STABILITY OF EUCLIDEAN SPACE UNDER RICCI FLOW

OLIVER C. SCHNÜRER, FELIX SCHULZE, AND MILES SIMON

ABSTRACT. We study the Ricci flow for initial metrics which are C^0 small perturbations of the Euclidean metric on $\mathbb{R}^n$. In the case that this metric is asymptotically Euclidean, we show that a Ricci harmonic map heat flow exists for all times, and converges uniformly to the Euclidean metric as time approaches infinity. In proving this stability result, we introduce a monotone integral quantity which measures the deviation of the evolving metric from the Euclidean metric. We also investigate the convergence of the diffeomorphisms relating Ricci harmonic map heat flow to Ricci flow.

1. INTRODUCTION

In this paper we investigate the evolution of a family of complete non-compact Riemannian manifolds $(\mathbb{R}^n, g(t))$ under Ricci flow

$$
\text{(RF)} \qquad
\begin{cases}
\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}(t) & \text{in } \mathbb{R}^n \times (0, \infty), \\
g(t) \to g_0 & \text{in } C^0_{loc} \text{ as } t \searrow 0,
\end{cases}
$$

where g_0 is a given initial metric on $\mathbb{R}^n$. We study the long-term behavior as $t \to \infty$ of solutions to (RF) for initial metrics g_0 which are C^0-close to the standard Euclidean metric h, in standard coordinates $h_{ij} = \delta_{ij}$. For analytic reasons, it is convenient to study the Ricci harmonic map heat flow which is a variant of the Ricci-DeTurck flow

$$
\text{(1.1)} \qquad
\begin{cases}
\frac{\partial}{\partial t} g_{ij}(x, t) = -2 R_{ij}(x, t) + \nabla_i V_j + \nabla_j V_i & \text{in } \mathbb{R}^n \times [0, \infty), \\
g_{ij}(t) \to (g_0)_{ij} & \text{in } C^0_{loc}(\mathbb{R}^n) \text{ as } t \searrow 0,
\end{cases}
$$

where $V_i = g_{ik} \left({}^g\Gamma^k_{rs} - {}^h\Gamma^k_{rs} \right) g^{rs}$, for a flat metric h, $V_i = g_{ik} {}^g\Gamma^k_{rs} g^{rs}$. If a family of metrics $(g(t))_{t \in (0, \infty)}$ solves (1.1), we call it a solution to the h-flow with initial metric g_0. Precise definitions are to be found in Section 2. Note that the h-flow (1.1) and the Ricci flow (RF) are equivalent up to diffeomorphisms [12] in the case that the initial metric is smooth, and M is compact.

Definition 1.1. Let g_1 and g_2 denote two Riemannian metrics on a given manifold. We say that g_1 is δ-fair to g_2, if

$$
\delta^{-1} g_2 \le g_1 \le \delta \, g_2 \, .
$$

We call g_1 ε-close to g_2, if g_1 is $(1 + \varepsilon)$-fair to g_2.

We denote with $\mathcal{M}^k(\mathbb{R}^n, I)$ the space of families $(g(t))_{t \in I}$ of sections in the space of Riemannian metrics on $\mathbb{R}^n$ which are C^k on $\mathbb{R}^n \times I$. Similarly, we define $\mathcal{M}^\infty$, $\mathcal{M}^k_{loc}$ and use $\mathcal{M}^k(\mathbb{R}^n)$ if the metric does not depend on t. We wish to point out that we use C^k on non-compact sets to denote the space, where derivatives of order up to k are in L^∞. We also use C^k_{loc}.

Our existence result reads as follows

Theorem 1.2. *For every $\varepsilon > 0$ there exists $0 < \varepsilon_0(\varepsilon, n) \leq \varepsilon$, such that the following is true: If $g_0 \in \mathcal{M}^0(\mathbb{R}^n)$ is a Riemannian metric on $\mathbb{R}^n$ which is ε_0-close to the standard metric $h = \delta$, then there exists a solution $g \in \mathcal{M}^\infty\big(\mathbb{R}^n, (0, \infty)\big) \cap \mathcal{M}^0_{loc}\big(\mathbb{R}^n, [0, \infty)\big)$ to (1.1) such that $g(t)$ is ε-close to h for all t. As $t \to \infty$, the metrics $g(t)$ converge subsequentially in $\mathcal{M}^\infty_{loc}\big(\mathbb{R}^n\big)$ to a complete flat metric.*

In this situation, our main theorem addresses convergence to the Euclidean background metric.

Theorem 1.3. *Let $g \in \mathcal{M}^\infty\big(\mathbb{R}^n, (0, \infty)\big) \cap \mathcal{M}^0_{loc}\big(\mathbb{R}^n, [0, \infty)\big)$ be a solution to (1.1). Assume that $g(t)$ is $\tilde{\varepsilon}(n)$-close to the standard metric h for all $t \geq 0$, for some $\tilde{\varepsilon}(n)$ chosen sufficiently small, and that g_0 is $\varepsilon(r)$-close to h on $\mathbb{R}^n \setminus B_r(0)$ with $\varepsilon(r) \to 0$ as $r \to \infty$. Then*

$$g(t) \to h \quad in \ C^\infty \ as \ t \to \infty.$$

That is:

$$\sup_{\mathbb{R}^n} |g(t) - h| \to 0$$

as $t \to \infty$, and

$$\sup_{\mathbb{R}^n} \left| {}^h\nabla^k g(t) \right| \to 0$$

as $t \to \infty$ for all $k \in \{1, 2 \ldots\}$.

In this paper we will be concerned with the geometric quantities

$$(1.2) \qquad \varphi_m := g^{a_1 b_1} h_{b_1 a_2} g^{a_2 b_2} h_{b_2 a_3} \cdots g^{a_m b_m} h_{b_m a_1}$$

and

$$(1.3) \qquad \psi_m := g_{a_1 b_1} h^{b_1 a_2} g_{a_2 b_2} h^{b_2 a_3} \cdots g_{a_m b_m} h^{b_m a_1}.$$

If (λ_i) are the eigenvalues of (g_{ij}) with respect to (h_{ij}), we get

$$\varphi_m = \sum_{i=1}^n \frac{1}{\lambda_i^m} \quad \text{and} \quad \psi_m = \sum_{i=1}^n \lambda_i^m.$$

In particular

$$(1.4) \qquad \varphi_m + \psi_m - 2n = \sum_{i=1}^n \left(\frac{1}{\lambda_i^m} + \lambda_i^m - 2 \right) = \sum_{i=1}^n \frac{1}{\lambda_i^m} \left(\lambda_i^m - 1 \right)^2 \geq 0$$

is non-negative and vanishes precisely when $\lambda_i = 1$ for all $i \in \{1, \ldots, n\}$.

Theorem 1.4. *Let $n \geq 3$ and $g \in \mathcal{M}^\infty_{loc}\big(\mathbb{R}^n, [0, \infty)\big)$ be a solution to (1.1), satisfying all the conditions of Theorem 1.3. Then there exist $m = m(n) \in \mathbb{N}$ and*

$\tilde{\varepsilon}(n) > 0$, such that the following holds. If $1 \leq p < \frac{n}{2}$ can be chosen such that initially

$$(1.5) \qquad \int_{\mathbb{R}^n} (\varphi_m + \psi_m - 2n)^p < \infty,$$

then there exists a smooth family $(\varphi_t)_{t \geq 0}$ of diffeomorphisms of $\mathbb{R}^n$, $\varphi_0 = \mathrm{id}_{\mathbb{R}^n}$, such that for $\tilde{g}(t) := \varphi_t^* g(t)$ the family $(\tilde{g}(t))_{t \geq 0}$ is a smooth solution to (RF) satisfying

$$\tilde{g}(t) \to (\varphi_\infty)^* h \quad in \ \mathcal{M}^\infty(\mathbb{R}^n) \ as \ t \to \infty$$

for some smooth diffeomorphism φ_∞ of $\mathbb{R}^n$ which satisfies $\varphi_t \to \varphi_\infty$ in $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ as $t \to \infty$ and

$$|\varphi_\infty(x) - x| \to 0 \quad as \ |x| \to \infty.$$

Rewriting (1.5) in terms of a decay condition on $\varepsilon(r)$ in the closeness condition for g_0, see Theorem 1.3, yields

Corollary 1.5. *Let $n \geq 3$ and $g \in \mathcal{M}_{loc}^\infty(\mathbb{R}^n, [0, \infty))$ be a solution to (1.1), satisfying all the conditions of Theorem 1.3. If g_0 is ε_0-close to h for some $\varepsilon_0 = \varepsilon_0(n) > 0$ sufficiently small, and in addition there exist constants $C > 0$ and $\zeta > 0$ such that g_0 is $Cr^{-1-\zeta}$-close to h on $\mathbb{R}^n \setminus B_r(0)$, then the same conclusions as in Theorem 1.4 hold.*

Note that the solutions constructed in Theorem 1.2 satisfy the conditions of Theorem 1.3, but are not necessarily $\varepsilon(r)$-close to h at infinity. We point out that these solutions may depend on the subsequence chosen in the construction in Section 4. Note, however, that it is not clear, whether solutions to the initial value problem (1.1) are unique, as in general g_0 does not have bounded curvature or a well-defined Riemannian curvature tensor. Even if $g(t)$ is $\varepsilon(n)$-close to h on $\mathbb{R}^n$ for all $t > 0$, we do not know, whether such solutions are unique. If we study only solutions as constructed in Section 4, we can replace the $\varepsilon(r)$-closeness condition imposed on g_0 by a considerably weaker integrability condition.

Theorem 1.6. *Fix $p \geq 1$, $m = m(n) \in \mathbb{N}$ sufficiently large, and $\varepsilon_0 = \varepsilon_0(n) > 0$ sufficiently small. Let $g_0 \in \mathcal{M}^0(\mathbb{R}^n)$ be a Riemannian metric which is ε_0-close to the standard metric h. Let $g \in \mathcal{M}_{loc}^\infty(\mathbb{R}^n, (0, \infty))$ be a solution to (1.1) as constructed in Section 4. If for every $\delta > 0$ the integral $I_\delta^{m,p}(0)$, as defined in Theorem 6.1, is finite, then*

$$g(t) \to h \quad in \ \mathcal{M}^\infty(\mathbb{R}^n) \ as \ t \to \infty.$$

In the situation of Theorem 1.6, if $I_0^{m,p}(0)$ is finite, we can show that the diffeomorphisms φ_t stay bounded and converge in C_{loc}^∞ to a limiting diffeomorphism φ_∞.

Outline. In order to prove stability of Euclidean $\mathbb{R}^n$ under the h-flow, we proceed as follows.

