$$

(as always, modulo contact forms). Moreover, substituting the normalization $\overline{U}_{\overline{X}} = 0$ into the first order lifted recurrence formula

$$d\overline{U}_{\overline{X}} \equiv \overline{U}_{\overline{X}\,\overline{X}}\,\overline{\omega} + 2\,\overline{U}_{\overline{X}}\,\overline{\mu}^3 - \overline{U}_{\overline{X}}^2\,\overline{\mu}^4 + \overline{\mu}^5,$$

and comparing the result with the last equation in (3.21) combined with (3.12) and (3.22), yields

$$-\,\overline{U}_{\overline{X}\,\overline{X}}\,\overline{\omega} \equiv \overline{\mu}^5 = \delta^2\,\mu^3 \equiv -\,\delta^2\,\kappa\,\omega = -\,\delta^3\,\kappa\,\overline{\omega}.$$

This produces the partially normalized lifted formula for the second order derivative:

$$\overline{U}_{\overline{X}\,\overline{X}} = \delta^3\,\kappa.$$

Normalizing $\overline{U}_{\overline{X}\,\overline{X}} = 1$, we find

$$\delta = \kappa^{-1/3}, \qquad \text{and so} \qquad \overline{\omega} = \sqrt[3]{\kappa}\,\omega, \tag{3.23}$$

which relates the equi-affine arc length element, as derived directly in (2.27), to the Euclidean arc length element (3.10). Substituting the preceding normalizations into the order 2 recurrence formula

$$d\overline{U}_{\overline{X}\,\overline{X}} \equiv \overline{U}_{\overline{X}\,\overline{X}\,\overline{X}}\,\overline{\omega} + 3\,\overline{U}_{\overline{X}\,\overline{X}}\,\overline{\mu}^3 - 3\,\overline{U}_{\overline{X}}\,\overline{U}_{\overline{X}\,\overline{X}}\,\overline{\mu}^4,$$

produces

$$\overline{\mu}^3 \equiv -\tfrac{1}{3}\,\overline{U}_{\overline{X}\,\overline{X}\,\overline{X}}\,\overline{\omega}.$$

On the other hand, from (3.12, 21, 23),

$$\overline{\mu}^3 = \frac{d\delta}{\delta} - \delta\,\beta\,\mu^3 \equiv -\left(\frac{\kappa_s}{3\,\kappa} + \beta\,\kappa^{2/3}\right)\omega = -\left(\frac{\kappa_s}{3\,\kappa^{4/3}} + \beta\,\kappa^{1/3}\right)\overline{\omega},$$

where the s subscript denotes the Euclidean arc length derivative. Comparing the last two formulas, we find

$$\overline{U}_{\overline{X}\,\overline{X}\,\overline{X}} = \frac{\kappa_s}{3\,\kappa^{4/3}} + \beta\,\kappa^{1/3}.$$

Applying the last normalization $\overline{U}_{\overline{X}\,\overline{X}\,\overline{X}} = 0$ produces the final formula

$$\alpha = \kappa^{1/3}, \qquad \beta = -\,\frac{\kappa_s}{3\,\kappa^{5/3}}\,, \qquad \gamma = 0, \qquad \delta = \kappa^{-1/3}, \qquad a = b = 0, \tag{3.24}$$

now expressing the equi-affine parameters in terms of the Euclidean invariants. The final equi-affine moving frame will be given by the product of the group element (3.24) with the Euclidean group element provided by its moving frame (3.7, 9). The order 3 reduced recurrence formula

$$0 = d\overline{U}_{\overline{X}\overline{X}\overline{X}} \equiv \overline{U}_{\overline{X}\overline{X}\overline{X}\overline{X}}\,\overline{\omega} + 4\,\overline{U}_{\overline{X}\overline{X}\overline{X}}\,\overline{\mu}^3 - \left(4\,\overline{U}_{\overline{X}}\,\overline{U}_{\overline{X}\overline{X}\overline{X}} + 3\,\overline{U}^2_{\overline{X}\overline{X}} \right) \overline{\mu}^4 = \overline{U}_{\overline{X}\overline{X}\overline{X}\overline{X}}\,\overline{\omega} + 3\,\overline{\mu}^4,$$

