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ABOUT DERIVATIONS AND VECTOR-VALUED DIFFERENTIAL FORMS

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Introduction

Let M be a complex analytic manifold. With any holomorphic vector bundle $\mathbf{E}$ over M one can associate the vector bundle $\mathbf{A} = \bigwedge \mathbf{E}$ which is a bundle of Grassmann algebras. The corresponding sheaf of holomorphic sections $\mathcal{A}$ is a locally free analytic sheaf of commutative graded algebras. Let $\mathcal{T} = \mathcal{D}er\mathcal{A}$ be the sheaf of $\mathbb{C}$-derivations of $\mathcal{A}$. Then $\mathcal{T}$ is a sheaf of graded Lie algebras which can be considered as the tangent sheaf of the splittable supermanifold $(M, \mathcal{A})$. The cohomology algebra $H^*(M, \mathcal{T})$ with the bracket inherited from $\mathcal{T}$ is of great interest for the theory of complex analytic supermanifolds (see, e.g., [8]). To compute this algebra, it would be useful to have a fine resolution of $\mathcal{T}$ which is a sheaf of differential graded Lie algebras. Our goal is to construct such a resolution.

The classic case is the case where $\mathbf{E}$ is the cotangent bundle $\mathbf{T}(M)^*$ of M, and hence where $\mathcal{A} = \Omega$ is the sheaf of holomorphic differential forms on M. The derivations of the sheaf of differential forms were first determined by Frölicher and Nijenhuis in [2], where an explicit description of these derivations in terms of vector-valued differential forms was given. The resolution of $\mathcal{T} = \mathcal{D}er\,\Omega$, which is constructed here, can also be expressed in terms of vector-valued forms. We use this expression in the case where M is a compact Hermitian symmetric space. In particular, we compute the algebra $H^*(\mathbb{CP}^n, \mathcal{T})$.

1. Preliminaries

Let A be a graded algebra over $\mathbb{C}$. We write $|a|$ for the degree q of a homogeneous element $a \in A_q$. As usual, we call a *derivation of degree* p of A any $\mathbb{C}$-linear mapping $u : A \to A$ of degree $p = |u|$ satisfying the relation

$$u(ab) = u(a)b + (-1)^{|u||a|} a u(b).$$

The derivations of A form the graded Lie algebra $\operatorname{Der} A = \bigoplus_{p\in\mathbb{Z}} (\operatorname{Der} A)_p$, where $(\operatorname{Der} A)_p$ is the set of all derivations of degree p of A, and the bracket is given by

$$[u, v] = uv + (-1)^{|u||v|+1} vu.$$

If the graded algebra A is (associative and) commutative, then $\operatorname{Der} A$ is an A-module due to the rule

$$(au)(b) = au(b),\ u \in \operatorname{Der} A,\ a, b \in A.$$

If A is a bigraded algebra, then we denote by $\operatorname{Der} A$ its graded Lie algebra of derivations, assuming that A is endowed with the total degree. It can be easily proved that A is actually a bigraded algebra with respect to the natural bigrading.

The same definitions can be applied to the sheaves of graded algebras on a topological space M. In particular, if $\mathcal{A}$ is a sheaf of commutative graded algebras, then the sheaf $\mathcal{D}er\mathcal{A}$ of derivations of $\mathcal{A}$ is defined which is a sheaf of graded Lie algebras and a sheaf of $\mathcal{A}$-modules on M.

Let us consider the case where $A = \bigwedge E$; here E is a complex vector space of dimension m. This is a commutative graded algebra with the standard grading $A = \bigoplus_{p=0}^{m} A_p$, where $A_p = \bigwedge^p E$. Denote $W(E) =$

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Der A. These graded Lie algebras are well known. Being endowed with the natural $\mathbb{Z}_2$-grading, they form one of the "Cartan type" series of simple finite-dimensional complex Lie superalgebras (see [6]).

We need the well-known description of derivations from $W(E)$ in terms of multilinear forms. Any $u \in W(E)_p$ is determined by its restriction to $E = A_1$, which is an arbitrary linear mapping $E \to A_{p+1} = \bigwedge^{p+1} E$. Thus, $W(E)_p$ is isomorphic, as a vector space, to $\bigwedge^{p+1} E \otimes E^*$. Elements of the latter vector space can be considered as vector-valued $(p+1)$-forms on E^*, i.e., as skew-symmetric $(p+1)$-linear mappings $(E^*)^{p+1} \to E^*$. Let us denote by $i(\varphi) \in W(E)_p$ the derivation which corresponds to a vector-valued form $\varphi \in \bigwedge^{p+1} E \otimes E^*$. Considering A as the set of all skew-symmetric multilinear forms on E^*, we have

$$\begin{aligned} &i(\varphi)(a)(x_1, \dots, x_{p+q}) \\ &= \frac{1}{(p+1)!(q-1)!} \sum_{\alpha \in S_{p+q}} (\operatorname{sgn} \alpha) a(\varphi(x_{\alpha_1}, \dots, x_{\alpha_{p+1}}), x_{\alpha_{p+2}}, \dots, x_{\alpha_{p+q}}) \end{aligned} \tag{1}$$

for $x_k \in E^*$. In fact, one can easily verify that the right-hand side of (1) determines a derivation u from $(\operatorname{Der} A)_p$. Choose a base $\xi_1, \dots, \xi_m$ of E and denote by $\xi_1^*, \dots, \xi_m^*$ the dual base of E^*. Then $\varphi = \sum_{j=1}^m \varphi_j \otimes \xi_j^*$, where $\varphi_j \in \bigwedge^{p+1} E$. Clearly, $i(\varphi)(\xi_j) = \varphi_j$. On the other hand, $u(\xi_j)(x_1, \dots, x_{p+1}) = \varphi_j(x_1, \dots, x_{p+1})$, and hence $i(\varphi) = u$.

Clearly, the derivations $\frac{\partial}{\partial \xi_j} = i(\xi_j^*) \in W(E)_{-1}$, $j = 1, \dots, m$, form a base of the A-module $W(E)$. It follows that the derivations

$$\xi_{i_1} \dots \xi_{i_{p+1}} \frac{\partial}{\partial \xi_j}, \ i_1 < \dots < i_{p+1}, \ j = 1, \dots, m,$$

form a base of $W(E)_p$ over $\mathbb{C}$. In particular, we see that $W(E)_p \neq 0$ only for $-1 \leq p \leq m$.

One can also write

$$i(\varphi)(a) = a \,\bar{\wedge}\, \varphi, \ a \in A, \ \varphi \in A \otimes E^*.$$

A similar operation can be defined for two vector-valued forms of arbitrary degrees. For example, let $\varphi \in A_p \otimes E^*$, $\psi \in A_q \otimes E^*$ be given. Considering these tensors as E^*-valued p- and q-forms on E^*, we define the form $\varphi \,\bar{\wedge}\, \psi \in A_{p+q-1} \otimes E^*$ by

$$\begin{aligned} &(\varphi \,\bar{\wedge}\, \psi)(x_1, \dots, x_{p+q-1}) \\ &= \frac{1}{(p-1)!q!} \sum_{\alpha \in S_{p+q-1}} (\operatorname{sgn} \alpha) \varphi(\psi(x_{\alpha_1}, \dots, x_{\alpha_q}), x_{\alpha_{q+1}}, \dots, x_{\alpha_{p+q-1}}) \end{aligned} \tag{2}$$

for $x_k \in E^*$. This operation can be used for expressing the bracket in $W(E)$. More precisely, define the bilinear operation $\{\ ,\ \}$ on $A \otimes E^*$ by

$$\{\varphi, \psi\} = \psi \,\bar{\wedge}\, \varphi - (-1)^{(p-1)(q-1)} \varphi \,\bar{\wedge}\, \psi$$

for $\varphi \in A_p \otimes E^*$, $\psi \in A_q \otimes E^*$. Then

$$i(\{\varphi, \psi\}) = [i(\varphi), i(\psi)].$$

In fact, using the above notation, we obtain

$$\begin{aligned} [i(\varphi), i(\psi)](\xi_j) &= i(\varphi) i(\psi)(\xi_j) - (-1)^{(p-1)(q-1)} i(\psi) i(\varphi) \xi_j \\ &= \psi_j \,\bar{\wedge}\, \varphi - (-1)^{(p-1)(q-1)} \varphi_j \,\bar{\wedge}\, \psi \\ &= (\psi \,\bar{\wedge}\, \varphi)_j - (-1)^{(p-1)(q-1)} (\varphi \,\bar{\wedge}\, \psi)_j \\ &= \{\varphi, \psi\}_j = \{\varphi, \psi\}(\xi_j). \end{aligned}$$

