Available online at www.sciencedirect.com
ScienceDirect
Chaos, Solitons and Fractals 38 (2008) 722–730
CHAOS SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

Symmetry group analysis and similarity solutions of variant nonlinear long-wave equations

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Accepted 2 January 2007

Abstract

Symmetry group properties and similarity solutions of the variant nonlinear long-wave equations in the form of system of nonlinear partial differential equations are analyzed. Lie symmetry group analysis of the variant nonlinear long-wave equations presents that the system has only two-parameter point symmetry group that corresponds to only traveling wave solutions. The symmetry groups yield the general reduced similarity form of the system, which is in the system of nonlinear ordinary differential equations. By using the improved tanh method the similarity solutions are obtained from the reduced system of equations. In addition, some graphical representations of the solitary and periodic solutions are presented.

1. Introduction

Symmetry group analysis based on the transformation groups, now known as Lie groups, is the most important solution method for the nonlinear problems in the literature. This approach is used to analysis the symmetries of the differential equations. Then, the corresponding symmetry groups can be used to simplify the analysis of the problems governing by the differential equations in the engineering science, mathematical physics, and mechanics. Lie groups characterize the symmetry of the differential equations and may be a point, a contact, and a potential or a nonlocal symmetry. It has also been verified that these kinds of groups can be represented by their infinitesimals that contain dependent variables, independent variables and the derivatives of dependent variables as arguments. In the last century, the application of the Lie groups has been developed by a number of mathematicians. Ovsiannikov [1], Olver [2], Ibragimov [3], and Bluman and Kumei [4] are some of the mathematicians who have huge number of studies in that field.

In this study, we analyze the symmetry groups and investigate the similarity solutions of variant nonlinear long-wave equations [5], which are nonlinear evolution equations, given by

$$\begin{aligned} &h_t(x,t) + u_x(x,t) + u(x,t)h_x(x,t) + h(x,t)u_x(x,t) + \delta^2 u_{xxx}(x,t) = 0, \\ &u_t(x,t) + h_x(x,t) + u(x,t)u_x(x,t) - \varepsilon^2 u_{xxt}(x,t) = 0, \end{aligned} \tag{1}$$

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0960-0779/$ - see front matter
doi:10.1016/j.chaos.2007.01.023

where $h(x,t)$ is the free surface, and $u(x,t)$ is the horizontal velocity component, δ, ε are constants. In the literature, some solutions of the system (1) were analyzed by the hyperbolic function method [6] and the double parameter hypothesis [7]. Inverse scattering theory [8], homogeneous balance method [9–12], tangent functions series method and the Jacobi elliptic expansion method [13,14] are some of methods to obtain the exact solutions of the nonlinear partial differential equations in the literature. In the study, we consider the Lie symmetry group method for the solution of the system (1). For this purpose, first, the Lie point symmetries will be analyzed, then the reductions forms will be found and then the invariant solutions of the original system of partial differential equations (PDE's) will be obtained from the solutions of reduced system of ordinary differential equations (ODE's) by using the improved tanh method.

2. Symmetry group analysis of variant nonlinear long-wave equations

In this section, we will explore the most general Lie group of point transformations, which leaves variant long-wave equations invariant (1).

Definition 1. We consider a scalar kth-order PDE represented by

$$\prod(\mathbf{x}, \mathbf{u}, \mathbf{u}_1, \ldots, \mathbf{u}_k) = 0, \tag{2}$$

$\mathbf{x} = (x_1, x_2, \ldots, x_m)$ denotes m independent variables, $\mathbf{u}$ denotes set of dependent (differential) variables, and $\mathbf{u}_j$ denotes set of corresponding to all jth-order partial derivatives of $\mathbf{u}$ with to variable $\mathbf{x}$. The infinitesimal generator of the one-parameter Lie group of transformations for equation (2.1) is

$$X = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u}, \tag{3}$$

where $\xi_i(x,u)$, $\eta(x,u)$ are the infinitesimals of (3), and the kth prolongation of the infinitesimal generator (3) [1–4,16] is

$$X_k = X + \eta_i^{(1)}(x,u,u_1)\frac{\partial}{\partial u_i} + \cdots + \eta^{(k)}_{i_1 i_2,\ldots,i_k}(x,u,u_1,\ldots,u_k)\frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}}, \tag{4}$$

