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The Geometry of a Quasilinear System of Two Partial Differential Equations Containing the First and the Second Partial Derivatives of Two Functions in Two Independent Variables

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Abstract—The geometry of the system of two partial differential equations containing the first and second partial derivatives of two functions in two independent variables is studied by using Élie Cartan's method of invariant forms and the group-theoretic method of extensions and enclosings due to G. F. Laptev (for finite groups) and A. M. Vasil'ev (for infinite groups). Systems of quasilinear equations with the first and second partial derivatives of two functions u and v in two independent variables x and y are classified.

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STATEMENT OF THE PROBLEM

Suppose given a system of two partial differential equations for two functions (u, v) of two independent variables (x, y), in which one equation is of order 1 and the other is of order 2, in the general form

$$\begin{aligned} F(x, y, u, v, u_x, u_y, v_x, v_y) &= 0, \\ \Phi(x, y, u, v, u_x, u_y, v_x, v_y, u_{xx}, u_{xy}, u_{yy}, v_{xx}, v_{xy}, v_{yy}) &= 0. \end{aligned} \tag{1}$$

This paper studies the geometry of this system of differential equations, i.e., according to Klein's general scheme, *properties of its set of integral manifolds invariant under transformations from the infinite group of point transformations*

$$\begin{aligned} x' &= f_1(x, y, u, v), \\ y' &= f_2(x, y, u, v), \\ u' &= f_3(x, y, u, v), \\ v' &= f_4(x, y, u, v), \end{aligned} \tag{2}$$

and classifies these systems.

The study uses Cartan's method of exterior forms [1]–[3] and the theory of immersed manifolds developed by Laptev [4] *for finite groups* and Vasil'ev [5], [6] *for infinite groups*.

To study the geometry of the given differential equations, we need a representation of the corresponding infinite group, which shows how this group transforms the variables involved in the equations.

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Namely, the group (2) transforms the variables involved in Eq. (1) according to formulas (2) supplemented by the equations

$$u'_x = \frac{\left(\dfrac{\partial f_3}{\partial x} + \dfrac{\partial f_3}{\partial u}u_x + \dfrac{\partial f_3}{\partial v}v_x\right)\left(\dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_2}{\partial u}u_y + \dfrac{\partial f_2}{\partial v}v_y\right)}{A} - \frac{\left(\dfrac{\partial f_3}{\partial y} + \dfrac{\partial f_3}{\partial u}u_y + \dfrac{\partial f_3}{\partial v}v_y\right)\left(\dfrac{\partial f_2}{\partial x} + \dfrac{\partial f_2}{\partial u}u_x + \dfrac{\partial f_2}{\partial v}v_x\right)}{A},$$

$$u'_y = \frac{\left(\dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_1}{\partial u}u_x + \dfrac{\partial f_1}{\partial v}v_x\right)\left(\dfrac{\partial f_3}{\partial y} + \dfrac{\partial f_3}{\partial u}u_y + \dfrac{\partial f_3}{\partial v}v_y\right)}{A} - \frac{\left(\dfrac{\partial f_1}{\partial y} + \dfrac{\partial f_1}{\partial u}u_y + \dfrac{\partial f_1}{\partial v}v_y\right)\left(\dfrac{\partial f_3}{\partial x} + \dfrac{\partial f_3}{\partial u}u_x + \dfrac{\partial f_3}{\partial v}v_x\right)}{A}, \tag{3}$$

$$A = \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial u}u_x + \frac{\partial f_1}{\partial v}v_x\right)\left(\frac{\partial f_2}{\partial y} + \frac{\partial f_2}{\partial u}u_y + \frac{\partial f_2}{\partial v}v_y\right) - \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial u}u_y + \frac{\partial f_1}{\partial v}v_y\right)\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial u}u_x + \frac{\partial f_2}{\partial v}v_x\right).$$

For $v'_{x'}$ and $v'_{y'}$, similar formulas hold. The derivatives $u'_{x'x'}$, $u'_{x'y'}$, $u'_{y'y'}$, $v'_{x'x'}$, $v'_{x'y'}$, and $v'_{y'y'}$ can be expressed in terms of the variables x, y, u, v, u_x, u_y, v_x, v_y, u_{xx}, u_{xy}, u_{yy}, v_{xx}, v_{xy}, and v_{yy}. We treat the derivatives u_x, u_y, v_x, v_y, u_{xx}, u_{xy}, u_{yy}, v_{xx}, v_{xy}, and v_{yy} as new variables transformed according to (3). Then the equations of the problem contain 14 variables, namely, $u_1 = x$, $u_2 = y$, $u_3 = u$, $u_4 = v$, $p_1 = u_x$, $p_2 = u_y$, $p_3 = v_x$, $p_4 = v_y$, $q_1 = u_{xx}$, $q_2 = u_{xy}$, $q_3 = u_{yy}$, $q_4 = v_{xx}$, $q_5 = v_{xy}$, and $q_6 = v_{yy}$. Specifying this system of equations reduces to specifying a submanifold in the representation space. The variables contained only in the first equation, that is, u_1, u_2, u_3, u_4, p_1, p_2, p_3, and p_4, determine an eight-dimensional space $M^{(8)}$, and the first equation determines a seven-dimensional surface $M^{(7)}$ in this space.