If we perturb the metric on $\mathbb{R}^n$, such that the perturbed metric is C^0-close to our original metric and the perturbation is small enough, this is preserved for all times. The perturbation is measured in terms of the deviation of the eigenvalues (λ_i) of the perturbed metric with respect to the Euclidean metric from one. We have

$$(1 + \varepsilon)^{-1} \leq \lambda_i \leq 1 + \varepsilon \quad for \quad i = 1, \ldots, n$$

everywhere for all times and $0 < \varepsilon = \varepsilon(n)$ provided that initially $(1 + \varepsilon_0)^{-1} \leq \lambda_i \leq 1 + \varepsilon_0$ for some $0 < \varepsilon \leq \varepsilon_0(n)$.

We have interior estimates for the gradient of the metric evolving under DeTurck flow. If we assume that $\lambda_1 \leq \ldots \leq \lambda_n$, we get the estimate

$$|\lambda_i(x,t) - \lambda_i(y,t)| \leq \frac{c}{\sqrt{t}} \cdot d_{\mathrm{Eucl.}}(x,y).$$

Thus, for large times, the eigenvalues λ_i are almost spatially constant in any Euclidean ball. It is, however, possible that these spatial constants vary in time. In particular, it is not clear up until now whether these constants converge as time tends to infinity.

Our perturbation of the Euclidean metric is small near infinity, i.e. all eigenvalues λ_i approach one near infinity. Note, however, that we do not impose a decay rate on our initial metric, at which the eigenvalues converge to one as we approach spatial infinity. Still, we obtain for all positive times that the eigenvalues converge to one at spatial infinity.

In order to get uniform control on the eigenvalues, we use the quantity considered in (1.4) for some $m \in \mathbb{N}$. It vanishes precisely when $\lambda_i = 1$ for all i and measures the deviation from $\lambda_i = 1$. The functions φ_m and ψ_m were initially considered by W.-X. Shi in [12]. The key estimate is to show essentially that $\int(\varphi_m + \psi_m - 2n)$ is non-increasing in time. Here we integrate over the manifold at a fixed time. For technical reasons we have to consider a more complicated quantity. For details we refer to the respective proofs in Section 6.

This implies convergence of the eigenvalues $\lambda_i(x,t)$ to 1 as $t \to \infty$, uniformly in x. For if this were not the case, we could pick points (x_k, t_k) with $t_k \to \infty$ such that at least one $\lambda_i(x_k, t_k)$ differs significantly from 1. If t_k is large enough, we find a big ball, where at least one eigenvalue differs considerably from 1. This yields an arbitrarily large contribution to $\int(\varphi_m + \psi_m - 2n)$ if t_k is big compared to $\max_i |\lambda_i(x_k, t_k) - 1|$, contradicting the monotonicity of $\int(\varphi_m + \psi_m - 2n)$. Thus $\lambda_i(x,t) \to 1$, uniformly in x as $t \to \infty$, follows.

Remark 1.7. Note that all the above results also hold if we replace $\mathbb{R}^n$ by $\mathbb{T}^k \times \mathbb{R}^{n-k}$ for $n - k \geq 1$, where $\mathbb{T}^k$ is a flat k-dimensional torus.

In two space dimensions, Ricci flow is a conformal flow given by the evolution equation

$$\frac{\partial}{\partial t} g = -Rg,$$

where R is the scalar curvature, see [6]. In this situation, we obtain (for details see Appendix A)

Theorem 1.8. *Let $g_0 = e^{-u_0} h$, where $h = \delta$ is the standard Euclidean metric, and $u_0 \in C^0\left(\mathbb{R}^2\right)$ such that*

$$\sup_{\mathbb{R}^2 \setminus B_r(0)} |u| \to 0 \quad as\ r \to \infty.$$

Then there exists a smooth solution $(g(t))_{t \in (0,\infty)}$, $g(t) = e^{-u(\cdot,t)} h$, to Ricci flow such that $u(\cdot,t) \to u_0$ in C^0_{loc} as $t \searrow 0$. Furthermore, as $t \to \infty$, we obtain that $u(\cdot,t) \to 0$ in $C^\infty\left(\mathbb{R}^2\right)$.

For two dimensional Ricci flow, L.-F. Wu studied long time behavior and convergence of solutions [16]. She studied initial complete metrics $g_0 = e^{u_0} h$ which have bounded curvature and satisfy $e^{-u_0} |Du_0|^2 < \infty$. In this case, she showed that a long time solution to Ricci flow exists and that it "converges smoothly in the sense of modified subsequences" to a smooth limiting metric as $t \to \infty$. A smooth family of metrics $(g(t))_{t \in [0,\infty)}$ on $\mathbb{R}^n$ "converges smoothly in the sense of modified subsequences" to a metric l on $\mathbb{R}^n$ as $t \to \infty$ if there exist diffeomorphisms $\varphi_i : \mathbb{R}^n \to \mathbb{R}^n$ and a sequence $t_1 < t_2 < \ldots$ with $t_i \to \infty$ as $i \to \infty$, such that $(\varphi_i)^*(g(t_i)) \to l$ smoothly on any fixed compact subset of $\mathbb{R}^n$. In particular, this does not imply uniform convergence on all of $\mathbb{R}^n$.

By imposing strong decay conditions on the curvature tensor at infinity, stability of flat space under Ricci flow was proved by W.-X. Shi [13]. Stability of compact flat manifolds under Ricci flow was studied by C. Guenther, J. Isenberg, D. Knopf and by N. Šešum [4, 15]. In the rotationally symmetric situation stability of flat Euclidean space was investigated by T. Oliynyk and E. Woolgar [10]. A. Chau and the first author obtained a stability result for the Kähler potential of stationary rotationally symmetric solitons of positive holomorphic bisectional curvature under Kähler-Ricci flow [1]. Uniqueness of solutions to Ricci flow with bounded curvature on non-compact manifolds is discussed in [2, 8]. J. Clutterbuck and the first two authors proved stability of convex rotationally symmetric translating solutions to mean curvature flow in [3]. Short time existence results for C^0-metrics were shown in [14] using similar techniques to this paper.

The rest of the paper is organized as follows. We introduce our notation in Section 2 and recall some evolution equations in Section 3. Existence of solutions for all times is shown, see Section 4. We prove interior closeness and a priori estimates in Section 5. In Section 6 we show that an integral quantity based on the expression in (1.4) is monotonically decreasing along the flow. Combining this with the interior gradient estimates, we obtain in Section 7 that the eigenvalues of $g_{ij}(x,t)$ with respect to the background metric converge uniformly to one as $t \to \infty$. An iteration scheme allows us to improve our gradient estimates for large times, see Section 8. In Section 9, we study the diffeomorphisms relating (RF) and (1.1), and show their convergence for large times. We address the proof of Theorem 1.6 in Section 10. Stability results for Ricci flow in two dimensions are addressed in Appendix A.

2. Notation

Let $g = (g_{ij})_{1 \le i, \, j \le n}$ be a Riemannian metric. By $(g(t))_{t \in (0, \infty)}$, we denote a family of metrics. We denote the inverse of (g_{ij}) by (g^{ij}) and use the Einstein summation convention for repeated upper and lower indices. If g_1 and g_2 are two metrics such that $(g_1)_{ij} \xi^i \xi^j \le (g_2)_{ij} \xi^i \xi^j$ for all $(\xi^i) \in \mathbb{R}^n$, we denote this by $g_1 \le g_2$. Unless otherwise stated, geometric quantities like covariant derivatives ∇_i and Christoffel symbols Γ_{ij}^k are computed with respect to the evolving metrics $g(t)$. We use indices

h to denote quantities depending on the metric h, e. g. ${}^h\nabla g$ denotes the covariant derivative of g with respect to h, i. e. a partial derivative. In short formulae, we also use ∇g instead. With the exception of the beginning of Section 4, h will always denote the standard metric on $\mathbb{R}^n$. On $\mathbb{R}^n$ we will always use coordinates such that $h_{ij} = \delta_{ij}$. A ball of radius r, centered at x, is denoted by $B_r(x)$. We will only use Euclidean balls. The norms $|\cdot|$ and ${}^h|\cdot|$ are computed with respect to the flat background metric h. The letter c denotes generic constants.

In formulae, where we use λ_i to denote the eigenvalues of g_{ij} with respect to h_{ij}, we will always assume that $h_{ij} = \delta_{ij}$ in the coordinate system chosen. Several times, we will use that ${}^h|g_{ij}(x,t) - g_{ij}(y,t)| \leq A$ implies that $|\lambda_i(x,t) - \lambda_i(y,t)| \leq A$ if we assume that $\lambda_1(\cdot, t) \leq \ldots \leq \lambda_n(\cdot, t)$. Similarly, ${}^h|g_{ij}(x,t) - h_{ij}| \leq A$ implies $|\lambda_i(x,t) - 1| \leq A$.

3. Evolution Equations

In this section, we collect some evolution equations from [12] and state some direct consequences.

Assume that in appropriate coordinates, we have at a fixed point and at a fixed time $h_{ij} = \delta_{ij}$, $g_{ij} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_i > 0$.

The evolution equation for the metric, computed with respect to a flat background metric, is, see [12, Lemma 2.1],

$$
\begin{aligned}
\frac{\partial}{\partial t} g_{ij} = {} & g^{ab}\,{}^h\nabla_a\,{}^h\nabla_b g_{ij} \\
& + \tfrac{1}{2} g^{ab} g^{pq} \left({}^h\nabla_i g_{pa}\,{}^h\nabla_j g_{qb} + 2\,{}^h\nabla_a g_{jp}\,{}^h\nabla_q g_{ib} - 2\,{}^h\nabla_a g_{jp}\,{}^h\nabla_b g_{iq} \right. \\
& \left. \qquad\qquad - 2\,{}^h\nabla_j g_{pa}\,{}^h\nabla_b g_{iq} - 2\,{}^h\nabla_i g_{pa}\,{}^h\nabla_b g_{jq} \right).
\end{aligned}
$$

Consider φ_m as defined in (1.2). The evolution equation of φ_m for a flat background metric h_{ij} is, see [12, Lemma 2.2 (70)],

$$
\begin{aligned}
\frac{\partial}{\partial t} \varphi_m = {} & g^{ab}\,{}^h\nabla_a\,{}^h\nabla_b \varphi_m \\
& - m \sum_{i,j,a} \frac{1}{\lambda_a} \left[\sum_{k=2}^{m} \left(\frac{1}{\lambda_i}\right)^k \left(\frac{1}{\lambda_j}\right)^{m+2-k} \right] \left({}^h\nabla_a g_{ij}\right)^2 \\
& - \sum_{i,q,k} \frac{m}{2\lambda_i^{m+1}\lambda_q\lambda_k} \left({}^h\nabla_k g_{iq} + {}^h\nabla_q g_{ik} - {}^h\nabla_i g_{qk}\right)^2.
\end{aligned}
$$