where

$$\begin{aligned} &\overset{(1)}{\underset{i}{\eta}} = D_i\eta - (D_i\xi_j)u_j, \quad i = 1,2,\ldots,m; \\ &\vdots \\ &\overset{(k)}{\underset{i_1\ldots i_k}{\eta}} = D_i \overset{(k-1)}{\underset{i_1 i_2 \ldots i_{k-1}}{\eta}} - (D_{i_k}\xi_j)u_{i_1 i_2 \ldots i_{k-1} j}, \quad i_l = 1,2,\ldots,m \text{ for } l = 1,2,\ldots,k \text{ with } k = 2,3,\ldots \end{aligned} \tag{5}$$

in terms of $[\xi(x,y), \eta(x,y)]$, $[\xi(x,y)$ denotes $(\xi_1(x,u), \xi_2(x,y), \ldots, \xi_m(x,y))$, and D is the total derivative operator defined as

$$D_i = \frac{\mathrm{D}}{\mathrm{D}x_i} = \frac{\partial}{\partial x_i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots + u_{ii_1 i_2 \ldots i_m}\frac{\partial}{\partial u_{i_1 i_2 \ldots i_m}} + \cdots, \quad u_i = \frac{\partial u}{\partial x_i}, \quad i = 1,2,\ldots,m \tag{6}$$

with summation over a repeated index.

Now, we consider the following Lie group of transformations with independent variables x, t; and dependent variables u, h

$$\tilde{x} = \tilde{x}(x,t,u,h;\beta), \quad \tilde{t} = \tilde{t}(x,t,u,h;\beta), \quad \tilde{u} = \tilde{u}(x,t,u,h;\beta), \quad \tilde{h} = \tilde{h}(x,t,u,h;\beta), \tag{7}$$

where β is the group parameter. The infinitesimal generator for the Lie group (4) can be expressed from the formula (3) in the following form:

$$X = \xi^x\frac{\partial}{\partial x} + \xi^t\frac{\partial}{\partial t} + \eta^u\frac{\partial}{\partial u} + \eta^h\frac{\partial}{\partial h} \tag{8}$$

in which $\xi^x, \xi^t, \eta^u, \eta^h$ are infinitesimal functions of group variables (independent and dependent variables).

Proposition 1. *The variant nonlinear long-wave equations have a two-parameter symmetry group, which are translations*

$$X = \alpha_1\frac{\partial}{\partial x} + \alpha_2\frac{\partial}{\partial t}, \tag{9}$$

where α_1 and α_2 are constants and called group parameters.

Proof. To calculate the Lie point symmetries of the governing equations (1), first we need to write the third prolongation of the infinitesimal generator given by (8) since the governing equations include at most third order partial derivatives. Due to the formula (4), the prolongation of the infinitesimal generator including the related terms is in the following form:

$$X_3 = \xi^x \frac{\partial}{\partial x} + \xi^t \frac{\partial}{\partial t} + \eta^u \frac{\partial}{\partial u} + \eta^h \frac{\partial}{\partial h} + \eta^u_x \frac{\partial}{\partial u_x} + \eta^u_t \frac{\partial}{\partial u_t} + \eta^u_{xx} \frac{\partial}{\partial u_{xx}} + \eta^u_{xx} \frac{\partial}{\partial u_{xxx}} + \eta^h_x \frac{\partial}{\partial h_x} + \eta^h_t \frac{\partial}{\partial h_t}, \tag{10}$$

where the formulas of terms $\eta^u_x, \eta^u_t, \eta^u_{xx}, \eta^u_{xxx}, \eta^h_x, \eta^h_t$ are given by the expression (5). Applying the third prolongation of the generator (1) to the first and the second equations in the system in (1) we reach the following determining equations [1–4,16–18]

$$\left(\eta^h_t + \eta^u_x + h\eta^u_x + u\eta^h_x + \delta^2\eta^u_{xxx}\right)\big|_{h_t=-u_x-(uh)_x-\delta^2 u_{xxx}} = 0, \left(\eta^u_t + \eta^h_x + u\eta^u_x - \varepsilon^2\delta^2\eta^u_{xxt}\right)\big|_{u_t=-h_x-uu_x-\varepsilon^2 u_{xxt}} = 0. \tag{11}$$

Calculating the needed terms in (6) and equating every one of the coefficients of independent terms to zero we find the following system of equations after considerable simplifications

$$\begin{aligned}
&\xi^x = \xi^x(x), \quad \xi^t = \xi^x(t), \quad \eta^u = \eta^u(u), \quad \eta^h = \eta^h(h),\\
&\eta^h - (1+h)\eta^h_h + (1+h)\eta^u_u - (1+u)\xi^x_x - \delta^2\xi^x_{xxx} = 0,\\
&\eta^u - u\xi^t_t = 0, \quad \eta^h_h + \eta^u_u + \xi^t_t + 3\xi^x_x = 0,\\
&\eta^h + h(\xi^t_t - \xi^x_x) = 0.
\end{aligned} \tag{12}$$

