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Lie Symmetries of the Equation $u_t(x,t) + g(u)u_x(x,t) = 0$

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Abstract. In this paper the invariance criterion is applied for the nonlinear equation

$$\frac{\partial}{\partial t}u(x,t) + g(u)\frac{\partial}{\partial x}u(x,t) = 0, \tag{0.1}$$

where $g(u)$ is a smooth function on u. Some particular set of Lie generators are given. In the case of inviscid Burger's equation [1]

$$\frac{\partial}{\partial t}u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t) = 0; \tag{0.2}$$

the Lie projectable symmetry algebra is determined, and the inviscid Burger's equation will be connected to some order differential equations. The obtained differential equations are solved and some exact solutions of (2) are found.

Keywords. Lie algebra, Burger's equation, symmetry group.

1. Introduction

The theory of Lie symmetry groups of differential equations was developed by Sophus Lie [2]. Such Lie groups are invertible point transformations of both the dependent and independent variables of the differential equations. The symmetry group methods provide an ultimate arsenal for analysis of differential equations and is of great importance to understand and to construct solutions of differential equations. Several applications of Lie groups in the theory of differential equations were discussed in the literature, the most important ones are: reduction of order of ordinary differential equations, construction of invariant solutions, mapping solutions to other solutions and the detection of linearizing transformations (for many other applications of Lie symmetries see.[3-10])

In this work we use the basic prolongation method and the infinitesimal criterion of invariance, we find some particular Lie point symmetries group of the

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nonlinear equation $\frac{\partial}{\partial t}u(x,t) + g(u)\frac{\partial}{\partial x}u(x,t) = 0$ where $g(u)$ is a smooth function on u. In the case of inviscid Burger's equation $\frac{\partial}{\partial t}u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t) = 0$ the Lie projectable symmetry algebra is determined, by using the reduction technic the inviscid Burger's equation will be connected to some order differential equations. The obtained differential equations are solved and some exact solutions of inviscid Burger's are found.

This work is organized as follows. In section 2 we recall some result needed to construct Lie point symmetries of a given system of differential equations. In section 3, we give the general form of an infinitesimal generator admitted by the equation (1). In section 4, the invariance condition is used and some particular solutions of defining equations are found. In section 5, we determine the group transformation corresponding to every infinitesimal generator obtained by projectable symmetries. Finally, we show how symmetries may be used to construct some exact solutions for the inviscid equation.

2. Symmetry Methods

In this section, we recall the general procedure for determining symmetries for any system of partial differential equations. To begin, let us consider the general case of a nonlinear system of partial differential equations of order n^{th} in p independent and q dependent variables is given as a system of equations:

$$\Delta_\nu(x, u^{(n)}) = 0. \qquad \nu = 1, \dots, l, \tag{2.1}$$

involving $x = (x^1, \dots, x^p), u = (u^1, \dots, u^q)$ and the derivatives of u with respect to x up to n, where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n.

We consider a one-parameter Lie group of infinitesimal transformations acting on the independent and dependent variables of the system Eqs. (2.1):

$$\tilde{x}^i = x^i + \epsilon\xi^i(x,u) + O(\epsilon^2), \qquad i = 1, \dots, p, \tag{2.2}$$

$$\tilde{u}^j = u^j + \epsilon\eta^j(x,u) + O(\epsilon^2), \qquad j = 1, \dots, q, \tag{2.3}$$

where ϵ is the parameter of the transformation and ξ^i, η^j are the infinitesimals of the transformations for the independent and dependent variables, respectively. The infinitesimal generator V associated with the above group of transformations can be written as

$$V = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{j=1}^{q} \eta^j(x,u)\frac{\partial}{\partial u^j}. \tag{2.4}$$

A symmetry of a differential equation is a transformation which maps solutions of the equation to other solutions.