coupled with (3.21), yields

$$-\tfrac{1}{3}\,\overline{U}_{\overline{X}\overline{X}\overline{X}\overline{X}}\,\overline{\omega} = \overline{\mu}^4 = \frac{d\beta}{\delta} + \frac{\beta\, d\delta}{\delta^2} - \left(\frac{1}{\delta^2} + \beta^2 \right) \mu^3.$$

Substituting for the group parameters via (3.24), we arrive at the well-known formula for the equi-affine curvature $\overline{\kappa} = \overline{U}_{\overline{X}\overline{X}\overline{X}\overline{X}}$ in terms of the Euclidean curvature:

$$\overline{\kappa} = \frac{\kappa_{ss}}{\kappa^{5/3}} - \frac{5\,\kappa_s^2}{3\,\kappa^{8/3}} + 3\,\kappa^{4/3}. \tag{3.25}$$

Remark: If we already know the local coordinate formula (2.31) for equi-affine curvature, we can immediately deduce the latter formula by applying invariantization with respect to the Euclidean moving frame, noting that $\iota(\overline{\kappa}) = \overline{\kappa}$, while, by the Euclidean recurrence formulae (3.14),

$$\iota(u_x) = 0, \qquad \iota(u_{xx}) = \kappa, \qquad \iota(u_{xxx}) = \kappa_s, \qquad \iota(u_{xxxx}) = \kappa_{ss} + 3\,\kappa^3.$$

A couple of pertinent remarks: First, as just noted above, if one has already computed a moving frame for G, there is no need to repeat the computation if the only goal is to express the G differential invariants in terms of the H differential invariants. Indeed, the Replacement Theorem can be immediately applied to express any G invariant quantity in terms of H invariant quantities. Second, observe that it was not necessary to use any sort of adapted basis for the infinitesimal generators of G, e.g., one that includes a basis for the Lie algebra of H.

4. Lie Pseudo–Groups.

The same underlying ideas work, when suitably re-interpreted, for Lie pseudo-group actions. In addition to the original references [**32**, **33**, **34**], the recent survey paper [**31**] may be profitably consulted for details on the following constructions.

Given a smooth (or, better, analytic) manifold M, let $\mathcal{D} = \mathcal{D}(M)$ denote the Lie pseudo-group of all local diffeomorphisms[†] $\phi\colon M \to M$. For $0 \le n \le \infty$, let $\mathcal{D}^{(n)} \subset \mathrm{J}^n(M,M)$ denote the bundle, or, more specifically, groupoid, [**19**, **38**], consisting of their n jets. The groupoid multiplication is provided by composition of jets (i.e., Taylor polynomials or, when $n = \infty$, series). Coordinates on $\mathcal{D}^{(n)}$ are denoted by $(z, Z^{(n)}) = (\ldots z^a \;\ldots\; Z^b_C \;\ldots\,)$, where the z^a, $a = 1, \ldots, m$, are source coordinates on M, while

† As before, our notation allows the domain of the diffeomorphism to be an open subset of the source space: $\operatorname{dom}\phi \subset M$.

the Z^b_C, $b = 1, \ldots, m$, $0 \leq \#C \leq n$, represent the target jet coordinates of a local diffeomorphism. There is an induced right action of $\mathcal{D}$ on $\mathcal{D}^{(n)}$ induced by right composition of local diffeomorphisms, and denoted by

$$\mathrm{R}_\phi(\mathrm{j}_n\psi|_z) = \mathrm{j}_n(\psi \circ \phi^{-1})|_{\phi(z)} \qquad \text{for} \qquad \phi \in \mathcal{D},\ z \in \operatorname{im}\phi.$$

