Invariants of geometric structures on differential systems (distributions)

by

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Chapter 1

Introduction

In this thesis we deal with classification of second order Monge-Ampere equations defined on a 2-dimensional smooth manifold M. These are partial differential equations of the form

$$F_1 + F_2\frac{\partial^2 u}{\partial q_1^2} + F_3\frac{\partial^2 u}{\partial q_2^2} + F_4\frac{\partial^2 u}{\partial q_1 \partial q_2} + F_5\left(\frac{\partial^2 u}{\partial q_1^2}\frac{\partial^2 u}{\partial q_2^2} - \left(\frac{\partial^2 u}{\partial q_1 \partial q_2}\right)^2\right) = 0, \qquad (1.1)$$

where the coefficients are functions in $q_1, q_2, u(q_1, q_2), \frac{\partial u}{\partial q_1}, \frac{\partial u}{\partial q_2}$ and q_1, q_2 are the coordinate functions on M. Monge-Ampere equations (with arbitrary number of variables) have occupied both physicists and mathematicians for several centuries. Some of the earliest results belong to Goursat ([G]) and Lie ([Lie]) at the end of the 19th century. Our investigation takes place in a geometrical framework with roots going back to Sophus Lie. The geometrical approach was proposed by V. Lychagin in [Ly1]. It gives us a way to represent the equations as geometrical structures, so called effective forms, on the manifold of 1-jets J^1M.

In chapter 2 we briefly recall the basic definitions from symplectic and contact geometry. Specially we introduce the strict contact manifold J^1M. The contact form on J^1M we denote by ω. This contact form induce a contact distribution $\mathcal{C}$ on J^1M. The last part of the chapter concerns with the geometrical representation of equation 1.1 in the strict contact manifold J^1M. We show that a given Monge-Ampere equation can be represented as a so called conformal effective 2-form $\theta \in \Lambda^2\mathcal{C}^*$ ([Ly1], [LRC]). We also give the pointwise classification of effective forms described in [LRC]. The chapter ends with the definition of generalized Monge-Ampere equations as a pair (ω, θ).

In chapter 3 we begin our investigation of a 5-dimensional strict contact manifold (M, ω) together with a conformal effective form θ. The rank 4 contact distribution induced from ω is denoted by Π. We start the chapter by recalling that any distribution induces the so called extending sequence of derived submodules. Next we introduce the operator $j : \Pi \longrightarrow \Pi$ induced by ω and θ. This operator splits the distribution

Π into two distributions Π^2_+ and Π^2_- of rank 2. We show that there are two canonical decompositions of the tangent bundle τ_M. In the last section of the chapter we define the Nijenhuis tensor $N_j \in \Lambda^2\Pi^* \otimes \Pi$ and we investigate its properties.

In chapter 4 we introduce the two invariants σ, R^σ_j of the generalized Monge-Ampere equation. We investigate how σ and R^σ_j relate to the Nijenhuis tensor and the distributions from the two decompositions of τ_M. The chapter ends with a classifying theorem where we state that nondegenerate, generalized Monge-Ampere equations defines a canonical frame on M.

Chapter 2

Preliminaries

In this chapter we establish a correspondence between Monge-Ampere equations and so called effective forms. We also introduce notions that we need in the main part of this thesis. In what follows we consider the $\mathbb{R}$-algebra $C^{\infty}(M)$. The module of sections of a bundle π over M is denoted by $C^{\infty}(\pi)$.

2.1 Symplectic Manifolds

Let V be a vector space. Then we call $\mu \in \Lambda^2 V^*$ **nondegenerate** if for any vector $X \in V$ there exists some vector $Y \in V$ such that $\mu(X, Y) \neq 0$.

Definition 2.1 *Let V be a vector space and let $\Omega \in \Lambda^2 V^*$ be nondegenerate. Then we call the pair (V, Ω) a **symplectic vector space** and we refer to Ω as the **symplectic form**.*

Definition 2.2 *Let (V, Ω) be a symplectic vector space and let W be a subspace in V. We say that W is a **Lagrangian subspace** if W is skew orthogonal to itself with respect to the symplectic form Ω. That is:*

$$W^{\perp_\Omega} = \{v \in V |\, \Omega(v, w) = 0 \ \forall\ w \in W\} = W.$$

As a consequence of the definition all symplectic vector spaces have to be even dimensional. Consider now a smooth manifold M. It is well known that the space of smooth functions over M, denoted $C^{\infty}(M)$, is an $\mathbb{R}$-algebra. For each point x in M, the space $\mu_x = \{f \in C^{\infty}(M) |\, f(x) = 0\}$ is an ideal in $C^{\infty}(M)$. This ideal defines the following filtration of ideals:

$$C^{\infty}(M) \supset \mu_x \supset \mu_x^2 \supset \mu_x^3 \supset \cdots.$$

The quotientspace

$$T_x^* M = \frac{\mu_x}{\mu_x^2}$$

is the cotangent space of M at the point x. T_x^*M is a vector space with the dimension equal to the dimension of M. The dual vector space T_xM is the tangent space of M at the point x.

Definition 2.3 *Let* $TM = \bigcup_{x\in M} T_xM$ *and* $T^*M = \bigcup_{x\in M} T_x^*M$. *We call the dual vector bundles*

$$\tau_M : TM \longrightarrow M$$

and

$$\tau_M^* : T^*M \longrightarrow M$$

the ***tangent*** *and* ***cotangent bundle*** *to* M *correspondingly.*

The $C^\infty(M)$-module of sections of the vector bundle $\Lambda^i\tau_M^*$ we denote by $\Omega^i(M)$, while the module of sections of the dual bundle $\Lambda^i\tau_M$ we denote by $\mathcal{D}_i(M)$.

Definition 2.4 *Let* M *be a* $2n$*-dimensional manifold and let* $\Omega \in \Omega^2(M)$ *be nondegenerate, i.e.* $\Omega_x \in \Lambda^2 T_x^*M$ *is nondegenerate for all* x *in* M. *Then we say that* (M,Ω) *is a* ***symplectic manifold****.*

Example 2.5 *Let* $M = \mathbb{R}^{2n}(q_1,\ldots,q_n,p_1,\ldots,p_n)$. *Then*

$$T_x^*M \simeq \mathbb{R}^{2n}(d_xq_1,\ldots,d_xq_n,d_xp_1,\ldots,d_xp_n)$$

and one can check that

$$\Omega_x = d_xq_1 \wedge d_xp_1 + \cdots + d_xq_n \wedge d_xp_n$$

defines a nondegenerate 2-form $\Omega \in \Omega^2(M)$*, such that* (M,Ω) *is a symplectic manifold.*

Definition 2.6 *The* ***vector bundle*** $\pi : E \longrightarrow B$ *is called* ***symplectic*** *if it is equipped with a symplectic structure* Ω_x *in fibers* $\pi^{-1}(x)$ *smoothly depending on* x.

Definition 2.7 *Let* (π,Ω) *be a symplectic vector bundle and let* γ *be a subbundle of* π. *We say that* γ *is a* ***Lagrangian subbundle*** *of* π *if every fiber in* γ *is a Lagrangian subspace of the corresponding fiber in* π.

A nondegenerate form $\Omega \in \Omega^2(M)$ defines the isomorphism

$$\begin{array}{rccccl} \Gamma: & \mathcal{D}_1(M) & \longrightarrow & \Omega^1(M) & & \\ & \xi & \longmapsto & i_\xi\Omega & = \Omega(\xi,\bullet) \end{array}$$

of $C^\infty(M)$-modules.

Definition 2.8 *Let*

$$\pi_1 : E_1 \longrightarrow B_1 \qquad \text{and} \qquad \pi_2 : E_2 \longrightarrow B_2$$

be vectorbundles and let us denote fibers by F_1, F_2 in E_1, E_2 respectively. By ***morphism of vectorbundles*** *$\pi_1 \longrightarrow \pi_2$, we mean a pair $\left(h, \widehat{h}\right)$ of smooth maps such that the diagram*

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{h}} & E_2 \\ \uparrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow{h} & B_2 \end{array}$$

commutes and

$$\widehat{h} : F_1(x) \longrightarrow F_2(h(x))$$

is a linear map for all points x in B_1.

When h is a diffeomorphism and $\widehat{h} : F_1(x) \longrightarrow F_2(h(x))$ is a linear isomorphism for all x in B_1, we say that $\left(h, \widehat{h}\right)$ is an isomorphism of vectorbundles. The following lemma is well known.

Lemma 2.9 *Let π_1, π_2 be the vectorbundles.*

$$\pi_1 : E_1 \longrightarrow B, \qquad\qquad \pi_2 : E_2 \longrightarrow B.$$

Let

$$\varphi : C^\infty(\pi_1) \longrightarrow C^\infty(\pi_2)$$

be $C^\infty(M)$-linear. Then φ is equivalent with a vectorbundle morphism

$$\widehat{\phi} : E_1 \longrightarrow E_2$$

over

$$\phi = Id : B \longrightarrow B.$$

From this lemma we conclude that Γ also can be considered as a morphism of vector bundles $\Gamma : \tau_M \longrightarrow \tau_M^*$.

$$\begin{array}{rcccl} \Gamma : & T_x M & \longrightarrow & T_x^* M & \\ & X & \longmapsto & i_X \Omega_x & = \Omega_x(X, \bullet) \end{array}$$

over $Id : M \longrightarrow M$. One can show that this is an isomorphism.

2.2 Contact Manifolds

Let M be a $(2n+1)$-dimensional smooth manifold. A 1-form $\omega \in \Omega^1(M)$ is said to be **nondegenerate** if the top form $\omega \wedge (d\omega)^n \in \Omega^{2n+1}(M)$ is nonzero.

Definition 2.10 *Let x be a point in M and let us denote a neighborhood in M containing x by $\mathcal{O}_x$. A distribution $\mathcal{C}$ on M is said to be a* ***contact distribution*** *if there for all points $x \in M$ exists a nondegenerate 1-form $\omega \in \Omega^1(\mathcal{O}_x)$ such that $\mathcal{C} = \ker(\omega)$.*

In this case $\mathcal{C}$ is also a subbundle of τ_M and ω can be taken as a local generator of $C^\infty(\mathrm{Ann}(\mathcal{C}))$, where $\mathrm{Ann}(\mathcal{C})$ is the distribution known as the annihilator of $\mathcal{C}$. $\mathrm{Ann}(\mathcal{C})$ is a subbundle of τ_M^* with rank 1 since $\mathcal{C}$ has rank $2n$.

Definition 2.11 *A smooth manifold M together with a contact distribution $\mathcal{C} \subset \tau_M$ is said to be a* ***contact manifold****.*

Definition 2.12 *A smooth manifold M together with a nondegenerate 1-form $\omega \in \Omega^1(M)$ is said to be a* ***strict contact manifold****.*

Since every nondegenerate 1-form $\omega \in \Omega^1(M)$ defines a contact distribution, any strict contact manifold is also a contact manifold. The opposite is not true. A contact distribution $\mathcal{C}$ does not define a unique nondegenerate 1-form ω. One can easily check that if ω annihilates $\mathcal{C}$, then so does $f \cdot \omega$ for any $f \in C^\infty(M)$.

Example 2.13 *Let $M = \mathbb{R}^{2n+1}(q_1, \ldots, q_n, u, p_1, \ldots, p_n)$. Then*

$$TM \simeq \mathbb{R}^{2n+1}(\partial_{q_1}, \ldots, \partial_{q_n}, \partial_u, \partial_{p_1}, \ldots, \partial_{p_n})$$

and

$$T^*M \simeq \mathbb{R}^{2n+1}(dq_1, \ldots, dq_n, du, dp_1, \ldots, dp_n).$$

The distribution

$$\mathcal{C} = \langle \partial_{q_1} + p_1\partial_u, \ldots, \partial_{q_n} + p_n\partial_u, \partial_{p_1}, \ldots, \partial_{p_n} \rangle$$

is a contact distribution on M. $\mathcal{C}$ is often referred to as the ***Cartan distribution****. One can check that the 1-form*

$$\omega = du - p_1 dq_1 - \cdots - p_n dq_n$$

annihilates $\mathcal{C}$ and that

$$\omega \wedge (d\omega)^n = dq_1 \wedge dp_1 \wedge dq_2 \wedge dp_2 \wedge \cdots \wedge dp_n \wedge du \neq 0,$$

hence ω is nondegenerate. This ω we call the ***Cartan form****.*

Definition 2.14 *Let $\Pi \subset \tau_M$ be a distribution. A submanifold $N \subset M$ is said to be an* ***integral manifold*** *of Π if $T_xN \subset \Pi_x$ for all points x in N.*

Definition 2.15 *Let (M, Π) be a contact manifold. An integral manifold N of Π is called* ***Legendrian*** *if*

$$2 \dim N = \dim M - 1.$$

Lemma 2.16 *Let (M, ω) be a strict contact manifold and $\mathcal{C}$ the contact distribution defined by ω, i.e. $\mathcal{C} = \ker(\omega)$. Then there exists a canonical "symplectic" form Ω on $\mathcal{C}$ induced by ω.*

Proof. This follows from the fact that ω is nondegenerate. That is:

$$\omega \wedge (d\omega)^n \neq 0$$

if $\dim(M) = 2n + 1$. Let

$$\tau_M = \left\langle \xi_1, \ldots, \xi_{2n+1} \right\rangle \qquad \text{and} \qquad \mathcal{C} = \langle \xi_1, \ldots, \xi_{2n} \rangle .$$

Then the following inequality must hold:

$$(\omega \wedge (d\omega)^n) \left(\xi_1, \cdots, \xi_{2n+1} \right) \neq 0$$

Since $\mathcal{C} = \ker(\omega)$, this is equivalent to

$$\omega \left(\xi_{2n+1} \right) \cdot (d\omega)^n (\xi_1, \cdots, \xi_{2n}) \neq 0,$$

which implies that $\Omega = d\omega|_{\mathcal{C}} \neq 0$. Thus Ω_x is a canonical symplectic structure on $\mathcal{C}_x$ for all points $x \in M$. ∎

2.3 Jet bundles π_k

We will now consider another important construction from the filtration of ideals

$$C^\infty(M) \supset \mu_x \supset \mu_x^2 \supset \mu_x^3 \supset \cdots$$

introduced above. From this we defined the cotangent space of the manifold M as the quotient $\frac{\mu_x}{\mu_x^2}$. Once more we are interested in quotients.

Definition 2.17 *We call the quotient*

$$J_x^k M = \frac{C^\infty(M)}{\mu_x^{k+1}}$$

the space of k-jets of functions at x in M.

Since $C^\infty(M)$ is an $\mathbb{R}$-algebra and μ_x^{k+1} is an ideal in $C^\infty(M)$, it is well known that $J_x^k M$ must be an $\mathbb{R}$-algebra. Let $\lambda \in \mathbb{R}$, $[f]_x^k, [g]_x^k \in J_x^k M$. Then one can check that

$$\lambda \cdot [f]_x^k = [\lambda f]_x^k \qquad [f]_x^k + [g]_x^k = [f+g]_x^k \qquad [f]_x^k \cdot [g]_x^k = [f \cdot g]_x^k$$

gives $J_x^k M$ an $\mathbb{R}$-algebra structure.

Definition 2.18 *We call the bundle*

$$\pi_k : J^k M \longrightarrow M$$

***the k-jet bundle over** M. Here $J^k M = \bigcup_{x\in N} J_x^k M$ is a smooth manifold.*

Every function $f \in C^\infty(M)$ defines a section in π_k. We denote the map relating functions and sections by j_k.

$$\begin{array}{rccc} j_k: & C^\infty(M) & \longrightarrow & C^\infty(\pi_k) \\ & f & \longmapsto & j_k(f) = [f]^k \end{array},$$

where

$$\begin{array}{rccc} [f]^k: & M & \longrightarrow & J^k M \\ & x & \longmapsto & [f]_x^k = f \bmod \mu_x^{k+1} \end{array}.$$

We say that $j_k(f)$ is the k-jet of the function f.

Remark 2.19 *$J^1 M$ is a strict contact manifold.*

2.4 Monge-Ampere equations and effective forms

The classical Monge-Ampere equation in two variables is a second order partial differential equation with two variables q_1, q_2 and one unknown function $u(q_1, q_2)$:

$$F_1 + F_2 \frac{\partial^2 u}{\partial q_1^2} + F_3 \frac{\partial^2 u}{\partial q_2^2} + F_4 \frac{\partial^2 u}{\partial q_1 \partial q_2} + F_5 \left(\frac{\partial^2 u}{\partial q_1^2} \frac{\partial^2 u}{\partial q_2^2} - \left(\frac{\partial^2 u}{\partial q_1 \partial q_2} \right)^2 \right) = 0, \qquad (2.1)$$

where $F_1, \ldots, F_5$ are functions of $q_1, q_2, u, \frac{\partial u}{\partial q_1}, \frac{\partial u}{\partial q_2}$.

From [Ly1] we have the following observation: Every differential n-form $\theta \in \Omega^n(J^1 M)$ defines a second order differential operator

$$\begin{array}{rcccc} \Delta_\theta: & C^\infty(M) & \longrightarrow & \Omega^n(M) & \\ & f & \longmapsto & j_1(f)^*(\theta) & = \theta|_{j_1(h)} \end{array}$$

called the Monge-Ampere operator. Here $j_1(f)^*$ denotes morphism

$$j_1(f)^* : \Omega^n\left(J^1 M\right) \longrightarrow \Omega^n(M)$$

of $C^\infty(M)$-modules induced from the section $j_1(f): M \longrightarrow J^1M$. The kernel of Δ_θ is equivalent to a second order partial differential equation in n variables.