The invariance of the system (2.1) under the infinitesimal transformations leads to the invariance condition [3, 11]

$$Pr^{(n)}V[\Delta_\nu(x, u^{(n)})] = 0. \nu = 1, \dots, l, \text{ whenever } \qquad \Delta_\nu(x, u^{(n)}) = 0, \tag{2.5}$$

where $Pr^{(n)}$ is the n^{th} order prolongation of the infinitesimal generator given by [3, 8, 9]

$$Pr^{(n)}V = V + \sum_{\alpha=1}^{q}\sum_{J} \varphi_{\alpha}^{J}(x, u^{(n)})\frac{\partial}{\partial u_{\alpha}^{J}}, \tag{2.6}$$

where $J = (j_1, \dots, j_k)$, and $1 \le j_k \le p$, $1 \le k \le n$,

$$\varphi_{\alpha}^{J}(x, u^{(n)}) = D_J(\varphi_\alpha - \sum_{i=1}^{p}\xi^i u_\alpha^i) + \sum_{i=1}^{p}\xi^i u_\alpha^{J,i},$$

where

$$u_\alpha^i = \frac{u_\alpha}{\partial x_i} \text{ and } u_\alpha^{J,i} = \frac{u_\alpha^J}{\partial x_i}.$$

3. Symmetries of Equation (0.1)

We consider the one parameter Lie group of infinitesimal transformations on (x, t, u),

$$\begin{aligned} \tilde{x} &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ \tilde{t} &= t + \epsilon\eta(x,t,u) + O(\epsilon^2), \\ \tilde{u} &= u + \epsilon\varphi(x,t,u) + O(\epsilon^2), \end{aligned} \tag{3.1}$$

where ϵ is the group parameter and ξ, η and φ are the infinitesimals of the transformations for the independent and dependent variables, respectively. We require a method which allows us to determine conditions on ξ, η and φ, so that the one parameter Lie group is a symmetry group of the PDE (1). The associated vector field is of the form:

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial t} + \varphi(x,t,u)\frac{\partial}{\partial u}. \tag{3.2}$$

One of the most important properties of these infinitesimal symmetries is that they form a Lie algebra under the usual Lie bracket.

4. Invariance Condition and Particular Solutions

The vector field V generates a one parameter symmetry group of the PDE (0.1) if and only if

$$Pr^{(1)}V[u_t + g(u)u_x] = 0, \qquad \text{whenever } u_t + g(u)u_x = 0; \tag{4.1}$$

here, $Pr^{(1)}V$ denotes the first prolongation of the vector field V given by (2.6),

$$Pr^{(1)}V = V + \varphi^x\frac{\partial}{\partial u_x} + \varphi^t\frac{\partial}{\partial u_t}, \tag{4.2}$$

with

$$\begin{aligned} \varphi^x &= D_x\varphi - u_x D_x\xi - u_t D_x\eta, \\ \varphi^t &= D_t\varphi - u_x D_t\xi - u_t D_t\eta, \end{aligned} \tag{4.3}$$

where D_x and D_t are the total derivatives with respect to x and t respectively, so the condition (4.1) is equivalent to,

$$[\varphi g^{'}(u)u_x + g(u)\varphi^x + \varphi^t]_{(u_t+g(u)u_x=0)} = 0; \tag{4.4}$$

and hence the condition (4.4) gives the set of defining equations:

$$\begin{aligned} \varphi g^{'}(u) + g(u)^2\eta_x + g(u)\eta_t - g(u)\xi_x - \xi_t &= 0; \\ g(u)\varphi_x + \varphi_t &= 0. \end{aligned} \tag{4.5}$$

As $g(u)$ is an arbitrary smooth function on u, so a particular solution for the system (4.5) is considered. Consequently, a set of infinitesimal generators $L_1, \dots, L_5$ are found :

Case 1 : $\xi = 1; \quad \eta = 0; \quad \varphi = 0;$

so

$$L_1 = \frac{\partial}{\partial x}. \qquad \text{(space translation)} \tag{4.6}$$

Case 2 : $\xi = 0; \quad \eta = 1; \quad \varphi = 0;$

then

$$L_2 = \frac{\partial}{\partial t}. \qquad \text{(time translation)} \tag{4.7}$$