From (12) it is easy to see that the only solution of this system is

$$\xi^x = \alpha_1, \quad \xi^t = \alpha_2, \quad \eta^u = 0, \quad \eta^h = 0 \tag{13}$$

which shows that the system of nonlinear PDE's has only a two-parameter symmetry group

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}. \qquad \square \tag{14}$$

3. Reduction form and similarity solutions of variant nonlinear long-wave equations

Definition 2. The order of differential equations can be reduced by one if the differential equation is invariant under a one-parameter Lie group point transformations. If a system of ODE's or PDE's is invariant under a Lie group of point transformations, then some special solutions of these equations can be found. These solutions are called *invariant* solutions. In the other words, they are invariant under some subgroups of the full group admitted by the system of ODE's, or PDE's. These solutions can be obtained from the solutions of the reduced system of differential equations with fewer independent variables.

To construct the reduced form of the system of PDE one needs to write the characteristic equation based on the symmetry groups (14)

$$\frac{dx}{\alpha_1} = \frac{dt}{\alpha_2} = \frac{du}{0} = \frac{dh}{0}. \tag{15}$$

From the solution of the characteristic equation we have

$$\xi = \alpha_2 x - \alpha_1 t, \quad u(x,t) = \tilde{u}(\xi), \quad h(x,t) = \tilde{h}(\xi) \tag{16}$$

which ξ is the similarity independent variable, $\tilde{u}(\xi)$ and $\tilde{u}(\xi)$ are the similarity dependent variables. The expressions in (16) confirm that the similarity solutions of variant shallow-water wave equations are only in the form of the traveling wave solutions. Then, if we substitute the similarity forms and the similarity variable into the original governing equations we obtain the reduced system of equations which are in the form of system of coupled ODE's

$$\begin{aligned}
&\alpha_1\tilde{h}'(\xi) - \alpha_2\tilde{u}'(\xi) - \alpha_2\tilde{h}(\xi)\tilde{u}'(\xi) - \alpha_2\tilde{u}(\xi)\tilde{h}'(\xi) - \delta^2\alpha_2^3\tilde{u}'''(\xi) = 0,\\
&\alpha_1\tilde{u}'(\xi) - \alpha_2\tilde{h}'(\xi) - \alpha_2\tilde{u}(\xi)\tilde{u}'(\xi) - \varepsilon^2\alpha_1\alpha_2^2\tilde{u}'''(\xi) = 0
\end{aligned} \tag{17}$$

To solve reduced system of nonlinear ODE's several methods can be considered. Here we consider the generalized tanh function method introduced in [15] to obtain the solutions of the reduced system (17). For this purpose, let us consider the solutions of system of ordinary differential equations (17) in the following forms:

$$\tilde{u}(\xi) = \sum_{i=0}^{n} a_i F^i(\xi), \quad \tilde{h}(\xi) = \sum_{i=0}^{m} b_i F^i(\xi), \tag{18}$$

where a_i, b_i are parameters and n and m are integers to be determined and F is the solution of the Riccati equation

$$F'(\xi) = A + BF(\xi) + CF^2(\xi), \tag{19}$$

where $()' := \mathrm{d}/\mathrm{d}\xi$ and $A,\ B,\ C$ are constants. By using the improved tanh method we consider all possible solutions of the Riccati equation (19) that tanh function satisfies. By substituting (18) and (19) into the reduced system of ODE's (17) we obtain algebraic system of equations in terms of $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ by equating all coefficients of the functions F^i to zero. By balancing the highest order derivative and highest nonlinear terms in the reduced system of ODE's (17) one can find $n = 2$ and $m = 3$. Then the solutions in (18) should be in the following forms:

$$\tilde{u}(\xi) = a_0 + a_1F(\xi) + a_2F^2(\xi), \quad \tilde{h}(\xi) = b_0 + b_1F(\xi) + b_2F^2(\xi) + b_3F^3(\xi). \tag{20}$$