Let M now be a complex manifold of dimension n, $\mathcal{F}$ its structure sheaf, and let $\mathbf{E}$ be a holomorphic vector bundle of rank m over M. Then we can construct the holomorphic bundle $\mathbf{A} = \bigwedge \mathbf{E}$ over M which is a bundle of commutative graded algebras. Let $\mathcal{E}$ and $\mathcal{A} = \bigwedge_{\mathcal{F}} \mathcal{E}$ be the corresponding locally free analytic sheaves of holomorphic sections. Then $\mathcal{A} = \bigoplus_{p=0}^{m} \mathcal{A}_p$, where $\mathcal{A}_p = \bigwedge_{\mathcal{F}}^{p} \mathcal{E}$ is a sheaf of commutative graded algebras. We denote $\mathcal{T} = \mathcal{D}er\, \mathcal{A}$. (In what follows, we denote by $\mathcal{D}er\, \mathcal{B}$ the sheaf of $\mathbb{C}$-derivations of a sheaf of $\mathbb{C}$-algebras $\mathcal{B}$.)

We include $\mathcal{T}$ in an exact sequence of locally free analytic sheaves on M (see [7]). In what follows, we omit the subscript $\mathcal{F}$ while denoting the tensor product over the sheaf $\mathcal{F}$. Assigning to any $u \in \mathcal{T}$ its restriction to $\mathcal{F} = \mathcal{A}_0$, we obtain a mapping

$$\alpha : \mathcal{T} \to \mathcal{H}om_{\mathbb{C}}(\mathcal{F}, \mathcal{A}) = \mathcal{A} \otimes \mathcal{E}nd_{\mathbb{C}}\mathcal{F}.$$

It can be easily proved that $\operatorname{Im} \alpha = \mathcal{A} \otimes \mathcal{D}er\, \mathcal{F} = \mathcal{A} \otimes \Theta$, where Θ is the tangent sheaf of M, and that $\alpha(\mathcal{T}_p) = \mathcal{A}_p \otimes \Theta$. In any local coordinate system $x_1, \ldots, x_n$ on M, the mapping α is expressed as follows:

$$\alpha(u) = \sum_{i=1}^{n} u(x_i) \otimes \frac{\partial}{\partial x_i}.$$

Clearly, $\operatorname{Ker} \alpha$ is the subsheaf $\mathcal{D}er_{\mathcal{F}} \mathcal{A}$ of the sheaf of graded Lie algebras $\mathcal{T}$ consisting of all $\mathcal{F}$-derivations. This is the sheaf of holomorphic sections of the holomorphic vector bundle $\mathbf{W}(\mathbf{E})$ with fibers $W(E_x)$, $x \in M$, associated with $\mathbf{E}$. We deduce from the above an injective sheaf homomorphism $i : \mathcal{A}_{p+1} \otimes \mathcal{E}^* \to \mathcal{T}_p$ such that $\operatorname{Im} i = (\mathcal{D}er_{\mathcal{F}} \mathcal{A})_p$. We write

$$i(\varphi)(a) = a \mathbin{\bar{\wedge}} \varphi, \; \varphi \in \mathcal{A} \otimes \mathcal{E}^*, \; a \in \mathcal{A}.$$

As a result, we get the exact sequence

$$0 \to \mathcal{A} \otimes \mathcal{E}^* \xrightarrow{i} \mathcal{T} \xrightarrow{\alpha} \mathcal{A} \otimes \Theta \to 0. \tag{3}$$

Here i is a homomorphism of sheaves of graded Lie algebras if we define the grading and the bracket $\{\ ,\ \}$ on the sheaf $\mathcal{A} \otimes \mathcal{E}^*$ as follows:

$$\begin{aligned} (\mathcal{A} \otimes \mathcal{E}^*)_p &= \mathcal{A}_{p+1} \otimes \mathcal{E}^*, \; p = -1, \ldots, m, \\ \{\varphi, \psi\} &= \psi \mathbin{\bar{\wedge}} \varphi - (-1)^{(|\varphi|-1)(|\psi|-1)} \varphi \mathbin{\bar{\wedge}} \psi, \end{aligned} \tag{4}$$

where the operation $\bar{\wedge}$ is defined by (2) pointwise. In particular, we see that $\mathcal{T}_p \neq 0$ only for $-1 \leq p \leq m$.

The extreme terms of (3) are locally free analytic sheaves on M. Notice that $\mathcal{T}$ has the same property; moreover, it is a locally free sheaf of modules over $\mathcal{A}$ (this is a well-known property of supermanifolds). In fact, consider a coordinate neighborhood U on M with local coordinates $x_1, \ldots, x_n$ such that $\mathbf{E}$ is trivial over U and choose a base $\xi_1, \ldots, \xi_m$ of local sections of $\mathcal{E}$ over U. Then $\mathcal{A}|U$ is identified with $\bigwedge_{\mathcal{F}|U}(\xi_1, \cdots, \xi_m)$. This allows us to define derivations $\frac{\partial}{\partial x_i} \in \mathcal{T}_0|U$, $i = 1, \ldots, n$, and thus to construct a local splitting $(\mathcal{A} \otimes \Theta)|U \to \mathcal{T}|U$ of the exact sequence (3). On the other hand, we have the derivations $\frac{\partial}{\partial \xi_j} \in \mathcal{T}_{-1}|U$, $j = 1, \ldots, m$, defined as for $W(E)$. We see from (3) that $\frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$, and $\frac{\partial}{\partial \xi_j}$, $j = 1, \ldots, m$, form a base of local sections of $\mathcal{T}$ over $\mathcal{A}$. Therefore, the derivations

$$\xi_{i_1} \ldots \xi_{i_{p+1}} \frac{\partial}{\partial \xi_j}, \; i_1 < \ldots < i_{p+1}, \; j = 1, \ldots, m,$$

$$\xi_{i_1} \ldots \xi_{i_p} \frac{\partial}{\partial x_j}, \; i_1 < \ldots < i_p, \; j = 1, \ldots, n,$$

form a base of local sections of $\mathcal{T}_p$ over $\mathcal{F}$.

We consider now the case where $\mathbf{E} = \mathbf{T}(M)^*$; here $\mathbf{T}(M)$ is the tangent bundle of M. Then $\mathcal{A}$ coincides with the sheaf Ω of holomorphic differential forms on M, and the sheaves $\mathcal{A} \otimes \Theta$ and $\mathcal{A} \otimes \mathcal{E}^*$ both coincide with the sheaf $\Omega \otimes \Theta$ of holomorphic vector-valued differential forms. Thus, the exact sequence (3) has the form

$$0 \to \Omega \otimes \Theta \xrightarrow{i} \mathcal{T} \xrightarrow{\alpha} \Omega \otimes \Theta \to 0. \tag{5}$$

It was found by Frölicher and Nijenhuis (see [2]) that this exact sequence splits globally. Actually, they define the mapping $l : \Omega \otimes \Theta \to \mathcal{T}$ by

$$l(\varphi) = [i(\varphi), d], \tag{6}$$

where d is the exterior differentiation, which is obviously a section of $\mathcal{T}_1$. It can be proved that $\alpha(l(\varphi)) = \varphi$, so that l is a splitting of (5). Hence there is the following decomposition into the direct sum of subalgebra sheaves (not ideals!):

$$\mathcal{T} = i(\Omega \otimes \Theta) \oplus l(\Omega \otimes \Theta).$$

More precisely,

$$\mathcal{T}_p = i(\Omega_{p+1} \otimes \Theta) \oplus l(\Omega_p \otimes \Theta) \simeq (\Omega_{p+1} \otimes \Theta) \oplus (\Omega_p \otimes \Theta).$$

By the above, $\Omega \otimes \Theta$ is a sheaf of graded Lie superalgebras under the grading and the bracket $\{\ ,\ \}$, defined by (4). In what follows, we call this bracket *algebraic.* In [2], another bracket $[\ ,\ ]$ was defined in $\Omega \otimes \Theta$, namely,

$$[\varphi, \psi] = \alpha([l(\varphi), l(\psi)]).$$

We call it the *FN-bracket.* Under this bracket and the grading

$$(\Omega \otimes \Theta)_p = \Omega_p \otimes \Theta,$$

the sheaf $\Omega \otimes \Theta$ is a sheaf of graded Lie algebras as well. We also have $l([\varphi, \psi]) = [l(\varphi), l(\psi)]$, and thus, l is a homomorphism of sheaves of graded Lie algebras. The following formula (see [2]) will also be important for us:

$$[i(\varphi), l(\psi)] = l(\psi \,\bar{\wedge}\, \varphi) + (-1)^q i([\varphi, \psi]), \tag{7}$$

where $\varphi \in \Omega \otimes \Theta$, $\psi \in \Omega_q \otimes \Theta$.

