Moving Frames and Joint Differential Invariants

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Abstract. This paper surveys the new, algorithmic theory of moving frames developed by the author and M. Fels. The method is used to classify joint invariants and joint differential invariants of transformation groups, and equivalence and symmetry properties of submanifolds. Applications in classical invariant theory, geometry, and computer vision are indicated.

1. Introduction.

First introduced by Gaston Darboux, the theory of moving frames ("repères mobiles") is most closely associated with the name of Élie Cartan, [**12**, **13**], who molded it into a powerful and algorithmic tool for studying the geometric properties of submanifolds and their invariants under the action of a transformation group. While applications to classical group actions are now ubiquitous in differential geometry, cf. [**23**, **46**], the theory and practice of the moving frame method for more general transformation groups has remained relatively undeveloped and poorly understood.

In the 1970's, several researchers, cf. [**21**, **20**, **14**, **25**], began the attempt to place Cartan's intuitive constructions on a firm theoretical foundation. A significant step was to

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begin the process of disassociating the theory of moving frames from reliance on frame bundles and connections. More recently, [**17**, **18**], Mark Fels and I formulated a new approach to the basic moving frame theory that can be systematically applied to general transformation groups. The key idea is to formulate a moving frame as an equivariant map to the transformation group. All classical moving frames can be reinterpreted in this manner, but the new approach applies in far wider generality. Cartan's construction of the moving frame by the process of normalization is interpreted with the choice of a cross-section to the group orbits. Building on these two simple ideas, one may algorithmically construct moving frames and complete systems of invariants for completely general group actions. Some important consequences include a new and more general proof of the fundamental theorem on classification of differential invariants, a general classification theorem for syzygies of differential invariants, and more general theorems on the equivalence, symmetry and rigidity properties of submanifolds.

New and significant applications of these results have been developed in a wide variety of directions. In [**36**, **1**], the theory was applied to produce new algorithms for solving the basic symmetry and equivalence problems of polynomials that form the foundation of classical invariant theory. In [**29**], the differential invariants of projective surfaces were classified and applied to generate integrable Poisson flows arising in soliton theory. In [**17**], the moving frame algorithm was extended to include infinite-dimensional pseudo-group actions. Faugeras, [**16**], initiated the applications of moving frames in computer vision. In [**11**], the characterization of submanifolds via their differential invariant signatures was applied to the problem of object recognition and symmetry detection, [**4**, **5**, **7**, **41**].

In this paper, I will concentrate on applications to joint invariant signatures, summarizing the results in [**18**, **38**], where full proofs, further details, and numerous additional examples appear. The moving frame method provides a direct route to the classification of joint invariants and joint differential invariants, establishing a geometric counterpart of what Weyl, [**45**], in the algebraic framework, calls the first main theorem for the transformation group. In computer vision, joint differential invariants have been proposed as noise-resistant alternatives to the standard differential invariant signatures, [**6**, **9**, **32**, **42**, **43**], but very few complete classifications were known, [**15**, **38**]. The approximation of higher order differential invariants by joint differential invariants and, generally, ordinary joint invariants leads to fully invariant finite difference numerical schemes, first proposed in [**10**, **11**, **3**]. Applications to the construction of invariant numerical algorithms and the theory of geometric integration, [**8**, **30**], are under development.

2. Moving Frames.

Throughout this paper, G will denote an r-dimensional Lie group acting smoothly on an m-dimensional manifold M. Let $G_S = \{g \in G \mid g \cdot S = S\}$ denote the *isotropy subgroup* of a subset $S \subset M$, and $G_S^* = \bigcap_{z \in S} G_z$ its *global isotropy subgroup*, which consists of those group elements which fix all points in S. The group G acts *freely* if $G_z = \{e\}$ for all $z \in M$, *effectively* if $G_M^* = \{e\}$, and *effectively on subsets* if $G_U^* = \{e\}$ for every open $U \subset M$. Local versions of these concepts are defined by replacing $\{e\}$ by a discrete subgroup of G. A non-effective group action can be replaced by an equivalent effective action of the quotient group G/G_M^*, and so we shall always assume that G acts locally effectively on subsets.

A group acts *semi-regularly* if all its orbits have the same dimension; in particular, an action is locally free if and only if it is semi-regular with r-dimensional orbits. The action is *regular* if, in addition, each point $x \in M$ has arbitrarily small neighborhoods whose intersection with each orbit is connected.

We begin with the fundamental definition, which has the important effect of decoupling the moving frame theory from any reliance on artificial frame bundle constructions. In geometrical situations, one can identify our moving frames with the standard versions, but these identifications break down for more general transformation groups.

Definition 2.1. A *moving frame* is a smooth, G-equivariant map $\rho : M \to G$.

The group G acts on itself by left or right multiplication. If $\rho(z)$ is any right-equivariant moving frame then $\widetilde{\rho}(z) = \rho(z)^{-1}$ is left-equivariant and conversely. All classical moving frames are left equivariant, but, in many cases, the right versions are easier to compute.

Theorem 2.2. *A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.*

Of course, most interesting group actions are *not* free, and therefore do not admit moving frames in the sense of Definition 2.1. There are two basic methods for converting a non-free (but effective) action into a free action. The first is to look at the product action of G on several copies of M, leading to joint invariants. The second is to prolong the group action to jet space, which is the natural setting for the traditional moving frame theory, and leads to differential invariants. Combining the two methods of prolongation and product will lead to joint differential invariants. In applications of symmetry constructions to numerical approximations of derivatives and differential invariants, one requires a unification of these different actions into a common framework, called "multispace", [**39**] — the simplest version is the blow-up construction of algebraic geometry, [**22**].

The practical construction of a moving frame is based on Cartan's method of *normalization*, [**26, 12**], which requires the choice of a (local) *cross-section* to the group orbits.

Theorem 2.3. *Let G act freely, regularly on M, and let K be a cross-section. Given $z \in M$, let $g = \rho(z)$ be the unique group element that maps z to the cross-section: $g \cdot z = \rho(z) \cdot z \in K$. Then $\rho : M \to G$ is a right moving frame for the group action.*

Given local coordinates $z = (z_1, \ldots, z_m)$ on M, let $w(g,z) = g \cdot z$ be the explicit formulae for the group transformations. The right moving frame $g = \rho(z)$ associated with a *coordinate cross-section* $K = \{ z_1 = c_1, \ldots, z_r = c_r \}$ is obtained by solving the *normalization equations*

$$w_1(g,z) = c_1, \qquad \ldots \qquad w_r(g,z) = c_r, \tag{2.1}$$

for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the coordinates $z = (z_1, \ldots, z_m)$.

Theorem 2.4. *If $g = \rho(z)$ is the moving frame solution to the normalization equations* (2.1), *then the functions*

$$I_1(z) = w_{r+1}(\rho(z), z), \qquad \ldots \qquad I_{m-r}(z) = w_m(\rho(z), z), \tag{2.2}$$

form a complete system of functionally independent invariants.

Definition 2.5. The *invariantization* of a scalar function $F: M \to \mathbb{R}$ with respect to a right moving frame ρ is the the invariant function $I = \iota(F)$ defined by $I(z) = F(\rho(z) \cdot z)$.