Let $\mathcal{G} \subset \mathcal{D}$ be a *regular Lie pseudo-group* acting on M, meaning that, for n sufficiently large, the pseudo-group jets $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ forms a subbundle, and the induced projection $\pi^{n+1}_n \colon \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ forms a fibration. The condition that $\mathcal{G}$ be a *Lie pseudo-group* requires that, again for $n \gg 0$, every local diffeomorphism $\phi \in \mathcal{D}$ that satisfies $\mathrm{j}_n\phi \subset \mathcal{G}^{(n)}$ belongs to the pseudo-group: $\phi \in \mathcal{G}$. (See [**11**] for how to complete a more general pseudo-group into a Lie pseudo-group with exactly the same invariants and local geometry.) We will regard the target coordinates on the subbundle $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ as a system of pseudo-group parameters, whose values will be recursively normalized during the course of the moving frame algorithm.

As with the submanifold jet bundle, we split the differential forms on $\mathcal{D}^{(\infty)}$ into horizontal and contact (or groupoid) components, and we let $d = d_M + d_G$ denote the corresponding decomposition of the differential. The right-invariant contact forms are interpreted as the *Maurer–Cartan forms* for the diffeomorphism pseudo-group, and an explicit basis μ^b_C, for $b = 1, \ldots, m$, $\#C \geq 0$, can be obtained by invariant differentiation, [**32**]. The corresponding Maurer–Cartan forms for the pseudo-group $\mathcal{G}$ are then obtained by restricting the diffeomorphism Maurer–Cartan forms to the pseudo-group jet groupoid.

A vector field on M is an infinitesimal generator of $\mathcal{G}$ provided the corresponding one-parameter group $\exp(t\,\mathbf{v}) \in \mathcal{G}$, where defined. In local coordinates, we write

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z)\,\frac{\partial}{\partial z^a}\,. \tag{4.1}$$

For each n, the n^{th} order jet of a vector field (4.1) is coordinatized by the derivatives of its coefficients, denoted $\zeta^{(n)} = (\ \ldots\ \zeta^b_C\ \ldots\ )$, with $1 \leq b \leq m$, $0 \leq \#C \leq n$. Infinitesimal generator jets are constrained by the *infinitesimal determining equations*

$$L^{(n)}(z, \zeta^{(n)}) = 0, \tag{4.2}$$

which forms a linear system of partial differential equations that prescribe the tangent space $T\mathcal{G}^{(n)} \subset T\mathcal{D}^{(n)}$.

We extend the *lift map* (2.1) so that it takes a vector field jet coordinate to the corresponding restricted diffeomorphism Maurer–Cartan form:

$$\lambda(\zeta^b_C) = \mu^b_C, \qquad \text{for} \qquad b = 1, \ldots, m, \qquad \#C \geq 0. \tag{4.3}$$

A fundamental result, established in [**32**], states that the restrictions of the diffeomorphism Maurer–Cartan forms to $\mathcal{G}^{(\infty)}$ are constrained by the system of linear algebraic relations obtained by formally applying the lift map to the infinitesimal determining equations (4.2):

$$L^{(n)}(Z, \mu^{(n)}) = 0. \tag{4.4}$$

To explicitly construct the Maurer–Cartan forms, we mimic the finite-dimensional lifted recurrence formulae (2.4). Consider the differentials of the target coordinates, which we view as functions on the diffeomorphism jet bundle $\mathcal{D}^{(\infty)}$, and thereby split into manifold and groupoid components:

$$dZ^a = d_M Z^a + d_G Z^a = \sigma^a + \mu^a, \qquad a = 1, \ldots, m. \tag{4.5}$$