We deduce for $g(t)$ that is ε-close to h that

$$
\begin{aligned}
\frac{\partial}{\partial t} \varphi_m \leq {} & g^{ij}\,{}^h\nabla_i\,{}^h\nabla_j \varphi_m \\
& - m(m-1)(1+\varepsilon)^{-(m+3)} \sum_{i,j,k} \left({}^h\nabla_k g_{ij}\right)^2.
\end{aligned}
$$

From the evolution equation of the metric, we get for ψ_m, as introduced in (1.3), the evolution equation

$$
\begin{aligned}
\frac{\partial}{\partial t}\psi_m &= m h^{a_1 b_1} g_{b_1 a_2} h^{a_2 b_2} g_{b_2 a_3} \cdots h^{a_m b_m} \frac{\partial}{\partial t} g_{b_m a_1} \\
&= m h^{a_1 b_1} g_{b_1 a_2} h^{a_2 b_2} g_{b_2 a_3} \cdots h^{a_m b_m} \left(g^{ij}\,^h\nabla_i\,^h\nabla_j g_{b_m a_1} \right) \\
&\quad + c(n) m \underbrace{h^{-1} * \cdots * h^{-1}}_{m} * \underbrace{g * \cdots * g}_{m-1} * g^{-1} * g^{-1} * {}^h\nabla g * {}^h\nabla g \\
&= g^{ij}\,^h\nabla_i\,^h\nabla_j \psi_m \\
&\quad - m \sum_{i,\,j,\,a} \left[\sum_{k=0}^{m-2} \lambda_i^k \lambda_j^{m-2-k} \right] \frac{1}{\lambda_a} \left({}^h\nabla_a g_{ij} \right)^2 \\
&\quad + c(n) m \underbrace{h^{-1} * \cdots * h^{-1}}_{m} * \underbrace{g * \cdots * g}_{m-1} * g^{-1} * g^{-1} * {}^h\nabla g * {}^h\nabla g,
\end{aligned}
$$

where $*$ indicates contractions and linear combinations of contractions. The factor $c(n)$ indicates that the number of the respective terms depends only on n. Therefore, we deduce that

$$
\begin{aligned}
\frac{\partial}{\partial t}\psi_m \leq\; & g^{ij}\,^h\nabla_i\,^h\nabla_j \psi_m \\
& - m(m-1)(1+\varepsilon)^{-(m-1)} \sum_{i,\,j,\,k} \left({}^h\nabla_k g_{ij} \right)^2 \\
& + c(n) m (1+\varepsilon)^{m+1} \sum_{i,\,j,\,k} \left({}^h\nabla_k g_{ij} \right)^2.
\end{aligned}
$$

4. Existence

In this section we prove long time existence of the h-flow for a smooth initial metric g_0 which is ε-close to $h = \delta$. Short time existence for Ricci-DeTurck flow on non-compact manifolds was first proved in [12]. Short time existence for an arbitrary smooth background metric h with bounded curvature, when g_0 is ε-close to h, was proved in [14], using similar techniques to those of W.-X. Shi. We also reprove the short time existence here for completeness. (note: some of the citations in [14] in the proof of the short time existence were incorrect.)

Lemma 4.1. *Let $(g(t))_{t\in[0,T)}$ be a smooth ε-close solution to the h-flow with flat background metric $h = \delta$ on a closed ball $D \subset \mathbb{R}^n$ with $0 < \varepsilon \leq \varepsilon(n)$ and $g(\cdot, t)|_{\partial D} = h(\cdot)|_{\partial D}$. Then*

$$
\sup_{D\times[0,T)} \left| {}^h\nabla^m g \right|^2 \leq c(n, m, D, T, g(0)).
$$

Proof. We proceed exactly as in the proof of Shi [12, Lemma 3.1]: where he uses $\tilde{g}$, we use h. The only other minor difference is that the dependence on $g(0)$ does not explicitly appear in Shi's paper [12, Lemma 3.1]. This is because he has $\tilde{g} = g(0)$. $\qquad\square$

Lemma 4.2. *Let $(g(t))_{t \in [0,T)}$ be a smooth solution to the h-flow on B_i, $i \in \mathbb{N}$, where h is the standard metric on $\mathbb{R}^n$. Then for all $\varepsilon > 0$ there exists $0 < \varepsilon_0(n, \varepsilon) < \varepsilon$, such that the following holds. If $g(0)$ is ε_0-close to h and $g(t) = h$ on $\partial B_i \times [0,T)$, then $g(t)$ is ε-close to h for all $t \in [0,T)$.*

Proof. Consider $\Phi := \varphi_m + \psi_m - 2n$ as in (1.4). It vanishes on $\partial D \times [0,T)$. The evolution equations in Section 3 imply that

(4.1) $$\frac{\partial}{\partial t}\Phi \leq g^{ij}{}^h\nabla_i{}^h\nabla_j\Phi$$

as long as $g(t)$ is $\tilde{\varepsilon}$-close to h for $\tilde{\varepsilon} > 0$ satisfying

$$c(n)(1 + \tilde{\varepsilon})^{2m} \leq (m - 1).$$

Choose $\tilde{\varepsilon} = \tilde{\varepsilon}(n)$ and $m = m(n)$ accordingly.

Fix $\delta > 0$ so that $\Phi \leq 2\delta$ implies, see (1.4), that the metric is ε-close to h, in particular $\tilde{\varepsilon}$-close to h. Now fix $\varepsilon_0 = \varepsilon_0(n) > 0$ such that $\varepsilon_0 < \varepsilon$ and so that $\Phi \leq \delta$ for every metric which is ε_0-close to h.

Consider the maximal time interval $I \subset [0,T)$ on which $\Phi \leq 2\delta$. We may assume that $I = [0, \tau]$ with $\tau < T$. This implies that $(g(t))_{t \in [0,\tau]}$ is $\tilde{\varepsilon}$-close to h (even ε-close to h). According to (4.1) and the maximum principle, $\max_{x \in D} \Phi(x,t)$ is non-increasing in t for all $t \in [0, \tau]$. Thus $\Phi(\tau) \leq \Phi(0) \leq \delta$. This contradicts the choice of τ. Therefore $I = [0,T)$ and $(g(t))_{t \in [0,T)}$ is ε-close to h. $\qquad\square$

In the remainder of this section, we will denote with $\varepsilon_0(n, \varepsilon) > 0$ the constant in Lemma 4.2. Note that although we allow for arbitrary $\varepsilon > 0$, it follows from the proof that $\varepsilon_0(n, \varepsilon) \leq \overline{\varepsilon}$ for some small $\overline{\varepsilon} = \overline{\varepsilon}(n) > 0$.

Theorem 4.3. *Let $\varepsilon > 0$ be given, and g_0 be smooth and $\varepsilon_0(\varepsilon, n)$-close to $h = \delta$ on a closed ball $D \subset \mathbb{R}^n$. Assume that $g_0 = h$ near ∂D. Then there exists a unique smooth solution $(g(t))_{t \in [0,\infty)}$ to the h-flow with $g(0) = g_0$ and $g(t)|_{\partial D} = h|_{\partial D}$. Furthermore, $(g(t))_{t \in [0,\infty)}$ is ε-close to h.*

Proof. Lemma 4.2 implies that a prospective solution $g(t)$ is ε-close to h as long as it exists. Therefore, we can apply the a priori bounds of Lemma 4.1 on a bounded time interval. We may then use the same arguments as in [9, Theorem 7.1, Chapter VII] to show that a smooth solution exists (the argument used there is based on the Leray-Schauder fixed point argument of [9, Theorem 6.1 of Chapter V]). Note that this argument works for every finite time interval. $\qquad\square$

Theorem 4.4. *Let $g \in \mathcal{M}^\infty(B_R, [0,T])$ be $\tilde{\varepsilon}$-close to h, solving (1.1), where $\tilde{\varepsilon} = \tilde{\varepsilon}(n) > 0$ is sufficiently small and $h = \delta$. Then*

$$\sup_{B_{\frac{R}{2}} \times [0,T]} \left|{}^h\nabla^m g\right|^2 \leq c(m, n, g(0)|_{B_R}, T).$$

Proof. This follows using the arguments of Shi [12, Lemma 4.1 and Lemma 4.2]. $\quad\square$

Theorem 4.5. *Let $\varepsilon > 0$ be given, and g_0 be smooth and $\varepsilon_0(n, \varepsilon)$-close to the standard Riemannian metric $h = \delta$ on $\mathbb{R}^n$. Then there exists a ε-close solution $g \in \mathcal{M}^\infty_{loc}(\mathbb{R}^n, [0,\infty))$ to the h-flow (1.1).*

Proof. Let B_i be the balls of radius i and center 0 (with respect to the Euclidean metric). Set ${}^i g_0 = \eta_i g_0 + (1 - \eta_i) h$, where $\eta_i : \mathbb{R}^n \to \mathbb{R}$ is smooth, and satisfies $\eta_i = 0$ on $B_i - B_{i-1}$, $\eta_i = 1$ on B_{i-2}, $0 \le \eta \le 1$ and $|\nabla^m \eta_i|^2 \le c(m, n)$. Note that ${}^i g_0$ is ε_0-close to h. Let ${}^i g(t) \in \mathcal{M}^\infty(B_i, [0, \infty))$ be the solution coming from the local existence theorem (Theorem 4.3) above. Using the interior estimates of Theorem 4.4 above, we see that the solutions all satisfy

$$\sup_{B_j \times [0,T]} \left| \nabla^m \left(\frac{d}{dt} \right)^k ({}^i g) \right|^2 \le c(j, g_0|_{B_{2j}}, m, k, n, T),$$

for all i big enough, and so, using a diagonal subsequence argument, we can find a subsequence which converges to a solution $g \in \mathcal{M}^\infty_{loc}(\mathbb{R}^n, [0, \infty))$ where the convergence ${}^i g \to g$ is uniform on $B_j \times [0, T]$ for every fixed $j \in \mathbb{N}$ and $T \in (0, \infty)$ and $g(0) = g_0$. $\qquad \square$

5. Interior Estimates

Note that the following lemma does not imply $\lambda_i(x, t) - 1 \to 0$ for $|x| \to \infty$ uniformly in t, i.e.

$$\lim_{R \to \infty} \sup_{x \in \mathbb{R}^n \setminus B_R} \sup_{i \in \{1, \ldots, n\}} \sup_{t \in [0, \infty)} |\lambda_i(x, t) - 1|$$

may be nonzero.