Case 3 : $\xi = x; \quad \eta = t; \quad \varphi = 0;$

hence

$$L_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}. \qquad \text{(scaling transformation)} \tag{4.8}$$

Case 4 : $\xi = u; \quad \eta = 0; \quad \varphi = 0;$

we find

$$L_4 = u\frac{\partial}{\partial x}. \tag{4.9}$$

Case 5 : $\xi = 0; \quad \eta = u; \quad \varphi = 0;$

in this case we get

$$L_5 = u\frac{\partial}{\partial t}. \tag{4.10}$$

In what follows we put $\varphi = h(u)$, and suppose that $g^{'} \neq 0$, we obtain

Case 6 : $\xi = x; \quad \eta = 0; \quad \varphi = h(u);$

the first equation in the system (4.5) becomes the differential equation

$$h(u)g^{'}(u) - g(u) = 0;$$

that implies

$$h(u) = \frac{g(u)}{g^{'}(u)}.$$

In addition to the vector fields $L_1, \dots, L_5$ admitted by all partial differential equations of the form (0.1), the technique illustrated above, shows us how to construct another infinitesimal symmetry admitted by a given partial differential equations of the form (0.1):

- $u_t + \exp(u)u_x = 0$. In this case $g(u) = \exp(u)$, then it admits the infinitesimal generator
$$V = x\frac{\partial}{\partial t} + \frac{\partial}{\partial u}.$$
- $u_t + u^m u_x = 0$, m nonzero integer. Then $g(u) = u^m$, so the infinitesimal generator is obtained to be
$$V = x\frac{\partial}{\partial t} + \frac{1}{m}u\frac{\partial}{\partial u}.$$
- $u_t + \frac{1}{1+u^2}u_x = 0$. Hence, $g(u) = \frac{1}{1+u^2}$, in this case we get
$$V = x\frac{\partial}{\partial t} - \frac{1+u^2}{2u}\frac{\partial}{\partial u}.$$

Case 7 : $\xi = 0; \quad \eta = x; \quad \varphi = h(u),$
then $h(u)$ must satisfy the equation
$$h(u)g^{'}(u) + g^2(u) = 0;$$
so
$$h(u) = -\frac{g^2(u)}{g^{'}(u)}.$$
As in the last case, we find for example:

- $u_t + \exp(u)u_x = 0$. So this equation admits the generator
$$V = x\frac{\partial}{\partial t} - \exp(u)\frac{\partial}{\partial u}.$$
- $u_t + u^m u_x = 0$, m nonzero integer, In this case we get
$$V = x\frac{\partial}{\partial t} - \frac{1}{m}u^{m+1}\frac{\partial}{\partial u}.$$
- $u_t + \frac{1}{1+u^2}u_x = 0$. That implies
$$V = x\frac{\partial}{\partial t} + \frac{1}{2u}\frac{\partial}{\partial u}.$$

To calculate all Lie point symmetries admitted by (0.1), we need to solve completely the system of defining equations. However, the system (4.5), may be more complicated then the studied equation (0.1) itself. In this paper we choose to solve partially the defining system for the equation (0.2). To do this, we suppose ξ and η depend only on x and t. So let
$$\xi = \xi(x,t); \tag{4.11}$$
and
$$\eta = \eta(x,t). \tag{4.12}$$
The resulting Lie point symmetries are called *projectable symmetries.*

The above conditions yields to the solution

$$\begin{aligned}\xi(x,t) &= -a_1xt + a_5t + a_3x^2 + a_6x + a_7;\\ \eta(x,t) &= a_3xt - a_4x - a_1t^2 + (a_6 - a_2)t + a_8;\\ \varphi(x,t,u) &= (-a_3t + a_4)u^2 + (a_3x + a_1t + a_2)u + (-a_1x + a_5);\end{aligned}$$

where $a_1, \ldots, a_8$ are arbitrary constants. Consequently, the inviscid Burger's equation admits the symmetry Lie algebra spanned by the vector fields :