It should be noted that all the considerations above can be carried over verbatim to the case where M is a differentiable manifold and $\mathbf{E}$ is a differentiable vector bundle over M. Notice that the setting considered by Frölicher and Nijenhuis in [2] was just the smooth one. In particular, in this situation, the operation $\bar{\wedge}$, the algebraic bracket, and the FN-bracket are defined.

2. Making the Resolution

Using the notation of the previous section, consider the sheaf $\mathcal{T}$ of derivations of the sheaf $\mathcal{A} = \bigwedge_{\mathcal{F}} \mathcal{E}$. Let us denote by $\Phi = \bigoplus_{p,q=0}^{n} \Phi^{p,q}$ the bigraded sheaf of smooth differential forms and by $\mathcal{F}_\infty = \Phi^{0,0}$ the sheaf of complex-valued smooth functions on M. We also denote by $\mathbf{T}_\infty(M)$ the complexified tangent bundle of the smooth manifold $(M, \mathcal{F}_\infty)$; it decomposes into the sum $\mathbf{T}^{1,0}(M) \oplus \mathbf{T}^{0,1}(M)$ of the components of types $(1,0)$ and $(0,1)$, respectively. Then $\mathbf{T}^{1,0}(M)$ is the smooth bundle corresponding to the holomorphic vector bundle $\mathbf{T}(M)$. Let $\Theta_\infty = \Theta^{1,0} \oplus \Theta^{0,1}$ be the corresponding sheaves of smooth vector fields. As in Sec. 1, we omit the subscript $\mathcal{F}$ in tensor products over the sheaf $\mathcal{F}$.

Since $\mathcal{T}$ is a locally free analytic sheaf (see Sec. 1), it can be considered as the sheaf of holomorphic sections of a vector bundle $\mathbf{ST}(\mathbf{E})$ over M (the *supertangent bundle* of $(M, \mathcal{O})$). Consider the standard Dolbeault–Serre resolution of $\mathcal{A}$, which is the sheaf $\mathcal{R} = \Phi^{0,*} \otimes \mathcal{T}$ of smooth $\mathbf{ST}$-valued differential forms of type $(0,*)$. This is a bigraded sheaf of modules over the sheaf $\mathcal{F}_\infty$ of complex-valued smooth functions on M, where the bigrading is defined by

$$\mathcal{R}_{p,q} = \Phi^{0,q} \otimes \mathcal{T}_p.$$

The coboundary operator $\bar{\partial}$ is given by

$$\bar{\partial}(\varphi \otimes u) = (\bar{\partial}\varphi) \otimes u;$$

it is of bidegree $(0,1)$.

We would like to provide $\mathcal{R}$ with a bracket coinciding on $\mathcal{T} = \mathcal{R}_{*,0} \cap (\operatorname{Ker} \bar{\partial})$ with the given one and such that $\bar{\partial}$ is a derivation (of total degree 1). Actually we will make another resolution $\mathcal{S}$ of $\mathcal{T}$ possessing the desired bracket and isomorphic to $\mathcal{R}$.

First, we consider the standard Dolbeault–Serre resolution of $\mathcal{A}$, which is the sheaf $\hat{\Phi} = \Phi^{0,*} \otimes \mathcal{A}$ of smooth $\mathbf{A}$-valued differential forms of type $(0,*)$. This is a bigraded sheaf of algebras, where the bigrading is defined by

$$\hat{\Phi}^{p,q} = \Phi^{0,q} \otimes \mathcal{A}_p,$$

and the multiplication is one of the tensor products of graded algebras. The coboundary operator $\bar{\partial}$ is given by

$$\bar{\partial}(\varphi \otimes a) = (\bar{\partial}\varphi) \otimes a;$$

it is of bidegree $(0,1)$. It can be easily proved that $\bar{\partial}$ is a derivation (of total degree 1).

Now, considering $\hat{\Phi}$ as a sheaf of graded algebras with respect to its total degree, we consider the sheaf of graded Lie algebras $\hat{\mathcal{T}} = \mathcal{D}er\, \hat{\Phi}$. We denote

$$\bar{D} = \operatorname{ad} \bar{\partial}.$$

Clearly, $\bar{D}$ is a derivation of degree 1 (and of bidegree $(0,1)$) of $\hat{\mathcal{T}}$, and

$$\bar{D}^2 = \frac{1}{2}[\bar{D}, \bar{D}] = \frac{1}{2} \operatorname{ad}[\bar{\partial}, \bar{\partial}] = 0.$$

By definition, we have

$$(\bar{D}u)(a) = \bar{\partial}u(a) - (-1)^{|u|}u(\bar{\partial}a), \; u \in \mathcal{S}, a \in \hat{\Phi}. \tag{8}$$

Set

$$\mathcal{S} = \{u \in \hat{\mathcal{T}} \mid u(\bar{f}) = u(d\bar{f}) = 0 \text{ for any } f \in \mathcal{F}\}.$$

It can readily be seen that $\mathcal{S}$ is a subsheaf of bigraded subalgebras and of $\mathcal{F}_\infty$-submodules of $\hat{\mathcal{T}}$. Further, for any $u \in \mathcal{S}$ and any local holomorphic $f \in \hat{\Phi}^{0,0} = \mathcal{F}_\infty$, by (8), we have

$$(\bar{D}u)(\bar{f}) = (\bar{D}u)(d\bar{f}) = 0,$$

and hence $\bar{D}(\mathcal{S}) \subset \mathcal{S}$.

Denote by $\mathcal{E}_\infty$ the sheaf of smooth sections of $\mathbf{E}$. Then the sheaf of algebras

$$\mathcal{A}_\infty = \bigwedge_{\mathcal{F}_\infty} \mathcal{E}_\infty$$

is the sheaf of smooth sections of $\mathbf{A}$. Also,

$$\Phi^{0,*} = \bigwedge_{\mathcal{F}_\infty} \Phi^{0,1},$$

$$\hat{\Phi} = \Phi^{0,*} \otimes_{\mathcal{F}_\infty} \mathcal{A}_\infty = \bigwedge_{\mathcal{F}_\infty} (\Phi^{0,1} \oplus \mathcal{E}_\infty).$$

Thus, $\hat{\Phi}$ is the sheaf of smooth sections of the vector bundle $\mathbf{T}^{0,1}(M) \oplus \mathbf{E}$. We can apply the arguments of Sec. 1 to $\hat{\Phi}$ bearing in mind the smooth setting.

In particular, we can include $\hat{\mathcal{T}}$ into an exact sequence of sheaves similar to (3) (this is sequence (12) to be studied in Sec. 3). It follows that $\hat{\mathcal{T}}$ is locally free over $\mathcal{F}_\infty$. To describe a base of local sections of $\hat{\mathcal{T}}$, we choose a coordinate neighborhood $U \subset M$ with holomorphic coordinates $x_1, \ldots, x_n$. Then $\frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$, and $\frac{\partial}{\partial \bar{x}_i}$, $i = 1, \ldots, n$, form bases of local sections of the sheaves $\Theta^{1,0}$ and $\Theta^{0,1}$, respectively. Denote $\eta_i = d\bar{x}_i$, $i = 1, \ldots, n$.