For the first PDE $(1)_1$ the corresponding algebraic equations are:

$$A\big(b_1(a_0\alpha_2 - \alpha_1) + \alpha_2\big(6ABa_2\alpha_2^2\delta^2 + a_1\big(1 + b_0 + B^2\alpha_2^2\delta^2 + 2AC\alpha_2^2\delta^2\big)\big)\big) = 0$$

$$\begin{aligned} &14AB^2a_2\alpha_2^3\delta^2 + B^3a_1\alpha_2^3\delta^2 + B\big(b_1(a_0\alpha_2 - \alpha_1) + a_1\alpha_2\big(1 + b_0 + 8AC\alpha_2^2\delta^2\big)\big) + \\ &2A(b_2(a_0\alpha_2 - \alpha_1) + \alpha_2(a_2 + a_2b_0 + a_1b_1 + 8ACa_2\alpha_2^2\delta^2)) = 0 \end{aligned} \tag{21}$$

$$\begin{aligned} &C(a_1\alpha_2 - b_1\alpha_1 + a_1b_0\alpha_2 + a_0b_1\alpha_2) + 8B^3a_2\alpha_2^3\delta^2 + 7B^2Ca_1\alpha_2^3\delta^2 + \\ &A\big(3a_2b_1\alpha_2 - 3b_3\alpha_1 + 3a_1b_2\alpha_2 + 3a_0b_3\alpha_2 + 8a_1C^2\alpha_2^3\delta^2\big) + \\ &2B\big(b_2(a_0\alpha_2 - \alpha_1) + \alpha_2\big(a_2 + a_2b_0 + a_1b_1 + 26ACa_2\alpha_2^2\delta^2\big)\big) = 0 \end{aligned} \tag{22}$$

$$\begin{aligned} &38B^2Ca_2\alpha_2^3\delta^2 + 3B\big(a_2b_1\alpha_2 - b_3\alpha_1 + a_1b_2\alpha_2 + a_0b_3\alpha_2 + 4a_1C^2\alpha_2^3\delta^2\big) + \\ &2\big(b_2(2Aa_2\alpha_2 - C\alpha_1 + Ca_0\alpha_2) + \alpha_2\big(2Aa_1b_3 + Ca_2 + Ca_2b_0 + Ca_1b_1 + 20AC^2a_2\alpha_2^2\delta^2\big)\big) = 0 \end{aligned} \tag{23}$$

$$\begin{aligned} &b_3(5Aa_2\alpha_2 - 3C\alpha_1 + 4Ba_1\alpha_2 + 3Ca_0\alpha_2) + \\ &\alpha_2\big(3Ca_1\big(b_2 + 2C^2\alpha_2^2\delta^2\big) + a_2\big(4Bb_2 + 3Cb_1 + 54BC^2\alpha_2^2\delta^2\big)\big) = 0 \end{aligned} \tag{24}$$

$$5Ba_2b_3\alpha_2 + 4C\alpha_2\big(a_1b_3 + a_2\big(b_2 + 6C^2\alpha_2^2\delta^2\big)\big) = 0 \tag{25}$$

$$5Ca_2b_3\alpha_2 = 0 \tag{26}$$

For the second PDE $(1)_2$ the corresponding algebraic equations are:

$$A\big(b_1\alpha_2 + 2A(3Ba_2 + Ca_1)\alpha_1\alpha_2^2\varepsilon^2 + a_1\big(a_0\alpha_2 + \alpha_1\big(B^2\alpha_2^2\varepsilon^2 - 1\big)\big)\big) = 0 \tag{27}$$

$$\begin{aligned} &16A^2Ca_2\alpha_1\alpha_2^2\varepsilon^2 + B\big(b_1\alpha_2 + a_1\big(a_0\alpha_2 - \alpha_1 + B^2\alpha_1\alpha_2^2\varepsilon^2\big)\big) + A\big(\alpha_2\big(a_1^2 + 2b_2 + 8BCa_1\alpha_1\alpha_2\varepsilon^2\big) \\ &\quad + 2a_2\big(a_0\alpha_2 - \alpha_1 + 7B^2\alpha_1\alpha_2^2\varepsilon^2\big)\big) = 0 \end{aligned} \tag{28}$$

$$\begin{aligned} &Ba_1^2\alpha_2 + \alpha_2(2Bb_2 + 3Ab_3 + Cb_1) + Ca_1\big(a_0\alpha_2 + \alpha_1\big(7B^2\alpha_2^2\varepsilon^2 - 1 + 8AC\alpha_2^2\varepsilon^2\big)\big) \\ &\quad + a_2\big(3Aa_1\alpha_2 + 8B^3\alpha_1\alpha_2^2\varepsilon^2 + 2B\big(a_0\alpha_2 - \alpha_1 + 26AC\alpha_1\alpha_2^2\varepsilon^2\big)\big) = 0 \end{aligned} \tag{29}$$