Example 2.20 (Linear wave 1) *Let $M = \mathbb{R}^2(q_1, q_2)$. Then $J^1M \simeq \mathbb{R}^5(q_1, q_2, u, p_1, p_2)$. Consider the form*

$$\theta = dq_1 \wedge dp_2 + dq_2 \wedge dp_1.$$

The corresponding Monge-Ampere operator has the form

$$\Delta_\theta(h) = \left(-\frac{\partial^2 h}{\partial q_1^2} + \frac{\partial^2 h}{\partial q_2^2}\right) dq_1 \wedge dq_2.$$

So the Monge-Ampere equation $\Delta_\theta(h) = 0$ becomes the linear wave equation:

$$\frac{\partial^2 h}{\partial q_2^2} - \frac{\partial^2 h}{\partial q_1^2} = 0.$$

Example 2.21 (Laplace) *Similar to the preceding example one can check that the form*

$$\theta = dq_1 \wedge dp_2 + dp_1 \wedge dq_2$$

corresponds to the Laplace equation

$$\frac{\partial^2 h}{\partial q_1^2} + \frac{\partial^2 h}{\partial q_2^2} = 0.$$

Definition 2.22 *Let M be an n-dimensional smooth manifold. We call*

$$\Delta_\theta(f) = 0 \quad ; \quad \theta \in \Omega^n M$$

the ***Monge-Ampere equation****. A solution of this equation is a Legendrian submanifold $L \subset J^1M$, such that $\theta|_L = 0$ and $\pi_1|_L$ is a diffeomorphism.*

To allow multivalued solutions, the following generalization is made:

Definition 2.23 *We define* ***generalized solutions*** *of Monge-Ampere equations to be Legendrian submanifolds $L \subset J^1M$, such that $\theta|_L = 0$.*

Example 2.24 *Let $M = \mathbb{R}^2(q_1, q_2)$. Then one can show that the form*

$$\theta = F_1 dq_1 \wedge dq_2 + F_2 dq_2 \wedge dp_1 + F_3 dq_1 \wedge dp_2 + F_4 dq_1 \wedge dp_1 - F_4 dq_2 \wedge dp_2 + F_5 dp_1 \wedge dp_2,$$

where the coefficients are functions in q_1, q_2, u, p_1 and p_2, corresponds to the Monge-Ampere equation

$$F_1 - F_2 \frac{\partial^2 u}{\partial q_1^2} + F_3 \frac{\partial^2 u}{\partial q_2^2} + 2F_4 \frac{\partial^2 u}{\partial q_1 \partial q_2} + F_5 \left(\frac{\partial^2 u}{\partial q_1^2}\frac{\partial^2 u}{\partial q_2^2} - \left(\frac{\partial^2 u}{\partial q_1 \partial q_2}\right)^2\right) = 0.$$

Although a given form θ defines an unique Monge-Ampere operator Δ_θ, the correspondence is not one-to-one. The correspondence is presented in [Ly1] and [Ly2]. We present the main steps of argumentation for the sake of completeness. We investigate the graded, super commutative, differential $\mathbb{R}$-algebra

$$\Omega^*\left(J^1M\right) = \bigoplus_{i=0}^{\dim J^1M} \Omega^i\left(J^1M\right).$$

Let ω denote the contact form in correspondence with the strict contact manifold J^1M, i.e. $\omega \in \Omega^1\left(J^1M\right)$ is nondegenerate.

Theorem 2.25 (Lepage) *Two k'th order Monge-Ampere operators Δ_θ and $\Delta_{\theta'}$ are equal if and only if the differential k-form $\theta' \in \Omega^k\left(J^1M\right)$ can be written in the form:*

$$\theta' = \theta + \alpha \wedge \omega + \beta \wedge d\omega$$

for some differential forms $\alpha \in \Omega^{k-1}\left(J^1M\right)$, $\beta \in \Omega^{k-2}\left(J^1M\right)$.

Proof. This statement with proof can be found in [Ly1] and [Ly2]. ∎
This theorem leads to the following definition.

Definition 2.26 *Let I^k be a subspace of $\Omega^k\left(J^1M\right)$ defined by:*

$$I^k = \left\{ \mu \in \Omega^k\left(J^1M\right) \middle|\, \mu = \alpha \wedge \omega + \beta \wedge d\omega \text{ for some } \alpha \in \Omega^{k-1}\left(J^1M\right), \beta \in \Omega^{k-2}\left(J^1M\right)\right\}$$

Then we call the ideal

$$I_{\mathcal{C}} = \bigoplus_k I^k$$

*in the exterior algebra $\Omega^*\left(J^1M\right)$, the **Cartan ideal**.*

It is easy to see that if $\mu \in I^k$, then $d\mu \in I^{k+1}$, so $I_{\mathcal{C}}$ is a graded, super commutative, differential ideal in $\Omega^*\left(J^1M\right)$. Thus the factor $\Omega^*\left(J^1M\right) \big/ I_{\mathcal{C}}$ must be a graded, super commutative, differential $\mathbb{R}$-algebra.

Definition 2.27 *We define **the algebra of effective forms** to be*

$$\Omega^*_{\text{eff}}\left(J^1M\right) = \frac{\Omega^*\left(J^1M\right)}{I_{\mathcal{C}}}.$$

The Lepage theorem gives us a one-to-one correspondence between effective forms and Monge-Ampere operators Δ_θ. But it still remains to define Monge-Ampere operators from effective forms. We need to find a canonical representative θ_0 in $\Omega^*\left(J^1M\right)$ from any given effective form $\theta_{\text{eff}} \in \Omega^*_{\text{eff}}\left(J^1M\right)$. Then we can define $\Delta_{\theta_{\text{eff}}}$ to be Δ_{θ_0}.

We know that the contact form ω induces a contact distribution $\mathcal{C} \subset \tau_{J^1M}$ and the symplectic structure $\Omega = d\omega|_{\mathcal{C}}$. Ω implies the isomorphism

$$\Gamma : \mathcal{C} \longrightarrow \mathcal{C}^*$$

as mentioned above. The exact sequences

$$0 \longrightarrow \mathcal{C} \longrightarrow \tau_{J^1M} \longrightarrow \left(\frac{\tau_{J^1M}}{\mathcal{C}}\right) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{C}^* \longrightarrow \tau^*_{J^1M} \longrightarrow \left(\frac{\tau_{J^1M}}{\mathcal{C}}\right)^* \longrightarrow 0$$

are well known. Using that $\left(\frac{\tau_{J^1M}}{\mathcal{C}}\right) \simeq \mathrm{Ann}\,(\mathcal{C}^*)$ and $\left(\frac{\tau_{J^1M}}{\mathcal{C}}\right)^* \simeq \mathrm{Ann}\,(\mathcal{C})$ we get the isomorphisms

$$\tau^*_{J^1M} \simeq \mathcal{C}^* \oplus \mathrm{Ann}\,(\mathcal{C}) \qquad \text{and} \qquad \tau_{J^1M} \simeq \mathrm{Ann}\,(\mathcal{C}^*) \oplus \mathcal{C}$$

These isomorphisms imply

$$\begin{aligned} \Lambda^k \tau^*_{J^1M} &\simeq \Lambda^k \mathcal{C}^* \oplus \Lambda^{k-1}\mathcal{C}^* \wedge \mathrm{Ann}\,(\mathcal{C})\,, \\ \Lambda^k \tau_{J^1M} &\simeq \mathrm{Ann}\,(\mathcal{C}^*) \wedge \Lambda^{k-1}\mathcal{C} \oplus \Lambda^k \mathcal{C}. \end{aligned}$$

Since the contact form ω can be taken as the generator of $\mathrm{Ann}\,(\mathcal{C})$, this leads to the isomorphism

$$\Omega^k_{\text{eff}}\left(J^1M\right) \simeq \frac{C^\infty\left(\Lambda^k\mathcal{C}^*\right)}{\{\beta \wedge d\omega|_{\mathcal{C}} |\, \beta \in C^\infty\left(\Lambda^{k-2}\mathcal{C}^*\right)\}}.$$

Let a be a point in J^1M. Let us investigate the space $\Lambda^k\left(C^*_a\right)$. We introduce the two operators

$$\begin{array}{rccc} \top : & \Lambda^i \mathcal{C}^*_a & \longrightarrow & \Lambda^{i+2}\mathcal{C}^*_a \\ & \theta & \longmapsto & \theta \wedge \Omega \end{array}$$

and

$$\begin{array}{rccc} \perp : & \Lambda^i \mathcal{C}^*_a & \longrightarrow & \Lambda^{i-2}\mathcal{C}^*_a \\ & \theta & \longmapsto & i_{X_\Omega}\theta \end{array},$$

where X_Ω denotes the canonical bivector field $\Lambda^2\Gamma^{-1}(\Omega)$.

Theorem 2.28 (Hodge-Lepage) *For any k-form $\theta \in \Lambda^k\mathcal{C}^*_a$, the following Hodge-Lepage decomposition holds:*

$$\theta = \theta_0 + \top\theta_1 + \top^2\theta_2 + \cdots,$$

where $\theta_i \in \Lambda^{k-2i}\mathcal{C}^*_a$ are uniquely determined forms satisfying $\perp\theta_i = 0$.

Proof. This statement with proof can be found in [Ly1] and [Ly2]. ■

From [Ly1] and [Ly2] we have an explicit expression for θ_0:

$$\theta_0 = \left(\sum_{i \geq 0} (-1)^i \frac{1}{(i+1)!i!} \top^i \bot^i \right) (\theta) .$$

So we have the isomorphism

$$\begin{aligned} \Omega^k_{\text{eff}} \left(J^1 M \right) &\simeq \left\{ \mu \in C^\infty \left(\Lambda^k \mathcal{C}^* \right) \middle| \bot \mu = 0 \right\} \stackrel{\text{def}}{=} C^\infty \left(\Lambda^k_{\text{eff}} \mathcal{C}^* \right) , \\ \theta \bmod I^k &\simeq \theta_0 = \left(\sum_{i \geq 0} (-1)^i \frac{1}{(i+1)!i!} \top^i \bot^i \right) (\theta) . \end{aligned}$$

This give us the one-to-one correspondence between Monge-Ampere operators of k'th order and effective forms $\theta_{\text{eff}} \in C^\infty \left(\Lambda^k_{\text{eff}} C^* \right)$.

Lemma 2.29 *A Monge-Ampere equation in k variables, defines a conformal effective form $\theta \in C^\infty \left(\Lambda^k_{\text{eff}} \mathcal{C}^* \right)$.*

Proof. We know the equivalence between effective forms $\theta_{\text{eff}} \in C^\infty \left(\Lambda^k_{\text{eff}} \mathcal{C}^* \right)$ and Monge-Ampere operators $\Delta_{\theta_0} : C^\infty (M) \longrightarrow \Omega^k (M)$. Remember that a Monge-Ampere equation is equivalent to the kernel of a Monge-Ampere operator:

$$\Delta_{\theta_0} (f) = 0.$$

Let $g \in \Omega^0 (J^1 M) = C^\infty (J^1 M)$. Then we know that

$$\Delta_{g \cdot \theta_0} (f) = j_1 (f)^* (g \cdot \theta_0) = j_1 (f)^* (g) \cdot j_1 (f)^* (\theta_0) = (g \circ j_1 (f)) \cdot \Delta_{\theta_0} (f) .$$

So we see that if g is a nonzero function, then

$$\ker (\Delta_{g \cdot \theta_0}) = \ker (\Delta_{\theta_0}) .$$

The claim follows. ■

Let $\mathbb{R}^* = \mathbb{R} \backslash \{0\}$. Then the Monge-Ampere equation $\Delta_\theta (f) = 0$ defines at every point a in $J^1 M$ the subset $\mathbb{R}^* \cdot \theta_0(a)$ in $\Lambda^k \mathcal{C}^*_a$, where θ_0 is the effective part of θ according to the Hodge-Lepage decompositon.

2.5 Classification of effective forms

Our main object in this thesis will be a 5-dimensional strict contact manifold. This corresponds to $J^1 M$, where M is 2-dimensional. Hence this is related to second order PDE's of Monge-Ampere type with 2 variables. A PDE like this corresponds to a conformal effective form $\theta \in C^\infty (\Lambda^2_{\text{eff}} \mathcal{C}^*)$. The following proposition from [Ly1] gives us a useful description of the space $C^\infty (\Lambda^2_{\text{eff}} \mathcal{C}^*)$.

Proposition 2.30 *When the Cartan distribution $\mathcal{C}$ has rank $2n$, the form $\omega \in \Lambda^{n-k}\mathcal{C}_a^*$ is effective if and only if $\top^{k+1}(\omega) = 0$.*

When $\mathcal{C}$ has rank 4, it follows that $\{\mu \in \Lambda^2\mathcal{C}^* | \top\mu = 0\} = \{\mu \in \Lambda^2\mathcal{C}^* | \perp\mu = 0\}$, so we can write

$$C^\infty\left(\Lambda^2_{\text{eff}}\mathcal{C}^*\right) = \left\{\mu \in C^\infty\left(\Lambda^2\mathcal{C}^*\right) \middle| \mu \wedge \Omega = 0\right\},$$

where $\Omega \in C^\infty(\Lambda^2\mathcal{C}^*)$ is the symplectic structure in $\mathcal{C}$.

In [Ly2] there is given a pointwise classification of effective forms $\theta \in C^\infty(\Lambda^2_{\text{eff}}\mathcal{C}^*)$. Let a be a point in J^1M and let $\Omega_a \in \Lambda^2\mathcal{C}_a^*$ be the symplectic form. Then by dimensional reasons $\theta_a \wedge \theta_a$ and $\Omega_a \wedge \Omega_a$ are proportional.

Definition 2.31 *We define the* ***Pfaffian*** *of a 2-form θ_a in $\Lambda^2\mathcal{C}_a^*$ by:*

$$\theta_a \wedge \theta_a = \text{Pf}(\theta_a)\,\Omega_a \wedge \Omega_a,$$

where Ω_a is the symplectic form in $\mathcal{C}_a$.

Definition 2.32 *An effective 2-form $\theta_a \in \Lambda^k\mathcal{C}_a^*$ is said to be*

- ***elliptic*** *if* $\text{Pf}(\theta_a) > 0$,
- ***hyperbolic*** *if* $\text{Pf}(\theta_a) < 0$,
- ***parabolic*** *if* $\text{Pf}(\theta_a) = 0$ *and* $\theta_a \neq 0$.

Example 2.33 *Let $M = \mathbb{R}^2(q_1, q_2)$ and $J^1M \simeq \mathbb{R}^5(q_1, q_2, u, p_1, p_2)$. Then one can show ([Ly2]) that an effective 2-form θ must have the form:*

$$\theta = F_1 dq_1 \wedge dq_2 + F_2 dq_2 \wedge dp_1 + F_3 dq_1 \wedge dp_2 + F_4 dq_1 \wedge dp_1 - F_4 dq_2 \wedge dp_2 + F_5 dp_1 \wedge dp_2.$$

And when $\Omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$ is the symplectic form, then

- *θ is hyperbolic when $F_1F_5 + F_2F_3 + F_4^2 > 0$,*
- *θ is elliptic when $F_1F_5 + F_2F_3 + F_4^2 < 0$,*
- *θ is parabolic when $F_1F_5 + F_2F_3 + F_4^2 = 0$.*

We will consider the pure hyperbolic case, that is when $\text{Pf}(\theta_a) < 0\ \forall\ a \in J^1M$. For short we will write $\text{Pf}(\theta) < 0$.

With any 2-form $\theta_a \in \Lambda^k\mathcal{C}_a^*$ we may associate an operator

$$j : \mathcal{C}_a \longrightarrow \mathcal{C}_a$$

by requiring that $\theta_a(X, Y) = \Omega_a(jX, Y)$ holds for all vectors X, Y in $\mathcal{C}_a$ ([LRC]). From the bilinearity of θ_a and Ω_a it follows that j is a linear operator.

Definition 2.34 *Let (M,ω) be a 5-dimensional strict contact manifold. Let Π be the contact distribution induced by ω, and let $\theta \in C^{\infty}(\Lambda^2\Pi^*)$ be effective. Then we say that (ω,θ) is a* ***generalized Monge-Ampere equation.*** *(ω,θ) is elliptic, hyperbolic or parabolic according to the effective form θ.*

Proposition 2.35 *For any effective 2-form $\theta_a \in \Lambda^k \mathcal{C}_a^*$ the equality*

$$j^2 = -\operatorname{Pf}(\theta_a)$$

holds.

Proof. The statement and proof can be found in [Ly2]. ■

In the hyperbolic case we see that $j^2 = |\operatorname{Pf}(\theta_a)| \neq 0$, so j is invertible, $j^{-1} = \frac{1}{|\operatorname{Pf}(\theta_a)|} j$, and hence it is a linear transformation of $\mathcal{C}_a$.

Example 2.36 (General j) *Let $M = \mathbb{R}^2(q_1,q_2)$ and $J^1M \simeq \mathbb{R}^5(q_1,q_2,u,p_1,p_2)$. In the case*

$$\theta = F_1 dq_1 \wedge dq_2 + F_2 dq_2 \wedge dp_1 + F_3 dq_1 \wedge dp_2 + F_4 dq_1 \wedge dp_1 - F_4 dq_2 \wedge dp_2 + F_5 dp_1 \wedge dp_2,$$
$$\Omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2,$$
$$\mathcal{C} = \langle \partial_{q_1} + p_1\partial_u, \partial_{q_2} + p_2\partial_u, \partial_{p_1}, \partial_{p_2} \rangle = \langle \mathcal{D}_{q_1}, \mathcal{D}_{q_2}, \partial_{p_1}, \partial_{p_2} \rangle,$$

then j has the form

$$j = \begin{bmatrix} F_4 & F_2 & 0 & -F_5 \\ F_3 & -F_4 & F_5 & 0 \\ 0 & F_1 & F_4 & F_3 \\ -F_1 & 0 & F_2 & -F_4 \end{bmatrix}$$

in the basis $\{\mathcal{D}_{q_1}, \mathcal{D}_{q_2}, \partial_{p_1}, \partial_{p_2}\}$ into $\mathcal{C}$.

Notation 2.37 *In later examples we will consequently write $\mathcal{D}_{q_1}$ and $\mathcal{D}_{q_2}$ instead of $\partial_{q_1} + p_1\partial_u$ and $\partial_{q_2} + p_2\partial_u$.*

Chapter 3

Hyperbolic M-A equations and 5-dimensional strict contact manifolds

In this chapter we discover two canonical rank 1 distributions, L^1 and χ, in τ_M. These subbundles imply two canonical decompositions of τ_M. In section 3 we introduce and investigate the Nijenhuis tensor N_j.