The two equations of the problem impose two constraints on the 14 variables. There remain 12 free variables. They determine a manifold $M^{(12)}$ in $M^{(14)}$, which we assume to be sufficiently smooth. To apply the method of exterior forms, we must specify this manifold by Pfaffian equations with respect to independent invariant linear forms of the corresponding infinite group. From the variables $u_1, u_2, \dots, q_6$ we pass to their differentials $du_1, du_2, \dots, dq_6$; then, changing the basis, we introduce the same number of new linearly independent forms. Instead of the differentials du_1, du_2, du_3, and du_4, we introduce forms ω^1, ω^2, ω^3, and ω^4; instead of the differentials dp_1, dp_2, dp_3, and dp_4, we introduce basis forms ω_1^3, ω_2^3, ω_1^4, and ω_2^4; and, instead of the differentials dq_1, dq_2, dq_3, dq_4, dq_5, and dq_6, we introduce forms ω_{11}^3, ω_{12}^3, ω_{22}^3, ω_{11}^4, ω_{12}^4, and ω_{22}^4. These new invariant forms of the group obey certain relations; namely, the exterior differentials of these forms satisfy the Lie–Cartan structure equations

$$\begin{aligned} D\omega^i &= [\omega_k^i \omega^k], \\ D\omega_k^i &= [\omega_l^i \omega_k^l] + [\omega_{kl}^i \omega^l], \qquad [\omega_{kl}^i \omega^k \omega^l] = 0, \\ D\omega_{kl}^i &= [\omega_{pl}^i \omega_k^p] + [\omega_{kp}^i \omega_l^p] - [\omega_{kl}^p \omega_p^i] + [\omega_{klp}^i \omega^p], \qquad [\omega_{klp}^i \omega^k \omega^l \omega^p] = 0, \quad i,k,l,p = 1,2,3,4. \end{aligned} \tag{4}$$

The differentials of the basic equations of the problem are

$$dF(u_1, u_2, u_3, u_4, p_1, p_2, p_3, p_4) = 0,$$
$$d\Phi(u_1, u_2, u_3, u_4, p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4, q_5, q_6) = 0,$$

or

$$\begin{aligned} &\frac{\partial F}{\partial u_1}\,du_1+\frac{\partial F}{\partial u_2}\,du_2+\frac{\partial F}{\partial u_3}\,du_3+\frac{\partial F}{\partial u_4}\,du_4 \\ &\qquad+\frac{\partial F}{\partial p_1}\,dp_1+\frac{\partial F}{\partial p_2}\,dp_2+\frac{\partial F}{\partial p_3}\,dp_3+\frac{\partial F}{\partial p_4}\,dp_4=0, \\ &\frac{\partial \Phi}{\partial u_1}\,du_1+\dots+\frac{\partial \Phi}{\partial p_1}\,dp_1+\dots+\frac{\partial \Phi}{\partial q_1}\,dq_1+\dots+\frac{\partial \Phi}{\partial q_6}\,dq_6=0. \end{aligned} \tag{5}$$

Let us pass from the differentials dp_1, dp_2, dp_3, dp_4, dq_1, dq_2, dq_3, dq_4, dq_5, and dq_6 to independent linear invariant forms of the group under consideration:

$$\begin{aligned} \widetilde{A}\omega_1^3+\widetilde{B}\omega_2^3+\widetilde{C}\omega_1^4+\widetilde{D}\omega_2^4+\widetilde{K}_i\omega^i&=0, \\ \overline{A}\omega_{11}^3+\overline{B}\omega_{22}^3+\overline{C}\omega_{11}^4+\overline{D}\omega_{22}^4+\alpha\omega_1^3+\beta\omega_2^3+\gamma\omega_2^4+e_i\omega^i&=0, \end{aligned} \qquad i=1,2,3,4. \tag{6}$$

Any *geometric object* invariantly related to the equation is a *point in the representation space of the given transformation group*. According to the general theory, all of them are obtained by means of *exterior differentiation, extension, and canonization of these equations*. If $\widetilde{C}\neq 0$, then the first equation in (6) can be solved with respect to ω_1^4:

$$\omega_1^4=A\omega_1^3+B\omega_2^3+C\omega_2^4+a_i\omega^i=0, \qquad i=1,2,3,4. \tag{7}$$

Canonization reduces (7) to $\omega_1^4=\omega_2^3$ (provided that $B\neq 0$). We consider the general case $B\neq 0$.