Invariantization amounts to restricting F to the cross-section, $I \mid K = F \mid K$, and then requiring that I be constant along the orbits. In particular, if $I(z)$ is an invariant, then $\iota(I) = I$, so invariantization defines a projection, depending on the moving frame, from functions to invariants.

3. Prolongation and Differential Invariants.

Traditional moving frames are obtained by prolonging the group action to the n^{th} order (extended) jet bundle $\mathrm{J}^n = \mathrm{J}^n(M,p)$ consisting of equivalence classes of p-dimensional submanifolds $S \subset M$ modulo n^{th} order contact; see [**34**; Chapter 3] for details. The n^{th} order *prolonged* action of G on J^n is denoted by $G^{(n)}$.

An n^{th} *order moving frame* $\rho^{(n)}: \mathrm{J}^n \to G$ is an equivariant map defined on an open subset of the jet space. In practical examples, for n sufficiently large, the prolonged action $G^{(n)}$ becomes regular and free on a dense open subset $\mathcal{V}^n \subset \mathrm{J}^n$, the set of *regular jets*.

Theorem 3.1. *An n^{th} order moving frame exists in a neighborhood of a point $z^{(n)} \in \mathrm{J}^n$ if and only if $z^{(n)} \in \mathcal{V}^n$ is a regular jet.*

Although there are no known counterexamples, for general (even analytic) group actions only a local theorem, [**40, 37**], has been established to date.

Theorem 3.2. *A Lie group G acts locally effectively on subsets of M if and only if for $n \gg 0$ sufficiently large, $G^{(n)}$ acts locally freely on an open subset $\mathcal{V}^n \subset \mathrm{J}^n$.*

We can now apply our normalization construction to produce a moving frame and a complete system of differential invariants in the neighborhood of any regular jet. Choosing local coordinates $z = (x,u)$ on M — considering the first p components $x = (x^1, \ldots, x^p)$ as independent variables, and the latter $q = m - p$ components $u = (u^1, \ldots, u^q)$ as dependent variables — induces local coordinates $z^{(n)} = (x, u^{(n)})$ on J^n with components u^α_J representing the partial derivatives of the dependent variables with respect to the independent variables, [**34, 35**]. We compute the prolonged transformation formulae

$$w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}, \qquad \text{or} \qquad (y, v^{(n)}) = g^{(n)} \cdot (x, u^{(n)})$$

by implicit differentiation of the v's with respect to the y's. For simplicity, we restrict to a coordinate cross-section by choosing $r = \dim G$ components of $w^{(n)}$ to normalize to constants:

$$w_1(g, z^{(n)}) = c_1, \qquad \ldots \qquad w_r(g, z^{(n)}) = c_r. \tag{3.1}$$

Solving the normalization equations (3.1) for the group transformations leads to the explicit formulae $g = \rho^{(n)}(z^{(n)})$ for the right moving frame. Moreover, substituting the moving frame formulae into the unnormalized components of $w^{(n)}$ leads to the *fundamental n^{th} order differential invariants*

$$I^{(n)}(z^{(n)}) = w^{(n)}(\rho^{(n)}(z^{(n)}), z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot z^{(n)}. \tag{3.2}$$

In terms of the local coordinates, the fundamental differential invariants will be denoted

$$H^i(x, u^{(n)}) = y^i(\rho^{(n)}(x, u^{(n)}), x, u), \qquad I_K^\alpha(x, u^{(k)}) = v_K^\alpha(\rho^{(n)}(x, u^{(n)}), x, u^{(k)}). \tag{3.3}$$

In particular, those corresponding to the normalization components (3.1) of $w^{(n)}$ will be constant, and are known as the *phantom differential invariants*.

Theorem 3.3. *Let $\rho^{(n)}\colon \mathrm{J}^n \to G$ be a moving frame of order $\leq n$. Every n^{th} order differential invariant can be locally written as a function $J = \Phi(I^{(n)})$ of the fundamental n^{th} order differential invariants. The function Φ is unique provided it does not depend on the phantom invariants.*

The *invariantization* of a differential function $F\colon \mathrm{J}^n \to \mathbb{R}$ with respect to the given moving frame is the differential invariant $J = \iota(F) = F \circ I^{(n)}$. As before, invariantization defines a projection, depending on the moving frame, from the space of differential functions to the space of differential invariants.

Example 3.4. Let us illustrate the theory with a very simple, well-known example: curves in the Euclidean plane. The orientation-preserving Euclidean group SE(2) acts on $M = \mathbb{R}^2$, mapping a point $z = (x, u)$ to

$$y = x\cos\theta - u\sin\theta + a, \qquad v = x\sin\theta + u\cos\theta + b. \tag{3.4}$$

For a parametrized curve $z(t) = (x(t), u(t))$, the prolonged group transformations

$$v_y = \frac{dv}{dy} = \frac{x_t\sin\theta + u_t\cos\theta}{x_t\cos\theta - u_t\sin\theta}\,, \qquad v_{yy} = \frac{d^2v}{dy^2} = \frac{x_t u_{tt} - x_{tt}u_t}{(x_t\cos\theta - u_t\sin\theta)^3}\,, \tag{3.5}$$

and so on, are found by successively applying implicit differentiation operator

$$D_y = \frac{1}{x_t\cos\theta - u_t\sin\theta}\,D_t \tag{3.6}$$

to v. The classical Euclidean moving frame for planar curves, [**23**], follows from the cross-section normalizations

$$y = 0, \qquad v = 0, \qquad v_y = 0. \tag{3.7}$$

Solving for the group parameters $g = (\theta, a, b)$ leads to the right-equivariant moving frame

$$\theta = -\tan^{-1}\frac{u_t}{x_t}\,, \qquad a = -\,\frac{x x_t + u u_t}{\sqrt{x_t^2 + u_t^2}}\,, \qquad b = \frac{x u_t - u x_t}{\sqrt{x_t^2 + u_t^2}}\,. \tag{3.8}$$

The inverse group transformation $g^{-1} = (\widetilde{\theta}, \widetilde{a}, \widetilde{b})$ is the classical left moving frame, [**12, 23**]: one identifies the translation component $(\widetilde{a}, \widetilde{b}) = (x, u) = z$ as the point on the curve, while the columns of the rotation matrix $\widetilde{R} = (\mathbf{t}, \mathbf{n})$ are the unit tangent and unit normal vectors. Substituting the moving frame normalizations (3.8) into the prolonged transformation formulae (3.5), results in the fundamental differential invariants

$$v_{yy} \longmapsto \kappa = \frac{x_t u_{tt} - x_{tt}u_t}{(x_t^2 + u_t^2)^{3/2}}\,, \qquad v_{yyy} \longmapsto \frac{d\kappa}{ds}\,, \qquad v_{yyyy} \longmapsto \frac{d^2\kappa}{ds^2} + 3\kappa^3, \tag{3.9}$$

where $D_s = (x_t^2 + u_t^2)^{-1/2} D_t$ is the arc length derivative — which is itself found by substituting the moving frame formulae (3.8) into the implicit differentiation operator (3.6). A complete system of differential invariants for the planar Euclidean group is provided by the curvature and its successive derivatives with respect to arc length: $\kappa, \kappa_s, \kappa_{ss}, \ldots$.