$$\begin{aligned} &2Aa_2^2\alpha_2 + \alpha_2\big(\big(a_1^2 + 2b_2\big)C + 3B\big(b_3 + 4a_1C^2\alpha_1\alpha_2\varepsilon^2\big)\big) + a_2(3Ba_1\alpha_2 + 40AC^2\alpha_1\alpha_2^2\varepsilon^2 \\ &\quad + 2C(a_0\alpha_2 - \alpha_1 + 19B^2\alpha_1\alpha_2^2\varepsilon^2)) = 0 \end{aligned} \tag{30}$$

$$\alpha_2(2Ba_2^2 + 3Ca_2(a_1 + 18BC\alpha_1\alpha_2\varepsilon^2) + 3C(b_3 + 2C^2a_1\alpha_1\alpha_2\varepsilon^2)) = 0 \tag{31}$$

$$2Ca_2\alpha_2(a_2 + 12C^2\alpha_1\alpha_2\varepsilon^2) = 0 \tag{32}$$

From solving the system of algebraic equations (21)–(32) one can obtain

$$\begin{aligned}
a_0 &= \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 - 2B^2\alpha_1^2\alpha_2^2\varepsilon^4 - 16AC\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2},\\
b_0 &= \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^4 - 2B^2\alpha_1^2\alpha_2^2\varepsilon^4\delta^2 - 16AC\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4},\\
a_1 &= -12BC\alpha_1\alpha_2\varepsilon^2, \quad b_1 = -6BC\alpha_2^2\delta^2, \quad a_2 = -12C^2\alpha_1\alpha_2\varepsilon^2, \quad b_2 = -6C^2\alpha_2^2\delta^2, \quad b_3 = 0.
\end{aligned} \tag{33}$$

Substituting (19) and (33) into (20) we have obtained the following multiple rational, solitary and periodic solutions for the system (1). These solutions could be getting in different cases as below

Case I: $A \neq 0,\ B \neq 0,\ C \neq 0$

$$\begin{aligned}
u(x,t) &= \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 - 2B^2\alpha_1^2\alpha_2^2\varepsilon^4 - 16AC\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2}\\
&-12BC\alpha_1\alpha_2\varepsilon^2\left(-C + \sqrt{4AB - C^2}\tan\left(\frac{1}{2}\left(\sqrt{4AB - C^2}(\alpha_2 x - \alpha_1 t) + \sqrt{4AB - C^2}c_0\right)\right)\Big/2B\right)\\
&+12C^2\alpha_1\alpha_2\varepsilon^2\left(-C + \sqrt{4AB - C^2}\tan\left(\frac{1}{2}\left(\sqrt{4AB - C^2}(\alpha_2 x - \alpha_1 t) + \sqrt{4AB - C^2}c_0\right)\right)\Big/2B\right)^2\\
h(x,t) &= \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^4 - 2B^2\alpha_1^2\alpha_2^2\varepsilon^4\delta^2 - 16AC\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4} -\\
&6BC\alpha_2^2\delta^2\left(-C + \sqrt{4AB - C^2}\tan\left(\frac{1}{2}\left(\sqrt{4AB - C^2}(\alpha_2 x - \alpha_1 t) + \sqrt{4AB - C^2}c_0\right)\right)\Big/2B\right) -\\
&6C^2\alpha_2^2\delta^2\left(-C + \sqrt{4AB - C^2}\tan\left(\frac{1}{2}\left(\sqrt{4AB - C^2}(\alpha_2 x - \alpha_1 t) + \sqrt{4AB - C^2}c_0\right)\right)\Big/2B\right)^2
\end{aligned} \tag{34}$$

where c_0 is a constant.

Case II: $A = 0,\ B \neq 0,\ C \neq 0$

$$\begin{aligned}
u(x,t) &= \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 - 2B^2\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2} + 12BC\alpha_1\alpha_2\varepsilon^2\left(\frac{C\exp(C(\alpha_2 x - \alpha_1 t) + Cd_0)}{B\exp(C(\alpha_2 x - \alpha_1 t) + Cd_0) - 1}\right)\\
&\quad + 12C^2\alpha_1\alpha_2\varepsilon^2\left(\frac{C\exp(C(\alpha_2 x - \alpha_1 t) + Cd_0)}{B\exp(C(\alpha_2 x - \alpha_1 t) + Cd_0) - 1}\right)^2,\\
h(x,t) &= \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^4 - 2B^2\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4} + 6BC\alpha_2^2\delta\left(\frac{C\exp(C(\alpha_2 x - \alpha_1 t) + Cd_0)}{B\exp(C(\alpha_2 x - \alpha_1 t) + Cd_0) - 1}\right)\\
&\quad + 6C^2\alpha_2^2\delta^2\left(\frac{C\exp(C(\alpha_2 x - \alpha_1 t) + Cd_0)}{B\exp(C(\alpha_2 x - \alpha_1 t) + Cd_0) - 1}\right)^2,
\end{aligned} \tag{35}$$

where d_0 is a constant.