3.1 The extending sequence of derived submodules

From any distribution $\Sigma \subset \tau_M$ we can construct the so called extending sequence of derived submodules. We know that $\mathcal{D}_1(M)$ is a $C^\infty(M)$-module. Consider a neighborhood (germ) $\mathcal{O}_x$ of x in M. We say that $\xi_1, \ldots, \xi_t \in \mathcal{D}_1(M)$ are local generators of $C^\infty(\Sigma)$ (in $\mathcal{O}_x$) if $\xi_1(y), \ldots, \xi_t(y)$ is a generating set of vectors in Σ_y for any $y \in \mathcal{O}_x$ and we write

$$C^\infty(\Sigma) = \langle \xi_1, \ldots, \xi_t \rangle .$$

Define

$$\mathcal{D}_1(\Sigma) = \{\eta \in \mathcal{D}_1(M) |\, \eta = f_1\xi_1 + \cdots + f_t\xi_t, \forall y \in \mathcal{O}_x\},$$

where $f_1, \ldots, f_t \in C^\infty(M)$. Then $\mathcal{D}_1(\Sigma)$ is a submodule of $\mathcal{D}_1(M)$. Now we define the $C^\infty(M)$-module $\mathcal{D}_{k+1}(\Sigma)$ to be generated by $\mathcal{D}_k(\Sigma)$ and all possible commutators $[\chi, \xi]$ where $(\chi, \xi) \in \mathcal{D}_k(\Sigma) \times \mathcal{D}_1(\Sigma)$.

$$\mathcal{D}_{k+1}(\Sigma) = [\mathcal{D}_k(\Sigma), \mathcal{D}_1(\Sigma)].$$

Then we get **the extending sequence of derived submodules**:

$$\mathcal{D}_1(\Sigma) \subset \mathcal{D}_2(\Sigma) \subset \cdots \subset \mathcal{D}_1(M).$$

In general $\mathcal{D}_k(\Sigma)$ is not a module of sections for some distribution, since the rank of its localization can vary from point to point. The module $\mathcal{D}_k(\Sigma)$ is **projective** if it is a direct summand in a free module. Then, in our situation, it is a module of sections for some distribution Σ_k. We call Σ_k the $(k-1)$-extension of Σ and we denote it $\partial^{(k-1)}\Sigma$.

How do we localize the module $\mathcal{D}_k(\Sigma)$? We define

$$\Sigma_{k,x} = \frac{\mathcal{D}_k(\Sigma)}{\mu_x \mathcal{D}_k(\Sigma)},$$

where μ_x is the ideal $\{f \in C^\infty(M) | f(x) = 0\}$ as before. It is well known that

$$T_x M \simeq \frac{\mathcal{D}_1(M)}{\mu_x \mathcal{D}_1(M)},$$

so we see that $\Sigma_{k,x}$ is a subspace of $T_x M$. Thus we get a sequence of vectorbundles:

$$\Sigma_1 \subseteq \Sigma_2 \subseteq \cdots \subseteq \tau_M.$$

From these vectorbundles we can define the associated vectorbundles

$$\Sigma^{[k+1]} = \frac{\Sigma_{k+1}}{\Sigma_k}$$

and get the graded bundle

$$\Sigma_1 \oplus \Sigma^{[2]} \oplus \cdots \oplus \Sigma^{[k]}$$

associated to the flag of $C^\infty(M)$-modules. This graded bundle is locally isomorphic to the tangent bundle τ_M when $\mathcal{D}_k(\Sigma) = \mathcal{D}_1(M)$ for some k.

Example 3.1 *Consider* $M = \mathbb{R}^3(x, u, p)$. *Let* $\Pi = \ker(du - pdx) = \langle \partial_x + p\partial_u, \partial_p \rangle$. *Since* $[\partial_p, \partial_x + p\partial_u] = \partial_u \notin \Pi$, *we have* $\mathcal{D}_2 = \mathcal{D}_1(M)$. *Thus we can identify* $\Pi^{[2]} = \frac{\Pi_2}{\Pi} = \langle \partial_u \rangle$ *and* $TM \simeq \Pi \oplus \Pi^{[2]} = \langle \partial_x + p\partial_u, \partial_p \rangle \oplus \langle \partial_u \rangle$.

3.2 Two canonical decompositions of τ_M

Let (M, ω) be a 5-dimensional strict contact manifold. We know that ω defines a rank 4 contact distribution Π and a symplectic structure $\Omega = d\omega|_\Pi$ upon Π. Let $\langle \theta \rangle$ be a conformal, hyperbolic, effective 2-form in $\Lambda^2\Pi^*$. We get a unique representative $\theta \in \langle \theta \rangle$ by requiring $\operatorname{Pf}(\theta) = -1$. From θ and Ω we define the automorphism

$$j : \Pi \longrightarrow \Pi$$

by requiring $\theta_x(X, Y) = \Omega_x(jX, Y)\ \forall X, Y \in \Pi_x\ \forall x \in M$. From proposition 2.35 it follows that

$$j^2 = -\operatorname{Pf}(\theta) = \mathbf{1},$$

so j is an almost product structure on Π. From the nondegenracy and effectivity of θ one can easily show that j must have the spectrum

$$\operatorname{spec}(j) = \{1, 1, -1, -1\},$$

and it follows from the nondegeneracy of θ and Ω that

$$\dim E_j(1) = \dim E_j(-1) = 2,$$

where E_j denotes the eigenspaces of j. Hence the vector space Π_x splits into $E_j(1) \oplus E_j(-1)$ at the point x. This leads to the lemma.

Lemma 3.2 *The automorphism $j : \Pi \longrightarrow \Pi$ splits the rank 4 distribution Π into the direct sum of two rank 2 distributions:*

$$\Pi = \Pi^2_+ \oplus \Pi^2_-,$$

where

$$\Pi^2_+ = \{X \in \Pi \mid jX = X\} = \ker(j-1)$$

and

$$\Pi^2_- = \{X \in \Pi \mid jX = -X\} = \ker(j+1).$$

So any vector $X \in \Pi_x$ can be written as $X_+ + X_- \in \Pi^2_+ \oplus \Pi^2_-$. It's easy to see that $\left(\frac{1}{2} + \frac{1}{2}j\right)$ and $\left(\frac{1}{2} - \frac{1}{2}j\right)$ are projections in Π onto Π^2_+ and Π^2_- respectively. Let us denote

$$P_+ = \frac{1}{2}(1+j) \qquad \text{and} \qquad P_- = \frac{1}{2}(1-j).$$

Lemma 3.3 *The distributions Π^2_+ and Π^2_- are skew orthogonal, $\Pi^2_+ \perp_\Omega \Pi^2_-$, that is*

$$\Omega(\xi, \eta) = 0 \ \forall \ \xi \in \Pi^2_+, \eta \in \Pi^2_-.$$

Proof. Let $\xi \in C^\infty\left(\Pi^2_+\right)$ and $\eta \in C^\infty\left(\Pi^2_-\right)$. So $j\xi = \xi$ while $j\eta = -\eta$. We observe that

$$\theta(\xi,\eta) = \Omega(j\xi,\eta) = \Omega(\xi,\eta) = -\Omega(\eta,\xi) = \Omega(j\eta,\xi) = \theta(\eta,\xi) = -\theta(\xi,\eta).$$

We conclude that $\Omega(\xi,\eta) = 0 \ \forall \ \xi \in \Pi^2_+, \ \eta \in \Pi^2_-$. ■

Recall the notion of integrable distributions.

Definition 3.4 *A distribution Π on the n-dimensional manifold M is said to be **integrable** if there exist coordinates $q_1, \ldots, q_n$ on M s.t. $\Pi = \langle \partial_{q_1}, \ldots, \partial_{q_r} \rangle$, $r = \operatorname{rank}(\Pi) \leq n$.*

Theorem 3.5 (Frobenius) *A distribution $\Pi \subset \tau_M$, generated by $\xi_1, \ldots, \xi_k \in \mathcal{D}_1(M)$ is integrable if*

$$\left[\xi_i, \xi_j\right] \in C^\infty(\Pi) \quad \forall i, j.$$

Proof. The theorem and proof is presented in [S]. ■

Lemma 3.6 *The distributions Π, Π^2_+ and Π^2_- are nonintegrable.*

Proof. This proof follows from the Cartan formula for 2-forms:

$$d\omega(\xi, \eta) = \xi(\omega(\eta)) - \eta(\omega(\xi)) - \omega([\xi, \eta]). \tag{3.1}$$

Here $\omega \in \Omega^1(M)$ is our contact form and $\xi, \eta \in \mathcal{D}_1(M)$. Let $\xi, \eta \in \mathcal{D}_1(M)$ be tangent to Π. Then the Cartan formula restricted to Π becomes

$$\Omega(\xi, \eta) = -\omega([\xi, \eta]). \tag{3.2}$$

Assume that $\xi \in \mathcal{D}_1(M)$ is tangent to Π^2_+. Ω is nondegenerate on Π, so there exist some $\eta \in \mathcal{D}_1(M)$ that is tangent to Π such that $\Omega(\xi, \eta) \neq 0$. Since $\Omega\left(\Pi^2_+, \Pi^2_-\right) = 0$, η cannot be tangent to Π^2_-. Thus if we decompose η according to the direct sum $C^\infty(\Pi) \simeq C^\infty\left(\Pi^2_+\right) \oplus_{C^\infty(M)} C^\infty\left(\Pi^2_-\right)$, that is

$$\eta = \eta_+ + \eta_- \in C^\infty\left(\Pi^2_+\right) \oplus_{C^\infty(M)} C^\infty\left(\Pi^2_-\right),$$

then η_+ must be nonzero. Hence

$$\Omega(\xi, \eta) = \Omega(\xi, \eta_+) = -\omega(\left[\xi, \eta_+\right]) \neq 0.$$

This implies that $\left[\xi, \eta_+\right]$ cannot be tangent to Π. We conclude that Π and Π^2_+ are nonintegrable. Similarly we can prove that Π^2_- is nonintegrable. ■

Corollary 3.7 *$\Omega|_{\Pi^2_+}$ and $\Omega|_{\Pi^2_-}$ are nondegenerate, hence $\Pi^2_\pm \subset \Pi$ are symplectic subbundles.*

Corollary 3.8 *$\left[\xi_\pm, \eta_\mp\right] \in C^\infty(\Pi)$ for all sections $\xi_\pm, \eta_\pm \in C^\infty\left(\Pi^2_\pm\right)$.*

Corollary 3.9 *There exists sections $\xi_\pm, \eta_\pm \in C^\infty\left(\Pi^2_\pm\right)$ such that $\left[\xi_\pm, \eta_\pm\right] \notin C^\infty(\Pi)$.*

The modules

$$\left[C^\infty\left(\Pi^2_+\right), C^\infty\left(\Pi^2_+\right)\right] \quad \text{and} \quad \left[C^\infty\left(\Pi^2_-\right), C^\infty\left(\Pi^2_-\right)\right]$$

are projective since $\left(\Pi^2_+, \Omega|_{\Pi^2_+}\right)$ and $\left(\Pi^2_-, \Omega|_{\Pi^2_-}\right)$ are 2-dimensional and "symplectic". Hence there exist distributions $\partial\Pi^2_+, \partial\Pi^2_- \subset \tau_M$ such that

$$C^\infty\left(\partial\Pi^2_\pm\right) = \left[C^\infty\left(\Pi^2_\pm\right), C^\infty\left(\Pi^2_\pm\right)\right].$$

The extensions $\partial\Pi^2_\pm$ must be of rank 3. Let us denote them

$$\Pi^3_\pm = \partial\Pi^2_\pm. \tag{3.3}$$

From corollary 3.9 it follows that $\Pi^3_\pm \not\subseteq \Pi$.

Proposition 3.10 *The distributions $\Pi^2_\pm$ and $\Pi^3_\mp$ have zero intersection:*

$$\Pi^2_\pm \cap \Pi^3_\mp = 0. \tag{3.4}$$

Proof. Let ξ_1, ξ_2 and η_1, η_2 in $\mathcal{D}_1(M)$ be generators of $C^\infty\left(\Pi^2_+\right)$ and $C^\infty\left(\Pi^2_-\right)$ in the neighborhood $\mathcal{O} \subset M$ containing x. Due to nonintegrability of Π^2_+ we know that $[\xi_1, \xi_2] \notin C^\infty(\Pi)$ in $\mathcal{O}$ for ξ_1, ξ_2 to be well defined as generators. Then $\xi_1, \xi_2, \xi_3 = [\xi_1, \xi_2]$ are local generators of $C^\infty\left(\partial\Pi^2_+\right)$. From this we get the following bases into the fibers $\Pi^2_{+,x}, \partial\Pi^2_{+,x}$ and $\Pi^2_{-,x}$:

$$\begin{aligned}
\Pi^2_{+,x} &= \operatorname{span}\{\xi_1(x), \xi_2(x)\} = \operatorname{span}\{X_1, X_2\}, \\
\partial\Pi^2_{+,x} &= \operatorname{span}\{\xi_1(x), \xi_2(x), \xi_3(x)\} = \operatorname{span}\{X_1, X_2, X_3\}, \quad \text{where } X_3 \notin \Pi_x, \\
\Pi^2_{-,x} &= \operatorname{span}\{\eta_1(x), \eta_2(x)\} = \operatorname{span}\{Y_1, Y_2\}.
\end{aligned}$$

Assume that $\partial\Pi^2_{+,x}$ and $\Pi^2_{-,x}$ have nonzero intersection, that is: $aX_1 + bX_2 + cX_3 \in \Pi^2_{-,x}$ for some $a, b, c \in \mathbb{R}$. For $aX_1 + bX_2 + cX_3$ to lie in Π_x we must require that $c = 0$ since Π_x is closed under addition. Hence $aX_1 + bX_2 + cX_3 \in \Pi^2_{-,x} \subset \Pi_x$ implies that $aX_1 + bX_2 = dY_1 + eY_2$. This contradicts the fact that $\Pi^2_{+,x} \cap \Pi^2_{-,x} = 0$, thus our assumption must be wrong. We conclude that $\partial\Pi^2_{+,x} \cap \Pi^2_{-,x} = 0$ for all x in $\mathcal{O}$. Since $\mathcal{O}$ is not specified we conclude that $\Pi^3_+ \cap \Pi^2_- = 0$. Similarly we can prove that $\Pi^3_- \cap \Pi^2_+ = 0$. ■

Corollary 3.11 $\Pi^3_+(x) \cap \Pi^3_-(x) \not\subseteq T_xM$ *and* $\dim(\Pi^3_+(x) \cap \Pi^3_-(x)) = 1$.

Proof. Obvious since T_xM is 5-dimensional and $\Pi^2_+ \cap \Pi^2_- = 0$. ■

Definition 3.12 *We define the rank 1 vectorbundle* $L^1 = \bigcup_{x \in M} \Pi^3_+(x) \cap \Pi^3_-(x)$.

So we have a canonical decomposition of τ_M:

$$\tau_M = \Pi \oplus L^1 = \Pi^2_+ \oplus \Pi^2_- \oplus L^1. \tag{3.5}$$

Example 3.13 (Linear wave 2) *Consider $M = \mathbb{R}^5(q_1, q_2, u, p_1, p_2)$ with the contact form $\omega = du - p_1 dq_1 - p_2 dq_2$. Then the contact distribution and symplectic form become*

$$\begin{aligned}
\Pi &= \langle \mathcal{D}_{q_1}, \mathcal{D}_{q_2}, \partial_{p_1}, \partial_{p_2} \rangle, \\
\Omega &= d\omega|_\Pi = dq_1 \wedge dp_1 + dq_2 \wedge dp_2.
\end{aligned}$$

We know that the linear wave equation

$$\frac{\partial^2 h}{\partial q_2^2} - \frac{\partial^2 h}{\partial q_1^2} = 0$$

corresponds to the hyperbolic effective form $\theta = dq_1 \wedge dp_2 + dq_2 \wedge dp_1$. *For the given basis into* Π *the operator* $j : \Pi \longrightarrow \Pi$ *has the matrix*

$$j = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We remember that $P_+ = \left(\frac{1}{2}j + \frac{1}{2}\right)$ *and* $P_- = \left(-\frac{1}{2}j + \frac{1}{2}\right)$ *are projections onto* Π^2_+ *and* Π^2_- *correspondingly. These projections lead to the following bases in* Π^2_+ *and* Π^2_-*:*

$$\begin{aligned} \Pi^2_+ &= \ker P_- = \langle \mathcal{D}_{q_1} + \mathcal{D}_{q_2}, \partial_{p_1} + \partial_{p_2} \rangle, \\ \Pi^2_- &= \ker P_+ = \langle \mathcal{D}_{q_1} - \mathcal{D}_{q_2}, \partial_{p_1} - \partial_{p_2} \rangle. \end{aligned}$$

From this we calculate the extensions:

$$\begin{aligned} \partial\Pi^2_- &= \Pi^3_- = \langle \mathcal{D}_{q_1} - \mathcal{D}_{q_2}, \partial_{p_1} - \partial_{p_2}, \partial_u \rangle, \\ \partial\Pi^2_+ &= \Pi^3_+ = \langle \mathcal{D}_{q_1} + \mathcal{D}_{q_2}, \partial_{p_1} + \partial_{p_2}, \partial_u \rangle. \end{aligned}$$

We see that

$$L^1 = \Pi^3_+ \cap \Pi^3_- = \langle \partial_u \rangle.$$

Example 3.14 (Nonlinear 1) *Consider the same contact manifold as described in the preceding example. Let* $S, T \in C^\infty(M)$. *The M-A equation*

$$-S\frac{\partial^2 u}{\partial q_1^2} + 2\frac{\partial^2 u}{\partial q_1 \partial q_2} - T\left(\frac{\partial^2 u}{\partial q_1^2}\frac{\partial^2 u}{\partial q_2^2} - \left(\frac{\partial^2 u}{\partial q_1 \partial q_2}\right)^2\right) = 0$$

corresponds to the hyperbolic effective form

$$\theta = dq_1 \wedge dp_1 - dq_2 \wedge dp_2 + S dq_2 \wedge dp_1 - T dp_1 \wedge dp_2.$$

The operator j *has the matrix*

$$j = \begin{bmatrix} 1 & S & 0 & T \\ 0 & -1 & -T & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & S & -1 \end{bmatrix}.$$

By the projections P_+ and P_- we find the distributions

$$\begin{aligned}\Pi_+^2 &= \langle \mathcal{D}_{q_1}, T\mathcal{D}_{q_2} - 2\partial_{p_1} - S\partial_{p_2}\rangle ,\\ \Pi_-^2 &= \langle S\mathcal{D}_{q_1} - 2\mathcal{D}_{q_2}, T\mathcal{D}_{q_2} - S\partial_{p_2}\rangle .\end{aligned}$$