Also, we can assume that $\mathbf{E}$ is trivial over U and choose a base $\xi_j,\ j = 1, \ldots, m$, of local sections of $\mathcal{E}$ in U. Then $\hat{\mathcal{T}}_{p,q}$ has the following base of local sections:

$$\xi_{i_1} \ldots \xi_{i_{p+1}} \eta_{k_1} \ldots \eta_{k_q} \frac{\partial}{\partial \xi_j},\ \xi_{i_1} \ldots \xi_{i_{p+1}} \eta_{k_1} \ldots \eta_{k_q} \frac{\partial}{\partial \eta_j},\ \xi_1 < \ldots < i_{p+1},\ k_1 < \ldots < k_q,$$
$$\xi_{i_1} \ldots \xi_{i_p} \eta_{k_1} \ldots \eta_{k_q} \frac{\partial}{\partial x_i},\ \xi_{i_1} \ldots \xi_{i_p} \eta_{k_1} \ldots \eta_{k_q} \frac{\partial}{\partial \bar{x}_i},\ \xi_1 < \ldots < i_p,\ k_1 < \ldots < k_q.$$

The definition of $\mathcal{S}$ implies that $\mathcal{S}_{p,q}$ is the locally free subsheaf of $\hat{\mathcal{T}}_{p,q}$ with the base of local sections

$$\begin{gathered} \xi_{i_1} \ldots \xi_{i_{p+1}} \eta_{k_1} \ldots \eta_{k_q} \frac{\partial}{\partial \xi_j},\ i_1 < \ldots < i_{p+1},\ k_1 < \ldots < k_q, \\ \xi_{i_1} \ldots \xi_{i_p} \eta_{k_1} \ldots \eta_{k_q} \frac{\partial}{\partial x_i},\ i_1 < \ldots < i_p,\ k_1 < \ldots < k_q. \end{gathered} \tag{9}$$

We are now going to compare the sheaves $\mathcal{R}$ and $\mathcal{S}$. Restricting any $u \in \hat{\mathcal{T}}$ to the subsheaf $\mathcal{A}_\infty = 1 \otimes \mathcal{A}_\infty$ of $\hat{\mathcal{T}}$, we obtain a homomorphism $\gamma : u \mapsto u|\mathcal{A}_\infty$ of $\hat{\mathcal{T}}$ to $\mathcal{H}om_{\mathbb{C}}(\mathcal{A}_\infty, \hat{\Phi})$. We have the following identification:

$$\mathcal{H}om_{\mathbb{C}}(\mathcal{A}_\infty, \hat{\Phi}) = \Phi^{0,*} \otimes_{\mathcal{F}_\infty} \mathcal{E}nd_{\mathbb{C}} \mathcal{A}_\infty.$$

In fact, any $\mathbb{C}$-homomorphism $h : \mathcal{A}_\infty \to \hat{\Phi} = \Phi^{0,*} \otimes_{\mathcal{F}_\infty} \mathcal{A}_\infty$ can be locally written in the form $h(a) = \sum_k \varphi_k \otimes h_k(a),\ a \in \mathcal{A}_\infty$, where φ_k is a fixed base of local sections of $\Phi^{0,*}$ (e.g., which formed by the forms $\eta_{k_1} \ldots \eta_{k_q}$) and $h_j \in \mathcal{E}nd_{\mathbb{C}} \mathcal{A}_\infty$. It can be easily proved that $\operatorname{Im} \gamma$ coincides with $\Phi^{0,*} \otimes_{\mathcal{F}_\infty} \mathcal{D}er \mathcal{A}_\infty$ under this identification.

Note that there is a natural injection $\Theta \to \Theta^{1,0} \subset \Theta_\infty$, which, written in local coordinates, maps $\frac{\partial}{\partial x_i} \in \Theta$ into the "formal derivative" $\frac{\partial}{\partial x_i}$ acting in $\mathcal{F}_\infty$. Similarly, we obtain an injection $\mathcal{T} \to \mathcal{D}er \mathcal{A} \to \mathcal{D}er \mathcal{A}_\infty$ which extends any $u = \sum_i a_i \frac{\partial}{\partial x_i} + \sum_j b_j \frac{\partial}{\partial \xi_j}$ to the derivation of $\mathcal{A}_\infty$ expressed by the same formula. It follows that $\mathcal{R} = \Phi^{0,*} \otimes \mathcal{T} \subset \Phi^{0,*} \otimes_{\mathcal{F}_\infty} \mathcal{D}er \mathcal{A}_\infty$.

Theorem 1. *The mapping $\gamma : \hat{\mathcal{T}} \to \Phi^{0,*} \otimes_{\mathcal{F}_\infty} \mathcal{D}er \mathcal{A}_\infty$ determines an isomorphism of bigraded sheaves of $\mathcal{F}_\infty$-modules $\gamma : \mathcal{S} \to \mathcal{R}$ satisfying the condition $\gamma \circ \bar{D} = \bar{\partial} \circ \gamma$.*

The inverse isomorphism γ^{-1} maps $\mathcal{T} = 1 \otimes \mathcal{T} \subset \mathcal{R}$ onto the subsheaf $\tilde{\mathcal{T}} = \{u \in \mathcal{S}_{*,0} | [\bar{\partial}, u] = 0\}$ graded by $\tilde{\mathcal{T}}_p = \tilde{\mathcal{T}} \cap \mathcal{S}_{p,0}$.

If we identify $\tilde{\mathcal{T}}$ with $\mathcal{T}$ with the help of γ, then the differential graded sheaf $(\mathcal{S}, \bar{D})$ is a fine resolution of $\mathcal{T}$, and for any fixed p, $-1 \le p \le m$, the differential graded sheaf $(\mathcal{S}_{p,*}, \bar{D})$ is a fine resolution of $\mathcal{T}_p$.

Proof. We can use the local coordinates, which are introduced above. Consider the base of local sections of $\mathcal{S}_{p,q}$ over $\mathcal{F}_\infty$ given by (9). Clearly,

$$\gamma(\xi_{i_1} \ldots \xi_{i_{p+1}} \eta_{k_1} \ldots \eta_{k_q} \frac{\partial}{\partial \xi_j}) = \eta_{k_1} \ldots \eta_{k_q} \otimes (\xi_{i_1} \ldots \xi_{i_{p+1}} \frac{\partial}{\partial \xi_j}),$$
$$\gamma(\xi_{i_1} \ldots \xi_{i_p} \eta_{k_1} \ldots \eta_{k_q} \frac{\partial}{\partial x_i}) = \eta_{k_1} \ldots \eta_{k_q} \otimes (\xi_{i_1} \ldots \xi_{i_p} \frac{\partial}{\partial x_i}).$$

But these elements form a base of local sections of $\mathcal{R}_{p,q}$. Hence $\gamma : \mathcal{S} \to \mathcal{R}$ is an isomorphism, preserving the bidegrees.

By (8), for any $u \in \hat{\mathcal{T}}$, we have

$$(\bar{D}u)(x_i) = \bar{\partial}u(x_i),\ i = 1, \ldots, n,$$
$$(\bar{D}u)(\xi_j) = \bar{\partial}u(\xi_j),\ j = 1, \ldots, m.$$

If $u \in \mathcal{S}$, then $(\bar{D}u)(\bar{x}_i) = 0,\ i = 1, \dots, n$, and hence

$$\gamma(\bar{D}u) = \sum_i \bar{\partial}u(x_i)\frac{\partial}{\partial x_i} + \sum_j \bar{\partial}u(\xi_j)\frac{\partial}{\partial \xi_j} =$$

$$\bar{\partial}(\sum_i u(x_i)\frac{\partial}{\partial x_i} + \sum_j u(\xi_j)\frac{\partial}{\partial \xi_j}) = \bar{\partial}\gamma(u).$$

This completes the proof of the first assertion. The other is obvious.

Remark. As we see from Theorem 1, the construction of the resolution $(\mathcal{S}, \bar{D})$ solves the question posed in the Introduction. Instead of $\mathcal{S}$, one can consider the resolution $(\mathcal{R}, \bar{\partial})$ endowed with the bracket $[\,,\,]$ obtained by transferring the bracket from $\mathcal{S}$ with the help of γ. An elementary calculation shows that this transferred bracket in $\mathcal{R}$ is expressed by

$$[\varphi \otimes u, \psi \otimes v] = (-1)^{|u||\psi|}(\varphi\psi) \otimes [u, v] + \varphi u(\psi)v - (-1)^{|\varphi\otimes u||\psi\otimes v|}\psi v(\varphi)u, \quad \varphi, \psi \in \Phi^{0,*},\ u, v \in \mathcal{T}, \tag{10}$$

where we identify $\Phi^{0,*}$ with $\Phi^{0,*} \otimes 1 \subset \hat{\mathcal{T}}$ and $\mathcal{T}$ with $1 \otimes \mathcal{T} \subset \mathcal{R}$.