Lemma 5.1. *Let $g \in \mathcal{M}^\infty_{loc}(\mathbb{R}^n, (0, \infty))$ be a solution to (1.1) with g_0 as in Theorem 1.3. Then the eigenvalues $(\lambda_i(x, t))$ of $(g_{ij}(x, t))$ with respect to $(h_{ij}(x))$ uniformly tend to 1 in bounded time intervals,*

$$\lim_{|x| \to \infty} \lambda_i(x, t) = 1.$$

This Lemma is a direct consequence of the following interior closeness estimate.

Lemma 5.2. *Let $g \in \mathcal{M}^\infty_{loc}(\mathbb{R}^n, (0, \infty))$ be a solution to (1.1) which is $\tilde{\varepsilon}(n)$-close to the Euclidean background metric h. Assume that g_0 is ε-close to h on some ball $B_R(x_0)$ for some $\varepsilon \le \tilde{\varepsilon}(n)/2$. Then there is a constant $\gamma(n) > 0$ such that $g(t)$ is 2ε-close to h on $B_{R/2}(x_0)$ for $t \in [0, \gamma R^2]$.*

Proof. According to the estimates in Section 3, there exists $\varepsilon(n) > 0$ and $m(n) \in \mathbb{N}$ such that

$$\frac{\partial}{\partial t}(\varphi_m + \psi_m - 2n) \le g^{ab}{}^h\nabla_a {}^h\nabla_b(\varphi_m + \psi_m - 2n) - \sum_{i,j,k} \left({}^h\nabla_k g_{ij} \right)^2 .$$

If the background metric h is the standard flat metric on $\mathbb{R}^n$, we define for $\lambda > 0$

$$g^\lambda(x, t) = g\left(\frac{x}{\lambda}, \frac{t}{\lambda^2} \right).$$

If g is a solution to the h-flow, g^λ is also a solution to the h-flow as h is the standard metric on $\mathbb{R}^n$. Moreover, g^λ is ε-close to h, if g is ε-close to h.

By scaling with the factor $\lambda = 1/R$ and translation we can assume that $R = 1$ and $x_0 = 0$. Let $\eta \in C_c^\infty(B_1)$ be such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on $B_{1/2}$ and let

$$\zeta := \eta \cdot (\varphi_m + \psi_m - 2n) .$$

We compute

$$\frac{\partial}{\partial t}\zeta \leq g^{ab}{}^{h}\nabla_a{}^{h}\nabla_b\zeta - 2g^{ab}{}^{h}\nabla_a\eta\,{}^{h}\nabla_b(\varphi_m + \psi_m - 2n)$$

$$- (\varphi_m + \psi_m - 2n)g^{ab}{}^{h}\nabla_a{}^{h}\nabla_b\eta - \eta\sum_{i,\,j,\,k}\left({}^{h}\nabla_k g_{ij}\right)^2$$

$$\leq g^{ab}{}^{h}\nabla_a{}^{h}\nabla_b\zeta + c\;,$$

since $\left|{}^{h}\nabla(\varphi_m + \psi_m - 2n)\right|^2 \leq c\sum_{i,\,j,\,k}\left({}^{h}\nabla_k g_{ij}\right)^2$ and $|\nabla\eta|^2/\eta \leq C(n, \|\eta\|_{C^2})$. So by the maximum principle the maximum of ζ grows at most linearly, which implies the stated estimate. $\qquad\square$

By scaling we want to extend a priori derivative estimates for the metric to balls of any radius. If the background metric h is not necessarily flat, the third author [14] obtained the following

Theorem 5.3. *Let $R > 0$. Let h be a complete background metric of bounded curvature. Fix $T = T(n, R, h) > 0$ sufficiently small. Fix a point x_0. Let g be a solution to the h-flow on ${}^{h}B_R(x_0) \times (0, T)$ which is $\tilde{\varepsilon}$-close to the background metric h for $\tilde{\varepsilon} = \tilde{\varepsilon}(n) > 0$ fixed sufficiently small, where ${}^{h}B_R(x_0)$ denotes a geodesic ball of radius R with respect to the metric h. Then*

$$\left|{}^{h}\nabla^i g(x, t)\right| \leq \frac{c(n, i, R)}{t^{i/2}}$$

for all $(x, t) \in {}^{h}B_{R/2}(x_0) \times (0, T)$ and all $i \in \mathbb{N}$.

Applying Theorem 5.3 to g^λ as defined in the proof of Lemma 5.2, we get directly

Corollary 5.4. *Let h denote the standard flat metric on $\mathbb{R}^n$ and assume that $\varepsilon = \varepsilon(n) > 0$ and $\gamma = \gamma(n) > 0$ are chosen sufficiently small. If $g : \mathbb{R}^n \times (0, T)$ solves the h-flow in $B_R(x_0) \times \left(0, \gamma R^2\right)$ for some $x_0 \in \mathbb{R}^n$, $R > 0$, and if g is ε-close to h on $B_R(x_0) \times \left(0, \gamma R^2\right)$, then we get the a priori estimates*

$$\left|{}^{h}\nabla^i g(x, t)\right| \leq \frac{c(n, i)}{t^{i/2}}$$

for all $(x, t) \in B_{R/2}(x_0) \times \left(0, \gamma R^2\right)$ and all $i \in \mathbb{N}$. We set $c_1 := c(n, 1)$.

Proof of Theorem 1.2. Let us first assume that $g_0 \in \mathcal{M}^\infty_{loc}(\mathbb{R}^n)$. Theorem 4.5 implies that an ε-close solution $g \in \mathcal{M}^\infty_{loc}(\mathbb{R}^n, (0, \infty))$ to (1.1) exists. The interior decay estimates of Corollary 5.4 and Arzelà-Ascoli imply that the metrics $g(t)$ converge subsequentially in $\mathcal{M}^\infty_{loc}$ to a complete flat metric as $t \to \infty$.

We approximate $g_0 \in \mathcal{M}_0(\mathbb{R}^n)$ by smooth metrics ${}^i g_0$ preserving the ε_0-closeness. Thus we obtain solutions ${}^i g \in \mathcal{M}^\infty_{loc}$ which are ε-close to h. In view of the interior a priori estimates we obtain a limiting solution $g \in \mathcal{M}^\infty_{loc}(\mathbb{R}^n, (0, \infty))$.

Note that $g(t) \to g_0$ in $\mathcal{M}^0_{loc}$ as $t \searrow 0$: Fix a point $x_0 \in \mathbb{R}^n$ and use $g_0(x_0)$ as a flat background metric on $\mathbb{R}^n$. Thus the metrics ${}^i g$ are again solutions to the h-flow with $h = g_0(x_0)$. Hence we can apply the interior closeness estimates of Lemma 5.2

in order to see that all metrics $^i g$ attain their initial values uniformly in i. That implies that $g(t) \to g_0$ as $t \searrow 0$ locally uniformly. $\qquad\square$

6. Integral Estimates

In this section, we are once again concerned with the quantity $\varphi_m + \psi_m - 2n$ as introduced in (1.4).

Theorem 6.1. *Fix $\delta > 0$, $m = m(n) \in \mathbb{N}$, and $p \geq 1$. Let $g \in \mathcal{M}_{loc}^\infty (\mathbb{R}^n, (0, \infty))$ be a solution to (1.1) which is $\tilde{\varepsilon}(n, m)$-close to the standard Euclidean metric h for some $1 \geq \tilde{\varepsilon} = \tilde{\varepsilon}(n, m) > 0$ sufficiently small. Let g_0 be as in Theorem 1.3. Then the integral*

$$I(t) \equiv I_\delta^{m,p}(t) := \frac{1}{p} \int\limits_{\mathbb{R}^n} (\varphi_m + \psi_m - 2n - \delta)_+^p \, dx \equiv \frac{1}{p} \int\limits_{\mathbb{R}^n} \Phi_{m,\delta}^p$$

is non-increasing in time.

Proof. As g is $\tilde{\varepsilon}$-close to h, $\varphi_m + \psi_m - 2n$ is uniformly bounded above. According to Lemma 5.1, we see that $\varphi_m + \psi_m - 2n \to 0$ for $|x| \to \infty$ and t bounded above. Thus the positive part of $\varphi_m + \psi_m - 2n - \delta$ is contained in a compact set for every bounded time interval and $I(t)$ is finite there.

Assume that in appropriate coordinates, we have at a fixed point $h_{ij} = \delta_{ij}$, $g_{ij} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_i > 0$.

Recall from Section 3 that we can estimate in the evolution equation of $\varphi_m + \psi_m$ for metrics ε-close to h as follows

$$\frac{\partial}{\partial t}(\varphi_m + \psi_m) \leq g^{ij}{}^h\nabla_i {}^h\nabla_j(\varphi_m + \psi_m)$$

$$- m(m-1)(1+\varepsilon)^{-(m-1)} \sum_{i,\,j,\,k} \left({}^h\nabla_k g_{ij}\right)^2$$

$$+ c(n)m(1+\varepsilon)^{m+1} \sum_{i,\,j,\,k} \left({}^h\nabla_k g_{ij}\right)^2.$$

We now want to show that $I(t)$ is decreasing in time for m and ε chosen properly. Note that $\Phi_{m,\delta}^p$ is Lipschitz continuous in space and time, and the support of $\Phi_{m,\delta}^p$ is contained in $B_R(0)$ for some $R > 0$ on bounded time intervals. This yields that $I(t)$ is Lipschitz continuous in time as well and thus also absolutely continuous. By Sard's theorem we know that for almost every $\delta > 0$ the sets

$$U_\delta \equiv U_\delta(t) := \{x \in \mathbb{R}^n \mid \varphi_m(x,t) + \psi_m(x,t) - 2n > \delta\}$$

have a smooth boundary for almost every t. For such a δ and $0 \leq t_1 < t_2$, we can compute

$$I(t_2) - I(t_1) = \int\limits_{t_1}^{t_2} \frac{1}{p} \frac{d}{d\tau} \int\limits_{\mathbb{R}^n} \Phi_{m,\delta}^p \, dx \, d\tau$$

$$= \int\limits_{t_1}^{t_2} \int\limits_{U_\delta} \Phi_{m,\delta}^{p-1} \frac{\partial}{\partial\tau}(\varphi_m + \psi_m) \, dx \, d\tau \,,$$

as $\Phi_{m,\delta}$ is compactly supported as long as t is finite. Since the boundary of U_δ is smooth we may estimate and integrate by parts

$$I(t_2) - I(t_1) \leq \int\limits_{t_1}^{t_2} \int\limits_{U_\delta} \Phi_{m,\delta}^{p-1} \left(g^{ij\,h}\nabla_i{}^h\nabla_j(\varphi_m + \psi_m) \right) \, dx \, d\tau$$

$$- \int\limits_{t_1}^{t_2} \int\limits_{U_\delta} m(m-1)(1+\varepsilon)^{-(m-1)} \Phi_{m,\delta}^{p-1} \sum_{i,\,j,\,k} \left({}^h\nabla_k g_{ij} \right)^2 \, dx \, d\tau$$