Their horizontal components are

$$\sigma^a = d_M Z^a = \sum_{b=1}^{m} Z^a_b \, dz^b, \qquad a = 1, \ldots, m, \tag{4.6}$$

with Z^a_b denoting first order jet coordinates on $\mathcal{D}^{(1)}$. Their contact components

$$\mu^a = d_G Z^a = dZ^a - \sum_{b=1}^{m} Z^a_b \, dz^b, \qquad a = 1, \ldots, m, \tag{4.7}$$

are the zero$^{\text{th}}$ order contact forms. Right-invariance of the target coordinates and the Cartesian product structure implies right-invariance of both the horizontal and zero$^{\text{th}}$ order contact forms, and hence the latter are identified as the order 0 Maurer–Cartan forms for the pseudo-group. The higher order Maurer–Cartan forms are obtained by invariant differentiation with respect to the horizontal coframe (4.6). More explicitly, let $\mathbb{D}_{Z^1}, \ldots, \mathbb{D}_{Z^m}$ be the dual total derivative operators, satisfying

$$d_M F = \sum_{a=1}^{m} (\mathbb{D}_{Z^a} F) \, dZ^a \tag{4.8}$$

for any $F \colon \mathcal{D}^{(\infty)} \to \mathbb{R}$. The higher order Maurer–Cartan forms are found by repeated total differentiation: $\mu^b_A = \mathbb{D}_{Z^{a_1}} \cdots \mathbb{D}_{Z^{a_k}} \mu^b$.

We are interested in the induced action of the pseudo-group $\mathcal{G}$ on p-dimensional submanifolds $S \subset M$. As before, we denote the submanifold jet bundle of order $0 \leq n \leq \infty$ by $\mathrm{J}^n = \mathrm{J}^n(M, p)$. The formulae for the prolonged action of $\mathcal{G}$ on J^n can be found by implicit differentiation with respect to the lifted horizontal coframe (2.7). Let $\mathcal{H}^{(n)} \to \mathrm{J}^n$ denote the pullback of the pseudo-group jet groupoid $\mathcal{G}^{(n)} \to M$ via the projection $\widetilde{\pi}^n_0 \colon \mathrm{J}^n \to M$. Given a choice of independent variables, the variational bicomplex structure on J^∞ induces a three-way splitting of the differential $d = d_J + d_G = d_H + d_V + d_G$ on the prolonged pseudo-group bundle (groupoid) $\mathcal{H}^{(\infty)} \to \mathrm{J}^\infty$, thus endowing it with the structure of a tricomplex. The lifts of functions and differential forms on J^∞ to $\mathcal{H}^{(\infty)}$ is defined, as before, by applying the target pull-back and then eliminating any Maurer–Cartan forms.

The pseudo-group version of the *lifted recurrence formula* can now be stated; see [**33**; Theorem 19] for details and a proof. We will use $\mathbf{v}$ to also denote the prolongation of the pseudo-group infinitesimal generator (4.1) to the submanifold jet spaces, acting on differential functions and differential forms by Lie differentiation.

Lemma 4.1. *Let ω be a differential form on J^∞, and $\Omega = \lambda(\omega)$ be its lift to $\mathcal{H}^{(\infty)}$. Then*

$$d_G \lambda(\omega) = \lambda[\mathbf{v}(\omega)], \qquad \text{and hence} \qquad d\lambda(\omega) = \lambda[d\omega + \mathbf{v}(\omega)]. \tag{4.9}$$

Equations (4.5) are particular cases of the lifted recurrence formula:

$$dZ^a = \lambda(dz^a + \zeta^a) = \sigma^a + \mu^a, \qquad a = 1, \ldots, m, \tag{4.10}$$

since, by definition, $\lambda(dz^a) = d_M Z^a$ and $\lambda(\zeta^a) = \mu^a$. Also, since every Lie group action is a Lie pseudo-group (of finite type), the finite-dimensional lifted recurrence formulae are particular cases of the pseudo-group version (4.9).

We explain the recursive moving frame algorithm for prolonged pseudo-group actions in the context of a well-studied example, [**18**, **33**]. The general construction proceeds in analogy to the finite-dimensional case discussed at the end of Section 2, relying on the lifted recurrence formulae and their recursive reductions.