Case III: $A = C = 1,\ B = 0$

$$\begin{aligned}
a_0 &= \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 - 16\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2}, \quad b_0 = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^4 - 16\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4}, \quad a_1 = b_1 = 0,\\
a_2 &= -12\alpha_1\alpha_2\varepsilon^2, b_2 = -6\alpha_2^2\delta^2,\\
u(x,t) &= \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 - 16\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2} - 12\alpha_1\alpha_2\varepsilon^2\tan^2(\alpha_2 x - \alpha_1 t),\\
h(x,t) &= \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^4 - 16\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4} - 6\alpha_2^2\delta^2\tan^2(\alpha_2 x - \alpha_1 t).
\end{aligned} \tag{36}$$

Case IV: $A = C = -1,\ B = 0$

$$\begin{aligned}
&a_0 = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 - 16\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2}, \quad b_0 = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^4 - 16\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4}, \quad a_1 = b_1 = 0,\\
&a_2 = -12\alpha_1\alpha_2\varepsilon^2, b_2 = -6\alpha_2^2\delta^2,\\
&u(x,t) = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 - 16\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2} - 12\alpha_1\alpha_2\varepsilon^2\cot^2(\alpha_2 x - \alpha_1 t),\\
&h(x,t) = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^4 - 16\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4} - 6\alpha_2^2\delta^2\cot^2(\alpha_2 x - \alpha_1 t).
\end{aligned} \tag{37}$$

Case V: $A = 1,\ C = -1,\ B = 0$

$$\begin{aligned}
&a_0 = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 + 16\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2}, \quad b_0 = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^4 + 16\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4},\\
&a_2 = -12\alpha_1\alpha_2\varepsilon^2, b_2 = -6\alpha_2^2\delta^2, a_1 = b_1 = 0,\\
&u(x,t) = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 + 16\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2} - 12\alpha_1\alpha_2\varepsilon^2\tanh^2(\alpha_2 x - \alpha_1 t),\\
&u(x,t) = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 + 16\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2} - 12\alpha_1\alpha_2\varepsilon^2\coth^2(\alpha_2 x - \alpha_1 t),\\
&h(x,t) = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^4 + 16\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4} - 6\alpha_2^2\delta^2\tanh^2(\alpha_2 x - \alpha_1 t),\\
&h(x,t) = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^4 + 16\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4} - 6\alpha_2^2\delta^2\coth^2(\alpha_2 x - \alpha_1 t).
\end{aligned} \tag{38}$$

Case VI: $A = C = \frac{1}{2},\ B = 0$

$$\begin{aligned}
&a_0 = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 - 4\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2}, b_0 = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^{42} - 4\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4},\\
&a_1 = 0, b_1 =^2, a_2 = -3\alpha_1\alpha_2\varepsilon^2, b_2 = -\frac{3}{2}\alpha_2^2\delta^2, b_3 = 0,\\
&u(x,t) = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 - 4\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2} - 3\alpha_1\alpha_2\varepsilon^2(\sec(\alpha_2 x - \alpha_1 t) + \tan(\alpha_2 x - \alpha_1 t))^2,\\
&u(x,t) = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 - 4\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2} - 3\alpha_1\alpha_2\varepsilon^2(\csc(\alpha_2 x - \alpha_1 t) + \cot(\alpha_2 x - \alpha_1 t))^2,\\
&h(x,t) = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^{42} - 4\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4} - \frac{3}{2}\alpha_2^2\delta(\sec(\alpha_2 x - \alpha_1 t) + \tan(\alpha_2 x - \alpha_1 t))^2,\\
&h(x,t) = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^{42} - 4\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4} - \frac{3}{2}\alpha_2^2\delta(\csc(\alpha_2 x - \alpha_1 t) + \cot(\alpha_2 x - \alpha_1 t))^2.
\end{aligned} \tag{39}$$