Lemma (of Cartan [3])**.** *If $2r$ linear forms f_i, φ_i from an n-dimensional ring* $\operatorname{Re}[u]$ *satisfy the identity*

$$[f_1\varphi_1]+[f_2\varphi_2]+\dots+[f_r\varphi_r]=0$$

and the forms f_i constitute a system of rank r, then the forms φ_k admit linear expressions in terms of f_i with symmetric matrix coefficients.

The exterior differentiation of Eq. (7) and the application of Cartan's lemma yield

$$\begin{aligned} -\omega_3^4&=\omega_2^1+a_{11}\omega_1^3+a_{12}\omega_2^3+a_{13}\omega_2^4+a_{1,i+3}\omega^i=0, \\ -\omega_4^4&=-\omega_1^1+\omega_2^2-\omega_3^3+a_{12}\omega_1^3+a_{22}\omega_2^3+a_{23}\omega_2^4+a_{2,i+3}\omega^i=0, \\ \omega_4^3&=-\omega_1^2+a_{13}\omega_1^3+a_{23}\omega_2^3+a_{33}\omega_2^4+a_{3,i+3}\omega^i=0, \\ \omega_{12}^3&=\omega_{11}^4+\cdots, \\ \omega_{23}^3&=\omega_{13}^4+\cdots, \\ \omega_{24}^3&=\omega_{14}^4+\cdots, \end{aligned} \qquad i=1,2,3,4. \tag{8}$$

The coefficient matrix is symmetric. After canonization, some of the coefficients a_{ij} can be killed. The remaining coefficients form a geometric object; these are a_{11}, a_{12}, a_{22}, a_{23}, and a_{33}. To study the geometric meaning of the geometric objects which we obtain in what follows, we need another interpretation of the 8-space $M^{(8)}$. Namely, consider the 4-space $M^{(4)}$. We associate each point (u_1, u_2, u_3, u_4) in this space with the two vectors

$$\begin{aligned} \vec{e}_1&=\overline{M}_{u_1}+\overline{M}_{u_3}\cdot p_1+\overline{M}_{u_4}\cdot p_3, \\ \vec{e}_2&=\overline{M}_{u_2}+\overline{M}_{u_3}\cdot p_2+\overline{M}_{u_4}\cdot p_4. \end{aligned}$$

These two vectors determine a plane. Each point (u_1, $\dots$, u_4, p_1, $\dots$, p_4) of the 8-space $M^{(8)}$ is associated with the point $M(u_1,u_2,u_3,u_4)$ and a 2-plane. It is easy to see that, for the integral manifold of the first initial equation

$$u=u(x,y), \quad v=v(x,y), \qquad p_1=\frac{\partial u}{\partial x}, \quad p_2=\frac{\partial u}{\partial y}, \quad p_3=\frac{\partial v}{\partial x}, \quad p_4=\frac{\partial v}{\partial y},$$

this 2-plane is a tangent element.

The first equation

$$\omega_1^4 = \omega_2^3$$

determines the submanifold of such two-dimensional elements. In this 4-space, take the point

$$\omega^i = 0, \qquad i = 1, 2, 3, 4.$$

We refer to it as the *zero point* of the space. The transformations from the group under consideration that leave this point invariant form the *centroaffine group*. Consider the structure equation (4) for the invariant forms of the problem under consideration. If $\omega^i = 0$, then these equations are

$$D\omega_k^i = [\omega_l^i \omega_k^l], \qquad i = 1, 2, 3, 4. \tag{9}$$

The remaining equations disappear. The equations (9) are the *structure equations of the centroaffine group*. Each fixed point $\omega^i = 0$ determine a 4-manifold of 2-planes. The *projective interpretation* of this pattern is as follows. The intersection of the centroaffine space with an improper hyperplane not passing through the zero point is a projective space with structure equations (9) and a moving frame (a tetrahedron) $A_1A_2A_3A_4$ whose infinitesimal transformations have the form

$$dA_i = \omega_i^k A_k, \qquad i, k = 1, 2, 3, 4. \tag{10}$$

The intersection of the 3-manifold of 2-planes in the centroaffine space determined by Eq. (7) with an improper 3-hyperplane not passing through the zero point is a *line complex*. To each line from this complex we assign a tetrahedron $A_1A_2A_3A_4$ whose edge coincides with this line. Geometrically, the canonization $A = 0$, $C = 0$, $B = 1$ of the equation corresponds to the choice of a tetrahedron $A_1A_2A_3A_4$ attached to the line from the complex in the most convenient way. Namely, the vanishing of the coefficient A means that the plane $A_2A_1A_3$ of the tetrahedron is tangent to the cone of rays from the complex with vertex at A_2; the vanishing of C corresponds to the plane $A_1A_2A_4$ being tangent to the cone of rays with vertex at A_1. The coefficient B is set equal to 1 by normalizing the coordinates of the tetrahedron vertex A_4. The object $(a_{11}, a_{12}, a_{22}, a_{23}, a_{33})$ characterizes this complex.