3. Exact Sequences

Here we return to the exact sequence (3) constructed in Sec. 1 and apply it to the study of the resolutions $\mathcal{R}$ and $\mathcal{S}$. Clearly, (3) leads to the following exact sequence formed by the Dolbeault–Serre resolutions of our sheaves:

$$0 \to \Phi^{0,*} \otimes \mathcal{A} \otimes \mathcal{E}^* \xrightarrow{\mathrm{id}\otimes i} \Phi^{0,*} \otimes \mathcal{T} \xrightarrow{\mathrm{id}\otimes \alpha} \Phi^{0,*} \otimes \mathcal{A} \otimes \Theta \to 0.$$

In the notation of Sec. 2, it is written as follows:

$$0 \to \hat{\Phi} \otimes \mathcal{E}^* \xrightarrow{\mathrm{id}\otimes i} \mathcal{R} \xrightarrow{\mathrm{id}\otimes \alpha} \hat{\Phi} \otimes \Theta \to 0. \tag{11}$$

This is an exact sequence of sheaves of complexes if we define the coboundary operators $\bar{\partial}$ in the boundary terms in the usual way:

$$\bar{\partial}(\varphi \otimes u) = (\bar{\partial}\varphi) \otimes u,\ u \in \mathcal{A} \otimes \mathcal{E}^* \text{or } \mathcal{A} \otimes \Theta.$$

On the other hand, the arguments of Sec. 1, being applied to the smooth vector bundle $\mathbf{T}^{0,1}(M) \oplus \mathbf{E}$, give the following exact sequence, which is similar to (3):

$$0 \to \hat{\Phi} \otimes_{\mathcal{F}_\infty} (\Theta^{0,1} \oplus \mathcal{E}^*_\infty) \xrightarrow{j} \hat{\mathcal{T}} \xrightarrow{\beta} \hat{\Phi} \otimes_{\mathcal{F}_\infty} \Theta_\infty \to 0. \tag{12}$$

The description (9) of the base of local sections of $\mathcal{S}$ implies $(\operatorname{Im} j) \cap \mathcal{S} = j(\hat{\Phi} \otimes_{\mathcal{F}_\infty} \mathcal{E}^*) = j(\hat{\Phi} \otimes_{\mathcal{F}} \mathcal{E}^*)$ and $\beta(\mathcal{S}) = \hat{\Phi} \otimes_{\mathcal{F}_\infty} \Theta^{1,0} = \hat{\Phi} \otimes \Theta$. Thus, (12) gives the exact sequence

$$0 \to \hat{\Phi} \otimes \mathcal{E}^* \xrightarrow{j} \mathcal{S} \xrightarrow{\beta} \hat{\Phi} \otimes \Theta \to 0. \tag{13}$$

Proposition 1. *The diagram*

$$\begin{array}{ccccccccc} 0 \to & \hat{\Phi} \otimes \mathcal{E}^* & \xrightarrow{j} & \mathcal{S} & \xrightarrow{\beta} & \hat{\Phi} \otimes \Theta & \to 0 \\ & \| & & \downarrow{\scriptstyle\gamma} & & \| & \\ 0 \to & \hat{\Phi} \otimes \mathcal{E}^* & \xrightarrow{1\otimes i} & \mathcal{R} & \xrightarrow{1\otimes \alpha} & \hat{\Phi} \otimes \Theta & \to 0 \end{array} \tag{14}$$

is commutative. The mapping $\mathrm{id}\otimes i$ *is a homomorphism of sheaves of algebras if we endow* $\hat{\Phi} \otimes \mathcal{E}^* \subset \hat{\Phi} \otimes_{\mathcal{F}_\infty} (\Theta^{0,1} \oplus \mathcal{E}^*_\infty)$ *with the algebraic bracket* $\{\ ,\ \}$ *and* $\mathcal{R}$ *with the bracket* (10).

Proof. The proof of the commutativity is straightforward by using the local coordinates. The second assertion follows from the fact that j is a homomorphism of sheaves of algebras.

Remark. Clearly, the subsheaf $\hat{\Phi} \otimes \mathcal{E}^* \subset \hat{\Phi} \otimes_{\mathcal{F}_\infty} (\Theta^{0,1} \oplus \mathcal{E}^*_\infty)$ is closed under the algebraic bracket. This bracket is defined as in (4), where $\hat{\Phi} \otimes \mathcal{E}^* = (\Phi^{0,*} \otimes \mathcal{A}) \otimes \mathcal{E}^*$ is considered as the sheaf of $\mathbf{E}^*$-valued forms on $\mathbf{E}^* \oplus \mathbf{T}^{0,1}(M)$ and the operation $\bar{\wedge}$ between two forms is defined by

$$(\varphi \,\bar{\wedge}\, \psi)(u_1, \dots, u_{r+p-1}, v_1, \dots, v_{s+q})$$
$$= \frac{1}{(p-1)!q!r!s!} \sum_{\substack{\alpha \in S_{p+q-1} \\ \beta \in S_{r+s-1}}} (\operatorname{sgn} \alpha)(\operatorname{sgn} \beta) \varphi(\psi(u_{\alpha(1)}, \dots, u_{\alpha(r)}, \tag{15}$$
$$v_{\beta(1)}, \dots, v_{\beta(s)}), u_{\alpha(r+1)}, \dots, u_{\alpha(r+p-1)}, v_{\beta(s+1)}, \dots, v_{\beta(s+q)}),$$

for $\varphi \in \hat{\Phi}_{p,q} \otimes \mathcal{E}^*, \psi \in \hat{\Phi}_{r,s} \otimes \mathcal{E}^*, u_i \in \mathcal{E}^*, v_j \in \Theta^{0,1}$.

Now we turn to the special case where $\mathbf{E} = \mathbf{T}^*(M)$. Clearly, in this case, $\hat{\Phi} = \Phi^{0,*} \otimes \Omega = \Phi$ and $\hat{\Phi}^{p,q} = \Phi^{p,q}$. Hence $\hat{\mathcal{T}} = \mathcal{D}er\Phi$. The exact sequence (12) is a smooth analogue of (5). Denoting j and β by i and α again, we write it in the form

$$0 \to \Phi \otimes_{\mathcal{F}_\infty} \Theta_\infty \xrightarrow{i} \hat{\mathcal{T}} \xrightarrow{\alpha} \Phi \otimes_{\mathcal{F}_\infty} \Theta_\infty \to 0. \tag{16}$$

By [2], there is the splitting $l : \Phi \otimes_{\mathcal{F}_\infty} \Theta_\infty \to \hat{\mathcal{T}}$ of (16) given by (6).

Consider now the sequence (13); in our case it has the form

$$0 \to \Phi \otimes \Theta \xrightarrow{i} \mathcal{S} \xrightarrow{\alpha} \Phi \otimes \Theta \to 0. \tag{17}$$

Its boundary terms are the standard resolutions of the sheaf $\Omega \otimes \Theta$ of holomorphic vector-valued forms, first considered in [3]. Note that l is a splitting of (17) as well. In fact, we see at once from the definition of $\mathcal{S}$ that $[d, \mathcal{S}] \subset \mathcal{S}$, and therefore, $l(\Phi \otimes \Theta) = [i(\Phi \otimes \Theta), d] \subset \mathcal{S}$.

We also see that l is a homomorphism of complexes. In fact, for any $\varphi \in \Phi \otimes \Theta$, using (6), we obtain the graded Jacobi identity and the relation $[\bar{\partial}, d] = 0$:

$$\bar{D}(l(\varphi)) = [\bar{\partial}, [i(\varphi), d]] = [[\bar{\partial}, i(\varphi)], d] = [i(\bar{\partial}\varphi), d] = l(\bar{\partial}\varphi).$$

As a result, we have the following theorem.