The extensions become:

$$\Pi_+^3 = \langle \mathcal{D}_{q_1}, T\mathcal{D}_{q_2} - 2\partial_{p_1} - S\partial_{p_2}, \mathcal{D}_{q_1}(T)\mathcal{D}_{q_2} + 2\partial_u - \mathcal{D}_{q_1}(S)\partial_{p_2}\rangle$$

and

$$\begin{aligned}\Pi_-^3 &= \langle S\mathcal{D}_{q_1} - 2\mathcal{D}_{q_2}, T\mathcal{D}_{q_2} - S\partial_{p_2}, \left(S\frac{\partial S}{\partial p_2} - T\mathcal{D}_{q_2}(S)\right)\mathcal{D}_{q_1} + \left(S\mathcal{D}_{q_1}(T) - 2\mathcal{D}_{q_2}(T)\right)\mathcal{D}_{q_2}\\ &\quad -2S\partial_u + \left(2\mathcal{D}_{q_2}(S) - S\mathcal{D}_{q_1}(S)\right)\partial_{p_2}\rangle .\end{aligned}$$

One can calculate the intersection to be

$$L^1 = \left\langle \left(S\mathcal{D}_{q_1}(T) - T\mathcal{D}_{q_1}(S) - \mathcal{D}_{q_2}(T) + \frac{\partial S}{\partial p_2}\right)\mathcal{D}_{q_1} - \mathcal{D}_{q_1}(T)\mathcal{D}_{q_2} - 2\partial_u + \mathcal{D}_{q_1}(S)\partial_{p_2}\right\rangle .$$

It is well known that the contact form $\omega \in \Omega^1(M)$ defines a unique vector field in $\mathcal{D}_1(M)$ by the intersection

$$\ker(d\omega) \cap \{\xi \in \mathcal{D}_1(M) | \, i_\xi(\omega) = 1\} .$$

This vector field is said to be the **characteristic vector field** or the **Reeb vector field** of the contact structure. Let us denote it by X_1. Hence

$$i_{X_1} d\omega = 0 \qquad \text{and} \qquad \omega(X_1) = 1.$$

Obviously this vector field is not tangent to the contact distribution. X_1 generates a subbundle χ of τ_M:

$$\chi = \langle X_1\rangle .$$

This gives us the second canonical decomposition of τ_M:

$$\tau_M = \Pi_+^2 \oplus \Pi_-^2 \oplus \chi.$$

Example 3.15 *In the case $M = \mathbb{R}^5(q_1, q_2, u, p_1, p_2)$ with the contact form $\omega = du - p_1 dq_1 - p_2 dq_2$, one can check that the characteristic vector field X_1 is $\partial_u \in \mathcal{D}_1(M)$.*

3.3 The Nijenhuis tensor

The decompositions $\tau_M = \Pi \oplus \chi$ and $\tau_M = \Pi \oplus L^1$ induce the vectorbundle morphisms

$$\begin{array}{llll} P_{L^1}: & \Pi \oplus L^1 & \longrightarrow & \Pi \\ & X = X_\Pi + X_{L^1} & \longmapsto & X_\Pi \end{array} \qquad \begin{array}{llll} P_\chi: & \Pi \oplus \chi & \longrightarrow & \Pi \\ & Y = Y_\Pi + Y_\chi & \longmapsto & Y_\Pi \end{array}$$

$$\begin{array}{llll} P^{L^1}: & \Pi \oplus L^1 & \longrightarrow & L^1 \\ & X = X_\Pi + X_{L^1} & \longmapsto & X_{L^1} \end{array} \qquad \begin{array}{llll} P^\chi: & \Pi \oplus \chi & \longrightarrow & \chi \\ & Y = Y_\Pi + Y_\chi & \longmapsto & Y_\chi \end{array}$$

So we get the morphisms

$$P_{L^1} : \mathcal{D}_1(M) \longrightarrow C^\infty(\Pi) \quad \text{and} \quad P_\chi : \mathcal{D}_1(M) \longrightarrow C^\infty(\Pi),$$

$$P^{L^1} : \mathcal{D}_1(M) \longrightarrow C^\infty\left(L^1\right) \quad \text{and} \quad P^\chi : \mathcal{D}_1(M) \longrightarrow C^\infty(\chi)$$

of $C^\infty(M)$-modules.

Remark 3.16 *We can represent the projections P_χ and P^χ as the tensors $P_\chi = 1 - X_1 \otimes \omega$ and $P^\chi = X_1 \otimes \omega$.*

$$\begin{array}{rcl} P_\chi(\xi) & = & \xi - \omega(\xi) X_1, \\ P^\chi(\xi) & = & \omega(\xi) X_1. \end{array}$$

In the following we need to extend $j \in \Pi^* \otimes \Pi$. We have several possibilities. Two of them are more natural then the others: We can extend by zero along the characteristic vector field X_1 or along L^1. We choose to extend along the characteristic vector field.

Definition 3.17 *Define $\widetilde{j} \in T^*M \otimes TM$ in the following way:*

$$\widetilde{j} = \begin{cases} j & on\ \Pi \\ 0 & on\ \chi \end{cases} . \tag{3.6}$$

So $\widetilde{j}(X) = \widetilde{j}(X_\Pi + X_\chi) = j(X_\Pi)$. We notice that $P_\chi \circ \widetilde{j}(X) = j(X_\Pi) = j \circ P_\chi(X)$, hence

$$P_\chi \circ \widetilde{j} = j \circ P_\chi. \tag{3.7}$$

Definition 3.18 *Let $\xi, \eta \in C^\infty(\Pi)$. We define the **Nijenhuis tensor** to be:*

$$N_j(\xi,\eta) = P_\chi\left([j\xi, j\eta] - \widetilde{j}[j\xi,\eta] - \widetilde{j}[\xi, j\eta] + [\xi,\eta]\right). \tag{3.8}$$

Remark 3.19 *It is not clear why we should extend j along χ and project with P_χ in the definition of N_j. It seems like we could equally well extend j along L^1 and then define N_j by the projection P_{L^1} instead of P_χ. This is not an option however, since the Nijenhuis tensor will turn out to be identically zero.*

Lemma 3.20 *N_j is a tensor:*

$$N_j \in \Lambda^2\Pi^* \otimes \Pi. \tag{3.9}$$

Proof. The proof is presented in appendix A. ■

Proposition 3.21 *The Nijenhuis tensor has the following properties:*

i) $N_j(X,Y) = -N_j(Y,X)$, *hence* $N_j(X,X) = 0$.

ii) $N_j(kX,Y) = kN_j(X,Y)$; $k \in \mathbb{R}$.

iii) $N_j(X_1 + X_2, Y) = N_j(X_1,Y) + N_j(X_2,Y)$.

iv) $N_j(jX,Y) = N_j(X,jY) = -jN_j(X,Y)$.

Here X and Y are vectors in Π, but these properties also hold for sections in Π.

Proof. The first 3 properties - skew symmetry and bilinearity - follows directly from the definition. We prove only the last property.

Let $X, Y \in \Pi$. First we calculate $N_j(jX,Y)$:

$$\begin{aligned} N_j(jX,Y) &= P_\chi\left(\left[j^2X, jY\right] - \widetilde{j}\left[j^2X,Y\right] - \widetilde{j}\left[jX,jY\right] + \left[jX,Y\right]\right) \\ &= P_\chi\left(\left[X,jY\right] - \widetilde{j}\left[X,Y\right] - \widetilde{j}\left[jX,jY\right] + \left[jX,Y\right]\right) \\ &= P_\chi\left(\left[X,jY\right]_\Pi - j\left[X,Y\right]_\Pi - j\left[jX,jY\right]_\Pi + \left[jX,Y\right]_\Pi + \left[X,jY\right]_\chi + \left[jX,Y\right]_\chi\right) \\ &= P_\chi\left(\left[X,jY\right]_\Pi - j\left[X,Y\right]_\Pi - j\left[jX,jY\right]_\Pi + \left[jX,Y\right]_\Pi\right). \end{aligned}$$

Second we calculate $N_j(X,jY)$:

$$\begin{aligned} N_j(X,jY) &= P_\chi\left(\left[jX, j^2Y\right] - \widetilde{j}\left[jX,jY\right] - \widetilde{j}\left[X,j^2Y\right] + \left[X,jY\right]\right) \\ &= P_\chi\left(\left[jX,Y\right] - \widetilde{j}\left[jX,jY\right] - \widetilde{j}\left[X,Y\right] + \left[X,jY\right]\right) \\ &= N_j(jX,Y). \end{aligned}$$

Finally we calculate $-jN_j(X,Y)$:

$$\begin{aligned}
-jN_j(X,Y) &= -j\circ P_\chi\left([jX,jY]-\widetilde{j}\,[jX,Y]-\widetilde{j}\,[X,jY]+[X,Y]\right)\\
&= -P_\chi\circ\widetilde{j}\left([jX,jY]-\widetilde{j}\,[jX,Y]-\widetilde{j}\,[X,jY]+[X,Y]\right)\\
&= -P_\chi\left(\widetilde{j}\,[jX,jY]-\widetilde{j}^2\,[jX,Y]-\widetilde{j}^2\,[X,jY]+\widetilde{j}\,[X,Y]\right)\\
&= -P_\chi\left(j\,[jX,jY]_\Pi-j^2\,[jX,Y]_\Pi-j^2\,[X,jY]_\Pi+j\,[X,Y]_\Pi\right)\\
&= -P_\chi\left(j\,[jX,jY]_\Pi-[jX,Y]_\Pi-[X,jY]_\Pi+j\,[X,Y]_\Pi\right)\\
&= P_\chi\left(-j\,[jX,jY]_\Pi+[jX,Y]_\Pi+[X,jY]_\Pi-j\,[X,Y]_\Pi\right)\\
&= N_j(jX,Y).
\end{aligned}$$

■

These properties coincide with the properties of the Nijenhuis tensor for the 4-dimensional symplectic case described in [Kr2], hence the name is justified. The proposition leads to the following corollaries:

Corollary 3.22 $N_j\left(\Pi^2_\pm,\Pi^2_\mp\right)=0.$

Proof. Let $X\in\Pi^2_+$ and $Y\in\Pi^2_-$. Then $jX=X$ and $jY=-Y$. From proposition 3.21 we get:

$$N_j(X,Y)=N_j(jX,Y)=N_j(X,jY)=N_j(X,-Y)=-N_j(X,Y).$$

So we see that $N_j\left(\Pi^2_\pm,\Pi^2_\mp\right)=0$. ■

Corollary 3.23 $N_j\left(\Pi^2_+,\Pi^2_+\right)\subseteq\Pi^2_-.$

Proof. Let $X\in\Pi^2_+$ and $Y\in\Pi^2_+$. Then $jX=X$ and $jY=Y$. From proposition 3.21 we get:

$$jN_j(X,Y)=-N_j(jX,Y)=-N_j(X,Y).$$

This implies that $N_j(X,Y)\in\Pi^2_-$. ■

Corollary 3.24 $N_j\left(\Pi^2_-,\Pi^2_-\right)\subseteq\Pi^2_+.$

Proof. Let $X\in\Pi^2_-$ and $Y\in\Pi^2_-$. Then $jX=-X$ and $jY=-Y$. From proposition 3.21 we get:

$$jN_j(X,Y)=-N_j(jX,Y)=-N_j(-X,Y)=N_j(X,Y).$$

This implies that $N_j(X,Y)\in\Pi^2_+$. ■

One can easily check that if ξ_+, η_+ and ξ_-, η_- generate Π^2_+ and Π^2_- correspondingly, then the following formulas hold:

$$\begin{aligned} N_j\left(\xi_+, \eta_+\right) &= 4P_- \circ P_\chi\left[\xi_+, \eta_+\right], \\ N_j\left(\xi_-, \eta_-\right) &= 4P_+ \circ P_\chi\left[\xi_-, \eta_-\right]. \end{aligned}$$

So we can write the Nijenhuis tensor like

$$N_j(\xi, \eta) = \begin{cases} 4P_\mp \circ P_\chi[\xi, \eta] & \text{when } \xi, \eta \in C^\infty\left(\Pi^2_\pm\right), \\ 0 & \text{when } \xi, \eta \in C^\infty\left(\Pi^2_\pm\right) \times C^\infty\left(\Pi^2_\mp\right). \end{cases} \tag{3.10}$$

Lemma 3.25 *The Nijenhuis tensor is identical to zero if and only if the vectorbundles χ and L^1 coincide.*

Proof. Let ξ_+, η_+ and ξ_-, η_- be local generating sections in Π^2_+ and Π^2_- correspondingly and let ℓ be a generating section in L^1. Then the characteristic vector field X_1 can be decomposed into the given generators as

$$X_1 = a_1\xi_+ + a_2\eta_+ + a_3\xi_- + a_4\eta_- + a_5\ell,$$

where $a_i \in C^\infty(M)$. Assume that $\left[\xi_+, \eta_+\right] \in C^\infty\left(\Pi^2_+\right) \oplus C^\infty\left(L^1\right)$ decomposes into $\left[\xi_+, \eta_+\right] = A\left[\xi_+, \eta_+\right]_+ + B\ell$ where $A, B \in C^\infty(M)$. Then we get

$$\begin{aligned} N_j\left(\xi_+, \eta_+\right) &= 4P_- \circ P_\chi\left(A\left[\xi_+, \eta_+\right]_+ + B\ell\right) = 4BP_- \circ P_\chi(\ell) \\ &= 4BP_- \circ P_\chi\left(\frac{1}{a_5}X_1 - \frac{a_1}{a_5}\xi_+ - \frac{a_2}{a_5}\eta_+ - \frac{a_3}{a_5}\xi_- - \frac{a_4}{a_5}\eta_-\right) \\ &= 4BP_-\left(\frac{a_1}{a_5}\xi_+ - \frac{a_2}{a_5}\eta_+ - \frac{a_3}{a_5}\xi_- - \frac{a_4}{a_5}\eta_-\right) = -4B\left(\frac{a_3}{a_5}\xi_- + \frac{a_4}{a_5}\eta_-\right). \end{aligned}$$

Assume that $\left[\xi_-, \eta_-\right] \in C^\infty\left(\Pi^2_-\right) \oplus C^\infty\left(L^1\right)$ decomposes into $\left[\xi_-, \eta_-\right] = \alpha\left[\xi_-, \eta_-\right]_- + \beta\ell$, where $\alpha, \beta \in C^\infty(M)$.

$$\begin{aligned} N_j\left(\xi_-, \eta_-\right) &= 4P_+ \circ P_\chi\left(\alpha\left[\xi_-, \eta_-\right]_- + \beta\ell\right) = 4\beta P_+ \circ P_\chi(\ell) \\ &= 4\beta P_+ \circ P_\chi\left(\frac{1}{a_5}X_1 - \frac{a_1}{a_5}\xi_+ - \frac{a_2}{a_5}\eta_+ - \frac{a_3}{a_5}\xi_- - \frac{a_4}{a_5}\eta_-\right) \\ &= 4\beta P_+\left(-\frac{a_1}{a_5}\xi_+ - \frac{a_2}{a_5}\eta_+ - \frac{a_3}{a_5}\xi_- - \frac{a_4}{a_5}\eta_-\right) = -4\beta\left(\frac{a_1}{a_5}\xi_+ + \frac{a_2}{a_5}\eta_+\right). \end{aligned}$$

We know that both B and β must be nonzero since Π^2_+ and Π^2_- are nonintegrable. So we conclude that $N_j = 0 \Longleftrightarrow a_1, a_2, a_3, a_4 = 0$. ■

Example 3.26 (Linear wave 3) *We continue our study of $M = \mathbb{R}^5(q_1, q_2, u, p_1, p_2)$ with the contact form $\omega = du - p_1dq_1 - p_2dq_2$, the contact distribution $\Pi = \langle \mathcal{D}_{q_1}, \mathcal{D}_{q_2}, \partial_{p_1}, \partial_{p_2}\rangle$*

and the hyperbolic effective form $\theta = dq_1 \wedge dp_2 + dq_2 \wedge dp_1$. *We know that* $X_1 = \partial_u$ *and* $L^1 = \Pi^3_+ \cap \Pi^3_- = \operatorname{span}\{\partial_u\}$ *in this case, so we conclude by the preceding lemma that* $N_j \equiv 0$.

Example 3.27 (Nonlinear 2) *Consider the same contact manifold as described in the preceding example. We know that the contact form* $\omega = du - p_1 dq_1 - p_2 dq_2$ *and the hyperbolic effective form*

$$\theta = dq_1 \wedge dp_1 - dq_2 \wedge dp_2 + S dq_2 \wedge dp_1 - T dp_1 \wedge dp_2$$

correspond to

$$\begin{aligned}
\Pi^2_+ &= \langle \mathcal{D}_{q_1}, T\mathcal{D}_{q_2} - 2\partial_{p_1} - S\partial_{p_2} \rangle, \qquad \Pi^2_- = \langle S\mathcal{D}_{q_1} - 2\mathcal{D}_{q_2}, T\mathcal{D}_{q_2} - S\partial_{p_2} \rangle, \\
j &= \begin{bmatrix} 1 & S & 0 & T \\ 0 & -1 & -T & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & S & -1 \end{bmatrix}, \\
L^1 &= \left\langle \left(S\mathcal{D}_{q_1}(T) - T\mathcal{D}_{q_1}(S) - \mathcal{D}_{q_2}(T) + \frac{\partial S}{\partial p_2} \right) \mathcal{D}_{q_1} - \mathcal{D}_{q_1}(T)\mathcal{D}_{q_2} - 2\partial_u + \mathcal{D}_{q_1}(S)\partial_{p_2} \right\rangle.
\end{aligned}$$

Let us denote

$$\begin{aligned}
\xi_+ &= \mathcal{D}_{q_1}, & \eta_+ &= T\mathcal{D}_{q_2} - 2\partial_{p_1} - S\partial_{p_2}, \\
\xi_- &= S\mathcal{D}_{q_1} - 2\mathcal{D}_{q_2}, & \eta_- &= T\mathcal{D}_{q_2} - S\partial_{p_2}.
\end{aligned}$$

Since $X_1 = \partial_u$ *we can calculate the Nijenhuis tensor by equation 3.10:*

$$\begin{aligned}
N_j(\xi_+, \eta_+) &= 4P_- \circ P_\chi [\xi_+, \eta_+] \\
&= 4P_- \circ P_\chi (\mathcal{D}_{q_1}(T)\mathcal{D}_{q_2} + 2\partial_u - \mathcal{D}_{q_1}(S)\partial_{p_2}) \\
&= 4P_- (\mathcal{D}_{q_1}(T)\mathcal{D}_{q_2} - \mathcal{D}_{q_1}(S)\partial_{p_2}) \\
&= 2(T\mathcal{D}_{q_1}(S) - S\mathcal{D}_{q_1}(T))\mathcal{D}_{q_1} + 4\mathcal{D}_{q_1}(T)\mathcal{D}_{q_2} - 4\mathcal{D}_{q_1}(S)\partial_{p_2}.
\end{aligned}$$

Similarly one can check that

$$\begin{aligned}
N_j(\xi_-, \eta_-) &= 4P_+ \circ P_\chi [S\mathcal{D}_{q_1} - 2\mathcal{D}_{q_2}, T\mathcal{D}_{q_2} - S\partial_{p_2}] \\
&= 2S\left(S\mathcal{D}_{q_1}(T) - T\mathcal{D}_{q_1}(S) - 2\mathcal{D}_{q_2}(T) + 2\frac{\partial S}{\partial p_2} \right)\mathcal{D}_{q_1}.
\end{aligned}$$

It is clear that the Nijenhuis tensor defines a vectorbundle by its image. We will only study the cases when N_j defines a vectorbundle with constant rank. This is clearly not always the case.

