Chapter 10

THE GENERAL ELASTICITY PROBLEM IN SOLIDS

In Chapters 3-5 and 8-9, we have developed equilibrium, kinematic and constitutive equations for a general three-dimensional elastic deformable solid body. In this chapter, we will summarize these equations for the one-, two- and three-dimensional states of stress. We will find that in each case, a system of equations will be obtained that must be solved with appropriate boundary conditions for the particular problem being addressed. The solution procedure for the general case will also be addressed in this chapter.

10.1 Necessary Equations for a 3-D Stress State

For a linear elastic solid under *static* equilibrium, we can now summarize the following three sets of equations for any 3-D body:

1. **Static Equilibrium Equations** (Conservation of Linear Momentum)

$$\begin{aligned}
x\text{-component:} \quad & \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho g_x = 0 \\
y\text{-component:} \quad & \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho g_y = 0 \quad \Leftarrow \quad \textbf{\textit{3 equations}} \\
z\text{-component:} \quad & \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho g_z = 0
\end{aligned} \tag{10.1}$$

2. **Constitutive Equations** for a linear elastic isotropic material are given by (no thermal effects included):

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \quad \Leftarrow \quad \textbf{\textit{6 equations}} \\
\varepsilon_{yz} &= \frac{1+\nu}{E}\sigma_{yz} \\
\varepsilon_{xz} &= \frac{1+\nu}{E}\sigma_{xz}
\end{aligned} \tag{10.2}$$

Alternately, (10.2) may be inverted to yield:

$$
\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\,\varepsilon_{xx}+\nu\varepsilon_{yy}+\nu\varepsilon_{zz}\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx}+(1-\nu)\,\varepsilon_{yy}+\nu\varepsilon_{zz}\right] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx}+\nu\varepsilon_{yy}+(1-\nu)\,\varepsilon_{zz}\right] \\
\sigma_{xy}=\sigma_{yx} &= \frac{E}{1+\nu}\varepsilon_{xy} \\
\sigma_{xz}=\sigma_{zx} &= \frac{E}{1+\nu}\varepsilon_{xz} \\
\sigma_{yz}=\sigma_{zy} &= \frac{E}{1+\nu}\varepsilon_{yz}
\end{aligned}
\tag{10.3}
$$

3. **Kinematics (Strain-Displacement) equations** for small strain are given by

$$
\begin{aligned}
[\varepsilon] &= \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix} \\
&\Leftarrow \quad \textbf{\textit{6 equations}}
\end{aligned}
\tag{10.4}
$$

Consequently, for the general 3-D linear elasticity problem, we have a *system of 15 governing partial differential equations*:

- 3 equilibrium (linear momentum) equations for $\boldsymbol{\sigma}$
- 6 constitutive equations ($[\boldsymbol{\sigma}] = [\mathbf{C}]\,\{\boldsymbol{\varepsilon}\}$)
- 6 kinematic (strain) equations ($\{\boldsymbol{\varepsilon}\}$ = function of displacement gradients)

that relate the *15 unknown variables*

- stresses (6),
- strains (6) and
- displacements (3).

The 15 governing equations are coupled partial differential equations that must be solved simultaneously. As in the solution of any differential equation, *boundary conditions must be specified to solve this boundary value problem.* These boundary conditions must be *either* displacements or tractions on *every* point of the boundary.

10.2 Necessary Equations for a 2-D Stress State

There are many *special cases* of the general elasticity problem that are *of practical interest* (and can be solved!). These include ***Plane Stress and Plane Strain*** problems.