Thereby, *classifying the first-order equation of the problem under consideration at a point reduces to classifying complexes*. The classification of complexes is known; therefore, the geometric meaning of the first equation is known as well. For example, if $a_{11} = a_{12} = a_{22} = a_{23} = a_{33} = 0$, then the corresponding complex is *linear*. In this case, we can consider the system of equations

$$\begin{gathered} \omega_1^4 = \omega_2^3, \\ \overline{A}\omega_{11}^3 + \overline{B}\omega_{22}^3 + \overline{C}\omega_{11}^4 + \overline{D}\omega_{22}^4 + \alpha\omega_1^3 + \beta\omega_2^3 + \gamma\omega_2^4 + e_i\omega^i = 0, \end{gathered} \qquad i = 1, 2, 3, 4. \tag{11}$$

We can solve the second equation with respect to various three-index forms (in the general case) and consider various canonizations of the resulting equations. The geometric meaning of each canonization is clarified by considering a new interpretation of the representation space of the group of transformations with respect to which the equations of the problem are studied. Under this new interpretation, the generating element consists of a point, a plane element, and a second-order element passing through the plane element. A point in the representation space and a plane element determine a 3-manifold of second-order elements. Indeed, the six second-order derivatives satisfy three equations, two of which are extensions of the first equation and one is the given equation. Let us introduce the notion of a characteristic. A *characteristic* is a linear element through which there passes more two-dimensional integral elements than through the neighboring elements. The initial conditions of the system cannot be set along characteristic directions.

Assuming that $\overline{C} \neq 0$ and solving the second equation of system (11) with respect to ω_{11}^4, we obtain a special case of the system of differential equations considered at the beginning of the paper. Indeed, further canonization leads only to the following cases:

1) $\omega_{11}^4 = \omega_{22}^3$, which corresponds to three different real characteristics;

2) $\omega_{11}^4 = 0$, which corresponds to two coinciding characteristics and one characteristic different from them.

Assuming that $\overline{B} \neq 0$ and solving the second equation with respect to ω^3_{22}, we again obtain a special case, because canonization yields the new equation

1) $\omega^3_{22} = 0$, which corresponds to two coinciding characteristics and one characteristic different from them.

If $\overline{A} \neq 0$, the second equation of system (11) can be solved with respect to ω^3_{11}. In this case, canonization leads to a system of general type:

1) $\omega^3_{11} = \omega^3_{22}$ corresponds to three different real characteristics;

2) $\omega^3_{11} = \omega^4_{11}$ corresponds to two coinciding characteristics and one characteristic different from them;

3) $\omega^3_{11} = 0$ corresponds to three coinciding characteristics.

A system of general type is also obtained if $\overline{D} \neq 0$ and the second equation in (11) is solved with respect to ω^4_{22}. In what follows, we consider this case. Canonizing such a system, we reduce the second equation to one of the following equations:

1) $\omega^4_{22} = \omega^4_{11}$, which corresponds to three different real characteristics;

2) $\omega^4_{22} = \omega^3_{11}$, which corresponds to three different characteristics (two of which are complex conjugate);

3) $\omega^4_{22} = \omega^3_{22}$, which corresponds to two coinciding characteristics and one characteristic different from them;

4) $\omega^4_{22} = 0$, which corresponds to three coinciding characteristics.

This is a primary classification of the system. Consider the first system (of the most general form, with three different real characteristics). We have

$$\begin{aligned} \omega^4_1 &= \omega^3_2, \\ \omega^4_{22} &= \omega^4_{11}. \end{aligned} \tag{12}$$

Further extension and, possibly, canonization yields the following equations under the linear complex condition $a_{11} = a_{12} = a_{22} = a_{23} = 0$:

$$\begin{aligned} \omega^1_2 &= b_{11}\omega^3_{11} + b_{12}\omega^4_{11} + b_{13}\omega^3_{22} + a\omega^3_1 + b\omega^3_2 + e\omega^4_2 + b_{1,i+6}\omega^i, \\ \omega^2_1 &= b_{12}\omega^3_{11} + b_{22}\omega^4_{11} + b_{23}\omega^3_{22} - e\omega^3_1 + g\omega^3_2 + f\omega^4_2 + b_{2,i+6}\omega^i, \\ \omega^1_1 - \omega^2_2 &= b_{13}\omega^3_{11} + b_{23}\omega^4_{11} + b_{33}\omega^3_{22} - b\omega^3_1 + c\omega^3_2 + g\omega^4_2 + b_{3,i+6}\omega^i, \end{aligned} \tag{13}$$

$$\begin{aligned} -2\omega^4_{13} &= \dots, \\ 2\omega^4_{23} - \omega^1_{22} - 2\omega^4_{14} + \omega^1_{11} &= \dots, \\ 2\omega^4_{24} - \omega^2_{22} + \omega^2_{11} &= \dots, \end{aligned} \tag{14}$$

