B.7 Lie Groups and Differential Equations

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The applications of Lie groups to solve differential equations dates back to the original work of Sophus Lie, who invented Lie groups for this purpose. The modern era begins with Birkhoff (1950), and was forged into a key tool of applied mathematics by Ovsiannikov (1982). Basic references are (Hydon, 2000; Olver, 1993, 1995).

First we review the geometric approach to systems of differential equations. We begin with a smooth m-dimensional manifold M; the reader will not experience any significant loss of generality by taking $M = \mathbb{R}^m$. Solutions will be identified as p-dimensional (smooth) submanifolds $S \subset M$. Local coordinates on M include a choice of independent variables $x = (x^1, \ldots, x^p)$, and dependent variables $u = (u^1, \ldots, u^q)$, where $p + q = m$, and so a (transverse) submanifold is given as the graph of a function $u = f(x)$. The derivatives of the dependent variables are represented by $u^\alpha_J = \partial_J f^\alpha(x)$, where $\alpha = 1, \ldots, q$ and J is a multi-index of order $0 \leq |J| \leq n$. These form a system of local coordinates, collectively denoted by $(x, u^{(n)})$, on the n-th order (extended) jet bundle $\mathrm{J}^n \to M$. The jet bundle can be defined as the set of equivalence classes of p-dimensional submanifolds $S \subset M$, where $S \sim \widetilde{S}$ define the same equivalence class or *jet* $\mathrm{j}_n S \in \mathrm{J}^n$ at common point $z \in S \cap \widetilde{S}$ whenever the two submanifolds have n-th order contact at z. A system of differential equations $\Delta_\nu(x, u^{(n)}) = 0$, $\nu = 1, \ldots, s$ is *regular* if the Jacobian matrix of the Δ_ν has maximal rank s at all $(x, u^{(n)})$ that satisfy the system. A regular system can be viewed as a submanifold $\mathcal{S}_\Delta = \{\Delta_\nu(x, u^{(n)}) = 0\} \subset \mathrm{J}^n$. A classical (smooth) solution is, thus, a submanifold $S \subset M$ whose jet $\mathrm{j}_n S \subset \mathcal{S}_\Delta$. The system is *locally solvable* if there exists a smooth solution passing through each point $(x, u^{(n)}) \in \mathcal{S}_\Delta$.

Let G be an r-dimensional Lie group acting smoothly on M. Since G preserves contact between submanifolds, there is an induced action, denoted $G^{(n)}$, on J^n, called the n-th order *prolonged* action, which tells us how G acts on the derivatives of functions. The action defines a *symmetry group* of a system of differential equations $\mathcal{S}_\Delta$ if it maps solutions to solutions. Assuming local solvability this occurs if and only if $\mathcal{S}_\Delta$ is a $G^{(n)}$-invariant subset of J^n.

A connected Lie group action is entirely determined by its *infinitesimal generators*, which are vector fields on the manifold M and can be identified with the Lie algebra $\mathfrak{g}$ (often denoted by $\mathfrak{g}$ in the literature) of G. Each vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u)\frac{\partial}{\partial u_J^\alpha} \in \mathfrak{g},$$

generates a one-parameter subgroup. The infinitesimal generator of the corresponding n-th prolonged one-parameter subgroup is a vector field

$$\mathrm{pr}\,\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{j=\#J} \varphi_J^\alpha(x,u^{(j)})\frac{\partial}{\partial u_J^\alpha},$$

on J^n. There is an explicit formula for the coefficients φ_J^α of the prolongation pr $\mathbf{v}$ in terms of the derivatives of the coefficients ξ^i, φ^α of $\mathbf{v}$. This *prolongation formula*, coupled with the following *infinitesimal symmetry criterion* allows us to explicitly compute the symmetry groups of almost any systems of differential equations. Indeed, there now exist a wide range of computer algebra packages for performing this computation, (Hereman, 1994).