$$+ \int\limits_{t_1}^{t_2} \int\limits_{U_\delta} c(n)m(1+\varepsilon)^{m+1} \Phi_{m,\delta}^{p-1} \sum_{i,\,j,\,k} \left({}^h\nabla_k g_{ij} \right)^2 \, dx \, d\tau$$

$$= \int\limits_{t_1}^{t_2} \int\limits_{\partial U_\delta} \Phi_{m,\delta}^{p-1} \nu_i g^{ij\,h}\nabla_j \Phi_{m,\delta} \, d\sigma \, dt$$

$$- \int\limits_{t_1}^{t_2} \int\limits_{U_\delta} (p-1) \Phi_{m,\delta}^{p-2} g^{ij\,h}\nabla_i \Phi_{m,\delta}{}^h\nabla_j \Phi_{m,\delta} \, dx \, d\tau$$

$$- \int\limits_{t_1}^{t_2} \int\limits_{U_\delta} \Phi_{m,\delta}^{p-1} \left({}^h\nabla_i g^{ij} \right){}^h\nabla_j \Phi_{m,\delta} \, dx \, d\tau$$

$$+ \int\limits_{t_1}^{t_2} \int\limits_{U_\delta} (-m(m-1))(1+\varepsilon)^{-m+1} \Phi_{m,\delta}^{p-1} \sum_{i,\,j,\,k} \left({}^h\nabla_k g_{ij} \right)^2 \, dx \, d\tau$$

$$+ \int\limits_{t_1}^{t_2} \int\limits_{U_\delta} c(n)m(1+\varepsilon)^{m+1} \Phi_{m,\delta}^{p-1} \sum_{i,\,j,\,k} \left({}^h\nabla_k g_{ij} \right)^2 \, dx \, d\tau$$

$$\leq m \int\limits_{t_1}^{t_2} \int\limits_{U_\delta} \Phi_{m,\delta}^{p-1} \sum_{i,\,j,\,k} \left({}^h\nabla_k g_{ij} \right)^2 \cdot$$

$$\cdot \left(-(m-1)(1+\varepsilon)^{-m+1} + c(n)(1+\varepsilon)^{m+3} \right) \, dx \, d\tau \,,$$

where we used that ${}^h\left| {}^h\nabla \Phi_{m,\delta} \right| \leq c\,m\,(1+\varepsilon)^{m+1} \cdot {}^h\left| {}^h\nabla g_{ij} \right|$ and that ${}^h\nabla \Phi_{m,\delta}$ is antiparallel to the outer unit normal ν of U_δ along ∂U_δ. Note that the integration by parts involving the exponent $p-2$ above is justified by applying the divergence

theorem on sets U_{δ_i} for a sequence $\delta_i \searrow \delta$, where we can assume by Sard's theorem that ∂U_{δ_i} is smooth for all i and that $\partial U_{\delta_i} \to \partial U_\delta$. If m and ε are such that

$$1 + c(n)(1 + \varepsilon)^{2m+2} \leq m,$$

we obtain that the right hand side is nonpositive and our theorem follows for such a δ. Fix $\tilde{\varepsilon} > 0$ accordingly. Since

$$I_{m,p}^{\delta_i}(t) \to I_{m,p}^\delta(t)$$

for $\delta_i \to \delta$ we obtain the above monotonicity for all δ. $\qquad\square$

Corollary 6.2. *Let $g \in \mathcal{M}_{loc}^\infty(\mathbb{R}^n, (0, \infty))$ be a solution to (1.1) with g_0 as in Theorem 1.3. Let m, p, and $\tilde{\varepsilon}$ be as in Theorem 6.1. If $I_0^{m,p}(0)$ is finite, then $I_0^{m,p}(t)$ is non-increasing and thus*

$$I^{m,p}(t) := \frac{1}{p} \int_{\mathbb{R}^n} (\varphi_m + \psi_m - 2n)^p \, dx = I_0^{m,p}(t) \leq I_0^{m,p}(0) < \infty.$$

Proof. Theorem 6.1 implies that

$$I_\delta^{m,p}(t) \leq I_\delta^{m,p}(0) \leq I_0^{m,p}(0)$$

for every $\delta > 0$. For $\delta \searrow 0$, we get

$$I_\delta^{m,p}(t) \to I_0^{m,p}(t)$$

and our claim follows. $\qquad\square$

7. CONVERGENCE OF EIGENVALUES

Lemma 7.1. *Let $g \in \mathcal{M}_{loc}^\infty(\mathbb{R}^n, (0, \infty))$ be a solution to (1.1) with g_0 as in Theorem 1.3, such that $g(t)$ is $\tilde{\varepsilon}(n)$-close to h for all t. Then the eigenvalues $(\lambda_i)_{1 \leq i \leq n}$ of $g(t)$ with respect to the background metric h converge uniformly to one as $t \to \infty$,*

$$\sup_{x \in \mathbb{R}^n} |\lambda_i(x, t) - 1| \to 0 \quad \text{as } t \to \infty.$$

Proof. Denote the eigenvalues of g_{ij} with respect to h_{ij} by (λ_i). It is convenient to assume that $\lambda_1 \leq \ldots \leq \lambda_n$.

Assume that the lemma were false. Then we could find $\zeta > 0$ and a sequence $(x_k, t_k)_{k \in \mathbb{N}}$ in $\mathbb{R}^n \times [0, \infty)$ with $t_k \to \infty$ so that

$$(7.1) \qquad \max_{i \in \{1, \ldots, n\}} |\lambda_i(x, t) - 1| \geq 2\zeta.$$

The strategy of the proof is to use Corollary 5.4 with $i = 1$ to show that we find a controlled spatial neighborhood of (x_k, t_k) such that (7.1) is fulfilled with 2ζ replaced by ζ, i.e. the eigenvalues are not all close to one there. Therefore, $\varphi_m + \psi_m - 2n$ is estimated from below by a positive constant in that neighborhood. For large values of t, these neighborhoods can be chosen arbitrarily large. Thus $I_\delta^{m,p}$ becomes large, if δ is chosen small enough. This contradicts Theorem 6.1.

Here we present the details: let $B_R(x_k)$ denote a Euclidean ball around x_k. According to Corollary 5.4, applied with $i = 1$, we obtain for $x \in \tilde{B}_R(x_k)$ that the eigenvalues $(\lambda_i(x, t_k))$ differ at most by $c \cdot R \cdot t_k^{-1/2}$ from the eigenvalues $(\lambda_i(x_k, t_k))$,

$$\sup_{i \in \{1, \dots, n\}} |\lambda_i(x_k, t_k) - \lambda_i(x, t_k)| \leq \frac{c_1 R}{\sqrt{t_k}}.$$

We deduce that in a ball of radius at least R around x_k with $R := \frac{1}{c_1} \zeta \sqrt{t_k}$, we have

$$\max_{i \in \{1, \dots, n\}} |\lambda_i(x, t_k) - 1| \geq \zeta.$$

Note that this implies

$$\max_{i \in \{1, \dots, n\}} |\lambda_i^m(x, t) - 1| \geq \zeta.$$

So at least one eigenvalue differs significantly from one in that ball. As our solution is $\tilde{\varepsilon}(n)$-close to h with $0 < \tilde{\varepsilon}(n) \leq 1$, we still have

$$\tfrac{1}{2} \leq (1 + \tilde{\varepsilon})^{-1} \leq \lambda_i, \, \lambda_i^{-1} \leq 1 + \tilde{\varepsilon} \leq 2.$$

We obtain in $B_R(x_k) \times \{t_k\}$

$$\varphi_m + \psi_m - 2n = \sum_{i=1}^{m} \frac{1}{\lambda_i^m} \left(\lambda_i^m - 1\right)^2 \geq \frac{1}{2^m} \zeta^2.$$

We define $\delta := \frac{1}{2} \frac{1}{2^m} \zeta^2$ and get in $B_R(x_k) \times \{t_k\}$

$$\varphi_m + \psi_m - 2n \geq 2\delta,$$
$$(\varphi_m + \psi_m - 2n - \delta)_+ \geq \delta.$$

This allows to estimate $I_\delta^{m,p}(t_k)$ from below as follows

$$I_\delta^{m,p}(t_k) \geq \quad \frac{1}{p} \int\limits_{B_{\frac{1}{c_1} \zeta \sqrt{t_k}}(x_k)} (\varphi_m + \psi_m - 2n - \delta)_+^p$$

$$\geq \frac{1}{p} \cdot \delta^p \cdot c(n) \cdot \left(\frac{1}{c_1} \zeta \sqrt{t_k}\right)^n.$$

We deduce that $I_\delta^{m,p}(t_k) \to \infty$ for $t_k \to \infty$. This contradicts Theorem 6.1 which implies that

$$I_\delta^{m,p}(t_k) \leq I_\delta^{m,p}(0)$$

as our initial decay assumption guarantees that for fixed $m \in \mathbb{N}_+$ as in Theorem 6.1, $p \geq 1$, and $\delta > 0$, the integral $I_\delta^{m,p}(0)$ is finite. □

Proof of Theorem 1.3. Lemma 7.1 implies that $g(t) \to h$ in C^0 as $t \to \infty$. The convergence of the derivatives $\overset{h}{\nabla}{}^m g(t)$ for all $m \in \mathbb{N}$ is a consequence of Corollary 5.4. □

If $I_0^{m,p}(0)$ is finite for some m, p as in Section 6, such a calculation implies even a decay rate in time of

$$\sup_{x \in \mathbb{R}^n} |\lambda_i(x, t) - 1|.$$

Lemma 7.2. *Let $g \in \mathcal{M}_{loc}^{\infty}(\mathbb{R}^n, (0, \infty))$ be a solution to (1.1) with g_0 as in Theorem 1.3, such that $g(t)$ is $\tilde{\varepsilon}(n)$-close to h for all t. Then the eigenvalues $(\lambda_i)_{1 \leq i \leq n}$ of $g(t)$ with respect to the flat background metric h converge to 1 as $t \to \infty$:*

$$\sup_{x \in \mathbb{R}^n} \sup_{i \in \{1, \ldots, n\}} |\lambda_i(x,t) - 1| \leq c(n,m,p,c_1) \cdot (I_0^{m,p}(0))^{\frac{1}{2p+n}} \cdot \left(\frac{1}{t}\right)^{\frac{n}{2(2p+n)}}$$

if $I_0^{m,p}(0) < \infty$.