$$\omega^4_{22i} - \omega^4_{11i} = 0. \tag{15}$$

The coefficients of the three-index forms form a geometric object, and those of the two-index forms are relative invariants of the group under consideration (provided that the point $\omega^i = 0$, where $i = 1, 2, 3, 4$, is fixed). Using these objects, we can refine the classification of the system. Namely, if a point, a plane element, and a two-dimensional element are fixed, then the three characteristics in the intersection with a line from the complex determine a triple of points. We can always assume that these points are A_1, A_2, and $A_1 + A_2$. If only an initial point and a plane element are fixed, then such triples form a three-parameter family. Thus, properties of the system at a point are determined by those of the geometric configuration consisting of a complex of lines on each of which a three-parameter manifold of triples of points is defined. If

$$a_{11} = a_{12} = a_{22} = a_{23} = a_{33} = 0, \qquad b_{ij} = 0 \quad \text{for } i = 1,2,3,4, \qquad a = b = c = -e = -f = g = 0$$

(the object vanishes), and all relative invariants vanish, then, on each ray, these triples are fixed, and the improper 3-hyperplane contains a fixed line through which three planes cutting out this triple of points on each line from the complex pass. Therefore, we can perform classification, first, by decreasing the arbitrariness of these triples on each ray of the complex, i.e., by reducing the rank of the matrix

$$\begin{pmatrix} a & b & e \\ -e & g & f \\ -b & c & g \end{pmatrix}, \tag{16}$$

and secondly, according to the existence on a line A_1A_2 in the complex of points $A_1 + A_2$ sweeping out only a 2-surface rather the entire space under the motion of the line. The problem leads to the fourth-order equation

$$ah^4 - 2bh^3 + (2eA_1 + A_2c)h^2 + 2gh - f = 0, \tag{17}$$

which suggests the following general classification:

1) the line contains no such points at all;

2) there are four different points;

3) various cases of the coincidence of such points;

4) the equation holds identically.

Since we consider a system of differential equations for which all of the three characteristics are real and different, it follows that there remain only three possibilities for Eq. (17):

1) the equation has no roots (i.e., none of the points on the line AA_2 from the complex describes a surface under the motion of the ray);

2) there are four different roots, i.e., four points A_1, A_2, $A_1 + A_2$, and A, describing surfaces (all of these surfaces are different);

3) there exists one double root and two different roots.

At a “double” point, only the coincidence of the point A with the point A_1, A_2, or $A_1 + A_2$ can occur (by assumption, the points A_1, A_2, and $A_1 + A_2$ cannot coincide). There are no other possible cases. Conditions for the points A_1 $(h_1 = 0)$, A_2 $(h_2 = \infty)$, and $A_1 + A_2$ $(h_3 = 1)$ to sweep out a surface under the motion of a line from the complex are

$$f = 0, \qquad a - 2b + 2e + c + 2g - f = 0, \tag{18}$$

respectively.

If these conditions hold simultaneously, then the coordinates of the fourth point A sweeping out a surface are determined by

$$h_4 = \frac{-g}{b} = -\lambda, \qquad \frac{g}{b} = \lambda = \text{const.}$$

The differential equations of the system under consideration at the point, i.e., when

$$\omega^i = 0 \qquad \text{for} \quad i = 1,2,3,4,$$

transform into the Pfaffian equations

$$\begin{aligned} \omega_1^4 &= \omega_2^3, \\ \omega_4^1 &= \omega_3^2, \end{aligned} \tag{19}$$

$$\begin{aligned} \omega_2^1 &= b\omega_2^3 + e\omega_2^4, \\ \omega_1^2 &= -e\omega_1^3 + \lambda b\omega_2^3, \\ \omega_1^1 - \omega_2^2 &= -b\omega_1^3 + 2(e + (\lambda - 1))b\omega_2^3 + \lambda b\omega_2^4 \end{aligned} \tag{20}$$

$$\begin{aligned} \omega_3^4 &= -\omega_2^1, \\ \omega_4^3 &= -\omega_1^2, \\ \omega_4^4 - \omega_3^3 &= \omega_1^1 - \omega_2^2. \end{aligned} \tag{21}$$

The classification of this system can be refined by reducing the rank of the matrix of coefficients on the right-hand sides of Eqs. (13):

$$D = \det \begin{pmatrix} 0 & b & e \\ -e & b & 0 \\ -b & 2(e + (-1)b) & b \end{pmatrix}. \tag{22}$$