Case VII: $A = \frac{1}{2},\ C = -\frac{1}{2},\ B = 0$

$$\begin{aligned}
&a_0 = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 + 4\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2}, \quad b_0 = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^{42} + 4\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4},\\
&a_1 = 0, b_1 = 0, a_2 = -3\alpha_1\alpha_2\varepsilon^2, b_2 = -\frac{3}{2}\alpha_2^2\delta^2, b_3 = 0,\\
&u(x,t) = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 + 4\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2} - 3\alpha_1\alpha_2\varepsilon^2(\coth(\alpha_2 x - \alpha_1 t) + \operatorname{csch}(\alpha_2 x - \alpha_1 t))^2,\\
&u(x,t) = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 + 4\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2} - 3\alpha_1\alpha_2\varepsilon^2(\tanh(\alpha_2 x - \alpha_1 t) + i\operatorname{sech}(\alpha_2 x - \alpha_1 t))^2,\\
&h(x,t) = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^{42} + 4\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4} - \frac{3}{2}\alpha_2^2\delta(\coth(\alpha_2 x - \alpha_1 t) + \operatorname{csch}(\alpha_2 x - \alpha_1 t))^2,\\
&h(x,t) = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^{42} + 4\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4} - \frac{3}{2}\alpha_2^2\delta(\tanh(\alpha_2 x - \alpha_1 t) + i\operatorname{sech}(\alpha_2 x - \alpha_1 t))^2,
\end{aligned} \tag{40}$$

where $i = \sqrt{-1}$.

Case VIII: $A = C = -\frac{1}{2},\ B = 0$

$$\begin{aligned}
&a_0 = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 - 4\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2}, \quad b_0 = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^{42} - 4\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4},\\
&a_1 = 0, b_1 = 0, a_2 = -3\alpha_1\alpha_2\varepsilon^2, b_2 = -\frac{3}{2}\alpha_2^2\delta^2, b_3 = 0,\\
&u(x,t) = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 + 4\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2} - 3\alpha_1\alpha_2\varepsilon^2(\sec(\alpha_2 x - \alpha_1 t) - \tan(\alpha_2 x - \alpha_1 t))^2,\\
&u(x,t) = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2 + 4\alpha_1^2\alpha_2^2\varepsilon^4}{2\alpha_1\alpha_2\varepsilon^2} - 3\alpha_1\alpha_2\varepsilon^2(\csc(\alpha_2 x - \alpha_1 t) - \cot(\alpha_2 x - \alpha_1 t))^2,\\
&h(x,t) = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^{42} + 4\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4} - \frac{3}{2}\alpha_2^2\delta(\sec(\alpha_2 x - \alpha_1 t) - \tan(\alpha_2 x - \alpha_1 t))^2,\\
&h(x,t) = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^{42} + 4\alpha_1^2\alpha_2^2\varepsilon^4\delta^2}{4\alpha_1^2\varepsilon^4} - \frac{3}{2}\alpha_2^2\delta(\csc(\alpha_2 x - \alpha_1 t) - \cot(\alpha_2 x - \alpha_1 t))^2.
\end{aligned} \tag{41}$$

Case IX: $A = B = 0,\ C \neq 0$

$$\begin{aligned}
&a_0 = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2}{2\alpha_1\alpha_2\varepsilon^2}, \quad b_0 = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^4}{4\alpha_1^2\varepsilon^4}, \quad a_1 = b_1 = b_3 = 0, \quad a_2 = -12C^2\alpha_1\alpha_2\varepsilon^2, \quad b_2 = -6C^2\alpha_2^2\delta^2,\\
&u(x,t) = \frac{2\alpha_1^2\varepsilon^2 - \alpha_2^2\delta^2}{2\alpha_1\alpha_2\varepsilon^2} - 12C^2\alpha_1\alpha_2\varepsilon^2\left(\frac{1}{C(\alpha_2 x - \alpha_1 t) + e_0}\right)^2,\\
&h(x,t) = \frac{\alpha_2^2\delta^4 - 4\alpha_1^2\varepsilon^4}{4\alpha_1^2\varepsilon^4} - 6C^2\alpha_2^2\delta^2\left(\frac{1}{C(\alpha_2 x - \alpha_1 t) + e_0}\right)^2,
\end{aligned} \tag{42}$$

where e_0 is a constant.