The one caveat is that the first prolongation of SE(2) is only locally free on J^1 since a 180° rotation has trivial first prolongation. The even derivatives of κ with respect to s change sign under a 180° rotation, and so only their absolute values are fully invariant. The ambiguity can be removed by including the second order constraint $v_{yy} > 0$ in the derivation of the moving frame. Extending the analysis to the full Euclidean group E(2) adds in a second sign ambiguity which can only be resolved at third order. See [**38**] for complete details.

Example 3.5. Let $n \neq 0, 1$. In classical invariant theory, the planar actions

$$y = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad \overline{u} = (\gamma x + \delta)^{-n} u, \tag{3.10}$$

of $G = \mathrm{GL}(2)$ play a key role in the equivalence and symmetry properties of binary forms, when $u = q(x)$ is a polynomial of degree $\leq n$, [**24, 36, 1**]. We identify the graph of the function $u = q(x)$ as a plane curve. The prolonged action on such graphs is found by implicit differentiation:

$$v_y = \frac{\sigma u_x - n\gamma u}{\Delta \sigma^{n-1}}, \qquad v_{yy} = \frac{\sigma^2 u_{xx} - 2(n-1)\gamma\sigma u_x + n(n-1)\gamma^2 u}{\Delta^2 \sigma^{n-2}},$$
$$v_{yyy} = \frac{\sigma^3 u_{xxx} - 3(n-2)\gamma\sigma^2 u_{xx} + 3(n-1)(n-2)\gamma^2\sigma u_x - n(n-1)(n-2)\gamma^3 u}{\Delta^3 \sigma^{n-3}},$$

and so on, where $\sigma = \gamma p + \delta$, $\Delta = \alpha\delta - \beta\gamma \neq 0$. On the regular subdomain

$$\mathcal{V}^2 = \{uH \neq 0\} \subset \mathrm{J}^2, \qquad \text{where} \qquad H = u u_{xx} - \frac{n-1}{n} u_x^2$$

is the classical Hessian covariant of u, we can choose the cross-section defined by the normalizations

$$y = 0, \qquad v = 1, \qquad v_y = 0, \qquad v_{yy} = 1.$$

Assume that $u > 0$, $H > 0$, and solving for the group parameters gives the right moving frame formulae

$$\alpha = u^{(1-n)/n}\sqrt{H}, \quad \beta = -x\, u^{(1-n)/n}\sqrt{H}, \quad \gamma = \tfrac{1}{n}\, u^{(1-n)/n}, \quad \delta = u^{1/n} - \tfrac{1}{n}\, x\, u^{(1-n)/n}. \tag{3.11}$$

Substituting the normalizations (3.11) into the higher order transformation rules gives us the differential invariants, the first two of which are

$$v_{yyy} \longmapsto J = \frac{T}{H^{3/2}}, \qquad v_{yyyy} \longmapsto K = \frac{V}{H^2}, \tag{3.12}$$

where

$$T = u^2 u_{xxx} - 3\,\frac{n-2}{n}\,u u_x u_{xx} + 2\,\frac{(n-1)(n-2)}{n^2}\,u_x^3,$$

$$V = u^3 u_{xxxx} - 4\,\frac{n-3}{n}\,u^2 u_x u_{xx} + 6\,\frac{(n-2)(n-3)}{n^2}\,u {u_x}^2 u_{xx} - 3\,\frac{(n-1)(n-2)(n-3)}{n^3}\,u_x^4,$$

and can be identified with classical covariants, which may be constructed using the basic transvectant process of classical invariant theory, cf. [**24, 36**]. Using $J^2 = T^2/H^3$ as the fundamental differential invariant will remove the ambiguity caused by the square root. As in the Euclidean case, higher order differential invariants are found by successive application of the normalized implicit differentiation operator $D_s = u H^{-1/2} D_x$ to the fundamental invariant J.

4. Recurrence Formulae and Syzygies.

As we noted in the preceding examples, substituting the moving frame normalizations into the implicit differentiation operators $D_{y^1}, \ldots, D_{y^p}$ associated with the transformed independent variables gives the fundamental invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ that map differential invariants to differential invariants.

Theorem 4.1. *If $\rho^{(n)}\colon \mathrm{J}^n \to G$ is an n^{th} order moving frame, then, for any $k \geq n+1$, a complete system of k^{th} order differential invariants can be found by successively applying the invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ to the non-constant (non-phantom) fundamental differential invariants $I^{(n+1)}$ of order at most $n+1$.*

Thus, the moving frame provides two methods for computing higher order differential invariants. The first is by normalization — plugging the moving frame formulae into the higher order prolonged group transformation formulae. The second is by invariant differentiation of the lower order invariants. These two processes lead to different differential invariants; for instance, see the last formula in (3.9). The fundamental *recurrence formulae*

$$\mathcal{D}_j H^i = \delta^i_j - L^i_j, \qquad\qquad \mathcal{D}_j I^\alpha_K = I^\alpha_{K,j} - M^\alpha_{K,j}, \tag{4.1}$$

connecting the normalized and the differentiated invariants (3.3) are of critical importance for the development of the theory, and in applications too.

The A remarkable fact, [**18, 19**], is that the *correction terms* L^i_j, $M^\alpha_{K,j}$ can be effectively computed, *without knowledge of the explicit formulae for the moving frame or the normalized differential invariants.* Let

$$\operatorname{pr} \mathbf{v}_\kappa = \sum_{i=1}^{p} \xi^i_\kappa(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{k=\#J\geq 0} \varphi^\alpha_{J,\kappa}(x,u^{(k)})\,\frac{\partial}{\partial u^\alpha_J}\,, \qquad \kappa = 1,\ldots,r,$$

be a basis for the Lie algebra $\mathfrak{g}^{(n)}$ of infinitesimal generators of $G^{(n)}$. The coefficients $\varphi^\alpha_{J,\kappa}(x,u^{(k)})$ are given by the standard *prolongation formula* for vector fields, cf. [**34, 35**], and are assembled as the entries of the n^{th} order *Lie matrix*

$$\mathbf{L}_n(z^{(n)}) = \begin{pmatrix} \xi^1_1 & \ldots & \xi^p_1 & \varphi^1_1 & \ldots & \varphi^q_1 & \ldots & \varphi^\alpha_{J,1} & \ldots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ \xi^1_r & \ldots & \xi^p_r & \varphi^1_r & \ldots & \varphi^q_r & \ldots & \varphi^\alpha_{J,r} & \ldots \end{pmatrix}. \tag{4.2}$$