Proof. Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$, and $i_0 \in \{1, \ldots, n\}$. Define

$$\zeta := \tfrac{1}{2}|\lambda_{i_0}(x_0, t_0) - 1|.$$

We want to estimate ζ from above. As in the proof of Lemma 7.1, we obtain that

$$\zeta \leq \max_{i \in \{1, \ldots, n\}} \max_{x \in B_{\frac{1}{c_1} \zeta \sqrt{t_0}}(x_0)} |\lambda_i(x, t_0) - 1|,$$

$$\varphi_m + \psi_m - 2n \geq \frac{1}{2^m} \zeta^2,$$

$$I_0^{m,p}(0) \geq I_0^{m,p}(t_0)$$

$$\geq \frac{1}{p} \int_{B_{\frac{1}{c_1} \zeta \sqrt{t_0}}(x_0)} (\varphi_m + \psi_m - 2n)^p$$

$$\geq \frac{1}{p} \left(\frac{1}{2^m} \zeta^2\right)^p \cdot c(n) \cdot \left(\frac{1}{c_1} \zeta \sqrt{t_0}\right)^n,$$

and thus

$$(2\zeta)^{2p+n} \leq c(n,m,p,c_1) \cdot t_0^{-n/2} \cdot I_0^{m,p}(0).$$

Our claim follows. $\qquad\square$

8. Improved C^1-Estimates

Based on an iteration of two steps, we can improve our a priori estimates. So far, we have obtained a priori estimates of the form

$$|g - h|_{C^0} \leq \frac{c\left(n, m, p, c_1(n), I_0^{m,p}\right)}{t^{\frac{n}{2(2p+n)}}} \equiv \frac{c_0}{t^{\frac{n}{2(2p+n)}}},$$

see Lemma 7.2, and

$$(8.1) \qquad \left|{}^h\nabla^i g\right| \leq \frac{c(i,n)}{t^{i/2}} \equiv \frac{c_i}{t^{i/2}}$$

for every $i \in \mathbb{N}$, see Lemma 5.4. The two steps to improve these are:

Step 1: Interpolation inequalities of the form $\|Du\|_{L^\infty}^2 \leq c \cdot \|u\|_{L^\infty} \cdot \|D^2 u\|_{L^\infty}$ and an iteration argument as in [11, Lemma C.2] can be applied to a metric g satisfying (8.1) and

$$|g - h|_{C^0} \leq \frac{a}{t^\alpha}$$

for some constant a. We obtain for any $0 < \beta < \alpha$ the estimate

$$\left|{}^h\nabla^i g\right| \leq \frac{c((c_k), a, \alpha, \beta)}{t^{\frac{i}{2}+\beta}}.$$

Step 2: Recall that for solutions of the h-flow satisfying

$$\left|{}^h\nabla g\right| \le \frac{c}{t^\gamma}.$$

Arguing as in Lemma 7.2 yields

$$|g - h|_{C^0} \le \frac{c}{t^{\gamma \frac{n}{2p+n}}}.$$

Iteration: We start with

$$\left|{}^h\nabla g\right| \le \frac{c}{t^{\gamma_0}}.$$

According to Corollary 5.4 (with $i = 1$), we may take $\gamma_0 = \frac{1}{2}$. So step 2 yields

$$|g - h|_{C^0} \le \frac{C_1}{t^{\alpha_1}}$$

with $\alpha_1 = \frac{n}{2(2p+n)}$. By combining steps 1 and 2, we get

$$|g - h|_{C^0} \le \frac{C_2(\delta)}{t^{\alpha_2}}$$

with $\alpha_2 = \left(\frac{1}{2} + (1 - \delta)\alpha_1\right)\frac{n}{2p+n}$ for any fixed $0 < \delta < 1$. We also get

$$|g - h|_{C^0} \le \frac{C_k(\delta)}{t^{\alpha_k}}$$

with $\alpha_{k+1} = \left(\frac{1}{2} + (1 - \delta)\alpha_k\right)\frac{n}{2p+n}$ for all $k \in \mathbb{N}$.

It is easy to check that

$$\frac{n}{2(2p+n)} = \alpha_1 < \alpha_2 < ... < \frac{n}{2(2p+\delta n)}.$$

As α_k is increasing in k, we see that $\lim \alpha_k$ exists. Passing to the limit in the defining equation for α_k yields

$$\lim_{k\to\infty} \alpha_k = \frac{n}{2(2p+\delta n)}.$$

Now $\delta > 0$ can be chosen arbitrarily small. So we obtain for any $0 < \zeta \ll 1$

$$|g - h|_{C^0} \le \frac{c(\zeta)}{t^{\frac{n}{4p}-\zeta}}.$$

We can finally apply step 1 once again in order to obtain a derivative bound.

Thus we have proved the following

Lemma 8.1. *Let g be as in Lemma 7.2. Then we obtain for every $0 < \zeta \ll 1$ the estimates*

$$|g - h|_{C^0} \le \frac{c(\zeta)}{t^{\frac{n}{4p}-\zeta}}$$

and for every $i \in \mathbb{N}$

(8.2) $$\left|{}^h\nabla^i g\right| \le \frac{c(\zeta)}{t^{\frac{i}{2}+\frac{n}{4p}-\zeta}}.$$

9. Construction of a Solution to Ricci Flow

In this section we construct a solution to Ricci flow for smooth initial metrics g_0 on $\mathbb{R}^n$ which are ε-close to the standard metric δ. It is well known that on a compact manifold, we can recover the Ricci flow from a solution to the h-flow using time dependent diffeomorphisms [7, 14]. We use the a priori estimates of the previous section and construct diffeomorphisms in our set up which allow us to construct a solution to the Ricci flow (similar to the compact case).

Lemma 9.1. *Let $g \in \mathcal{M}_{loc}^\infty(\mathbb{R}^n, [0, \infty))$ be a solution to (1.1) which is $\tilde{\varepsilon}(n)$-close to h (for all t with $\tilde{\varepsilon}$ as in Theorem 1.2). Then there exists a smooth family of diffeomorphisms $(\varphi(\cdot, t))_{t \in [0,\infty)}$, $\varphi_t \equiv \varphi(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n$, with $\varphi(x, 0) = x$, such that $(\varphi_t)^*(g(t))$ solves* (RF).

Proof. Let ${}^i\varphi : B_i \times [0, \infty) \to B_i \subset \mathbb{R}^n$ be the solutions to the ordinary differential equations

$$
\begin{cases}
\frac{\partial}{\partial t}({}^i\varphi)^\alpha(x, t) = \eta_i(\varphi(x, t))V^\alpha(\varphi(x, t), t), & (x, t) \in B_i \times [0, \infty), \\
{}^i\varphi(x, 0) = x, & x \in B_i,
\end{cases}
$$

where $V^\alpha(y, t) := g^{\beta\gamma}\left({}^g\Gamma_{\beta\gamma}^\alpha - {}^h\Gamma_{\beta\gamma}^\alpha\right)(y, t)$, and $\eta_i : \mathbb{R}^n \to \mathbb{R}$ is a smooth cut-off function $0 \leq \eta \leq 1$ with $\eta|_{B_{\frac{i}{2}}} \equiv 1$ and $\eta|_{\mathbb{R}^n - B_i} \equiv 0$. Notice that ${}^iV := \eta_i V$ is smooth and compactly supported in B_i. Hence the $\varphi_i(\cdot, t) : B_i \to B_i$ are diffeomorphisms.

Using again Corollary 5.4 we see that for $x \in B_{i/2}$ and small times

$$
\left|{}^h\nabla g(x, t)\right| \leq \frac{c(n)}{\sqrt{t}}.
$$

This implies that

$$
\left|\frac{\partial}{\partial t}({}^i\varphi)^\alpha(x, t)\right| = \left|{}^iV^\alpha({}^i\varphi(x, t), t)\right| \leq \frac{c(n)}{\sqrt{t}}
$$

(here we have used that $g(t)$ is ε-close to h). In particular, this gives us

$$
\left|({}^i\varphi)^\alpha(x, t) - x\right| \leq \int_0^t \left|\frac{\partial}{\partial \tau}({}^i\varphi)^\alpha(x, \tau)\right| d\tau \leq c(n)\sqrt{t}.
$$

This implies that ${}^i\varphi(\cdot, t)|_{B_j}$ is independent of i for i big enough, j fixed and t bounded above. In particular we obtain, taking a (diagonal sub-sequence) limit, a well defined solution $\varphi : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ to the equation

$$
(9.1) \qquad
\begin{cases}
\frac{\partial}{\partial t}(\varphi)^\alpha(x, t) = V^\alpha(\varphi(x, t), t), & (x, t) \in \mathbb{R}^n \times [0, \infty), \\
\varphi(x, 0) = x, & x \in \mathbb{R}^n,
\end{cases}
$$

and thus $\varphi_t = \varphi(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism for all t. According to [12] $(\varphi_t)^*(g(t))$ solves (RF). $\qquad \square$

In the situation that we have the improved decay estimates of Lemma 8.1 we show that the diffeomorphisms φ_t converge to a diffeomorphism $\varphi_\infty : \mathbb{R}^n \to \mathbb{R}^n$ as $t \to \infty$ for $n \geq 3$ as long as $p \geq 1$ is small enough.

Theorem 9.2. *Let $g \in \mathcal{M}_{loc}^{\infty}(\mathbb{R}^n, [0, \infty))$ be a solution to (1.1) and m, p, and $\tilde{\varepsilon}(m, n)$ all be as in Theorem 6.1. Assume that (1.5) holds. For $n \geq 3$, φ_t converges in C_{loc}^{∞} to a diffeomorphism $\varphi_{\infty} : \mathbb{R}^n \to \mathbb{R}^n$ as $t \to \infty$.*

Proof. Step 1: Fix p with $\frac{n}{2} > p \geq 1$ and $\zeta > 0$ such that $\alpha := \frac{1}{2} + \frac{n}{4p} - \zeta > 1$. In the following, our constants c may depend on n, p, ζ, m, and $I_0^{m,p}$ as defined in Theorem 6.1.