Example 4.2. Consider the intransitive Lie pseudo-group $\mathcal{G}$ given by

$$X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}, \tag{4.11}$$

where $f \in \mathcal{D}(\mathbb{R})$ is an arbitrary local diffeomorphism, on $M = \mathbb{R}^3 \setminus \{u = 0\}$. The corresponding jet coordinates $f, f_x, f_{xx}, \ldots$, will serve to parametrize the pseudo-group jet bundles $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}(M)$. The infinitesimal generator has the form

$$\mathbf{v} = \xi\, \partial_x + \eta\, \partial_y + \varphi\, \partial_u = a(x)\, \partial_x - a'(x)\, u\, \partial_u, \tag{4.12}$$

where $a = a(x)$ is an arbitrary function of x, being the general solution to the linearized determining equations

$$\xi_x = -\varphi_u, \qquad \xi_y = \xi_u = \eta_x = \eta_y = \eta_u = \varphi_y = 0. \tag{4.13}$$

Each infinitesimal generator coefficient jet is thus parametrized by the jet coordinates of $a(x)$, namely $a, a_x, a_{xx}, \ldots$. We use

$$\alpha = \lambda(a), \qquad \alpha_X = \lambda(a_x), \qquad \alpha_{XX} = \lambda(a_{xx}), \ \ldots\ ,$$

to denote the corresponding Maurer–Cartan forms, whose formulae will be determined shortly. Note that the lifts of the infinitesimal generator coefficients are

$$\lambda(\xi) = \lambda(a) = \alpha, \qquad \lambda(\eta) = 0, \qquad \lambda(\varphi) = \lambda(-u\, a_x) = -U\, \alpha_X. \tag{4.14}$$

As in (4.5), we begin by computing the differentials of the target coordinates, and separating them into manifold and groupoid components:

$$\begin{aligned}
dX &= d_M X + d_G X = (X_x\, dx + X_y\, dy + X_u\, du) + (dX - X_x\, dx - X_y\, dy - X_u\, du) \\
&\qquad\qquad\quad = f_x\, dx + (df - f_x\, dx), \\
dY &= d_M Y + d_G Y = Y_x\, dx + Y_y\, dy + Y_u\, du + (dY - Y_x\, dx - Y_y\, dy - Y_u\, du) = dy, \qquad (4.15) \\
dU &= d_M U + d_G U = U_x\, dx + U_y\, dy + U_u\, du + (dU - U_x\, dx - U_y\, dy - U_u\, du) \\
&= \frac{f_x\, du - f_{xx}\, u\, dx}{f_x^2} - \frac{u\,(df_x - f_{xx}\, dx)}{f_x^2} = \frac{f_x\, du - f_{xx}\, u\, dx}{f_x^2} - U\, \frac{df_x - f_{xx}\, dx}{f_x}\,.
\end{aligned}$$

On the other hand, the lifted recurrence formula (4.10) are

$$dX = \sigma^1 + \lambda(\xi) = \sigma^1 + \alpha, \qquad dY = \sigma^2 + \lambda(\eta) = \sigma^2, \qquad dU = \sigma^3 + \lambda(\varphi) = \sigma^3 - U\, \alpha_X,$$

and hence

$$\alpha = df - f_x\, dx, \qquad \alpha_X = \frac{df_x - f_{xx}\, dx}{f_x}\,. \tag{4.16}$$

The invariant horizontal coframe is

$$\sigma^1 = d_M X = f_x\, dx, \qquad \sigma^2 = d_M Y = dy, \qquad \sigma^3 = d_M U = \frac{f_x\, du - f_{xx}\, u\, dx}{f_x^2}\,, \tag{4.17}$$

with corresponding dual invariant differentiation operators

$$\mathbb{D}_X = \frac{1}{f_x}\,\mathbb{D}_x + \frac{f_{xx}\, u}{f_x^2}\,\mathbb{D}_u, \qquad \mathbb{D}_Y = \mathbb{D}_y, \qquad \mathbb{D}_U = f_x\,\mathbb{D}_u. \tag{4.18}$$