The rank of $\mathbf{L}_n(z^{(n)})$ equals the dimension of the orbit through $z^{(n)}$. The *invariantized Lie matrix* is obtained by $\mathbf{I}_n = \iota(\mathbf{L}_n) = \mathbf{L}_n(I^{(n)})$, replacing the jet coordinates $z^{(n)} = (x, u^{(n)})$ by the corresponding fundamental differential invariants (3.2). We perform a Gauss–Jordan row reduction on the matrix $\mathbf{I}_n$ so as to reduce the $r \times r$ minor whose columns correspond to the normalization variables $z_1, \ldots, z_r$ in (3.1) to an $r \times r$ identity matrix — let $\mathbf{K}_n$ denote the resulting matrix of differential invariants. Further, let $\mathbf{Z}(x, u^{(n)}) = (D_i z_\kappa)$ denote the $p \times r$ matrix whose entries are the total derivatives of the normalization coordinates $z_1, \ldots, z_r$, and $\mathbf{W} = \iota(\mathbf{Z}) = \mathbf{Z}(I^{(n)})$ its invariantization. The main result is that the correction terms in (4.1) are the entries of the matrix product

$$\mathbf{W} \cdot \mathbf{K}_n = \mathbf{M}_n = \begin{pmatrix} L_1^1 & \ldots & L_1^p & M_1^1 & \ldots & M_1^q & \ldots & M_{K,1}^\alpha & \ldots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ L_r^1 & \ldots & L_p^p & M_r^1 & \ldots & M_r^q & \ldots & M_{K,r}^\alpha & \ldots \end{pmatrix}. \tag{4.3}$$

Example 4.2. The infinitesimal generators of the planar Euclidean group SE(2) are

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_u, \qquad \mathbf{v}_3 = -u\partial_x + x\partial_u.$$

Prolonging these vector fields to J^5, we find the fifth order Lie matrix

$$\mathbf{L}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -u & x & 1+u_x^2 & 3u_x u_{xx} & M_3 & M_4 & M_5 \end{pmatrix}, \tag{4.4}$$

where

$$M_3 = 4u_x u_{xxx} + 3u_{xx}^2, \qquad M_4 = 5u_x u_{xxxxx} + 10u_{xx}u_{xxx}, \\ M_5 = 6u_x u_{xxxx} + 15u_{xx}u_{xxxx} + 10u_{xxx}^2.$$

Under the normalizations (3.7), the fundamental differential invariants are

$$y \longmapsto J = 0, \qquad v \longmapsto I = 0, \qquad v_y \longmapsto I_1 = 0, \qquad v_{yy} \longmapsto I_2 = \kappa, \tag{4.5}$$

and, in general, $v_k = D_y^k v \longmapsto I_k$; see (3.9). The recurrence formulae will express each normalized differential invariant I_k in terms of arc length derivatives of $\kappa = I_2$. Using (4.5), the invariantized Lie matrix takes the form

$$\iota(\mathbf{L}_5) = \mathbf{I}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3\,\kappa^2 & 10\,\kappa\, I_3 & 15\,\kappa\, I_4 + 10 I_3^2 \end{pmatrix}.$$

Since our chosen cross-section (3.7) is based on the jet coordinates x, u, u_x that index the first three columns of $\mathbf{I}_5$ is already in the appropriate row-reduced form, and so $\mathbf{K}_5 = \mathbf{I}_5$. Differentiating the normalization variables and then invariantizing produces the matrices

$$\mathbf{Z} = \begin{pmatrix} 1 & u_x & u_{xx} \end{pmatrix}, \qquad \iota(\mathbf{Z}) = \mathbf{W} = \begin{pmatrix} 1 & 0 & I_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \kappa \end{pmatrix}.$$

Therefore, the fifth order correction matrix is

$$\mathbf{M}_5 = \mathbf{W} \cdot \mathbf{K}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 3\kappa^3 & 10\kappa^2 I_3 & 15\kappa^2 I_4 + 10\kappa I_3^2 \end{pmatrix},$$

whose entries are the required the correction terms. The recurrence formulae (4.1) can then be read off in order:

$$D_s J = D_s(0) = 1 - 1, \qquad D_s I = D_s(0) = 0 - 0,$$
$$D_s I_1 = D_s(0) = 0 - 0, \qquad D_s I_2 = D_s \kappa = I_3 - 0,$$
$$D_s I_3 = I_4 - 3\kappa^3, \qquad D_s I_4 = I_5 - 10\kappa^2 I_3, \qquad D_s I_5 = I_6 - 15\kappa^2 I_4 - 10\kappa\, I_3^2,$$

We conclude that the higher order normalized differential invariants are given in terms of arc length derivatives of the curvature κ by

$$I_2 = \kappa, \qquad I_3 = \kappa_s, \qquad I_4 = \kappa_{ss} + 3\,\kappa^3,$$
$$I_5 = \kappa_{sss} + 19\,\kappa^2 \kappa_s, \qquad I_6 = \kappa_{ssss} + 34\,\kappa^2 \kappa_{ss} + 48\,\kappa \kappa_s^2 + 45\,\kappa^4 \kappa_s,$$

and so on. The direct derivation of these and similar formulae is, needless to say, considerably more tedious. Even sophisticated computer algebra systems have difficulty owing to the appearance of rational algebraic functions in many of the expressions.

A *syzygy* is a functional dependency $H(\,\ldots\,\mathcal{D}_J I_\nu\,\ldots\,) \equiv 0$ among the fundamental differentiated invariants. In Weyl's algebraic formulation of the "Second Main Theorem" for the group action, [**45**], syzygies are defined as algebraic relations among the joint invariants. Here, since we are classifying invariants up to functional independence, there are no algebraic syzygies, and so the classification of differential syzygies is the proper setting for the Second Main Theorem in the geometric/analytic context. See [**18, 38**] for examples and applications.

Theorem 4.3. *A generating system of differential invariants consists of* a) *all non-phantom differential invariants H^i and I^α coming from the un-normalized zero*$^{\text{th}}$ *order jet coordinates y^i, v^α, and* b) *all non-phantom differential invariants of the form $I^\alpha_{J,i}$ where I^α_J is a phantom differential invariant. The fundamental syzygies among the differentiated invariants are*

(i) $\mathcal{D}_j H^i = \delta^i_j - L^i_j$, *when H^i is non-phantom,*

(ii) $\mathcal{D}_J I^\alpha_K = c - M^\alpha_{K,J}$, *when I^α_K is a generating differential invariant, while $I^\alpha_{J,K} = c$ is a phantom differential invariant, and*

(iii) $\mathcal{D}_J I^\alpha_{LK} - \mathcal{D}_K I^\alpha_{LJ} = M^\alpha_{LJ,K} - M^\alpha_{LK,J}$, *where I^α_{LK} and I^α_{LJ} are generating differential invariants and $K \cap J = \varnothing$ are disjoint and non-zero.*

All other syzygies are all differential consequences of these generating syzygies.

5. Equivalence and Signatures.

Two submanifolds $S, \overline{S} \subset M$ are said to be *equivalent* if $\overline{S} = g \cdot S$ for some $g \in G$. A *symmetry* of a submanifold is a group transformation that maps S to itself, and so is an element $g \in G_S$. As emphasized by Cartan, [**12**], the solution to the equivalence and symmetry problems for submanifolds is based on the functional interrelationships among the fundamental differential invariants restricted to the submanifold.