In the general case, this determinant does not identically vanish; therefore, the rank of this matrix is equal to 3. The arbitrariness of the triples of points on rays from the complex is not violated if each ray from the complex contains four points sweeping out a surface under a displacement of this ray. The rank of the matrix decreases to 2 if either

$$e = 0, \qquad b = -e \qquad \text{or} \qquad \lambda b + e = 0.$$

Thus, if one of conditions (18) holds, then the points A_1, A_2, $A_1 + A_2$, and A on the line from the complex sweep out a surface, while the degree of arbitrariness of triples of points in the intersection of the characteristics with the line from the complex decreases to 2:

$$e = 0, \qquad b = -e \quad \text{or} \quad \lambda b + e = 0. \tag{23}$$

In [7], the case of three different characteristics on each integral manifold was considered. In the projective space associated with the tangent space to $M^{(4)}$ at an arbitrary point, such a system determines a complex of lines. Each line formed by lines from the complex contains a family of triples of points, which depends on three parameters in the general case and on fewer parameters in special cases. In [7], a classification of such geometric patterns was suggested. In particular, it turned out that simplest is the case in which the *complex is linear* and the three characteristic points on each of its lines are uniquely determined as the intersection points of the given line with three constant planes passing through one line belonging to the complex. In this case, the vector of frames associated with the system at a point satisfies the differential equations

$$\begin{aligned} dA_1 &= \widetilde{\omega} A_1 + \widetilde{\omega}_1^3 A_3 + \widetilde{\Omega} A_4, \\ dA_3 &= \widetilde{\Theta} A_3, \\ dA_2 &= \widetilde{\omega} A_2 + \widetilde{\Omega} A_3 + \widetilde{\omega}_2^4 A_4, \\ dA_4 &= \widetilde{\Theta} A_4 \end{aligned} \tag{24}$$

(the vectors A_1 and A_2 belong to a plane element; its characteristic directions are determined by the vectors A_1, A_2, and $A_1 - A_2$; the line in the intersection of the three planes is determined by the vectors

A_3 and A_4). In addition, the following differential equations hold:

$$\begin{aligned} d\omega^1 &= [\omega\omega^1] + \cdots, \\ d\omega^2 &= [\omega\omega^2] + \cdots, \\ d\omega^3 &= [\omega_1^3\omega^1] + [\Omega\omega^2] + [\Theta\omega^3] + \cdots, \\ d\omega^4 &= [\Omega\omega^1] + [\omega_2^4\omega^2] + [\Theta\omega^4] + \cdots, \end{aligned} \tag{25}$$

where the forms contained in (24) are obtained from the corresponding unmarked forms in (25) by fixing a point in $M^{(4)}$. The terms not written out are linear combinations of the forms ω^i. In the case under consideration, studying the system reduces to studying a g-structure in $M^{(4)}$. The group g leaves invariant the linear complex and the three planes specified above in the improper projective space, and the family of frames is determined by Eqs. (24).

We consider the even more special case in which the terms not written out in (25) reduce to zero, i.e., the first structure function of the g-structure under consideration vanishes:

$$\begin{aligned} d\omega^1 &= [\omega\omega^1], \\ d\omega^2 &= [\omega\omega^2], \\ d\omega^3 &= [\omega_1^3\omega^1] + [\Omega\omega^2] + [\Theta\omega^3], \\ d\omega^4 &= [\Omega\omega^1] + [\omega_2^4\omega^2] + [\Theta\omega^4]. \end{aligned} \tag{26}$$

Comparing (26) with (4), we obtain

$$\widetilde{\omega} = \widetilde{\omega}_1^1 = \widetilde{\omega}_2^2, \qquad \widetilde{\Omega} = \widetilde{\omega}_3^3 = \widetilde{\omega}_4^4; \tag{27}$$

the remaining forms $\widetilde{\omega}_i^k$ vanish. Differentiating Eq. (26), we obtain

$$\begin{aligned} [\Delta\omega_1^3\omega^1] + [\Delta\Omega\omega^2] + [d\Theta\omega^3] &= 0, \\ [\Delta\Omega\omega^1] + [\Delta\omega_2^4\omega^2] + [d\Theta\omega^4] &= 0, \\ [d\omega\omega^1] &= 0, \\ [d\omega\omega^2] &= 0, \end{aligned} \tag{28}$$

where

$$\begin{aligned} \Delta &= d\omega_1^3 - [\omega_1^3\omega] - [\Theta\omega_1^3], \\ \Delta\Omega &= d\Omega - [\Omega\omega] - [\Theta\Omega], \\ \omega_2^4 &= d\omega_2^4 - [\omega_2^4\omega] - [\Theta\omega_2^4]. \end{aligned} \tag{29}$$

To Eqs. (28), we first apply the generalized Cartan lemma and then Cartan's lemma proper. Taking into account the fact that the equations of the system give three equations for the forms and the characteristic directions are determined by the equations

$$\omega^1 = 0, \qquad \omega^2 = 0, \qquad \omega^1 + \omega^2 = 0$$

and introducing the notation

$$\omega_{12}^3 = \omega_{11}^4 = \omega_{22}^3 = \omega_{12}^4 = \Omega_0,$$

we obtain

$$\begin{aligned} \Delta\omega_1^3 &= [\omega_{11}^3\omega^1] + [\Omega_0\omega^2] + A_{1,ik}^3[\omega^i\omega^k], \\ \Delta\omega_2^4 &= [\Omega_0\omega^1] + [\omega_{22}^4\omega^2] + A_{2,ik}^4[\omega^i\omega^k], \\ \Delta\Omega &= [\Omega_0(\omega^1+\omega^2)] + B_{ik}[\omega^i\omega^k], \\ \Delta\Theta &= A_{ik}[\omega^i\omega^k]. \end{aligned} \tag{30}$$