We wish to show that

$$|\varphi_t(x) - x| \leq c \quad \forall \, t \in [0, \infty).$$

Corollary 5.4 implies that

$$\left| \frac{d}{dt} \varphi_t(x) \right| \leq \frac{c}{t^{1/2}}.$$

Hence $|\varphi_t(x) - x| \leq c$ for all $0 \leq t \leq 1$. So we may assume that $t > 1$. The evolution equation (9.1) for φ_t and the fact that $|\nabla g| \leq \frac{c}{t^{\alpha}}$ imply that

$$\left| \frac{d}{dt} \varphi_t(x) \right| \leq \frac{c}{t^{\alpha}}$$

and so we see that (upon integrating from 1 to $t > 1$) $|\varphi_1(x) - \varphi_t(x)| \leq c$. By the triangle inequality we get

$$|\varphi_t(x) - x| \leq |\varphi_t(x) - \varphi_1(x)| + |\varphi_1(x) - x| \leq c.$$

For later use, note that this implies for $F_t : \mathbb{R}^n \to \mathbb{R}^n$ defined by $F_t := \varphi_t^{-1}$

$$|F_t(x) - x| \leq c.$$

Step 2: Using the estimate $\left| \frac{d}{dt} \varphi_t(x) \right| \leq \frac{c}{t^{\alpha}}$ from Step 1, we see that for fixed $x \in \mathbb{R}^n$, $\varphi_t(x) \to y$ as $t \to \infty$ for some $y \in \mathbb{R}^n$. We define $\varphi_{\infty}(x) = \lim_{t \to \infty} \varphi_t(x)$. All the derivatives of φ_t (for $t > 1$) satisfy

$$\frac{\partial}{\partial t} \log \left(1 + \left| {}^h \nabla^m \varphi_t(x) \right|^2 \right) \leq \frac{c \left(\varphi_1|_{B_r(y)}, n, m, p, \zeta, (c_k), I_0^{m,p} \right)}{t^{\alpha}} \quad \forall \, x \in B_r(y)$$

for some $\alpha > 1$ and

$$\sup_{x \in B_r(y)} \left| {}^h \nabla^m \varphi_t(x) \right|^2 \leq c \left(\varphi_1|_{B_r(y)}, n, m, p, \zeta, (c_k), I_0^{m,p} \right).$$

This may be seen as follows (for the rest of this argument, we set $\nabla := {}^h \nabla$). Differentiating gives for V as in (1.1)

$$(9.2) \qquad |\nabla^m V(x, t)| \leq c_m \sum_{k=1}^{m+1} \sum_{\substack{i_1 + \cdots + i_k = m+1 \\ 1 \leq i_1, \ldots, i_k \leq m+1}} \left| \nabla^{i_1} g \right| \cdots \left| \nabla^{i_k} g \right|,$$

$$(9.3)$$

$$|\nabla^m (V(\varphi(x, t), t))| \leq c_m \sum_{i=1}^{m} \sum_{\substack{j_1 + \cdots + j_i = m \\ 1 \leq j_1, \ldots, j_i \leq m}} \left| \left(\nabla^i V \right)(\varphi(x, t), t) \right| \cdot \left| \nabla^{j_1} \varphi \right| \cdots \left| \nabla^{j_i} \varphi \right|.$$

Then

$$
\frac{\partial}{\partial t} \log\left(1 + |\nabla^m \varphi_t(x)|^2\right) = \frac{\frac{\partial}{\partial t} |\nabla^m \varphi_t(x)|^2}{\left(1 + |\nabla^m \varphi_t(x)|^2\right)}
$$

$$
\leq c \, \frac{|\nabla^m \varphi_t(x)| \, |\nabla^m \frac{\partial}{\partial t} \varphi_t(x)|}{\left(1 + |\nabla^m \varphi_t(x)|^2\right)}
$$

$$
= c \, \frac{|\nabla^m \varphi_t(x)| \, |\nabla^m (V(\varphi_t, t))(x)|}{\left(1 + |\nabla^m \varphi_t(x)|^2\right)}.
$$

Substituting (9.2) and (9.3) into the above, using the estimates for $|\nabla^k g|$ and arguing inductively (for m) leads to the estimate.

Hence $\varphi_\infty : \mathbb{R}^n \to \mathbb{R}^n$ is C^∞ and $\varphi_t \to \varphi_\infty$ converges in C^∞_{loc} as $t \to \infty$.

Step 3: We wish to show that φ_∞ is a diffeomorphism. Let F_t be the inverse of φ_t. Letting $l(t) := \varphi_t^* g(t)$, we know that l solves Ricci flow on $\mathbb{R}^n$. But then we get in view of (8.2)

$$
\left| \,^l \frac{\partial}{\partial t} l \right| (x, t) = 2 \cdot \,^l |\operatorname{Ric}(l)|(x, t) = 2 \cdot \,^g |\operatorname{Ric}(g)|(\varphi_t(x), t) \leq \frac{c}{t^{\frac{3}{2}}} \quad \text{for } t \geq 1,
$$

which is integrable (from 1 to ∞). This implies that $l(t)$ converges locally uniformly to a well defined continuous Riemannian metric l_∞ as $t \to \infty$ in view of [5, Lemma 14.2]. In particular, there must exist a c with $\frac{1}{c} h \leq l(x, t) \leq c\, h$ for all $t \in [0, \infty)$.

Now using the definition of l we get

$$
\frac{1}{c} \delta_{\alpha\beta} \leq l_{\alpha\beta}(y, t) = \frac{\partial \varphi^s}{\partial y^\alpha}(y, t) \frac{\partial \varphi^k}{\partial y^\beta}(y, t) \, g_{sk}(\varphi_t(y), t)
$$

$$
\leq (1 + \tilde{\varepsilon}) \frac{\partial \varphi^s}{\partial y^\alpha} \frac{\partial \varphi^k}{\partial y^\beta}(y, t) \delta_{sk}
$$

$$
= (1 + \tilde{\varepsilon}) \, (D\varphi) \, (D\varphi)^T (y, t).
$$

In particular, we see that $\det (D\varphi_t)^2 (y) \geq \frac{1}{(1+\tilde{\varepsilon})c} > 0$, where Df is the Jacobian of f. Taking the limit as $t \to \infty$ we get $\det (D\varphi_\infty)^2 (y) \geq \frac{1}{(1+\tilde{\varepsilon})c} > 0$. As $\det (D\varphi_0) = 1$, we see that $\det (D\varphi_\infty)(y) \geq \frac{1}{\sqrt{(1+\tilde{\varepsilon})c}} > 0$ for all $y \in \mathbb{R}^n$. Hence φ_∞ is an immersion. In view of the derivative estimates for $|\nabla^m \varphi_t|$, the function $\varphi_\infty : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth diffeomorphism if it is bijective. Recall that $F_t \equiv F(\cdot, t) = (\varphi_t)^{-1}$ for $t < \infty$. The estimates for φ_t above ensure that there exists a function $F_\infty : \mathbb{R}^n \to \mathbb{R}^n$ such that $F_t \to F_\infty$ in C^∞_{loc} as $t \to \infty$.

As φ_t and F_t are smooth diffeomorphisms with estimated derivatives, $\varphi_t \to \varphi_\infty$ and $F_t \to F_\infty$, we obtain

$$
(F_\infty \circ \varphi_\infty)(x) = \lim_{t \to \infty} (F_t \circ \varphi_t)(x) = x \quad \text{for all } x \in \mathbb{R}^n.
$$

Thus φ_∞ is injective. Similarly, we see that φ_∞ is surjective. Hence φ_∞ is a diffeomorphism. $\qquad\square$

Proof of Theorem 1.4. In Lemma 9.1 we have constructed diffeomorphisms φ_t so that $(\varphi_t^* g(t))_{t \in [0, \infty)}$ solves (RF) if $(g(t))_{t \in [0, \infty)}$ solves (1.1). Define $\tilde{g}(t) := \varphi_t^* g(t)$.

According to Theorem 1.3, we get $g(t) \to h$. Theorem 9.2 implies that the diffeomorphisms φ_t converge to a diffeomorphism φ_∞ for $t \to \infty$. Thus $\tilde{g}(t) \to \varphi_\infty^* h$ in C^∞ as $t \to \infty$. $\qquad\square$

We can also show that the diffeomorphisms φ_t are close to the identity near infinity, uniformly in t.

Lemma 9.3. *Let $g \in \mathcal{M}_{loc}^\infty(\mathbb{R}^n, [0, \infty))$ be a solution to (1.1) as in Theorem 9.2 and $(\varphi_t)_{t \in [0, \infty]}$ be the diffeomorphisms of $\mathbb{R}^n$ constructed before. Then for every $\eta > 0$ there exists an $R > 0$ such that*

$$\sup_{\mathbb{R}^n \setminus B_R(0)} |\varphi_t(x) - x| \leq \eta$$

for all $t \in [0, \infty]$.

Proof. We first want to show that for any given $\eta' > 0$ there exists an $R_1 > 0$ such that $g(t)$ is η'-close to $h = \delta$ on $\mathbb{R}^n \setminus B_{R_1}(0)$ for all t.

By Lemma 7.2 we can choose a $T > 0$ such that $g(t)$ is η'-close to h on all of $\mathbb{R}^n$ for $t \geq T$. Now choose $R_0 > 0$ such that $g(0)$ is $\eta'/2$-close to h on $\mathbb{R}^n \setminus B_{R_0}(0)$. Applying the interior closeness estimates, Lemma 5.2, we see that we can find $R_1 > R_0$ such that $g(t)$ is η'-close on $(\mathbb{R}^n \setminus B_{R_1}(0)) \times [0, T]$. Note that this implies

$$\sup_{\mathbb{R}^n \setminus B_{R_1}(0)} |g(t) - \delta| \leq \eta'$$

for all t. By Lemma 8.1 we have that

$$\left|{}^h\nabla g\right| \leq \frac{c}{t^{\frac{1}{2} + \frac{n}{4p} - \zeta}}$$

for $t > 0$. Now fix $t_1 > 0$ such that

$$\int_{t_1}^\infty \frac{c}{t^{\frac{1}{2} + \frac{n}{4p} - \zeta}}\, dt \leq \frac{\eta}{2}\ .$$

By interpolation we deduce from Lemma 5.4 that

$$\left|{}^h\nabla g\right| \leq \frac{\delta(\eta')}{\sqrt{t}} \quad \text{on } \mathbb{R}^n \setminus B_{R_1}(0)$$

for $t > 0$, and in particular for $0 < t \leq t_1$, where $\delta(\eta) \to 0$ as $\eta \to 0$. Arguing as in Step 1 of the proof of Theorem 9.2 we obtain the claimed estimate for some $R > R_1$. $\qquad\square$

10. Convergence Based on Integral Bounds

Proof of Theorem 1.6. In view of Corollary 5.4 and interpolation inequalities of the form $\|\nabla v\|_{C^0} \leq c \cdot \|v\|_{C^0} \cdot \|v\|_{C^2}$, it suffices to prove that $g(t) \to h$ in C^0 as $t \to \infty$.