Observe that $\alpha_X = \mathbb{D}_X \alpha$, as it should be; the higher order Maurer–Cartan forms are obtained by repeated differentiation, but since f only depends on x, the only nonzero ones are

$$\alpha_{XX} = \mathbb{D}_X^2 \alpha = \frac{df_{xx} - f_{xxx}\, dx}{f_x^2} - \frac{f_{xx}(\, df_x - f_{xx}\, dx)}{f_x^3}\,, \tag{4.19}$$

and its higher order X derivatives. The corresponding lifts of the infinitesimal generator coefficient jet coordinates are similarly obtained by successively applying the invariant differentiations (4.18) to (4.14); at first order,

$$\begin{aligned}
\mu_X &= \lambda(\xi_x) = \lambda(a_x) = \alpha_X, & \mu_Y &= \lambda(\xi_y) = 0, & \mu_U &= \lambda(\xi_u) = 0,\\
\nu_X &= \lambda(\eta_x) = 0, & \nu_Y &= \lambda(\eta_y) = 0, & \nu_U &= \lambda(\eta_u) = 0,\\
\psi_X &= \lambda(\varphi_x) = \lambda(-\, u\, a_{xx}) & \psi_Y &= \lambda(\varphi_y) = 0, & \psi_U &= \lambda(\varphi_u) = \lambda(-\, a_x)\\
&= -\, U\,\alpha_{XX}, & & & &= -\,\alpha_X.
\end{aligned} \tag{4.20}$$

The evident linear relations among the Maurer–Cartan forms on $\mathcal{G}$, namely,

$$\mu_X = -\,\psi_U, \qquad \mu_Y = \mu_U = \nu_X = \nu_Y = \nu_U = \psi_Y = 0,$$

follow from lifting the linearized determining equations (4.13).

We are interested in the induced action of (4.11) on surfaces $S \subset M$, which, for simplicity, we assume to be graphs of functions $u = h(x, y)$. (Adapting the constructions to parametrized surfaces is straightforward.) We thus further split the differential on the surface jet bundle $\mathrm{J}^\infty = \mathrm{J}^\infty(M, 2)$, into horizontal and vertical components: $d_J = d_H + d_V$. Again, for simplicity, we shall ignore the contact forms on the submanifold jet space, e.g., $\theta = du - u_x\, dx - u_y\, dx$, etc., in order to keep the expressions relatively short, although they are not so difficult to keep track of, and are useful for other purposes, e.g., invariant variational problems. The lifted horizontal coframe is

$$d_H X = \omega^1 = f_x\, dx, \qquad d_H Y = \omega^2 = dy, \tag{4.21}$$

and hence the dual implicit differentiations are

$$\mathrm{D}_X = \frac{1}{f_x}\,\mathrm{D}_x, \qquad \mathrm{D}_Y = \mathrm{D}_y, \tag{4.22}$$

which act on differential functions on J^∞.

Using (4.12), we construct the order 0 lifted recurrence formulae (4.9)

$$dX \equiv \omega^1 + \alpha, \qquad dY \equiv \omega^2, \qquad dU \equiv U_X\,\omega^1 + U_Y\,\omega^2 - U\,\alpha_X, \tag{4.23}$$

where U_X, U_Y denote the lifted jet coordinates, i.e., the prolongation of the pseudo-group action to first order surface jets. The order 0 cross-section sets $X = 0$, $U = 1$, which normalizes the first two pseudo-group parameters $f = 0$, $f_x = u$. Substituting these expressions into the Maurer–Cartan forms (4.16), we find

$$\alpha = -\,\omega^1 = -\,u\,dx, \qquad \alpha_X \equiv \frac{u_x - f_{xx}}{u}\,dx + \frac{u_y}{u}\,dy.$$