Definition 3.28 *We define* $N_j\left(\Pi^2_\mp, \Pi^2_\mp\right) = R^1_\pm \subseteq \Pi^2_\pm$.

Definition 3.29 *We call $\Pi_j = \mathrm{Im}\,(N_j) = R^1_+ \oplus R^1_-$ the* ***characteristic distribution*** *of the Nijenhuis tensor.*

Obviously Π_j projects onto R^1_+ and R^1_- under the projections P_+ and P_- correspondingly. We also notice that $N_j\,(\Pi_j, \Pi_j) = 0$.

Proposition 3.30 *Any submanifold N of the strict contact manifold M satisfying $TN = \Pi_j$ is Legendrian.*

Proof. Obviously N is an integral manifold of Π and $\dim N = \mathrm{rank}\,\Pi_j = 2 = \frac{\dim M - 1}{2}$, hence N satisfies definition 2.15. ■

One should also note that $TN = \Pi_j$ requires that Π_j is integrable.

From the properties of N_j it follows that there can be 3 main cases:

1. The Nijenhuis tensor is identically zero, $N_j \equiv 0$.
2. Π_j is a rank 1 distribution within Π^2_+ or Π^2_-.
3. Π_j is a rank 2 distribution.

We will consider case 3. From corollary 3.8 we know that $\partial\Pi_j$ must be a subbundle of Π. Under the assumption that $\partial\Pi_j$ has constant rank there are 4 cases:

- $\partial\Pi_j = \Pi_j$, i.e. Π_j is integrable.
- $\partial\Pi_j \neq \Pi_j$ and $\partial\Pi_j \cap \Pi^2_+ = \Pi^2_+$, i.e. the commutator of the two generating sections in Π_j is a section in Π^2_+.
- $\partial\Pi_j \neq \Pi_j$ and $\partial\Pi_j \cap \Pi^2_+ = \Pi^2_-$, i.e. the commutator of the two generating sections in Π_j is a section in Π^2_-.
- $\partial\Pi_j \neq \Pi_j$ and $\partial\Pi_j \cap \Pi^2_+$, $\partial\Pi_j \cap \Pi^2_+$ are both 1-dimensional, i.e. the commutator of the two generating sections in Π_j is neither a section in Π^2_+, Π^2_- nor Π_j.

The following picture shows how the fibers $\Pi_j\,(x)\,, \Pi^2_+\,(x)\,, \Pi^2_-\,(x)\,, L^1\,(x)$ and $\chi\,(x)$ relate in T_xM when Π_j has rank 2:

Example 3.31 (Nonlinear 3) *We continue example 3.27. We see that*

$$
\begin{aligned}
R^1_+ &= \langle \mathcal{D}_{q_1} \rangle = \left\langle \xi_+ \right\rangle , \\
R^1_- &= \langle \left(T\mathcal{D}_{q_1}(S) - S\mathcal{D}_{q_1}(T) \right) \mathcal{D}_{q_1} + 2\mathcal{D}_{q_1}(T)\,\mathcal{D}_{q_2} - 2\mathcal{D}_{q_1}(S)\,\partial_{p_2} \rangle = \left\langle \eta_- \right\rangle .
\end{aligned}
$$

Chapter 4

Invariants of hyperbolic M-A equations

We will now introduce three new invariants, the 1-form σ, the distribution Γ^1_σ and the vectorvalued 2-form R^σ_j, of the generalized hyperbolic Monge-Ampere equation (ω, θ). In the first two sections we construct and investigate these invariants. Section 3 contains the Main Theorem, stating that any nondegenerate equation (ω, θ) defines a canonical frame on M.

4.1 The canonical 1-form σ and the distribution Γ^1_σ

In this section we construct a canonical 1-form $\sigma \in C^\infty(\Pi^*)$ together with a rank 1 distribution Γ^1_σ from the forms $\omega \in \Omega^1(M)$ and $\Omega, i_j\Omega \in C^\infty(\Lambda^2\Pi^*)$. The construction is based on the properties of the de-Rham complex

$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \xrightarrow{d} \Omega^4(M) \xrightarrow{d} \Omega^5(M),$$

where M is our 5-dimensional manifold.

The canonical splitting

$$\tau^*_M = \Pi^* \oplus \chi^*$$

gives us the following isomorphisms:

$$\begin{aligned}
\Omega^1(M) &\simeq C^\infty(\Pi^*) \oplus_{C^\infty(M)} C^\infty(\chi^*), \\
\Omega^2(M) &\simeq C^\infty(\Lambda^2\Pi^*) \oplus_{C^\infty(M)} C^\infty(\Pi^* \wedge \chi^*), \\
\Omega^3(M) &\simeq C^\infty(\Lambda^3\Pi^*) \oplus_{C^\infty(M)} C^\infty(\Lambda^2\Pi^* \wedge \chi^*), \\
\Omega^4(M) &\simeq C^\infty(\Lambda^4\Pi^*) \oplus_{C^\infty(M)} C^\infty(\Lambda^3\Pi^* \wedge \chi^*).
\end{aligned}$$

Define the following two $C^{\infty}(M)$-linear maps:

$$\begin{array}{ccc} C^{\infty}(\Lambda^i\Pi^*) & \xrightarrow{\text{Ext}} & \Omega^i(M) \\ \mu & \longmapsto & \widetilde{\mu} \end{array} \quad \text{and} \quad \begin{array}{ccc} \Omega^i(M) & \xrightarrow{\text{Rst}} & C^{\infty}(\Lambda^i\Pi^*) \\ \omega & \longmapsto & \omega|_{\Pi} = \omega \bmod \chi^* \end{array}, \tag{4.1}$$

where

$$\widetilde{\mu} = \begin{cases} \mu & \text{on } \Lambda^i\Pi \\ 0 & \text{on } \Lambda^{i-1}\Pi \wedge \chi \end{cases}.$$

Definition 4.1 *Define the operator d_{Π} in the following way:*

$$d_{\Pi} = \begin{cases} \text{Rst} \circ d \circ \text{Ext} & ; \text{ on } C^{\infty}(\Lambda^i\Pi^*);\ i \geq 1, \\ \text{Rst} \circ d & ; \text{ on } C^{\infty}(M). \end{cases}$$

Then, for $i = 0, 1, 2, 3$ we have the following commutative diagram:

$$\begin{array}{ccc} \Omega^i(M) & \xrightarrow{d} & \Omega^{i+1}(M) \\ \uparrow{\scriptstyle Ext} & & \downarrow{\scriptstyle Rst} \\ C^{\infty}(\Lambda^i\Pi^*) & \xrightarrow{d_{\Pi}} & C^{\infty}(\Lambda^{i+1}\Pi^*) \end{array}$$

From the effective form θ we get the canonical 3-form $d_{\Pi}\theta \in C^{\infty}(\Lambda^3\Pi^*)$. From [Ly1] and [Ly2] we know that

$$\begin{array}{cccc} \top : & C^{\infty}(\Pi^*) & \longrightarrow & C^{\infty}(\Lambda^3\Pi^*) \\ & \alpha & \longmapsto & \alpha \wedge \Omega \end{array}$$

is an isomorphism. Hence there exists an unique 1-form $\sigma \in C^{\infty}(\Pi^*)$ such that $\top(\sigma) = d_{\Pi}\theta$.

Definition 4.2 *The canonical 1-form $\sigma \in C^{\infty}(\Pi^*)$ is defined to be the preimage of $d_{\Pi}\theta$ under the isomorphism $\top$.*

This 1-form induces a section in Π.

Definition 4.3 *We define the vector field $X_{\sigma} \in C^{\infty}(\Pi)$ by:*

$$i_{X_{\sigma}}\Omega = \sigma,$$

where Ω is the symplectic structure on Π.

Obviously X_{σ} is uniquely defined by σ since Ω is nondegenerate. We notice that

$$\sigma(X_{\sigma}) = 0.$$

Definition 4.4 *Define the distribution Γ^1_σ to be*

$$\Gamma^1_\sigma = \langle X_\sigma \rangle .$$

Example 4.5 *Consider $M = \mathbb{R}^5 (q_1, q_2, u, p_1, p_2)$ with the contact distribution $\Pi = \langle \mathcal{D}_{q_1}, \mathcal{D}_{q_2}, \partial_{p_1}, \partial_{p_2} \rangle$, the symplectic structure $\Omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$ and the general effective form*

$$\theta = F_1 dq_1 \wedge dq_2 + F_2 dq_2 \wedge dp_1 + F_3 dq_1 \wedge dp_2 + F_4 dq_1 \wedge dp_1 - F_4 dq_2 \wedge dp_2 + F_5 dp_1 \wedge dp_2,$$

where $F_1, \ldots, F_5 \in C^\infty(M)$. We know that $X_1 = \partial_u$. Let us identify

$$\begin{aligned} \chi^* &\simeq \operatorname{Ann}(\Pi) = \langle du - p_1 dq_1 - p_2 dq_2 \rangle , \\ \Pi^* &\simeq \operatorname{Ann}(\chi) = \langle dq_1, dq_2, dp_1, dp_2 \rangle . \end{aligned}$$

Then one can check that

$$\Lambda^3 \Pi^* = \langle dq_1 \wedge dq_2 \wedge dp_1, dq_1 \wedge dq_2 \wedge dp_2, dq_1 \wedge dp_1 \wedge dp_2, dq_2 \wedge dp_1 \wedge dp_2 \rangle ,$$

hence

$$\begin{aligned} d_\Pi \theta = & \left(\frac{\partial F_1}{\partial p_1} - \mathcal{D}_{q_2}(F_4) + \mathcal{D}_{q_1}(F_2) \right) dq_1 \wedge dq_2 \wedge dp_1 \\ & + \left(\frac{\partial F_1}{\partial p_2} - \mathcal{D}_{q_2}(F_3) - \mathcal{D}_{q_1}(F_4) \right) dq_1 \wedge dq_2 \wedge dp_2 \\ & + \left(\frac{\partial F_4}{\partial p_2} - \frac{\partial F_3}{\partial p_1} + \mathcal{D}_{q_1}(F_5) \right) dq_1 \wedge dp_1 \wedge dp_2 \\ & + \left(\frac{\partial F_2}{\partial p_2} + \frac{\partial F_4}{\partial p_1} + \mathcal{D}_{q_2}(F_5) \right) dq_2 \wedge dp_1 \wedge dp_2. \end{aligned}$$

Let $\sigma = \lambda_1 dq_1 + \lambda_2 dq_2 + \lambda_3 dp_1 + \lambda_4 dp_2 \in \Pi^*$. Then

$$\sigma \wedge \Omega = -\lambda_2 dq_1 \wedge dq_2 \wedge dp_1 + \lambda_1 dq_1 \wedge dq_2 \wedge dp_2 + \lambda_4 dq_1 \wedge dp_1 \wedge dp_2 - \lambda_3 dq_2 \wedge dp_1 \wedge dp_2.$$

By comparing $d_\Pi \theta$ with $\sigma \wedge \Omega$ we conclude that

$$\begin{aligned} \sigma = & \left(\frac{\partial F_1}{\partial p_2} - \mathcal{D}_{q_2}(F_3) - \mathcal{D}_{q_1}(F_4) \right) dq_1 - \left(\frac{\partial F_1}{\partial p_1} - \mathcal{D}_{q_2}(F_4) + \mathcal{D}_{q_1}(F_2) \right) dq_2 \\ & - \left(\frac{\partial F_2}{\partial p_2} + \frac{\partial F_4}{\partial p_1} + \mathcal{D}_{q_2}(F_5) \right) dp_1 + \left(\frac{\partial F_4}{\partial p_2} - \frac{\partial F_3}{\partial p_1} + \mathcal{D}_{q_1}(F_5) \right) dp_2. \end{aligned}$$

This leads to

$$\begin{aligned} X_\sigma &= -\left(\frac{\partial F_2}{\partial p_2}+\frac{\partial F_4}{\partial p_1}+\mathcal{D}_{q_2}(F_5)\right)\mathcal{D}_{q_1}+\left(\frac{\partial F_4}{\partial p_2}-\frac{\partial F_3}{\partial p_1}+\mathcal{D}_{q_1}(F_5)\right)\mathcal{D}_{q_2} \\ &\quad -\left(\frac{\partial F_1}{\partial p_2}-\mathcal{D}_{q_2}(F_3)-\mathcal{D}_{q_1}(F_4)\right)\partial_{p_1}+\left(\frac{\partial F_1}{\partial p_1}-\mathcal{D}_{q_2}(F_4)+\mathcal{D}_{q_1}(F_2)\right)\partial_{p_2}. \end{aligned}$$

In the case when θ has constant coefficients, e.g. in the case

$$\frac{\partial^2 h}{\partial q_1^2}-\frac{\partial^2 h}{\partial q_2^2}=0 \qquad \Longleftrightarrow \qquad \theta = dp_1\wedge dq_2+dp_2\wedge dq_1,$$

we see that $\sigma=0$. While in the case

$$-S\frac{\partial^2 u}{\partial q_1^2}+2\frac{\partial^2 u}{\partial q_1\partial q_2}-T\left(\frac{\partial^2 u}{\partial q_1^2}\frac{\partial^2 u}{\partial q_2^2}-\left(\frac{\partial^2 u}{\partial q_1\partial q_2}\right)^2\right)=0,$$

$$\theta = dq_1\wedge dp_1-dq_2\wedge dp_2+Sdq_2\wedge dp_1-Tdp_1\wedge dp_2,$$

we find that $\sigma=-\mathcal{D}_{q_1}(S)\,dq_2+\left(\mathcal{D}_{q_2}(T)-\frac{\partial S}{\partial p_2}\right)dp_1-\mathcal{D}_{q_1}(T)\,dp_2.$

4.2 The vectorvalued 2-form R_j^σ

In this section we define a vectorvalued 2-form $R_j^\sigma\in\Lambda^2\Pi^*\otimes\Pi$ from the 1-form σ and the Nijenhuis tensor. The properties of R_j^σ contain some useful information concerning the distribution Γ_σ^1. When investigating R_j^σ, we need to extend θ and Ω to $\widetilde{\theta},\widetilde{\Omega}\in\Omega^2(M)$ by the extension defined in equation 4.1. We can observe that

$$\widetilde{\Omega}=\begin{cases} d\omega & ;\ \text{on } \Pi \\ 0 & ;\ \text{on } \chi \end{cases} = d\omega.$$

The last equality follows from the definition of the characteristic vector field X_1 that generates χ.

Definition 4.6 *Let $\xi,\eta\in C^\infty(\Pi)$. We define the vector valued 2-form $R_j^\sigma\in\Lambda^2\Pi^*\otimes\Pi$ to be the following:*

$$R_j^\sigma(\xi,\eta)=N_j(\xi,\eta)+j\eta\sigma(\xi)-j\xi\sigma(\eta)+\eta\sigma(j\xi)-\xi\sigma(j\eta). \tag{4.2}$$

Proposition 4.7 *$R_j^\sigma(\xi,\eta)$ has the following properties:*

i) $R_j^\sigma(\xi,\eta)=-R_j^\sigma(\eta,\xi)$.

ii) $R_j^\sigma(k\xi,\eta)=kR_j^\sigma(\xi,\eta)$; $k\in\mathbb{R}$.

iii) $R_j^\sigma(\xi_1+\xi_2,\eta) = R_j^\sigma(\xi_1,\eta) + R_j^\sigma(\xi_2,\eta)$.

iv) $R_j^\sigma(j\xi,\eta) = R_j^\sigma(\xi,j\eta)$.

v) $2N_j(j\xi,\eta) = R_j^\sigma(j\xi,\eta) - jR_j^\sigma(\xi,\eta)$.

Proof. All five properties follow directly from the properties of the Nijenhuis tensor. We show only the last one.

$$\begin{aligned}
2N_j(j\xi,\eta) &= N_j(j\xi,\eta)+N_j(j\xi,\eta) = N_j(j\xi,\eta) - jN_j(\xi,\eta) \\
&= R_j^\sigma(j\xi,\eta) - j\eta\sigma(j\xi) + \xi\sigma(\eta) - \eta\sigma(\xi) + j\xi\sigma(j\eta) \\
&\quad -j\left(R_j^\sigma(\xi,\eta) - j\eta\sigma(\xi) + j\xi\sigma(\eta) - \eta\sigma(j\xi) + \xi\sigma(j\eta)\right) \\
&= R_j^\sigma(j\xi,\eta) - j\eta\sigma(j\xi) + \xi\sigma(\eta) - \eta\sigma(\xi) + j\xi\sigma(j\eta) \\
&\quad -jR_j^\sigma(\xi,\eta) + \eta\sigma(\xi) - \xi\sigma(\eta) + j\eta\sigma(j\xi) - j\xi\sigma(j\eta) \\
&= R_j^\sigma(j\xi,\eta) - jR_j^\sigma(\xi,\eta).
\end{aligned}$$

■

Theorem 4.8 *The formula*

$$R_j^\sigma = -2jX_\sigma \otimes \Omega \tag{4.3}$$

holds.

To prove this theorem we need some results.