Assume the conclusions were false. Then we could find $\delta > 0$ and points

$$(x_k, t_k, l_k) \in \mathbb{R}^n \times (0, \infty) \times \{1, \ldots, n\},$$

$k \in \mathbb{N}$, such that $t_k \to \infty$ as $k \to \infty$ and

$$|\lambda_{l_k}(x_k, t_k) - 1| \geq 4\delta.$$

By the construction of the solution $g(t)$ in Section 4 from solutions ${}^i g$, we find sequences $(i_j^k)_{j \in \mathbb{N}}$ with $i_j^k \to \infty$ as $j \to \infty$, such that for the eigenvalues $({}^i \lambda_l)_{1 \leq l \leq n}$ of ${}^i g$, we have

$$\left| {}^{i_j^k} \lambda_{l_k}(x_k, \, t_k) - 1 \right| \geq 3\delta.$$

Given any $R > 0$, Corollary 5.4 implies for k sufficiently large so that t_k is sufficiently large and for $j = j(k)$ sufficiently large so that $\nu_k - |x_k| \equiv i_j^k - |x_k|$ is sufficiently large that

(10.1) $$\inf_{x \in B_R(x_k)} \sup_{1 \leq l \leq n} \left| {}^{\nu_k} \lambda_l(x, \, t_k) - 1 \right| \geq 2\delta.$$

The condition that $\nu_k - |x_k|$ is sufficiently large ensures that the solutions ${}^{\nu_k} g$ are defined on all of $B_R(x_k)$. Define

$$ {}^i I_\delta^{m,p}(t) := \frac{1}{p} \int\limits_{B_i(0)} \left({}^i \varphi_m + {}^i \psi_m - 2n - \delta \right)_+^p \, dx, $$

where

$$ {}^i \varphi_m(x, t) := \sum_{l=1}^n \frac{1}{{}^i \lambda_l^m(x, t)} \quad \text{and} \quad {}^i \psi_m(x, t) := \sum_{l=1}^n {}^i \lambda_l^m(x, t). $$

By construction of the initial metrics ${}^i g_0$, we have for the eigenvalues $(\lambda_l(x))_{1 \leq l \leq n}$ of the metric g_0 with respect to h which we label such that $\lambda_1(x) \leq \ldots \leq \lambda_n(x)$ and similarly for the eigenvalues ${}^i \lambda_1(x) \leq \ldots \leq {}^i \lambda_n(x)$ of ${}^i g_0$ for $|x| \leq i$ the inequalities

$$ \begin{cases} \lambda_l(x) \leq {}^i \lambda_l(x) \leq 1 & \text{if } \lambda_l(x) \leq 1, \\ 1 \leq {}^i \lambda_l(x) \leq \lambda_l(x) & \text{if } 1 \leq \lambda_l(x), \end{cases} $$

which implies in particular the estimate $\left({}^i \lambda_l^m - 1 \right)^2 \leq \left(\lambda_l^m - 1 \right)^2$ for all $i, \, l, \, m \in \mathbb{N}$. We deduce that

$$ \begin{aligned} {}^i \varphi_m + {}^i \psi_m - 2n &= \sum_{l=1}^n \frac{1}{{}^i \lambda_l^m} \left({}^i \lambda_l^m - 1 \right)^2 \\ &\leq \sum_{l=1}^n \frac{\lambda_l^m}{\lambda_l^m} (1 + \varepsilon(n))^m \left(\lambda_l^m - 1 \right)^2 \\ &\leq (1 + \varepsilon(n))^{2m} \sum_{l=1}^n \frac{1}{\lambda_l^m} \left(\lambda_l^m - 1 \right)^2 \\ &= (1 + \varepsilon(n))^{2m} (\varphi_m + \psi_m - 2n). \end{aligned} $$

Thus

$$
\begin{aligned}
{}^{i}I_{\delta}^{m,p}(0) &= \frac{1}{p} \int\limits_{B_i(0)} \left({}^{i}\varphi_m + {}^{i}\psi_m - 2n - \delta\right)_+^p \, dx \Big|_{t=0} \\
&\leq (1 + \varepsilon(n))^{2m} \frac{1}{p} \int\limits_{B_i(0)} \left(\varphi_m + \psi_m - 2n - \delta\right)_+^p \, dx \Big|_{t=0} \\
&\leq (1 + \varepsilon(n))^{2m} \frac{1}{p} \int\limits_{\mathbb{R}^n} \left(\varphi_m + \psi_m - 2n - \delta\right)_+^p \, dx \Big|_{t=0} \\
&= (1 + \varepsilon(n))^{2m} I_{\delta}^{m,p}(0) < \infty
\end{aligned}
$$

(10.2)

is uniformly bounded in i.

The proof of Theorem 6.1 extends directly to our situation and ensures that

$$
{}^{i}I_{\delta}^{m,p}(t) \leq {}^{i}I_{\delta}^{m,p}(0) \quad \text{for } t \geq 0.
$$

Here, $\left({}^{i}\varphi_m + {}^{i}\psi_m - 2n - \delta\right)_+$ is compactly supported in $B_i(0)$ due to the boundary values imposed on ${}^{i}g$ in Theorem 4.3. Therefore, we do not have to use the local closeness estimates of Lemma 5.2 in order to justify the integration by parts employed in the proof of Theorem 6.1.

Using (10.1) and (10.2), we estimate for ε and m such that $(1 + \varepsilon(n))^m \leq 2$ as follows

$$
\begin{aligned}
(1 + \varepsilon(n))^{2m} I_{\delta}^{m,p}(0) &\geq {}^{\nu_k}I_{\delta}^{m,p}(0) \\
&\geq {}^{\nu_k}I_{\delta}^{m,p}(t_k) \\
&= \frac{1}{p} \int\limits_{B_{\nu_k}(0)} \left({}^{\nu_k}\varphi(x, t_k) + {}^{\nu_k}\psi(x, t_k) - 2n - \delta\right)_+^p \, dx \\
&\geq \frac{1}{p} \int\limits_{x \in B_R(x_k)} \left(\sum_{l=1}^{n} \frac{1}{{}^{\nu_k}\lambda_l^m(x, t_k)} \left({}^{\nu_k}\lambda_l^m(x, t_k) - 1\right)^2 - \delta\right)_+^p \, dx \\
&\geq \frac{1}{p} \int\limits_{B_R(x_k)} \left(\frac{1}{(1 + \varepsilon(n))^m} 4\delta - \delta\right)^p \\
&\geq \frac{1}{p} \int\limits_{B_R(x_k)} \delta^p \\
&= \frac{\delta^p}{p} c(n) R^n.
\end{aligned}
$$

As $R \geq 0$ can be chosen arbitrarily large, we arrive at a contradiction. This finishes the proof of Theorem 1.6. □

APPENDIX A. CONFORMAL RICCI FLOW IN TWO DIMENSIONS

Studying Ricci flow on $\mathbb{R}^2$ for conformally flat metrics $g(x,t) = e^{u(x,t)}\delta$ is equivalent [6, 16] to considering the evolution equation

$$(A.1) \qquad \frac{\partial}{\partial t}u = \Delta_g u = e^{-u}\Delta_\delta u \quad \text{on } \mathbb{R}^2 \times (0,\infty).$$

We are especially interested in initial conditions $u_0 : \mathbb{R}^2 \to \mathbb{R}$ fulfilling

$$(A.2) \qquad \sup_{\mathbb{R}^2 \setminus B_r(0)} |u_0| \to 0 \quad \text{as } r \to \infty.$$

We wish to point out that we do not assume in the following that u_0 fulfills an ε_0-closeness condition for a small constant $\varepsilon_0 > 0$, corresponding to $\sup|u_0|$ being small. It suffices to have $\sup|u_0|$ bounded. Note also that in two dimensions, we do not have to study Ricci harmonic map heat flow first before we can obtain results for Ricci flow.

In this situation, we get the following existence result.

Theorem A.1. *Let $u_0 \in C^0\left(\mathbb{R}^2\right)$ satisfy $\|u_0\|_{L^\infty} < \infty$. Then there exists $u \in C^\infty(\mathbb{R}^2 \times (0,\infty))$ solving (A.1) such that $u(\cdot,t) \to u_0$ in $C^0_{loc}\left(\mathbb{R}^2\right)$ as $t \searrow 0$. As $t \to \infty$, $u(\cdot,t)$ converges subsequentially in C^∞_{loc} to a complete flat metric.*

Idea of Proof. Mollifying u_0, we get a sequence $u_0^i \in C^\infty\left(\mathbb{R}^2\right)$, $i \in \mathbb{N}$, such that $u_0^i \to u_0$ in C^0_{loc} as $i \to \infty$, $\left\|u_0^i\right\|_{L^\infty} \leq \|u_0\|_{L^\infty}$, and $\left\|D^2 u_0^i\right\|_{L^\infty} + \left\|D u_0^i\right\|_{L^\infty} \leq c_i$. According to [16], there exist smooth solutions u^i for $t \geq 0$ solving (A.1) with $u^i(\cdot,0) = u_0^i$ such that $\left|u^i(x,t)\right| \leq \sup|u_0|$. The techniques of Krylov-Safonov, Schauder, and scaling imply $\left|D^k u^i\right| \leq \frac{c_k}{t^{k/2}}$. Thus we find a smooth solution u to (A.1). Considering $\psi = u^i(x,t) - u^i(x_0)$, we can argue as in Lemma 5.2 (using ψ^2 instead of $\varphi_m + \psi_m - 2n$) to see that for any $\delta > 0$ we find $i_0 = i_0(\delta, u_0)$ and $\zeta > 0$ such that $\left|u^i(x,t) - u^i(x_0)\right| \leq \delta$ for $|x - x_0|^2 + t \leq \zeta$ and $i \geq i_0$. Thus $u(\cdot,t) \to u_0$ in C^0_{loc} as $t \searrow 0$. Note that in the case of (A.2), this convergence is uniform in space. The remaining details are similar to the higher dimensional case. $\square$

Theorem A.2. *Let u_0 be as in Theorem A.1 and assume that (A.2) is fulfilled. Let u be the solution to (A.1) obtained there. Then $u(\cdot,t) \to 0$ in C^∞ as $t \to \infty$.*

Idea of Proof. The proof is similar to the proof of Theorem 1.3. Instead of Theorem 6.1, however, we use the following estimate for $p \geq \sup|u_0| + 1$ (and a similar argument for the negative part of u)

$$\frac{d}{dt}\int_{\mathbb{R}^2} \tfrac{1}{p}(u-\delta)_+^p \, dx = \int_{\{u>\delta\}} (u-\delta)^{p-1}e^{-u}\Delta_\delta u \, dx$$

$$= \int_{\{u>\delta\}} (u-\delta)^{p-2}((u-\delta) - p + 1)e^{-u}|\nabla u|^2 \, dx \leq 0.$$

$\square$

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OLIVER SCHNÜRER: FREIE UNIVERSITÄT BERLIN, ARNIMALLEE 6, 14195 BERLIN, GERMANY

E-mail address: `Oliver.Schnuerer@math.fu-berlin.de`

FELIX SCHULZE: FREIE UNIVERSITÄT BERLIN, ARNIMALLEE 6, 14195 BERLIN, GERMANY

E-mail address: `Felix.Schulze@math.fu-berlin.de`

MILES SIMON: UNIVERSITÄT FREIBURG, ECKERSTRASSE 1, 79104 FREIBURG I. BR., GERMANY

E-mail address: `msimon@gmx.de`