B.7.1 Theorem (SymGroupDEQ) *A connected group of transformations G is a symmetry group of the regular system of differential equations $\mathcal{S}_\Delta$ if and only if* pr $\mathbf{v}(\Delta_\nu)=0$, $\nu=1,\ldots,s$, *on $\mathcal{S}_\Delta$ for every $\mathbf{v} \in \mathfrak{g}$.*

Example

Consider the linear heat equation $u_t = u_{xx}$. Applying Theorem B.7.1, an infinitesimal symmetry $\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \varphi(x,t,u)\partial_u$ must satisfy $\tau_u = \tau_x = \xi_{uu} = 0$, $-\xi_u = -2\tau_{xu} - 3\xi_u$, $\varphi_{uu} = 2\xi_{xu}$, $\varphi_u - \tau_t = -\tau_{xx} + \varphi_u - 2\xi_x$, $-\xi_t = 2\varphi_{xu} - \xi_{xx}$, $\varphi_t = \varphi_{xx}$. The solution space to this overdetermined linear system of partial differential equations yields the symmetry algebra of the heat equation, with basis $\mathbf{v}_1 = \partial_x$, $\mathbf{v}_2 = \partial_t$, $\mathbf{v}_3 = u\partial_u$, $\mathbf{v}_4 = x\partial_x + 2t\partial_t$, $\mathbf{v}_5 = 2t\partial_x - xu\partial_u$, $\mathbf{v}_6 = 4xt\partial_x + 4t^2\partial_t - (x^2+2t)u\partial_u$, and $\mathbf{v}_\alpha = \alpha(x,t)\partial_u$, where $\alpha_t = \alpha_{xx}$. The corresponding one-parameter groups are, respectively, x and t translations, scaling in u, the scaling $(x,t) \mapsto (\lambda x, \lambda^2 t)$, Galilean boosts, an "inversional symmetry", and the addition of solutions stemming from the linearity of the equation. Each of these groups maps solutions to solutions, e.g., the inversional group tells us that if $u = f(x,t)$ is any solution, so is $u = \frac{1}{1+4\varepsilon t}\exp\left\{\frac{-\varepsilon x^2}{1+4\varepsilon t}\right\} f\left(\frac{x}{1+4\varepsilon t}, \frac{t}{1+4\varepsilon t}\right)$, for any $\varepsilon \in \mathbb{R}$. The constant solution $u = 1/\sqrt{2\pi}$ produces the fundamental solution at $(0, -(4\varepsilon)^{-1})$. Thus, the symmetry group provides an effective mechanism for computing a wide variety of new solutions from known solutions. Further applications—to finding explicit group-invariant solutions, to determining conservation laws, to solution, to classification of differential equations with given symmetry groups, and so on—are described below.

Generalized Symmetries

For ordinary or *point* symmetries, the coefficients ξ^i, φ^α of $\mathbf{v}$ depend only on x, u. Generalized or higher order symmetries, including contact transformations, allow dependence on the derivatives u_J^α as well. Higher order symmetries play an essential role in the study of integrable soliton equations, (Fokas, 1980; Mikhailov *et al.*, 184; Sanders and Wang, 1998). *Recursion operators* and *master symmetries* map symmetries to symmetries and thereby generate infinite hierarchies of generalized symmetries. The biHamiltonian structure theory of Magri (1978); Olver (1993), provides an method for constructing recursion operators.

Linearization of Partial Differential Equations

Any linear partial differential equation has an infinite-dimensional symmetry group: addition of solutions. A system of partial differential equations can be linearized if and only if it has an infinite-dimensional symmetry group of the proper form.

Noether's Theorems

A variational problem admitting a symmetry group G leads to a G-invariant system of Euler-Lagrange equations. Noether's first theorem, (Noether, 1918), associates a conservation law for the Euler-Lagrange equations with every one-parameter symmetry group of the variational problem. For instance, translation invariance leads to conservation of linear momentum, rotation invariance leads to conservation of angular momentum, and time translation invariance leads to conservation of energy. Noether's second theorem, of application in relativity and gauge theories, produces dependencies among the Euler-Lagrange equations arising from infinite-dimensional variational symmetry groups.