At this stage, we have already produced the (contact-)invariant horizontal coframe and dual differential operators:

$$\omega^1 = u\,dx, \qquad \omega^2 = dy, \qquad \mathrm{D}_X = \mathcal{D}_1 = \frac{1}{u}\,\mathrm{D}_x, \qquad \mathrm{D}_Y = \mathcal{D}_2 = \mathrm{D}_y. \tag{4.24}$$

On the other hand, upon normalization, $dX = dU = 0$, and so the order 0 lifted recurrence formulae (4.23) imply

$$U\,\alpha_X = U_X\,\omega^1 + U_Y\,\omega^2, \qquad \text{so} \qquad U_X = \frac{u_x - f_{xx}}{u^2}, \qquad U_Y = \frac{u_y}{u} = J,$$

the latter being a differential invariant, as it involves no pseudo-group parameters.

The order 1 normalization sets $U_X = 0$, whereby $f_{xx} = u_x$, which, referring back to (4.16), means that

$$\alpha_{XX} \equiv \frac{u_{xx} - f_{xxx}}{u^2}\,dx + \frac{u\,u_{xy} - u_x u_y}{u^3}\,dy. \tag{4.25}$$

On the other hand, the order 1 lifted recurrence formulae are

$$dU_X = U_{XX}\,\omega^1 + U_{XY}\,\omega^2 - U\,\alpha_{XX} - 2\,U_X\,\alpha_X, \qquad dU_Y = U_{XY}\,\omega^1 + U_{YY}\,\omega^2 - U_Y\,\alpha_X.$$

Upon normalization, $dU_X = 0$, and so the first of these, via (4.24) and (4.25), implies

$$\alpha_{XX} = U_{XX}\,\omega^1 + U_{XY}\,\omega^2, \quad \text{whence} \quad U_{XX} = \frac{u_{xx} - f_{xxx}}{u^3}, \quad U_{XY} = \frac{u\,u_{xy} - u_x u_y}{u^3},$$

while equating the second to

$$dU_Y = dJ = \mathcal{D}_1 J\,\omega^1 + \mathcal{D}_2 J\,\omega^2$$

results in

$$\mathcal{D}_1 J = J_1 = U_{XY} = \frac{u\,u_{xy} - u_x u_y}{u^3}, \qquad \mathcal{D}_2 J = J_2 - J^2, \quad \text{where} \quad J_2 = U_{YY} = \frac{u_{yy}}{u}.$$

We can clearly continue in this recursive fashion, reproducing the calculations in [**33**], but without the necessity of a priori computing the significantly more complicated full prolongation of the pseudo-group action on the submanifold jet space.

Example 4.3. Let us also revisit the equi-affine curve example, but now using the pseudo-group approach. This approach allows us to avoid finding the explicit formulas for the Maurer–Cartan forms on the Lie group, using the Maurer–Cartan contact forms on the corresponding pseudo-group jet bundle $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$ in their stead. Thus, consider the general infinitesimal generator

$$\mathbf{v} = \xi(x,u)\,\partial_x + \varphi(x,u)\,\partial_u, \tag{4.26}$$

where the coefficients satisfy the infinitesimal determining equations

$$\xi_x = \varphi_u, \qquad \xi_{xx} = \xi_{xu} = \xi_{uu} = \varphi_{xx} = \varphi_{xu} = \varphi_{uu} = 0. \tag{4.27}$$