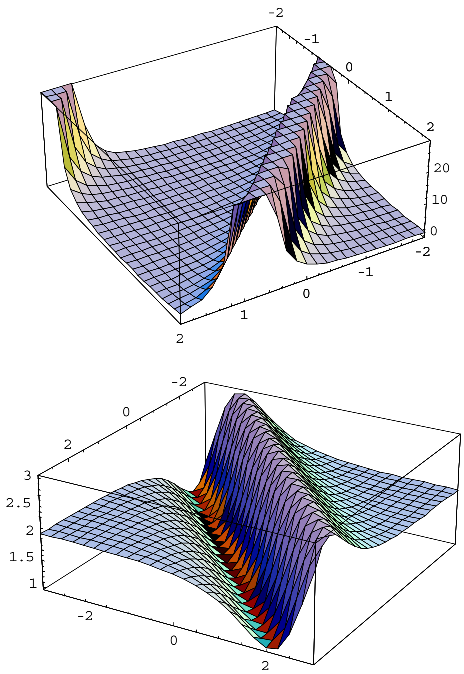

Fig. 1. The solitary wave solutions for the Case VI.

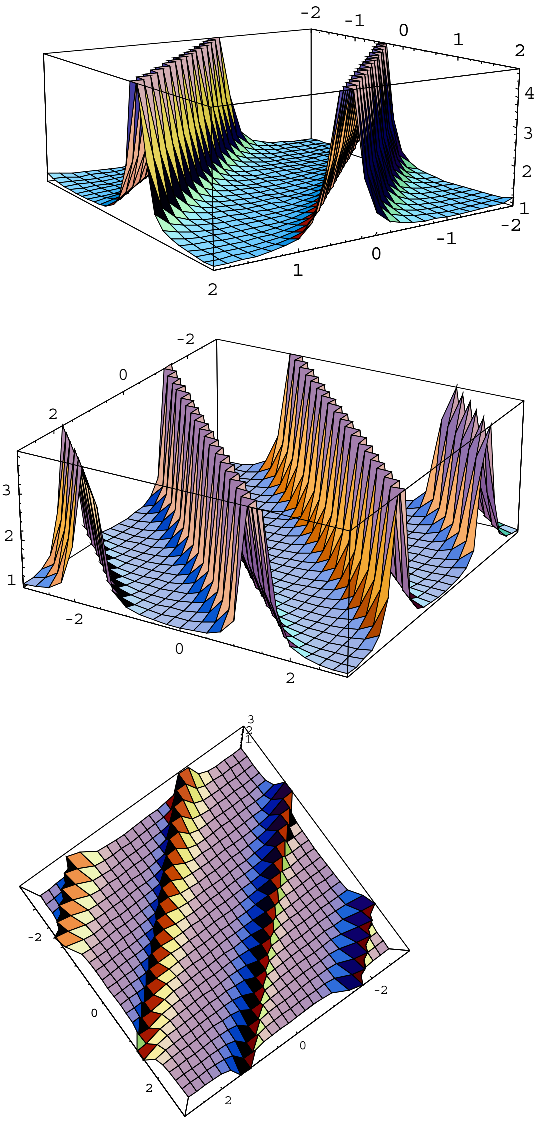

Fig. 2. The double periodic wave solutions for the Case III.

For example, if we consider the Case VI, and then the graphical representation of similarity solitary wave solutions for the variant nonlinear long-wave equations (1) for some specific values of parameters can be shown as in Fig. 1.

If we consider the Case III, and then the graphical representation of new similarity double periodic wave solutions for variant nonlinear the long-wave equations (1) for some specific values of parameters can be shown as in Fig. 2.

Other graphical representations for the similarity solutions for u and h can be analyzed by the similar manner presented above.

4. Conclusions

This study deals with the symmetry group analysis and invariant solutions of the variant nonlinear long-wave equations. In order to obtain exact analytic solutions of system of nonlinear differential equations, the most effective and important approach is the Lie symmetry group analysis. For this purpose, the most general symmetry groups that leave invariant the system under consideration are investigated. By the standard application of the approach we first prove that the variant nonlinear long-wave equations have a two-parameter symmetry group that corresponds to the two symmetry groups. This result shows that only traveling wave solutions of the variant nonlinear long-wave equations can be obtained by using the symmetry group analysis. Using the characteristic equation, the new similarity independent variable and dependent similarity variables are found. Then, the reduced form of the original nonlinear partial differential equations is obtained as the system of nonlinear ordinary differential equations. The improved tanh method is used for obtaining the solutions of the reduced system of nonlinear ordinary differential equations. Thus, the similarity solutions of the variant nonlinear long-wave equations are obtained as a rich variety of exact analytic solutions. As examples, for some cases the graphical representations of the solutions corresponding to the solitary wave solutions and double periodic wave solutions are presented.

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