Theorem 2. *Assume that $\mathbf{E} = \mathbf{T}^*(M)$. The mappings i and l determine the splitting of the resolution $\mathcal{S}$ of $\mathcal{T}$ into the direct sum of two subsheaves of bigraded subalgebras:*

$$\mathcal{S} = i(\Phi \otimes \Theta) \oplus l(\Phi \otimes \Theta).$$

Here

$$\mathcal{S}_{p,q} = i(\Phi^{p+1,q} \otimes \Theta) \oplus l(\Phi^{p,q} \otimes \Theta),$$

and the bracket in the left summand is determined by the algebraic bracket in $\Phi \otimes \Theta$, while that in the right summand is determined by the FN-bracket. In the entire $\mathcal{S}$, relation (7) holds.

Corollary. *If $\mathbf{E} = \mathbf{T}^*(M)$, then*

$$H^*(M, \mathcal{T}) = i^*(H^*(\Gamma(M, \Phi \otimes \Theta), \bar{\partial})) \oplus l^*(H^*(\Gamma(M, \Phi \otimes \Theta), \bar{\partial}))$$
$$\simeq H^*(M, \Omega \otimes \Theta) \oplus H^*(M, \Omega \otimes \Theta).$$

The bigrading in $H^*(M, \mathcal{T})$ is given by

$$H^q(M, \mathcal{T}_p) \simeq H^q(M, \Omega^{p+1} \otimes \Theta) \oplus H^q(M, \Omega_p \otimes \Theta), \ p \geq -1, q \geq 0,$$

and the bracket $[\alpha, \beta]$, $\alpha, \beta \in H^*(M, \mathcal{T})$, is determined by the algebraic bracket of the vector-valued forms in the left summand, by the FN-bracket in the right one, and by (7) when α, β belong to different summands.

4. Invariant Cohomology of Compact Hermitian Symmetric Spaces

Let M be a simply connected compact Hermitian symmetric space. We can represent M as the coset space K/L, where K is a connected compact semisimple Lie group and L a connected symmetric subgroup of K, which is the stabilizer K_o of a point $o \in M$. It is known (see [5]) that the symmetry s at the point o belongs to the center of L. The complexification $G = K(\mathbb{C})$ also acts on M, and $M = G/P$, where $P = G_o$ is a parabolic subgroup of G. Let $\mathcal{T} = \mathcal{D}er\,\Omega$, where Ω is the sheaf of holomorphic differential forms on M. Clearly, G acts by automorphisms on the sheaves $\Omega, \mathcal{T}$ and hence on the bigraded cohomology algebra $H^*(M, \mathcal{T})$. The set of invariant cohomology classes $H^*(M, \mathcal{T})^G$ is, clearly, a bigraded subalgebra of $H^*(M, \mathcal{T})$. In this section, we discuss the problem of computing this subalgebra. The complete computation will be done in the simplest case where $M = \mathbb{CP}^n$.

We start by studying the cohomology $H^*(M, \Omega \otimes \Theta)$, where $\Omega \otimes \Theta$ is the sheaf of vector-valued holomorphic forms. In Sec. 1, two brackets, the algebraic bracket $\{\ ,\ \}$ and the FN-bracket $[\ ,\ ]$, were defined on this sheaf. Each of them leads to a structure of the bigraded algebra on $H^*(M, \Omega \otimes \Theta)$ and on the invariant part $H^*(M, \Omega \otimes \Theta)^G$, which is a graded Lie algebra with respect to the complete degree. Similar brackets are defined in the resolution $\Phi \otimes \Theta$ of $\Omega \otimes \Theta$, and the induced brackets on the cohomology of $(\Gamma(M, \Phi \otimes \Theta), \bar{\partial})$ coincide with the corresponding brackets in $H^*(M, \Omega \otimes \Theta)$ if we identify these two cohomology groups (see [4]).

The first step in the calculation of $H^*(M, \Omega \otimes \Theta)^G$ is the reduction to the study of G-invariant forms from $\Gamma(M, \Phi \otimes \Theta)$. Denote by δ the operator on $\Gamma(M, \Phi \otimes \Theta)$ conjugate to $\bar{\partial}$ (with respect to the K-invariant Hermitian metric on M) and by $\Box = \bar{\partial}\delta + \delta\bar{\partial}$ the Beltrami–Laplace operator. As usual, a form $\varphi \in \Gamma(M, \Phi \otimes \Theta)$ is called *harmonic* if $\Box\varphi = 0$. For a harmonic φ, we have $\bar{\partial}\varphi = 0$; any cohomology class contains precisely one harmonic form.

Proposition 2. *We have*

$$\Gamma(M, \Phi^r \otimes \Theta)^G = 0$$

whenever r is even.

Any $\varphi \in \Gamma(M, \Phi \otimes \Theta)^G$ is harmonic. Assigning to a form $\varphi \in \Gamma(M, \Phi \otimes \Theta)^G$ its cohomology class, we get an isomorphism of bigraded algebras $\lambda : \Gamma(M, \Phi \otimes \Theta)^G \to H^*(M, \Omega \otimes \Theta)^G$ both under the algebraic and the FN-brackets.

The FN-bracket in $H^*(M, \Omega \otimes \Theta)^G$ is identically 0.

Proof. For any form $\varphi \in \Gamma(M, \Phi^r \otimes \Theta)^G$, we have $s^*\varphi = \varphi$. Since $ds_o = -\mathrm{id}$, we obtain $(s^*\varphi)_o = (-1)^{r+1}\varphi_o$. If r is even, then $\varphi_o = 0$, and hence $\varphi = 0$. This proves the first assertion.

Moreover, in the same situation, we have $\bar{\partial}\varphi \in \Gamma(M, \Phi^{r+1} \otimes \Theta)^G$. If r is odd, then $\bar{\partial}\varphi = 0$. Similarly, $\delta\varphi = 0$, and hence φ is harmonic. It follows that $\lambda : \Gamma(M, \Phi \otimes \Theta)^G \to H^*(M, \Omega \otimes \Theta)^G$ is defined and injective. To prove that λ is surjective, assume that $\varphi \in \Gamma(M, \Phi \otimes \Theta)$ is a harmonic form representing a G-invariant cohomology class. Then, for any $k \in K$, the form $k^*\varphi$ is harmonic and lies in the same cohomology class as φ. Therefore, $k^*\varphi = \varphi$, $k \in K$, whence $\varphi \in \Gamma(M, \Phi \otimes \Theta)^G$.

Clearly, $\Gamma(M, \Phi \otimes \Theta)^G$ is a subalgebra under both brackets and λ is an isomorphism of algebras. The FN-bracket is 0, since $H^q(M, \Omega^p \otimes \Theta)^G = 0$ whenever $p + q$ is even.

Remark. Proposition 2 can be carried over to the cohomology $H^*(M, \mathcal{E}_\chi)$, where $\mathcal{E}_\chi$ is the sheaf of holomorphic sections of the homogeneous vector bundle $\mathbf{E}_\chi$ over M, determined by a holomorphic representation χ of P such that $\chi(s) = \mu\,\mathrm{id}$, $\mu^2 = 1$ (by the Schur lemma, this is true, e.g., when χ is irreducible). It can be proved that $H^p(M, \mathcal{E}_\chi)^G = 0$ whenever p is odd (even) for $\mu = 1$ (respectively, for $\mu = -1$). Hence it follows that if χ is completely reducible, then all forms from $\Gamma(M, \Phi^{0,*} \otimes \mathcal{E}_\chi)^G$ are harmonic (with respect to an appropriate K-invariant Hermitian metric on $\mathbf{T}^*(M) \otimes \mathbf{E}_\chi$), and the natural mapping $\lambda : \Gamma(M, \Phi^{0,*} \otimes \mathcal{E}_\chi)^G \to H^*(M, \mathcal{E}_\chi)^G$ is an isomorphism of graded vector spaces.

The next step is the reduction to invariants of the isotropy representation τ of P in the tangent space $T_o(M)$. The well-known Cartan principle of reducing invariants of a transitive action to invariants of the isotropy group gives

Proposition 3. *The mapping $\varphi \mapsto \varphi_o$ of $\Gamma(M, \Phi\otimes\Theta)$ onto $\bigwedge(T_o^{1,0}(M)\oplus T_o^{0,1}(M))^*\otimes T_o^{1,0}(M)$ determines an isomorphism of the bigraded vector spaces*

$$\Gamma(M,\Phi\otimes\Theta)^G \to (\bigwedge(T_o^{1,0}(M)\oplus T_o^{0,1}(M))^*\otimes T_o^{1,0}(M))^P$$

preserving the operations $\bar{\wedge}$ and $\{\ ,\ \}$.