Integration of Ordinary Differential Equations

Lie observed that virtually all the classical methods for solving ordinary differential equations (separable, homogeneous, exact, etc.) are instances of a general method for integrating ordinary differential equations that admit a symmetry group. An n-th order scalar ordinary differential equation admitting an n-dimensional solvable symmetry group can be integrated by quadrature. The associated conservation laws of variational problems and Hamiltonian systems doubles the order of symmetry reduction.

Symmetry Reduction of Partial Differential Equations

If the orbits of G are s-dimensional and transverse to the vertical fibers $\{x=c\}$, then the G-invariant solutions to a G-invariant system of differential equations are found by reducing to a system in $p-s$ variables. See (Bluman and Cole, 1969; Olver and Rosenau, 1987) for the nonclassical generalization, and (Anderson and Fels, 1997) for the nontransverse case, of importance in many physical systems, e.g., relativity, fluid mechanics.

Differential Invariants

A function $I\colon \mathrm{J}^n \to \mathbb{R}$ which is invariant under the action of $G^{(n)}$ is known as a differential invariant. Basic examples include curvature and torsion of curves, and the Gaussian and mean curvature of surfaces in three-dimensional Euclidean geometry. The differential invariants are the fundamental building blocks for constructing G-invariant differential equations, variational problems, etc., as well as solving the basic problems of equivalence and symmetry of submanifolds. For example, every Euclidean-invariant differential equation for space curves involves just the curvature, torsion and their arc-length derivatives: $F_\nu(\kappa, \kappa_s, \ldots, \tau, \tau_s, \ldots) = 0$.

Differential invariants are characterized by the infinitesimal invariance criterion $\mathrm{pr}\,\mathbf{v}(I) = 0$ for all $\mathbf{v} \in \mathfrak{g}$. Cartan's method of moving frames, (Cartan, 1935; Fels and Olver, 1999), forms an effective tool for producing complete systems of differential invariants. An n-th order *moving frame* is a smooth, G-equivariant map $\rho : \mathrm{J}^n \to G$, where G acts on itself by left multiplication. The most familiar case is the moving frame for a curve in $\mathbb{R}^3$, consisting of a point z on the curve together with the unit tangent $\vec{t}$, normal $\vec{n}$ and binormal $\vec{b}$ at z. These form a left-equivariant map $\rho : \mathrm{J}^2 \to \mathrm{E}(3)$ from the second jet space to the Euclidean group, where we interpret $z \in \mathbb{R}^3$ as the translation component and the 3×3 matrix $[\,\vec{t}, \vec{n}, \vec{b}\,] \in \mathrm{O}(3)$ as the rotation component of the group element. In general, a moving frame exists if and only if $G^{(n)}$ acts freely and regularly, which holds in all practical examples for $n \gg 0$. Normalization amounts to setting $r = \dim G$ components of the prolonged group transformations $(g^{(n)})^{-1} \cdot (x, u^{(n)})$ to be suitably chosen constants. Solving for the group parameters and substituting into the remaining components produces a complete system of differential invariants. In the case of space curves, these are the curvature, torsion and their successive derivatives with respect to arc length.

Symmetry Classification of Ordinary Differential Equations

Lie's classification of all finite-dimensional Lie groups acting on the plane, (Lie, 1924; Olver, 1995), along with their differential invariants and Lie determinants leads to a complete symmetry classification of scalar ordinary differential equations, and possible symmetry reductions.

Discrete Symmetries

Discrete symmetry groups also play an important role in differential equations, including Schwarz's theory of hypergeometric functions, Fuchsian and Kleinian groups, etc., (Hille, 1976). Discrete symmetries can often be determined from the continuous symmetry group, (Hydon, 2000).