A submanifold $S \subset M$ is called *regular* of order n at a point $z_0 \in S$ if its n^{th} order jet $\mathrm{j}_n S|_{z_0} \in \mathcal{V}^n$ is regular. Any order n regular submanifold admits a (locally defined) moving frame of that order — one merely restricts a moving frame defined in a neighborhood of z_0 to it: $\rho^{(n)} \circ \mathrm{j}_n S$. Thus, only those submanifolds having singular jets at arbitrarily high order fail to admit any moving frame whatsoever. The complete classification of such *totally singular submanifolds* appears in [**37**]; an analytic version of this result is:

Theorem 5.1. *Let G act effectively, analytically. An analytic submanifold $S \subset M$ is totally singular if and only if G_S does not act locally freely on S itself.*

Given a regular submanifold S, let $J^{(k)} = I^{(k)} \mid S = I^{(k)} \circ \mathrm{j}_k S$ denote the k^{th} order *restricted differential invariants*. The k^{th} order *signature* $\mathcal{S}^{(k)} = \mathcal{S}^{(k)}(S)$ is the set parametrized by the restricted differential invariants; S is called *fully regular* if $J^{(k)}$ has constant rank $0 \leq t_k \leq p = \dim S$. In this case, $\mathcal{S}^{(k)}$ forms a submanifold of dimension t_k — perhaps with self-intersections. In the fully regular case,

$$t_n < t_{n+1} < t_{n+2} < \cdots < t_s = t_{s+1} = \cdots = t \leq p,$$

where t is the *differential invariant rank* and s the *differential invariant order* of S.

Theorem 5.2. *Let $S, \overline{S} \subset M$ be regular p-dimensional submanifolds with respect to a moving frame $\rho^{(n)}$. Then S and $\overline{S}$ are (locally) equivalent, $\overline{S} = g \cdot S$, if and only if they have the same differential invariant order s and their signature manifolds of order $s+1$ are identical: $\mathcal{S}^{(s+1)}(\overline{S}) = \mathcal{S}^{(s+1)}(S)$.*

Example 5.3. A curve in the Euclidean plane is uniquely determined, modulo translation and rotation, from its curvature invariant κ and its first derivative with respect to arc length κ_s. Thus, the curve is uniquely prescribed by its *Euclidean signature curve* $\mathcal{S} = \mathcal{S}(C)$, which is parametrized by the two differential invariants (κ, κ_s). The Euclidean (and equi-affine) signature curves have been applied to the problems of object recognition and symmetry detection in digital images in [**11**].

Theorem 5.4. *If $S \subset M$ is a fully regular p-dimensional submanifold of differential invariant rank t, then its symmetry group G_S is an $(r-t)$–dimensional subgroup of G that acts locally freely on S.*

A submanifold with maximal differential invariant rank $t = p$ is called *nonsingular*. Theorem 5.4 says that these are the submanifolds with only discrete symmetry groups. The *index* of such a submanifold is defined as the number of points in S map to a single generic point of its signature, i.e., $\operatorname{ind} S = \min \{\# \, \sigma^{-1}\{\zeta\} \mid \zeta \in \mathcal{S}^{(s+1)}\}$, where $\sigma(z) = J^{(s+1)}(z)$ denotes the *signature map* from S to its order $s+1$ signature $\mathcal{S}^{(s+1)}$. Incidentally, a point on the signature is non-generic if and only if it is a point of self-intersection of $\mathcal{S}^{(s+1)}$. The index is equal to the number of symmetries of the submanifold, a fact that has important implications for the computation of discrete symmetries in computer vision, [**11**], and in classical invariant theory, [**1, 36**].

Theorem 5.5. *If S is a nonsingular submanifold, then its symmetry group is a discrete subgroup of cardinality $\# G_S = \operatorname{ind} S$.*

At the other extreme, a rank 0 or *maximally symmetric* submanifold has all constant differential invariants, and so its signature degenerates to a single point.

Theorem 5.6. *A regular p-dimensional submanifold S has differential invariant rank 0 if and only if it is an orbit,* $S = H \cdot z_0$, *of a p-dimensional subgroup* $H = G_S \subset G$.

For example, in planar Euclidean geometry, the maximally symmetric curves have constant Euclidean curvature, and are the circles and straight lines. Each is the orbit of a one-parameter subgroup of SE(2), which also forms the symmetry group of the orbit.

In equi-affine planar geometry, when $G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$ acts on planar curves, the maximally symmetric curves are the conic sections, which admit a one-parameter group of equi-affine symmetries. The straight lines are totally singular, and admit a three-parameter equi-affine symmetry group, which, in accordance with Theorem 5.1, does not act freely thereon. In planar projective geometry, with $G = \mathrm{SL}(3,\mathbb{R})$ acting on $M = \mathbb{RP}^2$, the maximally symmetric curves, having constant projective curvature, are the "W–curves" studied by Lie and Klein, [**27**, **28**].

In the case of binary forms studied in Example 3.5, the signature curve $\mathcal{S} = \mathcal{S}(q)$ of a function (polynomial) $u = q(x)$ is parametrized by the covariants J^2 and K, as given in (3.12). The following non-classical theorem solving the equivalence problem for complex-valued binary forms appears in [**33**, **36**, **1**].

Theorem 5.7. *Two nondegenerate complex-valued forms* $q(x)$ *and* $\overline{q}(x)$ *are equivalent under the group action* (3.10) *if and only if their signature curves are identical*: $\mathcal{S}(q) = \mathcal{S}(\overline{q})$.

If q and $\overline{q}$ are nonsingular polynomials and have identical signature curves, then each solution $\overline{x} = \varphi(x)$ of the two rational equations

$$J(x)^2 = \overline{J}(\overline{x})^2, \qquad K(x) = \overline{K}(\overline{x}). \tag{5.1}$$

will define an equivalence between q and $\overline{q}$. In particular, the theory guarantees φ is necessarily a linear fractional transformation!

Theorem 5.8. *A nondegenerate binary form* $q(x)$ *is maximally symmetric if and only if it satisfies the following equivalent conditions*:

(*a*) q *is complex-equivalent to a monomial* x^k, *with* $k \neq 0, n$.

(*b*) *The covariant* T^2 *is a constant multiple of* $H^3 \not\equiv 0$.

(*c*) *The signature set is a single point.*

(*d*) q *admits a one-parameter symmetry group.*

(*e*) *The graph of* q *coincides with the orbit of a one-parameter subgroup of* GL(2).

A binary form $q(x)$ *is nonsingular if and only if it is not complex-equivalent to a monomial if and only if it has a finite symmetry group.*

The symmetries of a nonsingular form can be explicitly determined by solving the rational equations (5.1) with $\overline{J} = J$, $\overline{K} = K$. See [**1**] for a MAPLE implementation of this method for computing discrete symmetries and classification of univariate polynomials.