$$\begin{aligned}V_1 &= \tfrac{\partial}{\partial x}; & V_5 &= -x\tfrac{\partial}{\partial t} + u^2\tfrac{\partial}{\partial u};\\ V_2 &= \tfrac{\partial}{\partial t}; & V_6 &= x^2\tfrac{\partial}{\partial x} + tx\tfrac{\partial}{\partial t} + (xu - tu^2)\tfrac{\partial}{\partial u};\\ V_3 &= x\tfrac{\partial}{\partial x} + t\tfrac{\partial}{\partial t}; & V_7 &= -t\tfrac{\partial}{\partial t} + u\tfrac{\partial}{\partial u};\\ V_4 &= t\tfrac{\partial}{\partial x} + \tfrac{\partial}{\partial u}; & V_8 &= -xt\tfrac{\partial}{\partial x} - t^2\tfrac{\partial}{\partial t} + (tu - x)\tfrac{\partial}{\partial u}.\end{aligned} \tag{4.13}$$

Their commutator $[V_i, V_j]$ table is

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
V_1	0	0	V_1	0	-V_2	$2V_3 + V_7$	0	-V_4
V_2	0	0	V_2	V_1	0	-V_5	-V_2	V_7 - V_3
V_3	-V_1	-V_2	0	0	0	V_6	0	V_8
V_4	0	-V_1	0	0	$2V_7 + V_3$	-V_8	V_4	0
V_5	V_2	0	0	-$2V_7$ - V_3	0	0	-V_5	V_6
V_6	-$2V_3$ - V_7	V_5	-V_6	V_8	0	0	0	0
V_7	0	V_2	0	-V_4	V_5	0	0	-V_8
V_8	V_4	V_3 - V_7	-V_8	0	-V_6	0	V_8	0

Table 1: Commutator $[V_i, V_j]$ table for the Lie algebra V_i and V_j

5. Transformed Solutions

To obtain the group transformation which is generated by the infinitesimal generators $V_1, \ldots, V_8$, we need to solve the system of first order ordinary differential equations,

$$\frac{d\tilde{x}}{d\epsilon} = \xi(\tilde{x}, \tilde{t}, \tilde{u}); \tag{5.1}$$

$$\frac{d\tilde{t}}{d\epsilon} = \eta(\tilde{x}, \tilde{t}, \tilde{u}); \tag{5.2}$$

$$\frac{d\tilde{u}}{d\epsilon} = \varphi(\tilde{x}, \tilde{t}, \tilde{u}); \tag{5.3}$$

with initial conditions

$$\tilde{x}(o) = x, \qquad \tilde{t}(0) = t, \qquad \tilde{u}(0) = u.$$

Exponentiating the infinitesimal symmetries of the inviscid Burger's equation (0.2), we get the one parameter groups G_k generated by V_k for $k = 1, \dots, 8$:

$$G_1 : \quad (x,t,u) \longmapsto (x+\epsilon, t, u); \tag{5.4}$$

$$G_2 : \quad (x,t,u) \longmapsto (x, t+\epsilon, u); \tag{5.5}$$

$$G_3 : \quad (x,t,u) \longmapsto (xe^{\epsilon}, e^{\epsilon}t, u); \tag{5.6}$$

$$G_4 : \quad (x,t,u) \longmapsto (t\epsilon + x, t, u+\epsilon); \tag{5.7}$$

$$G_5 : \quad (x,t,u) \longmapsto (x, -x\epsilon + t, \frac{u}{\epsilon u - 1}); \tag{5.8}$$

$$G_6 : \quad (x,t,u) \longmapsto (\frac{x}{1-x\epsilon}, \frac{t}{1-x\epsilon}, \frac{u}{t\epsilon u + 1 - x\epsilon}); \tag{5.9}$$

$$G_7 : \quad (x,t,u) \longmapsto (x, e^{-\epsilon}t, e^{\epsilon}u); \tag{5.10}$$

$$G_8 : \quad (x,t,u) \longmapsto (\frac{x}{1+t\epsilon}, \frac{t}{1+t\epsilon}, (1+t\epsilon)u - \epsilon x). \tag{5.11}$$