Note that since M is symmetric, τ is trivial on the unipotent radical of P. Hence the P-invariants in $T_o(M)$ coincide with the $L(\mathbb{C})$- or L-invariants.

In conclusion, we will study the example $M = \mathbb{CP}^n = \mathrm{SL}_{n+1}(\mathbb{C})/P$, where P is a subgroup of all matrices of the form

$$h = \begin{pmatrix} A & 0 \\ b & c \end{pmatrix},\ A\in \mathrm{GL}_n(\mathbb{C}),\ c = (\det A)^{-1}.$$

Thus, $P = \mathrm{SL}_{n+1}(\mathbb{C})_o$, where $o = (0 : \ldots : 0 : 1)$. We also have

$$K = \mathrm{SU}_{n+1},$$
$$L = \mathrm{S}(\mathrm{U}_n\times \mathrm{U}_1) = \{h\in P \mid A\in \mathrm{U}_n,\ b=0\},$$
$$L(\mathbb{C}) = \mathrm{S}(\mathrm{GL}_n(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})) = \{h\in P\mid A\in\mathrm{GL}_n(\mathbb{C}),\ b=0\}.$$

As a Cartan subalgebra of $\mathfrak{p}$, we use the subalgebra of all diagonal matrices $\mathrm{diag}(\lambda_1,\ldots,\lambda_{n+1})$, where $\lambda_1+\cdots+\lambda_{n+1}=0$.

The vector space $T_o^{1,0}(M)$ is identified with the subalgebra $\mathfrak{n}_+\subset\mathfrak{sl}_{n+1}(\mathbb{C})$ of all matrices of the form

$$\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix},$$

where u is an n-column. The restriction of the isotropy representation τ to $L(\mathbb{C})$ is given by

$$\tau\begin{pmatrix} A & 0 \\ 0 & c \end{pmatrix}(u) = c^{-1}Au.$$

If we identify $L(\mathbb{C})$ with $\mathrm{GL}_n(\mathbb{C})$ by the isomorphism $\begin{pmatrix} A & 0 \\ 0 & c \end{pmatrix}\mapsto A$, then

$$\tau|L(\mathbb{C}) = (\det\rho)\rho,$$

where ρ is the standard representation of $\mathrm{GL}_n(\mathbb{C})$.

The vector space $T_o^{0,1}(M)$ is identified with $T_o^{1,0}(M)^* = \mathfrak{n}_+^*$ by means of the K-invariant Hermitian metric on M, and $\mathfrak{n}_+^*$ with the subalgebra $\mathfrak{n}_-\subset\mathfrak{sl}_{n+1}(\mathbb{C})$ of all matrices of the form

$$\begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix},$$

where v is an n-row. (The pairing between $\mathfrak{n}_+$ and $\mathfrak{n}_-$ is given by the invariant inner product in $\mathfrak{sl}_{n+1}(\mathbb{C})$.) By Proposition 3,

$$H^q(M,\Omega^p\otimes\Theta)^G \simeq ((\bigwedge^p\mathfrak{n}_-)\otimes(\bigwedge^q\mathfrak{n}_+)\otimes\mathfrak{n}_+)^{L(\mathbb{C})}.$$

Clearly, the representations $\bigwedge^q\tau$, $\bigwedge^p\tau^*$ are irreducible, and

$$(\bigwedge^q\tau)|L(\mathbb{C}) = (\det\rho)^q\bigwedge^q\rho,$$
$$(\bigwedge^p\tau^*)|L(\mathbb{C}) = (\det\rho)^{-p}\bigwedge^p\rho^*.$$

Further, for $p \geq 1$, we have the following decomposition into the sum of two irreducible components:

$$(\bigwedge^p \tau^*)\tau | L(\mathbb{C}) = (\det \rho)^{1-p}(\bigwedge^p \rho^*)\rho = (\det \rho)^{1-p} \bigwedge^{p-1} \rho^* + (\det \rho)^{1-p}\sigma, \tag{18}$$

where the leading weight of σ is $\lambda_1 - \lambda_{n-p+1} - \ldots - \lambda_n$. It follows that

$$H^q(M, \Omega^p \otimes \Theta)^G = \begin{cases} \mathbb{C} & \text{for } q = p-1 \geq 2 \\ 0 & \text{for } q \neq p-1. \end{cases}$$

It is easy to see that for any $p \geq 1$, we can choose the following vector-valued form ω_p of type $(p, p-1)$ on $\mathfrak{n}_+ = T_o^{1,0}(M)$:

$$\omega_p(u_1, \ldots, u_p, v_1, \ldots, v_{p-1}) = (p-1)! \begin{vmatrix} u_1 & (u_1, v_1) & \ldots & (u_1, v_{p-1}) \\ u_2 & (u_2, v_1) & \ldots & (u_2, v_{p-1}) \\ \vdots & \vdots & & \vdots \\ u_p & (u_p, v_1) & \ldots & (u_p, v_{p-1}) \end{vmatrix}, \tag{19}$$

where $u_i \in \mathfrak{n}_+$, $v_j \in \mathfrak{n}_-$ and $(\ ,\ )$ is the invariant inner product. Clearly, $\omega_p \neq 0$ for $p = 1, \ldots, n$. The corresponding basic G-invariant vector-valued form on M and its cohomology class in $H^{p-1}(M, \Omega^p \otimes \Theta)^G$ will also be denoted by ω_p.

Now we are able to calculate the algebra $H^*(M, \mathcal{T})^G$. Clearly, the decomposition in Theorem 2 is G-invariant, and hence

$$H^*(M, \mathcal{T})^G \simeq H^*(M, \Omega \otimes \Theta)^G \oplus H^*(M, \Omega \otimes \Theta)^G,$$

the bigrading and the bracket being described in the corollary of this theorem. The above calculation implies that the only nonzero cohomology spaces $H^q(M, \mathcal{T}_p)^G$ are the following ones:

$$\begin{aligned} &H^p(M, \mathcal{T}_p)^G = \langle i^*(\omega_{p+1}) \rangle, \\ &H^p(M, \mathcal{T}_{p+1})^G = \langle l^*(\omega_{p+1}) \rangle, \ p = 0, 1, \ldots, n-1. \end{aligned}$$

In particular, $i^*(\omega_1) = \varepsilon \in H^0(M, \mathcal{T}_0)^G$ is the grading derivation of $\mathcal{T}$, and $l^*(\omega_1) = d \in H^0(M, \mathcal{T}_1)^G$ is the exterior differentiation.

Proposition 4. *The bracket operation in $H^*(\mathbb{CP}^n, \mathcal{T})^G$ is given by*

$$\begin{aligned} &[i^*(\omega_p), i^*(\omega_q)] = (q-p) i^*(\omega_{p+q-1}), \\ &[l^*(\omega_p), l^*(\omega_q)] = 0, \\ &[i^*(\omega_p), l^*(\omega_q)] = q\, l^*(\omega_{p+q-1}), \ p, q \geq 1. \end{aligned}$$

Proof. One uses the corollary of Theorem 2, Proposition 3, and the following relation:

$$\omega_p \bar{\wedge} \omega_q = p \omega_{p+q-1}, \ p, q \geq 1. \tag{20}$$

To prove (20), we expand the determinant in (19) with respect to the first row; we have

$$\begin{aligned} &\omega_p(u_1, \ldots, u_p, v_1, \ldots, v_{p-1}) \\ &= (p-1)(\omega_{p-1}(u_2, \ldots, u_p, v_1, \ldots, v_{p-1}), v_1) u_1 \\ &+ (p-1) \sum_{i=1}^{p-1} (-1)^i (u_1, v_i) \omega_{p-1}(u_2, \ldots, u_p, v_1, \ldots, \hat{v}_i, \ldots, v_{p-1}). \end{aligned} \tag{21}$$