Theorem 5.9. *If* $q(x)$ *is a binary form of degree* n *which is not complex-equivalent to a monomial, then its projective symmetry group has cardinality*

$$k \leq \begin{cases} 6n - 12 & \text{if } V = cH^2 \text{ for some constant } c, \text{ or} \\ 4n - 8 & \text{in all other cases.} \end{cases}$$

6. Joint Invariants and Joint Differential Invariants.

Consider now the joint action

$$g \cdot (z^0, \ldots, z^n) = (g \cdot z^0, \ldots, g \cdot z^n), \qquad g \in G, \quad z^0, \ldots, z^n \in M. \tag{6.1}$$

of the group G on the $(n+1)$-fold Cartesian product $M^{\times (n+1)} = M \times \cdots \times M$. An invariant $I(z^0, \ldots, z^n)$ of (6.1) is an $(n+1)$-*point joint invariant* of the original transformation group. In most cases of interest, although not in general, if G acts effectively on M, then, for $n \gg 0$ sufficiently large, the product action is free and regular on an open subset of $M^{\times (n+1)}$. Consequently, the moving frame method outlined in Section 2 can be applied to such joint actions, and thereby establish complete classifications of joint invariants and, via prolongation to Cartesian products of jet spaces, joint differential invariants. We will discuss two particular examples — planar curves in Euclidean geometry and projective geometry, referring to [**38**] for details.

Example 6.1. *Euclidean joint differential invariants.* Consider the proper Euclidean group SE(2) acting on oriented curves in the plane $M = \mathbb{R}^2$. We begin with the Cartesian product action on $M^{\times 2} \simeq \mathbb{R}^4$. Taking the simplest cross-section $x^0 = u^0 = x^1 = 0, u^1 > 0$ leads to the normalization equations

$$\begin{gathered} y^0 = x^0 \cos\theta - u^0 \sin\theta + a = 0, \qquad\qquad v^0 = x^0 \sin\theta + u^0 \cos\theta + b = 0, \\ y^1 = x^1 \cos\theta - u^1 \sin\theta + a = 0. \end{gathered} \tag{6.2}$$

Solving, we obtain a right moving frame

$$\theta = \tan^{-1}\left(\frac{x^1 - x^0}{u^1 - u^0}\right), \qquad a = -x^0 \cos\theta + u^0 \sin\theta, \qquad b = -x^0 \sin\theta - u^0 \cos\theta, \tag{6.3}$$

along with the fundamental interpoint distance invariant

$$v^1 = x^1 \sin\theta + u^1 \cos\theta + b \quad \longmapsto \quad I = \| \, z^1 - z^0 \, \|. \tag{6.4}$$

Substituting (6.3) into the prolongation formulae (3.5) leads to the the normalized first and second order joint differential invariants

$$v^k_y \quad \longmapsto \quad J_k = -\,\frac{(z^1 - z^0) \cdot z^k_t}{(z^1 - z^0) \wedge z^k_t}\,, \qquad v^k_{yy} \quad \longmapsto \quad K_k = -\,\frac{\| \, z^1 - z^0 \, \|^3 \, (z^k_t \wedge z^k_{tt})}{\big[\,(z^1 - z^0) \wedge z^0_t\,\big]^3}\,, \tag{6.5}$$

for $k = 0, 1$. Note that

$$J_0 = -\cot\phi^0, \qquad J_1 = +\cot\phi^1, \tag{6.6}$$

where $\phi^k = \sphericalangle(z^1 - z^0, z^k_t)$ denotes the angle between the chord connecting z^0, z^1 and the tangent vector at z^k, as illustrated in Figure 1. The modified second order joint differential invariant

$$\widehat{K}_0 = -\| \, z^1 - z^0 \, \|^{-3} \, K_0 = \frac{(z^0_t \wedge z^0_{tt})}{\big[\,(z^1 - z^0) \wedge z^0_t\,\big]^3} \tag{6.7}$$

equals the ratio of the area of triangle whose sides are the first and second derivative vectors z^0_t, z^0_{tt} at the point z^0 over the *cube* of the area of triangle whose sides are the

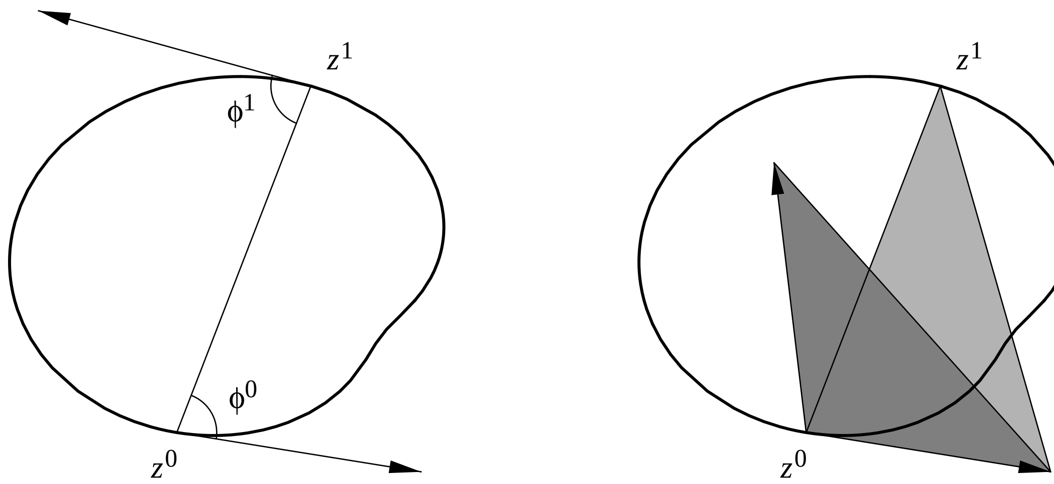

Figure 1. First and Second Order Joint Euclidean Differential Invariants.

chord from z^0 to z^1 and the tangent vector at z^0; see Figure 1. Interestingly, $\widehat{K}_0$ (but not K_0) is also an equi-affine joint differential invariant.

On the other hand, we can construct the joint differential invariants by invariant differentiation of the basic distance invariant (6.4). The normalized invariant differential operators are

$$D_{y^k} \quad \longmapsto \quad \mathcal{D}_k = - \frac{\| z^1 - z^0 \|}{(z^1 - z^0) \wedge z_t^k} \, D_{t^k} \, . \tag{6.8}$$

The recurrence formulae expressing the differentiated invariants in terms of the fundamental normalized joint differential invariants can either be found directly, or using an adaptation of the infinitesimal algorithm. The resulting recurrence formulae

$$\begin{aligned} \mathcal{D}_0 I &= -J_0, & \mathcal{D}_1 I &= J_1, \\ \mathcal{D}_0 J_0 &= K_0 - \frac{1+J_0^2}{I}\,, & \mathcal{D}_1 J_0 &= \frac{1+J_0^2}{I}\,, \\ \mathcal{D}_0 J_1 &= - \frac{1+J_1^2}{I}\,, & \mathcal{D}_1 J_1 &= K_1 + \frac{1+J_1^2}{I}\,. \end{aligned} \tag{6.9}$$

imply that *all* of the joint differential invariants can be obtained from the basic distance invariant by invariant differentiation.

Proposition 6.2. *Every two-point Euclidean joint differential invariant is a function of the interpoint distance* $I = \| z^1 - z^0 \|$ *and its invariant derivatives with respect to* (6.8).