Proposition 4.9 *Let $\Omega \in C^\infty(\Lambda^2\Pi^*)$ be the symplectic form and $\theta = i_j\Omega$ the effective form. Then the following formula holds:*

$$\begin{aligned}
d\widetilde{\theta}(\xi,\eta,\gamma) &= L_\xi\left(\widetilde{\Omega}\left(\widetilde{j}\eta,\gamma\right)\right) - L_{\widetilde{j}\xi}\left(\widetilde{\Omega}(\eta,\gamma)\right) + \widetilde{\Omega}\left(\left[\widetilde{j}\xi,\eta\right],\gamma\right) \\
&\quad -\widetilde{\Omega}\left(\widetilde{j}[\xi,\eta],\gamma\right) + \widetilde{\Omega}\left(\eta,\left[\widetilde{j}\xi,\gamma\right]\right) - \widetilde{\Omega}\left(\eta,\widetilde{j}[\xi,\gamma]\right) \\
&\quad +d\widetilde{\Omega}\left(\widetilde{j}\xi,\eta,\gamma\right)
\end{aligned} \tag{4.4}$$

for vector fields $\xi,\eta,\gamma \in \mathcal{D}_1(M)$.

Proof. The proof is presented in appendix B. ■

We know that the vectorbundle isomorphism $\tau_M \simeq \Pi \oplus \chi$ induces the $C^\infty(M)$-linear isomorphism

$$\mathcal{D}_1(M) \simeq C^\infty(\Pi) \oplus_{C^\infty(M)} C^\infty(\chi).$$

Restricting the proposition to $C^\infty(\Pi)$ we obtain the following corollary.

Corollary 4.10 *The formula*

$$d_\Pi\theta(\xi,\eta,\gamma) = L_\xi\left(\widetilde{\Omega}(j\eta,\gamma)\right) - L_{j\xi}\left(\widetilde{\Omega}(\eta,\gamma)\right) + \widetilde{\Omega}([j\xi,\eta],\gamma) - \widetilde{\Omega}\left(\widetilde{j}[\xi,\eta],\gamma\right) + \widetilde{\Omega}(\eta,[j\xi,\gamma]) - \widetilde{\Omega}\left(\eta,\widetilde{j}[\xi,\gamma]\right) \quad (4.5)$$

holds for $\xi,\eta,\gamma \in C^\infty(\Pi)$.

Proof. From the proposition we have:

$$d\widetilde{\theta}(\xi,\eta,\gamma) = L_\xi\left(\widetilde{\Omega}\left(\widetilde{j}\eta,\gamma\right)\right) - L_{\widetilde{j}\xi}\left(\widetilde{\Omega}(\eta,\gamma)\right) + \widetilde{\Omega}\left(\left[\widetilde{j}\xi,\eta\right],\gamma\right) - \widetilde{\Omega}\left(\widetilde{j}[\xi,\eta],\gamma\right) + \widetilde{\Omega}\left(\eta,\left[\widetilde{j}\xi,\gamma\right]\right) - \widetilde{\Omega}\left(\eta,\widetilde{j}[\xi,\gamma]\right) + d\widetilde{\Omega}\left(\widetilde{j}\xi,\eta,\gamma\right).$$

Since $\xi,\eta,\gamma \in C^\infty(\Pi)$, it follows from the definition of d_Π that $d\widetilde{\theta}(\xi,\eta,\gamma) = d_\Pi\theta(\xi,\eta,\gamma)$. We also know that $d\widetilde{\Omega}\left(\xi,\widetilde{j}\eta,\gamma\right) = d(d\omega)\left(\xi,\widetilde{j}\eta,\gamma\right) = 0$, so we get:

$$d_\Pi\theta(\xi,\eta,\gamma) = L_\xi\left(\widetilde{\Omega}(j\eta,\gamma)\right) - L_{j\xi}\left(\widetilde{\Omega}(\eta,\gamma)\right) + \widetilde{\Omega}([j\xi,\eta],\gamma) - \widetilde{\Omega}\left(\widetilde{j}[\xi,\eta],\gamma\right) + \widetilde{\Omega}(\eta,[j\xi,\gamma]) - \widetilde{\Omega}\left(\eta,\widetilde{j}[\xi,\gamma]\right).$$

■

Lemma 4.11 *Let* $\xi,\eta \in C^\infty(\Pi)$. *Then*

$$i_{N_j(\xi,\eta)}\theta = -d_\Pi\theta(\xi,\eta,\bullet) - d_\Pi\theta(j\xi,j\eta,\bullet). \quad (4.6)$$

Proof. Let $\xi,\eta,\gamma \in C^\infty(\Pi) \subset \mathcal{D}_1(M)$. From the preceding corollary we know that

$$-d_\Pi\theta(\xi,\eta,\gamma) = -L_\xi\left(\widetilde{\Omega}(j\eta,\gamma)\right) + L_{j\xi}\left(\widetilde{\Omega}(\eta,\gamma)\right) - \widetilde{\Omega}([j\xi,\eta],\gamma) + \widetilde{\Omega}\left(\widetilde{j}[\xi,\eta],\gamma\right) - \widetilde{\Omega}(\eta,[j\xi,\gamma]) + \widetilde{\Omega}\left(\eta,\widetilde{j}[\xi,\gamma]\right).$$

Since $j^2 = 1$ we also get:

$$-d_\Pi\theta(j\xi,j\eta,\gamma) = -L_{j\xi}\left(\widetilde{\Omega}(\eta,\gamma)\right) + L_\xi\left(\widetilde{\Omega}(j\eta,\gamma)\right) - \widetilde{\Omega}([\xi,j\eta],\gamma) + \widetilde{\Omega}\left(\widetilde{j}[j\xi,j\eta],\gamma\right) - \widetilde{\Omega}(j\eta,[\xi,\gamma]) + \widetilde{\Omega}\left(j\eta,\widetilde{j}[j\xi,\gamma]\right).$$

Considering their sum, we see:

$$\begin{aligned}-d_{\Pi}\theta(\xi,\eta,\gamma)-d_{\Pi}\theta(j\xi,j\eta,\gamma) &= \widetilde{\Omega}\left(\widetilde{j}\,[j\xi,j\eta]\,,\gamma\right)-\widetilde{\Omega}\left([j\xi,\eta]\,,\gamma\right)+\widetilde{\Omega}\left(\widetilde{j}\,[\xi,\eta]\,,\gamma\right)\\ &\quad -\widetilde{\Omega}\left([\xi,j\eta]\,,\gamma\right)-\widetilde{\Omega}\left(\eta,[j\xi,\gamma]\right)+\widetilde{\Omega}\left(j\eta,\widetilde{j}\,[j\xi,\gamma]\right)\\ &\quad +\widetilde{\Omega}\left(\eta,\widetilde{j}\,[\xi,\gamma]\right)-\widetilde{\Omega}\left(j\eta,[\xi,\gamma]\right).\end{aligned}$$

One can easily check that $\widetilde{\Omega}\left(\widetilde{j}X,Y\right)=\widetilde{\Omega}\left(X,\widetilde{j}Y\right)$ and since $\widetilde{j}j\equiv 1 \bmod \chi$, the last four terms vanish. So we are left with:

$$\begin{aligned}-d_{\Pi}\theta(\xi,\eta,\gamma)-d_{\Pi}\theta(j\xi,j\eta,\gamma) &= \widetilde{\Omega}\left(\widetilde{j}\,[j\xi,j\eta]\,,\gamma\right)-\widetilde{\Omega}\left([j\xi,\eta]\,,\gamma\right)-\widetilde{\Omega}\left([\xi,j\eta]\,,\gamma\right)\\ &\quad +\widetilde{\Omega}\left(\widetilde{j}\,[\xi,\eta]\,,\gamma\right)\\ &= \widetilde{\Omega}\left(\widetilde{j}\,[j\xi,j\eta]-[j\xi,\eta]-[\xi,j\eta]+\widetilde{j}\,[\xi,\eta]\,,\gamma\right)\\ &= \widetilde{\theta}\left([j\xi,j\eta]-\widetilde{j}\,[j\xi,\eta]-\widetilde{j}\,[\xi,j\eta]+[\xi,\eta]\,,\gamma\right),\end{aligned}$$

where we have used that $i_j\Omega=\theta \Longleftrightarrow i_{\widetilde{j}}\widetilde{\Omega}=\widetilde{\theta}$. Since $\widetilde{\theta}=0$ on χ we can write:

$$\begin{aligned}-d_{\Pi}\theta(\xi,\eta,\gamma)-d_{\Pi}\theta(j\xi,j\eta,\gamma) &= \theta\left(P_\chi\left([j\xi,j\eta]-\widetilde{j}\,[j\xi,\eta]-\widetilde{j}\,[\xi,j\eta]+[\xi,\eta]\right),\gamma\right)\\ &= \theta\left(N_j(\xi,\eta),\gamma\right).\end{aligned}$$

The claim follows. ■

Lemma 4.12 *Let $\xi,\eta,\gamma\in C^\infty(\Pi)$. Then the following identity holds:*

$$d_{\Pi}\theta(\xi,\eta,\gamma)=\Omega(\xi,\eta)\,\sigma(\gamma)+\Omega(\gamma,\xi)\,\sigma(\eta)+\Omega(\eta,\gamma)\,\sigma(\xi). \tag{4.7}$$

Proof. We know from the construction of σ that

$$d_{\Pi}\theta=\Omega\wedge\sigma.$$

Let $\xi,\eta,\gamma\in C^\infty(\Pi)$. Simple calculation gives the wanted result.

$$\begin{aligned}d_{\Pi}\theta(\xi,\eta,\gamma) &= i_\gamma i_\eta i_\xi(\Omega\wedge\sigma)=i_\gamma i_\eta\left(\Omega(\xi,\bullet)\wedge\sigma+\Omega\sigma(\xi)\right)\\ &= i_\gamma\left(\Omega(\xi,\eta)\,\sigma-\Omega(\xi,\bullet)\,\sigma(\eta)+\Omega(\eta,\bullet)\,\sigma(\xi)\right)\\ &= \Omega(\xi,\eta)\,\sigma(\gamma)-\Omega(\xi,\gamma)\,\sigma(\eta)+\Omega(\eta,\gamma)\,\sigma(\xi)\\ &= \Omega(\xi,\eta)\,\sigma(\gamma)+\Omega(\gamma,\xi)\,\sigma(\eta)+\Omega(\eta,\gamma)\,\sigma(\xi).\end{aligned}$$

■

Now we have what we need to prove **theorem 4.8**:

Proof. Let $\xi, \eta, \gamma \in C^{\infty}(\Pi)$. From lemma 4.11 and 4.12 it follows that

$$\begin{aligned}
\Omega(jN_j(\xi,\eta),\gamma) &= \theta(N_j(\xi,\eta),\gamma) = -d_{\Pi}\theta(\xi,\eta,\gamma) - d_{\Pi}\theta(j\xi,j\eta,\gamma) \\
&= -\Omega(\xi,\eta)\sigma(\gamma) - \Omega(\gamma,\xi)\sigma(\eta) - \Omega(\eta,\gamma)\sigma(\xi) - \Omega(j\xi,j\eta)\sigma(\gamma) \\
&\quad -\Omega(\gamma,j\xi)\sigma(j\eta) - \Omega(j\eta,\gamma)\sigma(j\xi) \\
&= -2\Omega(\xi,\eta)\sigma(\gamma) - \Omega(\gamma,\xi)\sigma(\eta) - \Omega(\gamma,j\xi)\sigma(j\eta) - \Omega(\eta,\gamma)\sigma(\xi) \\
&\quad -\Omega(j\eta,\gamma)\sigma(j\xi).
\end{aligned}$$

Thus

$$\begin{aligned}
-2\Omega(\xi,\eta)\sigma(\gamma) &= \Omega(jN_j(\xi,\eta),\gamma) + \Omega(\gamma,\xi)\sigma(\eta) + \Omega(\gamma,j\xi)\sigma(j\eta) + \Omega(\eta,\gamma)\sigma(\xi) \\
&\quad +\Omega(j\eta,\gamma)\sigma(j\xi) \\
&= \Omega(jN_j(\xi,\eta),\gamma) - \Omega(\sigma(\eta)\xi,\gamma) - \Omega(\sigma(j\eta)j\xi,\gamma) + \Omega(\sigma(\xi)\eta,\gamma) \\
&\quad +\Omega(\sigma(j\xi)j\eta,\gamma) \\
&= \Omega(jN_j(\xi,\eta) - \sigma(\eta)\xi - \sigma(j\eta)j\xi + \sigma(\xi)\eta + \sigma(j\xi)j\eta,\gamma) \\
&= \Omega\left(jR_j^{\sigma}(\xi,\eta),\gamma\right).
\end{aligned}$$

From this equation we find:

$$\begin{aligned}
\Omega\left(jR_j^{\sigma}(\xi,\eta),\gamma\right) &= -2\widetilde{\Omega}(\xi,\eta)\sigma(\gamma) = \sigma(-2\Omega(\xi,\eta)\cdot\gamma) \\
&= \Omega\left(X_{\sigma}, -2\widetilde{\Omega}(\xi,\eta)\cdot\gamma\right) \\
&= \Omega\left(-2\widetilde{\Omega}(\xi,\eta)\cdot X_{\sigma},\gamma\right).
\end{aligned}$$

Since Ω is nondegenerate, this implies that

$$jR_j^{\sigma}(\xi,\eta) = -2\Omega(\xi,\eta)\cdot X_{\sigma},$$

and the claim follows. ■

Remark 4.13 *We observe the following chain of relations:*

$$d_{\Pi}\theta = 0 \overset{1}{\Longleftrightarrow} \sigma = 0 \overset{2}{\Longleftrightarrow} X_{\sigma} = 0 \overset{3}{\Longleftrightarrow} R_j^{\sigma} = 0 \overset{4}{\Longleftrightarrow} N_j = 0.$$

The first equivalence follows from the construction of σ. The second follows from the construction of X_{σ}. The third one follows from theorem 4.8, since Ω is nondegenerate. And the fourth equivalence follows from proposition 4.7 v).

Lemma 4.14 $\Gamma_{\sigma}^{1} \subset \Pi_j \subset \ker(\sigma) = (\Gamma_{\sigma}^{1})^{\perp\Omega}$.

Proof. By the definition of X_{σ}, we have that $\Omega(X_{\sigma},\xi) = \sigma(\xi)$. So $\ker(\sigma) = (\Gamma_{\sigma}^{1})^{\perp\Omega}$.

Obviously $\sigma(X_\sigma) = 0$. From this it follows that

$$\begin{aligned}
\sigma(N_j(\xi,\eta)) &= \sigma(jN_j(j\eta,\xi)) = \sigma\left(jR_j^\sigma(j\eta,\xi) + \xi\sigma(j\eta) - j\eta\sigma(\xi) + j\xi\sigma(\eta) - \eta\sigma(j\xi)\right) \\
&= \sigma\left(jR_j^\sigma(j\eta,\xi)\right) + \sigma(j\eta)\sigma(\xi) - \sigma(\xi)\sigma(j\eta) + \sigma(\eta)\sigma(j\xi) - \sigma(j\xi)\sigma(\eta) \\
&= \sigma\left(jR_j^\sigma(j\eta,\xi)\right) = \sigma(-2\Omega(j\eta,\xi)X_\sigma) = 0.
\end{aligned}$$

Thus $\Pi_j \subset \ker(\sigma)$.

Let us show that $\Gamma_\sigma^1 \subset \Pi_j$. By construction, Π_j projects to R_+^1 and R_-^1, that is:

$$\begin{aligned}
P_-(\Pi_j) &= R_-^1, \\
P_+(\Pi_j) &= R_+^1.
\end{aligned}$$

Let X and Y be vectors in Π. From proposition 4.7 it follows that

$$\begin{aligned}
-2jN_j(X,Y) &= R_j^\sigma(jX,Y) - jR_j^\sigma(X,Y), \\
2N_j(X,Y) &= R_j^\sigma(X,Y) - jR_j^\sigma(jX,Y).
\end{aligned}$$

By adding and subtracting these equations we get:

$$\begin{aligned}
2(1-j)N_j(X,Y) &= (1-j)\left(R_j^\sigma(jX,Y) + R_j^\sigma(X,Y)\right) = (1-j)R_j^\sigma((1+j)X,Y), \\
2(1+j)N_j(X,Y) &= (1+j)\left(R_j^\sigma(X,Y) - R_j^\sigma(jX,Y)\right) = (1+j)R_j^\sigma((1-j)X,Y).
\end{aligned}$$

Using theorem 4.8, we see:

$$\begin{aligned}
(1-j)R_j^\sigma((1+j)X,Y) &= -2\Omega((1+j)X,Y)(1-j)jX_\sigma = 2\Omega(2P_+X,Y)(1-j)X_\sigma, \\
(1+j)R_j^\sigma((1-j)X,Y) &= -2\Omega((1-j)X,Y)(1+j)jX_\sigma = -2\Omega(2P_-X,Y)(1+j)X_\sigma.
\end{aligned}$$

Since Π_+^2 and Π_-^2 are skew orthogonal with respect to Ω, we can write

$$\begin{aligned}
P_-N_j(X,Y) &= \Omega(P_+X,P_+Y)(1-j)X_\sigma, \\
P_+N_j(X,Y) &= -\Omega(P_-X,P_-Y)(1+j)X_\sigma.
\end{aligned}$$

We see that $(1\pm j)X_\sigma \in R_\pm^1$. Since $X_\sigma = P_+X_\sigma + P_-X_\sigma$, we conclude that $X_\sigma \in R_+^1 \oplus R_-^1 = \Pi_j$. This implies that $\Gamma_\sigma^1 \subset \Pi_j$ as we were to prove. ∎

Corollary 4.15 $\Gamma_\sigma^1 \cap R_+^1 = 0$ *and* $\Gamma_\sigma^1 \cap R_-^1 = 0$.

Proof. We know the equations

$$P_-N_j(X,Y) = \Omega(P_+X,P_+Y)(1-j)X_\sigma$$

and

$$P_+N_j(X,Y) = -\Omega(P_-X,P_-Y)(1+j)X_\sigma.$$

Assume now that $\Gamma_\sigma^1 \cap R_+^1 \neq 0$, that is $\Gamma_\sigma^1 = R_+^1$. Then $P_- X_\sigma = 0$ must be true, hence

$$P_- N_j (X, Y) = 0$$

must hold for all vectors $X, Y \in \Pi$. We know that $P_- N_j \left(\Pi_+^2, \Pi_+^2\right) = R_-^1 \neq 0$, hence the assumption must be wrong and $\Gamma_\sigma^1 \cap R_+^1 = 0$ must be true. Similarly we can prove that $\Gamma_\sigma^1 \cap R_-^1 = 0$. ■

Now we have two decompositions of Π_j, $R_+^1 \oplus R_-^1$ and $\Gamma_\sigma^1 \oplus j\Gamma_\sigma^1$.