Starting with the group transformation formulae (2.10), the order 0 lifted recurrence formulae, in the pseudo-group form (4.9), are

$$\begin{gathered}(\alpha\,dx + \beta\,du) + (x\,d\alpha + u\,d\beta + da) = d_M X + d_G X = dX = d\lambda(x) = \lambda(dx+\xi) = \sigma^1 + \mu,\\ (\gamma\,dx + \delta\,du) + (x\,d\gamma + u\,d\delta + db) = d_M U + d_G U = dU = d\lambda(u) = \lambda(du+\varphi) = \sigma^2 + \nu,\end{gathered} \tag{4.28}$$

where the horizontal coframe is

$$\sigma^1 = d_M X = \alpha\,dx + \beta\,du, \qquad \sigma^2 = d_M U = \gamma\,dx + \delta\,du, \tag{4.29}$$

while the order 0 Maurer–Cartan forms are

$$\begin{aligned}\mu &= \lambda(\xi) = d_G X = dX - X_x\,dx - X_u\,du = x\,d\alpha + u\,d\beta + da,\\ \nu &= \lambda(\varphi) = d_G U = dU - U_x\,dx - U_u\,du = x\,d\gamma + u\,d\delta + db.\end{aligned} \tag{4.30}$$

Note that while these can be identified as X,U–dependent linear combinations of the Lie group-based Maurer–Cartan forms (2.12), we do not need to make this identification, since we only care about the eventually normalized Maurer–Cartan forms, and the pseudo-group contact forms serve equally well for this purpose. The higher order Maurer–Cartan contact forms are obtained by differentiation with respect to the invariant horizontal coframe (4.29), that is, by successively applying the dual differentiation operators

$$\mathbb{D}_X = \delta\,\mathbb{D}_x - \gamma\,\mathbb{D}_u, \qquad \mathbb{D}_U = -\,\beta\,\mathbb{D}_x + \alpha\,\mathbb{D}_u.$$

Thus, the order 1 Maurer–Cartan forms are

$$\begin{aligned}\mu_X &= \lambda(\xi_x) = \mathbb{D}_X\mu = \delta\,d\alpha - \gamma\,d\beta, &\qquad \nu_X &= \lambda(\varphi_x) = \mathbb{D}_X\nu = \delta\,d\gamma - \gamma\,d\delta,\\ \mu_U &= \lambda(\xi_u) = \mathbb{D}_U\mu = \alpha\,d\beta - \beta\,d\alpha, &\qquad \nu_U &= \lambda(\varphi_u) = \mathbb{D}_U\nu = \alpha\,d\delta - \beta\,d\gamma,\end{aligned} \tag{4.31}$$

while all second and higher order Maurer–Cartan forms are 0. This, and the evident identity $\mu_X = -\,\nu_U$, are in accordance with the lifted infinitesimal determining equations (4.27), as in (4.2). The order 1 contact Maurer–Cartan forms happen to coincide with the ordinary Maurer–Cartan forms (2.13), but this identification is not needed to complete the calculation.

To study the action of SA(2) on plane curves, we select x as the independent variable, and u as the dependent variable, and, as before, ignore contact forms, concentrating on the horizontal components with respect to our usual choice of independent and dependent variables. Solving the cross-section normalization equations $X = U = 0$ for the group parameters a, b as in (2.21), the reduced order 0 Maurer–Cartan forms become

$$\mu = -\sigma^1 \equiv -\omega, \qquad \nu = -\sigma^2 \equiv -U_X\,\omega,$$

where ω and U_X are as before, (2.15, 18). Next, using the usual vector field prolongation formula, [**23**], to determine

$$\mathbf{v}(u_x) = \varphi_x + (\varphi_u - \xi_x)\,u_x - \xi_u\,u_x^2,$$

the first order lifted recurrence formula (4.9) takes the form

$$dU_X = d\lambda(u_x) \equiv \lambda[\,u_x\,dx + \mathbf{v}(u_x)\,] = U_{XX}\,\omega + \mu_X + 2U_X\,\mu_X + U_X^2\nu_U, \tag{4.32}$$

which, in view of (4.31), coincides with first order recurrence formula in (2.20). From here on, the computations are essentially as before, and will not be reproduced. As noted above, the simplification is that we did not need to solve for the Lie group Maurer–Cartan forms to proceed with the algorithm, but rather could use the pseudo-group Maurer–Cartan contact forms obtained by applying the invariant differentiation operators to the order 0 Maurer–Cartan forms.

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