Consequently, if $u = f(x,t)$ is a solution of equation (0.2), so are the functions

$$f(x-\epsilon, t); \tag{5.12}$$

$$f(x, t-\epsilon); \tag{5.13}$$

$$f(xe^{-\epsilon}, e^{-\epsilon}t); \tag{5.14}$$

$$f(x - t\epsilon, t) + \epsilon; \tag{5.15}$$

$$\frac{f(x, t+\epsilon x)}{\epsilon f(x, t+\epsilon x) - 1}; \tag{5.16}$$

$$\frac{f(\frac{x}{1+\epsilon x}, \frac{t}{1+\epsilon t})}{t\epsilon f(\frac{x}{1+\epsilon x}, \frac{t}{1+\epsilon t)}) + 1 - \frac{\epsilon x}{1+\epsilon x}}; \tag{5.17}$$

$$e^{\epsilon} f(x, e^{\epsilon}t); \tag{5.18}$$

$$\frac{\epsilon}{1-\epsilon t} f(\frac{x}{1-\epsilon t}, \frac{t}{1-\epsilon t}) - \frac{\epsilon x}{1-\epsilon t}. \tag{5.19}$$

We see that these transformations don't give us an important solution from a known particular solution. However, to illustrate how this technique may be of great interest, we consider the one dimensional heat equation

$$u_t = u_{xx}. \tag{5.20}$$

It is known that its symmetry Lie algebra is spanned by the vector fields [3]

$$H_1 = \frac{\partial}{\partial x}; \qquad H_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t};$$

$$H_2 = \frac{\partial}{\partial t}; \qquad H_5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u};$$

$$H_3 = u\frac{\partial}{\partial u}; \qquad H_6 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2+2t)u\frac{\partial}{\partial u};$$

and the infinite dimensional algebra

$$H_\alpha = \alpha(x,t)\frac{\partial}{\partial u},$$

where α is an arbitrary solution of the heat equation. If we consider the vector field H_6, since it generates a symmetry group N_6, so if $u = f(x,t)$ is a solution of the heat equation, then is the function

$$\frac{1}{\sqrt{1+4\epsilon t}}\exp[\frac{-\epsilon x^2}{1+4\epsilon t}]f(\frac{x}{1+4\epsilon t},\frac{t}{1+4\epsilon t}).$$

If we let $u(x,t) = 1$ be a constant solution of the heat equation, we conclude that the function

$$u(x,t) = \frac{1}{\sqrt{1+4\epsilon t}}\exp\{\frac{-\epsilon x^2}{1+4\epsilon t}\};$$

is a solution.

In the above solution let $t \to -\frac{1}{4\epsilon}$, and set $\epsilon = \pi$, we conclude that from the constant solution 1, we find the fundamental solution,

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\exp\{-\frac{x^2}{4t}\}; \tag{5.21}$$

of the heat equation. Unfortunately, transformations (5.12)–(5.19) doesn't allow us to construct an important solution of equation (0.2) from a trivial solution as in the case of the heat equation. Hence, to apply the symmetry group method for the equation (0.2), will not be in the same direction as for equation (5.20), however, symmetry group method will be used to reduce the equation (0.2) to order differential equations, that is to reduce the number of independent variables.

6. Reduced Equations

The first advantage of symmetry group methods is to construct new solutions from known solutions. The second is when a nonlinear system of differential equations admits infinite symmetries, so it is possible to transform it to a linear system. Neither the first advantage nor the second will be investigated here, but symmetry group method will be applied to the inviscid Burger's equation to be connected directly to some order differential equations. To do this, a particular linear combinations of infinitesimals are considered and their corresponding invariants are determined.