4.3 The canonical $\{e\}$-structure

Let ξ_+ and η_- be generating sections in R_+^1 and R_-^1 in a neighborhood $\mathcal{O} \subset M$. We know that $\partial\Pi_j \subset \Pi$ has rank 3. Hence $\left[\xi_+, \eta_-\right]$ is a section in Π such that $P_\pm \left[\xi_+, \eta_-\right] \notin C^\infty \left(R_\pm^1\right)$. This means that $C^\infty \left(\Pi_+^2\right)$ and $C^\infty \left(\Pi_-^2\right)$ is locally generated by $\xi_+, P_+ \left[\xi_+, \eta_-\right]$ and $\eta_-, P_- \left[\xi_+, \eta_-\right]$ correspondingly. Assume that $\gamma_\pm \in \mathbb{R}{\cdot}P_\pm \left[\xi_+, \eta_-\right]$, then we can uniquely determine ξ_+, η_-, γ_+ and γ_- by imposing the conditions

$$N_j \left(\xi_+, \gamma_+\right) = \eta_-, \quad N_j \left(\eta_-, \gamma_-\right) = \xi_+, \quad \Omega \left(\xi_+, \gamma_+\right) = 1, \quad \Omega \left(\eta_-, \gamma_-\right) = 1.$$

This system is solvable since Π_+^2 and Π_-^2 are orthogonal with respect to N_j. So we have a basis frame on $\mathcal{O}$, namely

$$\left\{\xi_+, \gamma_+, \eta_-, \gamma_-, X_1\right\}.$$

Lemma 4.16 $\sigma\left(\gamma_+\right) = -\frac{1}{2}$, $\sigma\left(\gamma_-\right) = \frac{1}{2}$.

Proof. From theorem 4.8 and the properties of ξ_+, η_-, γ_+ and γ_- we have

$$\begin{aligned} R_j^\sigma \left(\xi_+, \gamma_+\right) &= N_j \left(\xi_+, \gamma_+\right) + \gamma_+ \sigma\left(\xi_+\right) - \xi_+ \sigma\left(\gamma_+\right) + \gamma_+ \sigma\left(\xi_+\right) - \xi_+ \sigma\left(\gamma_+\right) \\ &= \eta_- - 2\sigma\left(\gamma_+\right)\xi_+. \end{aligned}$$

We also know

$$R_j^\sigma \left(\xi_+, \gamma_+\right) = -2jX_\sigma \otimes \Omega\left(\xi_+, \gamma_+\right) = -2jX_\sigma.$$

Hence $\eta_- - 2\sigma\left(\gamma_+\right)\xi_+ = -2jX_\sigma$. Similarly for η_-, γ_- we have the two equations

$$\begin{aligned} R_j^\sigma \left(\eta_-, \gamma_-\right) &= N_j \left(\eta_-, \gamma_-\right) - \gamma_- \sigma\left(\eta_-\right) + \eta_- \sigma\left(\gamma_-\right) - \gamma_- \sigma\left(\eta_-\right) + \eta_- \sigma\left(\gamma_-\right) \\ &= \xi_+ + 2\sigma\left(\gamma_-\right)\eta_- \end{aligned}$$

and

$$R_j^\sigma \left(\eta_-, \gamma_-\right) = -2jX_\sigma \otimes \Omega\left(\eta_-, \gamma_-\right) = -2jX_\sigma.$$

Thus $\xi_+ + 2\sigma\left(\gamma_-\right)\eta_- = -2jX_\sigma$. So

$$X_\sigma = \sigma\left(\gamma_+\right)\xi_+ + \frac{1}{2}\eta_- = -\frac{1}{2}\xi_+ + \sigma\left(\gamma_-\right)\eta_-.$$

Since ξ_+ and η_- are linearly independent, we conclude that $\sigma\left(\gamma_+\right)\xi_+ = -\frac{1}{2}\xi_+$ and $\frac{1}{2}\eta_- = \sigma\left(\gamma_-\right)\eta_-$. ■

Definition 4.17 *Let us call a generalized Monge-Ampere equation (ω, θ)* ***nondegenerate*** *in a neighborhood $\mathcal{O}$ in M if:*

- $\operatorname{Im} N_j = \Pi_j$ *is a rank 2 distribution with no singularities within $\mathcal{O}$.*
- Π_j *is nonintegrable and the rank 3 distribution $\partial\Pi_j$ has no singularities within $\mathcal{O}$.*
- *Given generators $\xi_+, \eta_- \in C^\infty\left(R^1_+\right) \times C^\infty\left(R^1_-\right)$ of Π_j, the vector fields ξ_+, η_-, $P_+\left[\xi_+, \eta_-\right]$ and $P_-\left[\xi_+, \eta_-\right]$ are linearly independent in $\mathcal{O}$.*

Theorem 4.18 (Main theorem) *A nondegenerate generalized hyperbolic Monge-Ampere equation (ω, θ) defines a canonical $\{e\}$-structure, i.e. a field of basis frames $\{P_1, P_2, Q_1, Q_2, U\}$. This structure is a complete invariant, i.e. two nondegenerate hyperbolic Monge-Ampere equations are isomorphic if and only if the corresponding $\{e\}$-structures are. The classifying $\{e\}$-structure satisfies the following relations:*

$d\omega\left(\downarrow, \longrightarrow\right)$	P_1	P_2	Q_1	Q_2	U
P_1	0	0	1	0	0
P_2	0	0	0	1	0
Q_1	-1	0	0	0	0
Q_2	0	-1	0	0	0
U	0	0	0	0	0

X	P_1	P_2	Q_1	Q_2	U
$\omega(X)$	0	0	0	0	1
jX	P_1	$-P_2$	Q_1	$-Q_2$	$-$
$\sigma(X)$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-$

$\theta\left(\downarrow, \longrightarrow\right)$	P_1	P_2	Q_1	Q_2
P_1	0	0	1	0
P_2	0	0	0	-1
Q_1	-1	0	0	0
Q_2	0	1	0	0

$N_j\left(\downarrow, \longrightarrow\right)$	P_1	P_2	Q_1	Q_2
P_1	0	0	P_2	0
P_2	0	0	0	P_1
Q_1	$-P_2$	0	0	0
Q_2	0	$-P_1$	0	0

Proof. The tables for $\omega, d\omega, j, N_j$ and σ follow from the construction of the $\{e\}$-structure described in the beginning of this section. We have just renamed the generating sections:

$$P_1 = \xi_+, \qquad P_2 = \eta_-, \qquad Q_1 = \gamma_+, \qquad Q_2 = \gamma_-, \qquad U = X_1.$$

The relation $\theta(X, Y) = d\omega|_\Pi (jX, Y)$ implies the table for θ.

Given an $\{e\}$-structure, satisfying the tables in the theorem, we define the generalized Monge-Ampere equation (ω, θ). Note that the Nijenhuis tensor given by the table must coincide with the Nijenhuis tensor defined from the almost product structure j given by the table. Equivalently we can require that $d_\Pi\theta$ coincides with $d\omega|_\Pi \wedge \sigma$. Under the fulfillment of either of these conditions, a given $\{e\}$-structure $\{P_1, P_2, Q_1, Q_2, U\}$ uniquely determines the Monge-Ampere equation. ■

An $\{e\}$-structure, $\{e_i\}_{i=1}^5$, give us (potentially) a huge set of invariants of the equation, namely the set of structure coefficients c_{jk}^i from the decomposition $[e_j, e_k] = \sum_i c_{jk}^i e_i$.

Example 4.19 (Nonlinear 4) *We continue the study of the generalized Monge-Ampere equation*

$$(\omega, \theta) = (du - p_1 dq_1 - p_2 dq_2, dq_1 \wedge dp_1 - dq_2 \wedge dp_2 + S dq_2 \wedge dp_1 - T dp_1 \wedge dp_2)$$

represented on the manifold $M = \mathbb{R}^5 (q_1, q_2, u, p_1, p_2)$. *We see from example 3.31 that*

$$\begin{aligned} R_+^1 &= \langle \mathcal{D}_{q_1} \rangle = \langle \xi_+ \rangle, \\ R_-^1 &= \langle (T\mathcal{D}_{q_1}(S) - S\mathcal{D}_{q_1}(T)) \mathcal{D}_{q_1} + 2\mathcal{D}_{q_1}(T) \mathcal{D}_{q_2} - 2\mathcal{D}_{q_1}(S) \partial_{p_2} \rangle = \langle \eta_- \rangle. \end{aligned}$$

So the commutator $[\xi_+, \eta_-]$ *becomes*

$$[\xi_+, \eta_-] = (T\mathcal{D}_{q_1}^2(S) - S\mathcal{D}_{q_1}^2(T)) \mathcal{D}_{q_1} + 2\mathcal{D}_{q_1}^2(T) \mathcal{D}_{q_2} - 2\mathcal{D}_{q_1}^2(S) \partial_{p_2},$$

where $\mathcal{D}_{q_1}^2 = \mathcal{D}_{q_1} \circ \mathcal{D}_{q_1}$. *We see that* $\Pi_j = R_+^1 \oplus R_-^1$ *is nonintegrable. From the operator* j *in the previous example we can calculate that*

$$\begin{aligned} P_+ [\xi_+, \eta_-] &= 0, \\ P_- [\xi_+, \eta_-] &= [\xi_+, \eta_-]. \end{aligned}$$

Thus (ω, θ) *is degenerate according to our definition.*

Example 4.20 *Consider the Monge-Ampere equation*

$$Q - T\frac{\partial^2 u}{\partial q_1^2} + Q\frac{\partial^2 u}{\partial q_2^2} + 2\frac{\partial^2 u}{\partial q_1 \partial q_2} - T\left(\frac{\partial^2 u}{\partial q_1^2}\frac{\partial^2 u}{\partial q_2^2} - \left(\frac{\partial^2 u}{\partial q_1 \partial q_2}\right)^2\right) = 0.$$

Represented in $\mathbb{R}^5 (q_1, q_2, u, p_1, p_2)$ *with the contact form* $\omega = du - p_1 dq_1 - p_2 dq_2$, *this equation corresponds to the effective form*

$$\theta = Q dq_1 \wedge dq_2 + T dq_2 \wedge dp_1 + Q dq_1 \wedge dp_2 + dq_1 \wedge dp_1 - dq_2 \wedge dp_2 - T dp_1 \wedge dp_2.$$

One can check that

$$j = \begin{bmatrix} 1 & T & 0 & T \\ Q & -1 & -T & 0 \\ 0 & Q & 1 & Q \\ -Q & 0 & T & -1 \end{bmatrix}.$$

This leads to the splitting of $\Pi = \langle \mathcal{D}_{q_1}, \mathcal{D}_{q_2}, \partial_{p_1}, \partial_{p_2} \rangle$ *into the sum of*

$$\Pi_+^2 = \langle 2\mathcal{D}_{q_1} + Q\mathcal{D}_{q_2} - Q\partial_{p_2}, T\mathcal{D}_{q_1} + Q\partial_{p_1} \rangle$$

and

$$\Pi_-^2 = \langle \mathcal{D}_{q_2} - \partial_{p_2}, T\mathcal{D}_{q_1} + Q\partial_{p_1} - 2\partial_{p_2} \rangle .$$

Denote

$$\begin{array}{ll} \xi_+ = 2\mathcal{D}_{q_1} + Q\mathcal{D}_{q_2} - Q\partial_{p_2}, & \eta_+ = T\mathcal{D}_{q_1} + Q\partial_{p_1}, \\ \xi_- = \mathcal{D}_{q_2} - \partial_{p_2}, & \eta_- = T\mathcal{D}_{q_1} + Q\partial_{p_1} - 2\partial_{p_2}. \end{array}$$

Then

$$\begin{aligned} N_j \left(\xi_+, \eta_+\right) &= 4P_- \circ P_\chi \left[\xi_+, \eta_+\right] \\ &= \left\{-4Q\mathcal{D}_{q_1}(T) - 2Q^2\mathcal{D}_{q_2}(T) + 2Q^2\partial_{p_2}(T) - 4Q\partial_{p_1}(Q) \right. \\ &\quad \left. +2QT\mathcal{D}_{q_2}(Q) - 2QT\partial_{p_2}(Q)\right\} \left(\mathcal{D}_{q_2} - \partial_{p_2}\right). \end{aligned}$$

Similarly we find that

$$\begin{aligned} N_j \left(\xi_-, \eta_-\right) &= 4P_+ \circ P_\chi \left[\xi_-, \eta_-\right] \\ &= 4\left(\mathcal{D}_{q_2}(T) - \partial_{p_2}(T)\right)\mathcal{D}_{q_1} \\ &\quad +2\left(Q\left(\mathcal{D}_{q_2}(T) - \partial_{p_2}(T)\right) - T\left(\mathcal{D}_{q_2}(Q) - \partial_{p_2}(Q)\right)\right)\mathcal{D}_{q_2} \\ &\quad +4\left(\mathcal{D}_{q_2}(Q) - \partial_{p_2}(Q)\right)\partial_{p_1} \\ &\quad -2\left(Q\left(\mathcal{D}_{q_2}(T) - \partial_{p_2}(T)\right) - T\left(\mathcal{D}_{q_2}(Q) - \partial_{p_2}(Q)\right)\right)\partial_{p_2}. \end{aligned}$$

Let us denote

$$A = \mathcal{D}_{q_2}(T) - \partial_{p_2}(T) \qquad \textit{and} \qquad B = \mathcal{D}_{q_2}(Q) - \partial_{p_2}(Q).$$

Then we can write

$$R_+^1 = \langle 2A\mathcal{D}_{q_1} + (QA - TB)\mathcal{D}_{q_2} + 2B\partial_{p_1} - (QA - TB)\partial_{p_2} \rangle = \langle X_+ \rangle$$

and

$$R_-^1 = \langle \mathcal{D}_{q_2} - \partial_{p_2} \rangle = \langle Y_- \rangle .$$

One can check that $\Pi_j = R_+^1 \oplus R_-^1$ *is nonintegrable and that*

$$\begin{aligned} P_+ [X_+, Y_-] &= [X_+, Y_-], \\ P_- [X_+, Y_-] &= 0. \end{aligned}$$

So (ω, θ) *is degenerate according to our definition.*

Chapter 5

Appendix A

This appendix contains the proof of lemma 3.20. We should show that $N_j(\xi,\eta)(m) = N_j(\xi(m),\eta(m))$ for all sections ξ,η in Π. Let $e_1,\ldots,e_4$ be generators of $C^\infty(\Pi)$ in a neighborhood $\mathcal{O}_m \subset M$ containing the point m. Assume that $\langle e_1,e_2\rangle = \Pi^2_+$ and $\langle e_3,e_4\rangle = \Pi^2_-$. Consider sections $\xi,\widehat{\xi},\Delta\xi,\eta,\widehat{\eta},\Delta\eta \in C^\infty\left(\Pi^2_+\right)$ such that

$$\widehat{\xi} = \xi + \Delta\xi, \qquad \widehat{\eta} = \eta + \Delta\eta,$$

and

$$\widehat{\xi}(m) = \xi(m), \qquad \widehat{\eta}(m) = \eta(m),$$

i.e. $\Delta\xi(m) = \Delta\eta(m) = 0$. Assume that they decompose like

$$\begin{aligned} \xi &= A_1e_1 + \cdots + A_4e_4, \qquad & \Delta\xi = a_1e_1 + \cdots + a_4e_4, \\ \eta &= B_1e_1 + \cdots + B_4e_4, \qquad & \Delta\eta = b_1e_1 + \cdots + b_4e_4, \end{aligned}$$

where $A_i,a_i,B_i,b_i \in C^\infty(M)$ and $a_i(m) = b_i(m) = 0$ for $i = 1,\ldots,4$. Consider the following difference:

$$\begin{aligned} N_j\left(\widehat{\xi},\widehat{\eta}\right) - N_j(\xi,\eta) &= P_\chi\left(\left[j\widehat{\xi},j\widehat{\eta}\right] - \widetilde{j}\left[j\widehat{\xi},\widehat{\eta}\right] - \widetilde{j}\left[\widehat{\xi},j\widehat{\eta}\right] + \left[\widehat{\xi},\widehat{\eta}\right]\right) \\ &\quad -P_\chi\left([j\xi,j\eta] - \widetilde{j}[j\xi,\eta] - \widetilde{j}[\xi,j\eta] + [\xi,\eta]\right) \\ &= P_\chi\Big([j\xi,j\Delta\eta] - \widetilde{j}[j\xi,\Delta\eta] - \widetilde{j}[\xi,j\Delta\eta] + [\xi,\Delta\eta] \\ &\quad + [j\Delta\xi,j\eta] - \widetilde{j}[j\Delta\xi,\eta] - \widetilde{j}[\Delta\xi,j\eta] + [\Delta\xi,\eta] \\ &\quad + [j\Delta\xi,j\Delta\eta] - \widetilde{j}[j\Delta\xi,\Delta\eta] - \widetilde{j}[\Delta\xi,j\Delta\eta] + [\Delta\xi,\Delta\eta]\Big). \end{aligned}$$

We decompose into generating sections and use that $je_1 = e_1, je_2 = e_2$ while $je_3 = -e_3, je_4 = -e_4$.