A generic product curve $\mathbf{C} = C^0 \times C^1 \subset M^{\times 2}$ has joint differential invariant rank $2 = \dim \mathbf{C}$, and its joint signature $\mathcal{S}^{(2)}(\mathbf{C})$ will be a two-dimensional submanifold parametrized by the joint differential invariants I, J_0, J_1, K_0, K_1 of order ≤ 2. There will exist a (local) syzygy $\Phi(I, J_0, J_1) = 0$ among the three first order joint differential invariants.

Differentiating and using the recurrence formulae (6.9), we find

$$-J_0 \frac{\partial \Phi}{\partial I} + \left(K_0 - \frac{1+J_0^2}{I} \right) \frac{\partial \Phi}{\partial J_0} - \left(\frac{1+J_1^2}{I} \right) \frac{\partial \Phi}{\partial J_1} = 0,$$

$$J_1 \frac{\partial \Phi}{\partial I} + \left(\frac{1+J_0^2}{I} \right) \frac{\partial \Phi}{\partial J_0} + \left(K_1 + \frac{1+J_1^2}{I} \right) \frac{\partial \Phi}{\partial J_1} = 0.$$

Thus, the syzygies for second order joint differential invariant K_0, K_1 are uniquely determined, *provided* $\partial \Phi / \partial J_0 \neq 0$ and $\partial \Phi / \partial J_1 \neq 0$. The surface parametrized by the first order joint differential invariants I, J_0, J_1 can be used as a reduced signature set to characterize such (generic) product curves.

If the first order signature degenerates to a one-dimensional curve, then, locally,

$$J_0 = -\cot \phi^0 = \Phi_0(I), \qquad\qquad J_1 = \cot \phi^1 = \Phi_1(I). \tag{6.10}$$

Differentiating these two syzygies and using (6.9) leads to the four derived syzygies

$$\begin{aligned} K_0 &= -J_0\, \Phi_0'(I) + \frac{1+J_0^2}{I}\,, & \frac{1+J_0^2}{I} &= J_1\, \Phi_0'(I), \\ K_1 &= J_1\, \Phi_1'(I) - \frac{1+J_1^2}{I}\,, & -\frac{1+J_1^2}{I} &= -J_0\, \Phi_1'(I). \end{aligned} \tag{6.11}$$

The first of each pair prove that the second order joint differential invariants K_0, K_1 are also functionally dependent upon I, as determined by the syzygies (6.10).

Theorem 6.3. *A curve C or, more generally, a pair of curves $C_0, C_1 \subset \mathbb{R}^2$, is uniquely determined up to a Euclidean transformation by its reduced joint signature, which is parametrized by the first order joint differential invariants I, J_0, J_1. The curve(s) have a one-dimensional symmetry group if and only if their signature is a one-dimensional curve if and only if they are orbits of a common one-parameter subgroup (i.e., concentric circles or parallel straight lines); otherwise the signature is a two-dimensional surface, and the curve(s) have only discrete symmetries.*

For $n > 2$ points, we can use the two-point moving frame (6.3) to construct the additional joint invariants

$$y^k \;\longmapsto\; H_k = \|\, z^k - z^0 \,\| \;\cos \psi^k, \qquad\qquad v^k \;\longmapsto\; I_k = \|\, z^k - z^0 \,\| \;\sin \psi^k,$$

where $\psi^k = \sphericalangle(z^k - z^0, z^1 - z^0)$. Therefore, a complete system of joint invariants for SE(2) consists of the angles ψ^k, $k \geq 2$, and distances $\|\, z^k - z^0 \,\|$, $k \geq 1$. The other interpoint distances can all be recovered from these angles; vice versa, given the distances, and the sign of one angle, one can recover all other angles. In this manner, we establish a "First Main Theorem" for joint Euclidean differential invariants.

Theorem 6.4. *If $n \geq 2$, then every n-point joint* E(2) *differential invariant is a function of the interpoint distances $\|\, z^i - z^j \,\|$ and their invariant derivatives with respect to* (6.8). *For the proper Euclidean group* SE(2), *one must also include the sign of one of the angles, say $\psi^2 = \sphericalangle(z^2 - z^0, z^1 - z^0)$.*

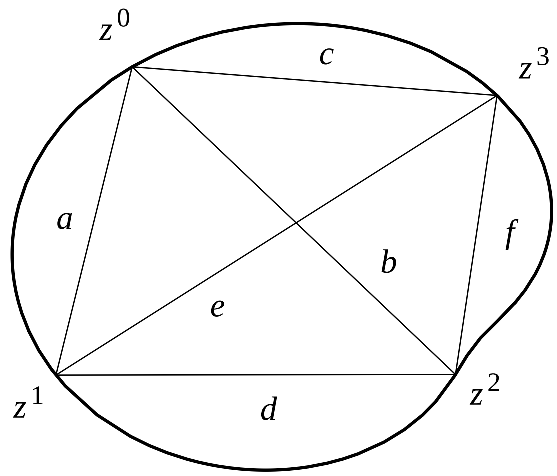

Figure 2. Four-Point Euclidean Curve Invariants.

Generic three-pointed Euclidean curves still require first order signature invariants. To create a Euclidean signature based entirely on joint invariants, we take four points z^0, z^1, z^2, z^3 on our curve $C \subset \mathbb{R}^2$. As illustrated in Figure 2, there are six different interpoint distance invariants

$$\begin{aligned} a &= \| z^1 - z^0 \|, & b &= \| z^2 - z^0 \|, & c &= \| z^3 - z^0 \|, \\ d &= \| z^2 - z^1 \|, & e &= \| z^3 - z^1 \|, & f &= \| z^3 - z^2 \|, \end{aligned} \tag{6.12}$$

which parametrize the joint signature $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}(C)$ that uniquely characterizes the curve C up to Euclidean motion. This signature has the advantage of requiring no differentiation, and so is not sensitive to noisy image data. There are two local syzygies

$$\Phi_1(a, b, c, d, e, f) = 0, \qquad \Phi_2(a, b, c, d, e, f) = 0, \tag{6.13}$$

among the the six interpoint distances. One of these is the universal Cayley–Menger syzygy

$$\det \begin{vmatrix} 2a^2 & a^2 + b^2 - d^2 & a^2 + c^2 - e^2 \\ a^2 + b^2 - d^2 & 2b^2 & b^2 + c^2 - f^2 \\ a^2 + c^2 - e^2 & b^2 + c^2 - f^2 & 2c^2 \end{vmatrix} = 0, \tag{6.14}$$

which is valid for all possible configurations of the four points, and is a consequence of their coplanarity, cf. [**2**, **31**]. The second syzygy in (6.13) is curve-dependent and serves to effectively characterize the joint invariant signature. Euclidean symmetries of the curve, both continuous and discrete, are characterized by this joint signature. For example, the number of discrete symmetries equals the signature index — the number of points in the original curve that map to a single, generic point in $\mathcal{S}$.