6.1. Reduction with V_3

The invescid Burger's equation is expressed in the coordinates (x,t,u), so to reduce this equation is to search for its form in specific coordinates. Those coordinates will be constructed by searching for independent invariants (Y,W) corresponding to the infinitesimal generator. So using the chain role, the expression of the equation in the new coordinate allows us to the reduced equation. In this first case we have

$$V_3 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}.$$

To determine independent invariants we need to solve the first partial differential equations :

$$x\frac{\partial k(x,t,u)}{\partial x} + t\frac{\partial k(x,t,u)}{\partial t} = 0,$$

so we solve the associated characteristic ordinary differential equation

$$\frac{dx}{x} = \frac{dt}{t};$$

hence, we obtain two independent invariants:

$$Y = \frac{x}{t}; \qquad \text{and} \qquad W = u;$$

then,

$$u_t = W' \times (-\frac{x}{t^2});$$

$$u_x = W' \times (\frac{1}{t});$$

the primes dentes the differentiation of W with respect Y. Substituting for u_t and u_x their expressions in the equation (0.1), we obtain the order ordinary differential equation

$$W'(-Y + W) = 0; \tag{6.1}$$

so the solutions of the equation (6.1) are

$$W = cte \qquad \text{and} \qquad W = Y;$$

consequently, we obtain that

$$u(x,t) = cte$$

and

$$u(x,t) = \frac{x}{t}$$

are solutions of the studied equation (0.2).

6.2. Reduction with $V_3 - V_4 + V_5$

The invariants are

$$Y = x + t;$$

$$W = \frac{(u+1)(2t - Y)}{u - 1};$$

the reduced equation is

$$W' + 1 = 0; \tag{6.2}$$

that implies

$$W = -Y + cte;$$

we obtain that

$$u(x,t) = \frac{x + cte}{t + cte}$$

is a solution of the equation (0.2).

6.3. Reduction with $V_4 - V_2$

The invariants are

$$Y = t^2 + 2x;$$

$$W = u + t;$$

the reduced differential equation is

$$(W^2)^{'} = 1; \tag{6.3}$$

its solutions are given by

$$W = \pm\sqrt{Y + cte};$$

we find that

$$u(x,t) = \pm(t^2 + 2x + cte)^{\frac{1}{2}} - t$$

is a solution of the equation (0.2).

6.4. Reduction with $V_4 + V_2$

The invariants are

$$Y = t^2 - 2x;$$

$$W = u - t;$$

as above, we reduce the equation (0.2) to the ordinary differential equation

$$(W^2)^{'} = 1; \tag{6.4}$$

we find that

$$u(x,t) = \pm(t^2 - 2x + cte)^{\frac{1}{2}} + t$$

is a solution of the equation (0.2).

6.5. Reduction with $V_5 - V_1$

The invariants are

$$Y = x^2 - 2t;$$

$$W = \frac{1}{u} - x;$$

we find the reduced equation is the same as in the last case:

$$(W^2)^{'} = 1; \tag{6.5}$$

we obtain that

$$u(x,t) = [x \pm (x^2 - 2t + cte)^{\frac{1}{2}}]^{-1}$$

is a solution of the equation (0.2).

6.6. Reduction with $V_5 + V_1$

The invariants are

$$Y = x^2 + 2t;$$
$$W = \frac{1}{u} + x;$$

also the reduced equation obtained here is the same as in the three last cases:

$$(W^2)' = 1; \tag{6.6}$$

then,

$$u(x,t) = [-x \pm (x^2 + 2t + cte)^{\frac{1}{2}}]^{-1}$$

is a solution of the equation (0.2).

Conclusion

By using here the basic prolongation method we have obtained some particular Lie point symmetries group of the nonlinear equation $\frac{\partial}{\partial t}u(x,t) + g(u)\frac{\partial}{\partial x}u(x,t) = 0$ where $g(u)$ is a smooth function on u. In the case of inviscid Burger's nonlinear equation $\frac{\partial}{\partial t}u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t) = 0$ the Lie projectable symmetry algebra is determined, by using the reduction technique the inviscid Burger's nonlinear equation will be connected to some order differential equations. This enables us to obtain a class of interesting solutions to these equation.

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