The transformations of forms

$$\omega_{11}^3 \to \omega_{11}^3 + u_i^3\omega^k, \qquad \omega_{22}^4 \to \omega_{22}^4 + u_i^4\omega^k, \qquad \Omega_0 \to \Omega_0 + v^i\omega^i,$$

which do not affect equations (30), reduce these equations to the form

$$\begin{aligned}
\Delta\omega^3_1 &= [\omega^3_{11}\omega^1] + [\Omega_0\omega^2] - A_{12}[\omega^2\omega^3] + B_{14}[\omega^2\omega^4] + A_{14}[\omega^3\omega^4],\\
\Delta\omega^4_2 &= [\Omega_0\omega^1] + [\omega^4_{22}\omega^2] + B_{23}[\omega^1\omega^3] + A_{12}[\omega^1\omega^4] - A_{23}[\omega^3\omega^4],\\
\Delta\Omega &= [\Omega_0(\omega^1+\omega^2)] + B_{23}[\omega^2\omega^3] + B_{14}[\omega^1\omega^4] + A[\omega^3\omega^4],\\
d\Theta &= A_{12}[\omega^1\omega^2] + A[\omega^1\omega^3] + A_{14}[\omega^1\omega^4] + A_{23}[\omega^2\omega^3] + A[\omega^2\omega^4] + A_{34}[\omega^3\omega^4],\\
d\omega &= K[\omega^1\omega^2].
\end{aligned} \tag{31}$$

Note that the system under examination is equivalent to the system of exterior differential equations

$$\omega^3 = 0, \quad \omega^4 = 0, \qquad [(\omega^3_1 - \Omega)\omega^1] = 0, \quad [(\omega^4_2 - \Omega)\omega^2] = 0, \quad [\Omega(\omega^1+\omega^2)] = 0 \tag{32}$$

on the manifold $M^{(7)}$. Taking the exterior differentials of Eq. (31) and reducing similar terms, we obtain

$$\begin{aligned}
&[\Delta\omega^3_{11}\omega^1] + [\Delta\Omega_0\omega^2] - [\Delta A_{12}\omega^2\omega^3] + [\Delta B_{14}\omega^2\omega^4] + [\Delta A_{14}\omega^3\omega^4] = 0,\\
&[\Delta\Omega_0\omega^1] + [\Delta\omega^4_{22}\omega^2] + [\Delta B_{23}\omega^1\omega^3] + [\Delta A_{12}\omega^1\omega^4] - [\Delta A_{23}\omega^3\omega^4] = 0,\\
&[\Delta\Omega_0(\omega^1+\omega^2)] + [\Delta B_{23}\omega^2\omega^3] + [\Delta B_{14}\omega^1\omega^4] + [\Delta A\omega^3\omega^4] = 0,\\
&[\Delta A_{12}\omega^1\omega^2] + [\Delta A(\omega^2\omega^4 - \omega^1\omega^3)] + [\Delta A_{14}\omega^1\omega^4]\\
&\qquad + [\Delta A_{23}\omega^2\omega^3] + [\Delta A_{34}\omega^3\omega^4] = 0,\\
&[\Delta K\omega^1\omega^2] = 0,
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
\Delta K &= dK + 2K\omega,\\
\Delta A_{34} &= dA_{34} + 2A_{34}\Theta,\\
\Delta A &= dA + A\omega + A\Theta + A_{34}\Omega,\\
\Delta A_{14} &= dA_{14} + A_{14}\omega + A_{14}\Theta + A_{34}\omega^3_1,\\
\Delta A_{23} &= dA_{23} + A_{23}\omega + A_{23}\Theta - A_{34}\omega^4_2,\\
\Delta A_{12} &= dA_{12} + 2A_{12}\omega + A_{14}\omega^4_2 - A_{23}\omega^3_1,\\
\Delta B_{14} &= dB_{14} + 2B_{14}\omega + A\omega^3_1 + A_{14}\Omega - 2A\Omega,\\
\Delta B_{23} &= dB_{23} + 2B_{23}\omega - A\omega^4_2 + A_{23}\Omega + 2A\Omega,\\
\Delta\Omega_0 &= d\Omega_0 - 2[\Omega_0\omega] + [\Omega_0\Theta] - \frac{1}{2}K[\Omega(\omega^1-\omega^2)] + \frac{1}{2}A_{12}[\Omega(\omega^1-\omega^2)]\\
&\qquad - \frac{1}{2}B_{23}[\omega^3_1(\omega^1-\omega^2)] + \frac{1}{2}B_{14}[\omega^4_2(\omega^1-\omega^2)] - 2A[\Omega\omega^4] + 2A[\Omega\omega^3],\\
\Delta\omega^3_{11} &= d\omega^3_{11} - 2[\omega^3_{11}\omega] + [\omega^3_{11}\Theta] + K[\omega^3_1\omega^2] - \frac{1}{2}K[\Omega\omega^2]\\
&\qquad - 2A_{12}[\omega^3_1\omega^2] + \frac{1}{2}A_{12}[\Omega\omega^2] + B_{14}[\Omega\omega^2] + \frac{1}{2}B_{14}[\omega^4_2\omega^2]\\
&\qquad - \frac{1}{2}B_{23}[\omega^3_1\omega^2] - 2A_{14}[\omega^3_1\omega^4] + A_{14}[\Omega\omega^3] + A[\omega^3_1\omega^3],\\
\Delta\omega^4_{22} &= d\omega^4_{22} - 2[\omega^4_{22}\omega] + [\omega^4_{22}\Theta] - K[\omega^4_2\omega^1] + \frac{1}{2}K[\Omega\omega^1]\\
&\qquad + 2A_{12}[\omega^4_2\omega^1] - \frac{1}{2}A_{12}[\Omega\omega^1] + B_{23}[\Omega\omega^1] - \frac{1}{2}B_{14}[\omega^4_2\omega^1]\\
&\qquad + \frac{1}{2}B_{23}[\omega^3_1\omega^1] - 2A_{23}[\omega^4_2\omega^3] + A_{23}[\Omega\omega^4] - A[\omega^4_2\omega^4].
\end{aligned}$$