$$
\begin{aligned}
N_j\left(\widehat{\xi},\widehat{\eta}\right) - N_j(\xi,\eta) &= \sum_{i=1}^{2}\sum_{k=1}^{2} P_\chi\Big([A_ie_i,b_ke_k] - \widetilde{j}\,[A_ie_i,b_ke_k] - \widetilde{j}\,[A_ie_i,b_ke_k] \\
&\quad + [A_ie_i,b_ke_k] + [a_ie_i,B_ke_k] - \widetilde{j}\,[a_ie_i,B_ke_k] - \widetilde{j}\,[a_ie_i,B_ke_k] \\
&\quad + [a_ie_i,B_ke_k] + [a_ie_i,b_ke_k] - \widetilde{j}\,[a_ie_i,b_ke_k] - \widetilde{j}\,[a_ie_i,b_ke_k] + [a_ie_i,b_ke_k]\Big) \\
&\quad + \sum_{i=1}^{2}\sum_{k=3}^{4} P_\chi\Big(-[A_ie_i,b_ke_k] - \widetilde{j}\,[A_ie_i,b_ke_k] + \widetilde{j}\,[A_ie_i,b_ke_k] \\
&\quad + [A_ie_i,b_ke_k] - [a_ie_i,B_ke_k] - \widetilde{j}\,[a_ie_i,B_ke_k] + \widetilde{j}\,[a_ie_i,B_ke_k] \\
&\quad + [a_ie_i,B_ke_k] - [a_ie_i,b_ke_k] - \widetilde{j}\,[a_ie_i,b_ke_k] + \widetilde{j}\,[a_ie_i,b_ke_k] + [a_ie_i,b_ke_k]\Big) \\
&\quad + \sum_{i=3}^{4}\sum_{k=1}^{2} P_\chi\Big(-[A_ie_i,b_ke_k] - \widetilde{j}\,[A_ie_i,b_ke_k] + \widetilde{j}\,[A_ie_i,b_ke_k] \\
&\quad + [A_ie_i,b_ke_k] - [a_ie_i,B_ke_k] - \widetilde{j}\,[a_ie_i,B_ke_k] + \widetilde{j}\,[a_ie_i,B_ke_k] \\
&\quad + [a_ie_i,B_ke_k] - [a_ie_i,b_ke_k] - \widetilde{j}\,[a_ie_i,b_ke_k] + \widetilde{j}\,[a_ie_i,b_ke_k] + [a_ie_i,b_ke_k]\Big) \\
&\quad + \sum_{i=3}^{4}\sum_{k=3}^{4} P_\chi\Big([A_ie_i,b_ke_k] + \widetilde{j}\,[A_ie_i,b_ke_k] + \widetilde{j}\,[A_ie_i,b_ke_k] \\
&\quad + [A_ie_i,b_ke_k] + [a_ie_i,B_ke_k] + \widetilde{j}\,[a_ie_i,B_ke_k] + \widetilde{j}\,[a_ie_i,B_ke_k] \\
&\quad + [a_ie_i,B_ke_k] + [a_ie_i,b_ke_k] + \widetilde{j}\,[a_ie_i,b_ke_k] + \widetilde{j}\,[a_ie_i,b_ke_k] + [a_ie_i,b_ke_k]\Big) \\
&= 2\sum_{i=1}^{2}\sum_{k=1}^{2} P_\chi\Big([A_ie_i,b_ke_k] - \widetilde{j}\,[A_ie_i,b_ke_k] + [a_ie_i,B_ke_k] \\
&\quad - \widetilde{j}\,[a_ie_i,B_ke_k] + [a_ie_i,b_ke_k] - \widetilde{j}\,[a_ie_i,b_ke_k]\Big) \\
&\quad + 2\sum_{i=3}^{4}\sum_{k=3}^{4} P_\chi\Big([A_ie_i,b_ke_k] + \widetilde{j}\,[A_ie_i,b_ke_k] + [a_ie_i,B_ke_k] \\
&\quad + \widetilde{j}\,[a_ie_i,B_ke_k] + [a_ie_i,b_ke_k] + \widetilde{j}\,[a_ie_i,b_ke_k]\Big).
\end{aligned}
$$

Recall that

$$[F\xi, G\eta] = FG[\xi,\eta] + FL_\xi(G)\,\eta - GL_\eta(F)\,\xi$$

holds for any $F, G \in C^{\infty}(M)$ and $\xi, \eta \in C^{\infty}\Pi$. Then we get

$$
\begin{aligned}
N_j\left(\widehat{\xi}, \widehat{\eta}\right) - N_j(\xi, \eta) &= 2\sum_{i=1}^{2}\sum_{k=1}^{2} P_{\chi}\left(A_i b_k [e_i, e_k] + A_i L_{e_i}(b_k) e_k - b_k L_{e_k}(A_i) e_i\right. \\
&\quad - A_i b_k \widetilde{j}[e_i, e_k] - A_i L_{e_i}(b_k) \widetilde{j} e_k + b_k L_{e_k}(A_i) \widetilde{j} e_i \\
&\quad + a_i B_k [e_i, e_k] + a_i L_{e_i}(B_k) e_k - B_k L_{e_k}(a_i) e_i \\
&\quad - a_i B_k \widetilde{j}[e_i, e_k] - a_i L_{e_i}(B_k) \widetilde{j} e_k + B_k L_{e_k}(a_i) \widetilde{j} e_i \\
&\quad + a_i b_k [e_i, e_k] + a_i L_{e_i}(b_k) e_k - b_k L_{e_k}(a_i) e_i \\
&\quad \left. - a_i b_k \widetilde{j}[e_i, e_k] - a_i L_{e_i}(b_k) \widetilde{j} e_k + b_k L_{e_k}(a_i) \widetilde{j} e_i\right) \\
&\quad + 2\sum_{i=3}^{4}\sum_{k=3}^{4} P_{\chi}\left(A_i b_k [e_i, e_k] + A_i L_{e_i}(b_k) e_k - b_k L_{e_k}(A_i) e_i\right. \\
&\quad + \widetilde{j} A_i b_k [e_i, e_k] + \widetilde{j} A_i L_{e_i}(b_k) e_k - \widetilde{j} b_k L_{e_k}(A_i) e_i \\
&\quad + a_i B_k [e_i, e_k] + a_i L_{e_i}(B_k) e_k - B_k L_{e_k}(a_i) e_i \\
&\quad + \widetilde{j} a_i B_k [e_i, e_k] + \widetilde{j} a_i L_{e_i}(B_k) e_k - \widetilde{j} B_k L_{e_k}(a_i) e_i \\
&\quad + \widetilde{j} a_i b_k [e_i, e_k] + \widetilde{j} a_i L_{e_i}(b_k) e_k - \widetilde{j} b_k L_{e_k}(a_i) e_i \\
&\quad \left. + a_i b_k [e_i, e_k] + a_i L_{e_i}(b_k) e_k - b_k L_{e_k}(a_i) e_i\right) \\
&= 2\sum_{i=1}^{2}\sum_{k=1}^{2} P_{\chi}\left(A_i b_k [e_i, e_k] - A_i b_k \widetilde{j}[e_i, e_k] + a_i B_k [e_i, e_k]\right. \\
&\quad - a_i B_k \widetilde{j}[e_i, e_k] + a_i b_k [e_i, e_k] - a_i b_k \widetilde{j}[e_i, e_k] \\
&\quad + 2\sum_{i=3}^{4}\sum_{k=3}^{4} P_{\chi}\left(A_i b_k [e_i, e_k] + A_i b_k \widetilde{j}[e_i, e_k] + a_i B_k [e_i, e_k]\right. \\
&\quad \left. + a_i B_k \widetilde{j}[e_i, e_k] + a_i b_k \widetilde{j}[e_i, e_k] + a_i b_k [e_i, e_k]\right).
\end{aligned}
$$

Evaluating at the point m we see that

$$N_j\left(\widehat{\xi}, \widehat{\eta}\right)(m) - N_j(\xi, \eta)(m) = 0.$$

So we conclude that N_j is a tensor.

Chapter 6

Appendix B

Here we prove proposition 4.4. Let $\xi, \eta, \gamma \in \mathcal{D}_1(M)$. Then we have:

$$\begin{aligned} d\widetilde{\theta}(\xi,\eta,\gamma) &= \left(i_\xi d\widetilde{\theta}\right)(\eta,\gamma) \\ &= \left(L_\xi \widetilde{\theta}\right)(\eta,\gamma) - \left(d i_\xi \widetilde{\theta}\right)(\eta,\gamma). \end{aligned}$$

Using

$$L_\xi\left(\widetilde{\theta}(\eta,\gamma)\right) = \left(L_\xi \widetilde{\theta}\right)(\eta,\gamma) + \widetilde{\theta}(L_\xi \eta,\gamma) + \widetilde{\theta}(\eta, L_\xi \gamma),$$

we get that

$$d\widetilde{\theta}(\xi,\eta,\gamma) = L_\xi\left(\widetilde{\theta}(\eta,\gamma)\right) - \widetilde{\theta}(L_\xi\eta,\gamma) - \widetilde{\theta}(\eta, L_\xi\gamma) - \left(d i_\xi \widetilde{\theta}\right)(\eta,\gamma).$$

Further

$$\widetilde{\theta} = i_{\widetilde{j}}\widetilde{\Omega} \Longleftrightarrow i_\xi \widetilde{\theta} = i_{\widetilde{j}\xi}\widetilde{\Omega}$$

implies

$$d\widetilde{\theta}(\xi,\eta,\gamma) = L_\xi\left(\widetilde{\Omega}\left(\widetilde{j}\eta,\gamma\right)\right) - \widetilde{\Omega}\left(\widetilde{j}L_\xi\eta,\gamma\right) - \widetilde{\Omega}\left(\widetilde{j}\eta, L_\xi\gamma\right) - \left(d i_{\widetilde{j}\xi}\widetilde{\Omega}\right)(\eta,\gamma).$$

As for $L_\xi\left(\widetilde{\theta}(\eta,\gamma)\right)$, we can write:

$$\begin{aligned} d\widetilde{\Omega}\left(\widetilde{j}\xi,\eta,\gamma\right) &= L_{\widetilde{j}\xi}\left(\widetilde{\Omega}(\eta,\gamma)\right) - \widetilde{\Omega}\left(L_{\widetilde{j}\xi}\eta,\gamma\right) - \widetilde{\Omega}\left(\eta, L_{\widetilde{j}\xi}\gamma\right) - \left(d i_{\widetilde{j}\xi}\widetilde{\Omega}\right)(\eta,\gamma), \\ d\widetilde{\Omega}\left(\xi,\widetilde{j}\eta,\gamma\right) &= L_\xi\left(\widetilde{\Omega}\left(\widetilde{j}\eta,\gamma\right)\right) - \widetilde{\Omega}\left(L_\xi\widetilde{j}\eta,\gamma\right) - \widetilde{\Omega}\left(\widetilde{j}\eta, L_\xi\gamma\right) - \left(d i_\xi\widetilde{\Omega}\right)\left(\widetilde{j}\eta,\gamma\right). \end{aligned}$$

Using these equations, we get:

$$\begin{aligned}
d\widetilde{\theta}\left(\xi,\eta,\gamma\right) &= d\widetilde{\Omega}\left(\xi,\widetilde{j}\eta,\gamma\right)+\widetilde{\Omega}\left(L_{\xi}\widetilde{j}\eta,\gamma\right)+\left(di_{\xi}\widetilde{\Omega}\right)\left(\widetilde{j}\eta,\gamma\right)-\widetilde{\Omega}\left(\widetilde{j}L_{\xi}\eta,\gamma\right)\\
&\quad+d\widetilde{\Omega}\left(\widetilde{j}\xi,\eta,\gamma\right)-L_{\widetilde{j}\xi}\left(\widetilde{\Omega}\left(\eta,\gamma\right)\right)+\widetilde{\Omega}\left(L_{\widetilde{j}\xi}\eta,\gamma\right)+\widetilde{\Omega}\left(\eta,L_{\widetilde{j}\xi}\gamma\right)\\
&= d\widetilde{\Omega}\left(\xi,\widetilde{j}\eta,\gamma\right)+d\widetilde{\Omega}\left(\widetilde{j}\xi,\eta,\gamma\right)+\widetilde{\Omega}\left(\left[\xi,\widetilde{j}\eta\right],\gamma\right)+\left(di_{\xi}\widetilde{\Omega}\right)\left(\widetilde{j}\eta,\gamma\right)\\
&\quad-\widetilde{\Omega}\left(\widetilde{j}\left[\xi,\eta\right],\gamma\right)-L_{\widetilde{j}\xi}\left(\widetilde{\Omega}\left(\eta,\gamma\right)\right)+\widetilde{\Omega}\left(\left[\widetilde{j}\xi,\eta\right],\gamma\right)+\widetilde{\Omega}\left(\eta,\left[\widetilde{j}\xi,\gamma\right]\right).
\end{aligned}$$

The infinitesimal Stokes formula:

$$di_{\xi}\widetilde{\Omega}=L_{\xi}\widetilde{\Omega}-i_{\xi}d\widetilde{\Omega},$$

gives us:

$$\begin{aligned}
d\widetilde{\theta}\left(\xi,\eta,\gamma\right) &= d\widetilde{\Omega}\left(\xi,\widetilde{j}\eta,\gamma\right)+d\widetilde{\Omega}\left(\widetilde{j}\xi,\eta,\gamma\right)+\widetilde{\Omega}\left(\left[\xi,\widetilde{j}\eta\right],\gamma\right)+\left(L_{\xi}\widetilde{\Omega}-i_{\xi}d\widetilde{\Omega}\right)\left(\widetilde{j}\eta,\gamma\right)\\
&\quad-\widetilde{\Omega}\left(\widetilde{j}\left[\xi,\eta\right],\gamma\right)-L_{\widetilde{j}\xi}\left(\widetilde{\Omega}\left(\eta,\gamma\right)\right)+\widetilde{\Omega}\left(\left[\widetilde{j}\xi,\eta\right],\gamma\right)+\widetilde{\Omega}\left(\eta,\left[\widetilde{j}\xi,\gamma\right]\right)\\
&= d\widetilde{\Omega}\left(\widetilde{j}\xi,\eta,\gamma\right)+\left(L_{\xi}\widetilde{\Omega}\right)\left(\widetilde{j}\eta,\gamma\right)\\
&\quad+\widetilde{\Omega}\left(\left[\xi,\widetilde{j}\eta\right],\gamma\right)-\widetilde{\Omega}\left(\widetilde{j}\left[\xi,\eta\right],\gamma\right)-L_{\widetilde{j}\xi}\left(\widetilde{\Omega}\left(\eta,\gamma\right)\right)+\widetilde{\Omega}\left(\left[\widetilde{j}\xi,\eta\right],\gamma\right)\\
&\quad+\widetilde{\Omega}\left(\eta,\left[\widetilde{j}\xi,\gamma\right]\right).
\end{aligned}$$

We then use that

$$L_{\xi}\left(\widetilde{\Omega}\left(\widetilde{j}\eta,\gamma\right)\right)=\left(L_{\xi}\widetilde{\Omega}\right)\left(\widetilde{j}\eta,\gamma\right)+\widetilde{\Omega}\left(L_{\xi}\widetilde{j}\eta,\gamma\right)+\widetilde{\Omega}\left(\widetilde{j}\eta,L_{\xi}\gamma\right),$$

and we get:

$$\begin{aligned}
d\widetilde{\theta}(\xi,\eta,\gamma) &= d\widetilde{\Omega}\left(\widetilde{j}\xi,\eta,\gamma\right)+L_{\xi}\left(\widetilde{\Omega}\left(\widetilde{j}\eta,\gamma\right)\right)-\widetilde{\Omega}\left(L_{\xi}\widetilde{j}\eta,\gamma\right)\\
&\quad -\widetilde{\Omega}\left(\widetilde{j}\eta,L_{\xi}\gamma\right)+\widetilde{\Omega}\left(\left[\xi,\widetilde{j}\eta\right],\gamma\right)-\widetilde{\Omega}\left(\widetilde{j}\left[\xi,\eta\right],\gamma\right)\\
&\quad -L_{\widetilde{j}\xi}\left(\widetilde{\Omega}(\eta,\gamma)\right)+\widetilde{\Omega}\left(\left[\widetilde{j}\xi,\eta\right],\gamma\right)+\widetilde{\Omega}\left(\eta,\left[\widetilde{j}\xi,\gamma\right]\right)\\
&= +d\widetilde{\Omega}\left(\widetilde{j}\xi,\eta,\gamma\right)+L_{\xi}\left(\widetilde{\Omega}\left(\widetilde{j}\eta,\gamma\right)\right)-\widetilde{\Omega}\left(\left[\xi,\widetilde{j}\eta\right],\gamma\right)\\
&\quad -\widetilde{\Omega}\left(\widetilde{j}\eta,\left[\xi,\gamma\right]\right)+\widetilde{\Omega}\left(\left[\xi,\widetilde{j}\eta\right],\gamma\right)-\widetilde{\Omega}\left(\widetilde{j}\left[\xi,\eta\right],\gamma\right)\\
&\quad -L_{\widetilde{j}\xi}\left(\widetilde{\Omega}(\eta,\gamma)\right)+\widetilde{\Omega}\left(\left[\widetilde{j}\xi,\eta\right],\gamma\right)+\widetilde{\Omega}\left(\eta,\left[\widetilde{j}\xi,\gamma\right]\right)\\
&= L_{\xi}\left(\widetilde{\Omega}\left(\widetilde{j}\eta,\gamma\right)\right)-L_{\widetilde{j}\xi}\left(\widetilde{\Omega}(\eta,\gamma)\right)+\widetilde{\Omega}\left(\left[\widetilde{j}\xi,\eta\right],\gamma\right)-\widetilde{\Omega}\left(\widetilde{j}\left[\xi,\eta\right],\gamma\right)\\
&\quad +\widetilde{\Omega}\left(\eta,\left[\widetilde{j}\xi,\gamma\right]\right)-\widetilde{\Omega}\left(\widetilde{j}\eta,\left[\xi,\gamma\right]\right)+d\widetilde{\Omega}\left(\widetilde{j}\xi,\eta,\gamma\right).
\end{aligned}$$

Since $\widetilde{\theta}$ and $\widetilde{\Omega}$ are skew symmetric, we have that:

$$\begin{aligned}
\widetilde{\Omega}\left(\widetilde{j}\eta,\left[\xi,\gamma\right]\right) &= i_{\widetilde{j}}\widetilde{\theta}(\eta,[\xi,\gamma])=-i_{\widetilde{j}}\widetilde{\theta}([\xi,\gamma],\eta)=-\widetilde{\Omega}\left(\widetilde{j}\left[\xi,\gamma\right],\eta\right)\\
&= \widetilde{\Omega}\left(\eta,\widetilde{j}\left[\xi,\gamma\right]\right).
\end{aligned}$$

So finally we get:

$$\begin{aligned}
d\widetilde{\theta}(\xi,\eta,\gamma) &= L_{\xi}\left(\widetilde{\Omega}\left(\widetilde{j}\eta,\gamma\right)\right)-L_{\widetilde{j}\xi}\left(\widetilde{\Omega}(\eta,\gamma)\right)+\widetilde{\Omega}\left(\left[\widetilde{j}\xi,\eta\right],\gamma\right)\\
&\quad -\widetilde{\Omega}\left(\widetilde{j}\left[\xi,\eta\right],\gamma\right)+\widetilde{\Omega}\left(\eta,\left[\widetilde{j}\xi,\gamma\right]\right)-\widetilde{\Omega}\left(\eta,\widetilde{j}\left[\xi,\gamma\right]\right)\\
&\quad +d\widetilde{\Omega}\left(\widetilde{j}\xi,\eta,\gamma\right),
\end{aligned}$$

just as we should prove.

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