Example 6.5. As a final example, we consider the joint differential invariants for curves in the real (or complex) projective plane. First, an application of the moving frame

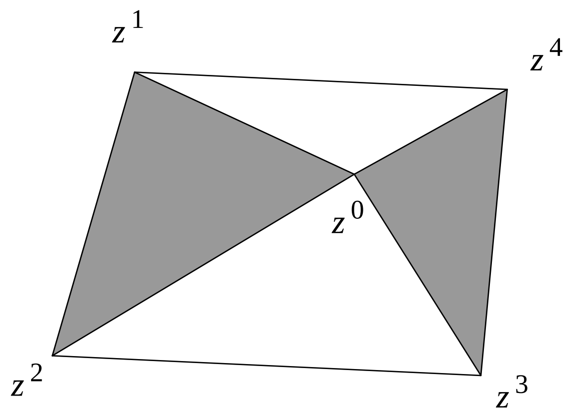

Figure 3. Projective Area Cross Ratio Invariant.

method for the joint projective action of $\mathrm{PSL}(m+1,\mathbb{R})$ on m-dimensional projective space $M = \mathbb{RP}^m$ leads to a complete classification of joint projective invariants.

Theorem 6.6. *Every joint invariant of the projective group is a function of the fundamental volume cross-ratios*

$$C(i_0,\ldots,i_{m-2};\, j,k,l,n) = \frac{[\, i_0,\ldots,i_{m-2}\ j\ k\,]\,[\, i_0,\ldots,i_{m-2}\ l\ n\,]}{[\, i_0,\ldots,i_{m-2}\ j\ l\,]\,[\, i_0,\ldots,i_{m-2}\ k\ n\,]}, \tag{6.15}$$

where the bracket notation

$$[\, i\ j\ \ldots\ m\,] = [\, z^i\ z^j\ \ldots\ z^n\,] = \det(\, z^i\ z^j, \ldots, z^n\,) \tag{6.16}$$

denotes the indicated parallelepiped volume in $\mathbb{R}^m$.

The projective invariance of the volume cross-ratios was noted by Veblen and Young, [**44**; §27]. Neither they, nor Weyl, who only briefly mentions this case, cf. [**45**; pp. 112–114], prove completeness or discuss minimal generating sets. In the one-dimensional case, we recover the usual cross-ratios. In the two-dimensional case, all five-point joint projective invariants are generated by

$$C(0;1,2,3,4) = \frac{[\,0\ 1\ 2\,]\,[\,0\ 3\ 4\,]}{[\,0\ 1\ 3\,]\,[\,0\ 2\ 4\,]}, \qquad C(1;0,2,3,4) = \frac{[\,0\ 1\ 2\,]\,[\,1\ 3\ 4\,]}{[\,0\ 1\ 3\,]\,[\,1\ 2\ 4\,]}. \tag{6.17}$$

For example, C(0;1,2,3,4) equals the product of the areas of the two shaded triangles divided by the product of the areas of the two white triangles in Figure 3.

A complete analysis of joint differential invariants and projective signatures appears in [**38**]. The following theorem summarizes the results. We use dots over bracket entries to indicate derivatives:

$$[\, i\ j\ k\,] = (z^j - z^i) \wedge (z^k - z^i), \quad [\, i\ j\ \dot{k}\,] = (z^j - z^i) \wedge \dot{z}^k, \quad [\, \dot{k}\ \ddot{m}\,] = [\, \dot{z}^k\ \ddot{z}^m\,] = \dot{z}^k \wedge \ddot{z}^m.$$

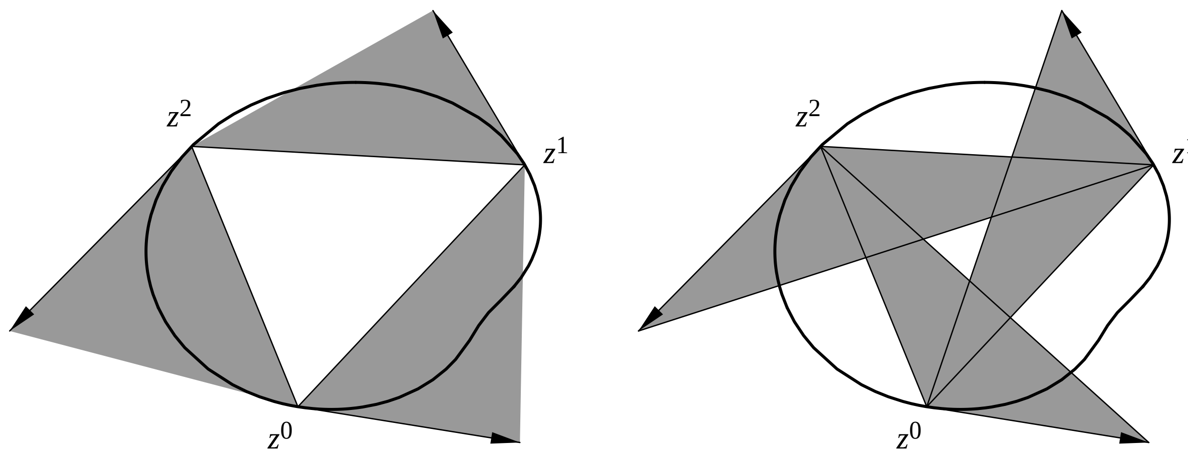

Figure 4. First Order Projective Joint Differential Invariant.

Theorem 6.7. *Every n-point joint projective differential invariant is obtained by invariant differentiation of the following joint differential invariants:*

$n \geq 5$ *The cross-ratio invariants*

$$C(0;1,3,k,2), \qquad C(1;0,3,k,2), \qquad k = 4, \ldots, n-1.$$

$n = 4$ *The first order derived cross-ratio invariants:*

$$C(i\,;\,j,k,l,\dot{m}) = \frac{[\,i\ j\ k\,]\,[\,i\ l\ \dot{m}\,]}{[\,i\ j\ l\,]\,[\,i\ k\ \dot{m}\,]}\,.$$

$n = 3$ *The first order three point joint differential invariant:*

$$\frac{[\,0\ 2\ \dot{0}\,]\,[\,0\ 1\ \dot{1}\,]\,[\,1\ 2\ \dot{2}\,]}{[\,0\ 1\ \dot{0}\,]\,[\,1\ 2\ \dot{1}\,]\,[\,0\ 2\ \dot{2}\,]}\,.$$

$n = 2$ *The second order two point joint differential invariant:*

$$\frac{[\,0\ 1\ \dot{0}\,]^3\,[\,\dot{1}\ \ddot{1}\,]}{[\,0\ 1\ \dot{1}\,]^3\,[\,\dot{0}\ \ddot{0}\,]}\,.$$

$n = 1$ *The projective curvature κ.*

The two-point joint differential invariant is a ratio of the Euclidean (actually equi-affine) joint differential invariants (6.7). The three point joint differential invariant is the ratio of the product of the three triangular areas in the first diagram over the product of the three triangular areas in the second diagram in Figure 4. The invariant differential operators depend on the number of points; see [**38**] for more details.

Additional cases, including plane curves under the equi-affine and affine groups, as well as curves and surfaces in $\mathbb{R}^3$ under Euclidean and equi-affine transformations, are investigated in detail in [**38**].

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