It is seen from these equations that K and A_{34} are *relative invariants* and the other coefficients form *linear geometric objects*, such as

$$(A_{34}, \lambda A + \mu A_{14} + \nu A_{23}), \qquad \lambda = \text{const}, \quad \mu = \text{const}, \quad \nu = \text{const},$$
$$(A_{34}, A_{14}, A, B_{14}), \qquad (A_{34}, A, A_{23}, B_{23}), \qquad (A_{34}, A_{14}, A_{23}, A_{12}),$$

and so on.

All coefficients together form a *linear geometric object* (*a second-order structure function of the g-structure*). Let us clarify the geometric meaning of the vanishing of these objects. The completely integrable system $\omega^1 = 0$, $\omega^2 = 0$ determines a *fiber bundle of the manifold* $M^{(4)}$ with basis manifold $M^{(2)}$, which we call the *space of independent variables*.

Each of the equations $\omega^1 = 0$, $\omega^2 = 0$, and $\omega^1 + \omega^2 = 0$ is completely integrable. They determine three families of curves on $M^{(2)}$. The quantity K is the *curvature of this three-web* in the sense of Blaschke [8]. Similarly, on each fiber of the bundle of $M^{(4)}$ (i.e., on the integral manifold of the system $\omega^1 = 0, \omega^2 = 0$), the equations $\omega^3 = 0$ and $\omega^4 = 0$ determine a *three-web with curvature* A_{34}.

The vanishing of the object

$$(A_{34}, \lambda A + \mu A_{14} + \nu A_{23})$$

is *necessary and sufficient for the complete integrability of the Pfaffian system*

$$\omega^1 = 0, \qquad \omega^2 = 0, \qquad \lambda\Omega + \mu\omega_1^3 + \nu\omega_2^4 = 0$$

in the manifold $M^{(7)}$. Similarly, the vanishing of the object

$$(A_{34}, A, A_{14}, A_{23}, B_{23} + A_{12})$$

is *necessary and sufficient for the complete integrability of the system* $\omega^1 = 0$, $\Omega - \omega_1^3 = 0$. In this case, the equation $[(\omega_1^3 - \Omega)\omega^1] = 0$ is an *independent differential equation in system* (32). Substituting its solution into the remaining equations of the system, we obtain a simpler system of differential equations; thus, the system *admits an intermediate integral*. For the equations $[(\omega_2^4 - \Omega)\omega^2] = 0$, a similar role is played by the vanishing of the object

$$(A_{34}, A, A_{14}, A_{23}, B_{23}, B_{14} - A_{12}),$$

and for the equation $[\Omega(\omega^1 + \omega^2)] = 0$, of the object

$$(A_{34}, A, A_{14}, A_{23}, B_{23}, B_{14}).$$

In particular, we obtain the following result.

Theorem. *If two of the Pfaffian systems*

$$\begin{matrix} \omega_1^3 - \Omega = 0, & \omega_2^4 - \Omega = 0, & \Omega = 0, \\ \omega^1 = 0, & \omega^2 = 0, & \omega^1 + \omega^2 = 0 \end{matrix}$$

for a system of the form under consideration are completely integrable, then so is the third system.

Such equations arise in solving various applied problems of hydrodynamics, atmospheric physics, and plasma physics. It is pertinent to mention equations obtained by Sobolev and his students. Systems of equations of the form under consideration are also applied to describe real-life physico-chemical processes. Related results are presented in Akramov's book [9]. Very interesting examples of such systems can be found in the book [10].

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