By (15), we have

$$(\omega_p \bar{\wedge} \omega_q)(u_1, \dots, u_{p+q-1}, v_1, \dots, v_{p+q-2})$$
$$= \frac{1}{(p-1)!^2 q!(q-1)!} \sum_{\substack{\alpha \in S_{p+q-1} \\ \beta \in S_{p+q-2}}} (\operatorname{sgn}\alpha)(\operatorname{sgn}\beta)$$
$$\times \omega_p(\omega_q(u_{\alpha(1)}, \dots, u_{\alpha(q)}, v_{\beta(1)}, \dots, v_{\beta(q-1)}), u_{\alpha(q+1)}, \dots, u_{\alpha(p+q-1)}, v_{\beta(q)}, \dots, v_{\beta(p+q-2)}).$$

Due to (21), this expression is the sum of p terms $\sum_{i=0}^{p-1} Q_i$, where

$$Q_0 = \frac{1}{(p-1)!(p-2)!q!(q-1)!} \sum_{\substack{\alpha \in S_{p+q-1} \\ \beta \in S_{p+q-2}}} (\operatorname{sgn}\alpha)(\operatorname{sgn}\beta)$$
$$\times (\omega_{p-1}(u_{\alpha(q+1)}, \dots, u_{\alpha(p+q-1)}, v_{\beta(q+1)}, \dots, v_{\beta(p+q-2)}), v_{\beta(q)})$$
$$\times \omega_q(u_{\alpha(1)}, \dots, u_{\alpha(q)}, v_{\beta(1)}, \dots, v_{\beta(q-1)}),$$
$$Q_i = \frac{(-1)^i}{(p-1)!(p-2)!q!(q-1)!} \sum_{\substack{\alpha \in S_{p+q-1} \\ \beta \in S_{p+q-2}}} (\operatorname{sgn}\alpha)(\operatorname{sgn}\beta)$$
$$\times (\omega_q(u_{\alpha(1)}, \dots, u_{\alpha(q)}, v_{\beta(1)}, \dots, v_{\beta(q-1)}), v_{\beta(q+i-1)})$$
$$\times \omega_{p-1}(u_{\alpha(q+1)}, \dots, u_{\alpha(p+q-1)}, v_{\beta(q)}, \dots, \widehat{v}_{\beta(q+i-1)}, \dots, v_{\beta(p+q-2)}),$$
$$i = 1, \dots, p-1.$$

Using the expansion with respect to the first $p-1$ rows, we see that

$$Q_0 = \omega_{p+q-1}(u_1, \dots, u_{p+q-1}, v_1, \dots, v_{p+q-2}).$$

To calculate Q_i, $i > 0$, we change the running element $\beta \in S_{p+q-2}$ by inserting

$$\beta = \gamma \circ (1, \dots, q) \circ (q, q+1, \dots, q+i-1), \ \gamma \in S_{p+q-2}.$$

Clearly, $\operatorname{sgn}\beta = (-1)^{q+i} \operatorname{sgn}\gamma$, and hence

$$Q_i = \frac{(-1)^q}{(p-1)!(p-2)!q!(q-1)!} \sum_{\substack{\alpha \in S_{p+q-1} \\ \beta \in S_{p+q-2}}} (\operatorname{sgn}\alpha)(\operatorname{sgn}\gamma)$$
$$\times (\omega_q(u_{\alpha(1)}, \dots, u_{\alpha(q)}, v_{\gamma(2)}, \dots, v_{\gamma(q)}), v_{\gamma(1)})$$
$$\times \omega_{p-1}(u_{\alpha(q+1)}, \dots, u_{\alpha(p+q-1)}, v_{\gamma(q+1)}, \dots, v_{\gamma(p+q-2)}).$$

Proceeding as in the case $i = 0$, we see that

$$Q_i = (-1)^q (p+q-1)!$$

$$\times \begin{vmatrix} (u_1, v_1) & \dots & u_1 & (u_1, v_{q+1}) & \dots & (u_1, v_{q+p-2}) \\ (u_2, v_1) & \dots & u_2 & (u_2, v_{q+1}) & \dots & (u_2, v_{q+p-2}) \\ \vdots & & \vdots & \vdots & & \vdots \\ (u_{p+q-1}, v_1) & \dots & u_{p+q-1} & (u_{p+q-1}, v_{q+1}) & \dots & (u_{p+q-1}, v_{q+p-2}) \end{vmatrix}$$

$$= \omega_{p+q-1}(u_1, \dots, u_{p+q-1}, v_1, \dots, v_{p+q-2})$$

for any $i \geq 1$. This proves (20).

Using the theorem of Bott [1] on the structure of an induced representation, we can also describe the whole algebra $H^*(\mathbb{CP}^n, \mathcal{T})$.

Theorem 3. *The Lie superalgebra* $H^0(\mathbb{CP}^n, \mathcal{T})$ *has the form*

$$H^0(\mathbb{CP}^n, \mathcal{T}) = H^0(\mathbb{CP}^n, \mathcal{T}_{-1}) \oplus H^0(\mathbb{CP}^n, \mathcal{T}_0) \oplus H^0(\mathbb{CP}^n, \mathcal{T}_1),$$

where

$$H^0(\mathbb{CP}^n, \mathcal{T}_{-1}) = i^*(\mathfrak{sl}_{n+1}(\mathbb{C})),$$
$$H^0(\mathbb{CP}^n, \mathcal{T}_0) = l^*(\mathfrak{sl}_{n+1}(\mathbb{C})) \oplus \langle \varepsilon \rangle,$$
$$H^0(\mathbb{CP}^n, \mathcal{T}_1) = \langle d \rangle,$$

and

$$[i^*(x), l^*(y)] = i^*([x,y]),\ x, y \in \mathfrak{sl}_{n+1}(\mathbb{C}).$$

The bigraded algebra $H^*(\mathbb{CP}^n, \mathcal{T})$ *is the semidirect sum of the subalgebra* $H^0(\mathbb{CP}^n, \mathcal{T})$ *and the ideal*

$$\bigoplus_{p+q\geq 2} H^q(\mathbb{CP}^n, \mathcal{T}_p) = \langle i^*(\omega_p), l^*(\omega_p) | p \geq 2 \rangle.$$

Proof. Clearly, $H^*(\mathbb{CP}^n, \mathcal{T})$ contains the bigraded algebra described in the formulation as a subalgebra. To prove the coincidence, it is sufficient to show that the G-module $H^*(\mathbb{CP}^n, \Omega_p \otimes \Theta)$ is trivial or irreducible for any $p \geq 0$. The homogeneous vector bundle $\bigwedge^p \mathrm{T}(M)^* \otimes \mathrm{T}(M)$ over $\mathbb{CP}^n$ is determined by the representation $\chi = (\bigwedge^p \tau^*)\tau$ of P. For $p = 0$, the representation $\chi = \tau$ is irreducible, while for $p \geq 1$, χ splits into two irreducible components (see (18)). The second summand has the leading weight

$$\Lambda = \lambda_1 - \lambda_{n-p+1} - \ldots - \lambda_n + (1-p)\lambda_{n+1}.$$

Let g be half of the sum of all positive roots of G and $\alpha = \lambda_{n-p+1} - \lambda_{n+1}$. Then $(\Lambda + g, \alpha) = 0$, and hence $\Lambda + g$ is singular. Therefore, our assertion follows from the above-mentioned theorem of Bott.

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REFERENCES

1. R. Bott, "Homogeneous vector bundles," *Ann. Math.*, **66**, 203-248 (1957).
2. A. Frölicher and A. Nijenhuis, "Theory of vector-valued differential forms, P.1. Derivations in the graded ring of differential forms," *Proc. Kon. Ned. Akad. Wet. Amsterdam*, **59**, 338-359 (1956).
3. A. Frölicher and A. Nijenhuis, "Some new cohomology invariants of complex manifolds," *Proc. Kon. Ned. Akad. Wet. Amsterdam*, **59**, 540-564 (1956).
4. R. Godement, *Topologie Algébrique et Théorie des Faisceaux*, Hermann (1958).
5. S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press (1978).
6. V. G. Kac, "Lie superalgebras," *Adv. Math.*, **26**, 8-97 (1977).
7. A. L. Onishchik, "Transitive Lie superalgebras of vector fields," *Reports Dep. Math. Univ. Stockholm*, **26**, 1-21 (1987).
8. A. L. Onishchik, *A spectral sequence for the tangent sheaf cohomology of a supermanifold*, Sfb 288, Preprint No. 148, Berlin (1994).