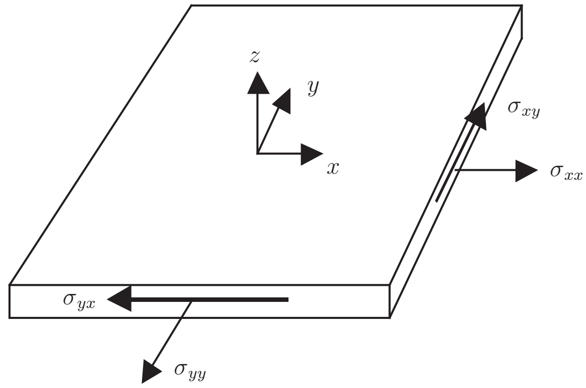


Figure 10.1: Plate in State of Plane Stress

Plane stress occurs in *thin* bodies which have non-zero stresses in one plane only (and all out-of-plane stresses are zero). For example, in the x-y plane, we have:

For a geometry that is plane stress in the x-y plane, we assume $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$. The stress tensor reduces to $[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and

1. **Static Equilibrium** becomes

$$\begin{aligned} x\text{-component:} \quad & \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \rho g_x = 0 \\ y\text{-component:} \quad & \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho g_y = 0 \end{aligned} \tag{10.5}$$

2. **Constitutive Equations** for a linear elastic isotropic material *in plane stress condition* become (have to substitute plane stress requirement of $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$ into general equations for strain):

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu\sigma_{yy}] \\ \varepsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu\sigma_{xx}] \\ \varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\ \varepsilon_{zz} &= -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \end{aligned} \tag{10.6}$$

Alternately, (10.6) may be inverted to yield:

$$\begin{aligned} \sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}] \\ \sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy} + \nu\varepsilon_{zz}] \\ \sigma_{zz} &= 0 \\ \sigma_{xy} &= \frac{E}{(1+\nu)}\varepsilon_{xy} \\ \sigma_{xz} &= 0 \\ \sigma_{yz} &= 0 \end{aligned} \tag{10.7}$$

3. **Kinematics (Strain-Displacement) equations** become

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yx} & 0 \\ \varepsilon_{xy} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & 0 \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{\partial u_y}{\partial y} & 0 \\ 0 & 0 & \frac{\partial u_z}{\partial z} \end{bmatrix} \tag{10.8}$$

Note that for 2-D plane stress, we have 8 unknowns (3 stresses, 3 strains, and 2 displacements).

10.3 Necessary Equations for a 1-D Stress State

Consider the case where a uniaxial load is applied to a uniaxial bar oriented in the x direction. For this one-dimensional state of stress, only σ_{xx} exists and $\sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$. We have the following set of necessary equations:

1. **Static Equilibrium** becomes

$$x\text{-component: } \frac{d\sigma_{xx}}{dx} + \rho g_x = 0 \tag{10.9}$$

2. **Constitutive Equation** for a linear elastic isotropic material becomes:

$$\begin{aligned} \sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}] \\ \varepsilon_{xx} &= \frac{\sigma_{xx}}{E}, \\ \varepsilon_{yy} &= -\frac{\nu\sigma_{xx}}{E}, \\ \varepsilon_{zz} &= -\frac{\nu\sigma_{xx}}{E} \end{aligned} \tag{10.10}$$

3. **Kinematics (Strain-Displacement) equation** becomes

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & 0 & 0 \\ 0 & \frac{\partial u_y}{\partial y} & 0 \\ 0 & 0 & \frac{\partial u_z}{\partial z} \end{bmatrix} \tag{10.11}$$

For the 1-D case, we have 3 unknowns: 1 stress (σ_{xx}), 1 strain (ε_{xx}) and 1 displacement (u_x). The other non-zero strain and displacement components appearing in equations (10.10) and (10.11) can be written in terms of the three unknown quantities listed previously. It must be noted that while there is only one non-zero stress (σ_{xx}), this stress produces non-zero strains normal to the direction of loading (ε_{yy} and ε_{zz})) due to the Poisson's ratio effect.

10.3.1 1-D Special Cases

All of the general 3-D equations relating stress, strain and deformation may be specialized to some important 1-D cases which involve long, slender geometries like beams, rods, bars, tubes, etc. in which the stress state is 1-D. In the following chapters, we will consider in detail three special cases: bars in tension, bars in torsion and beam bending. Typical examples of these are shown below.

- Bar with axial force only
- Bar (or pipe) in torsion
- Beam in bending

In each case, we will develop expressions for the appropriate displacement (or twist) and stress within the body as function of position within the body.

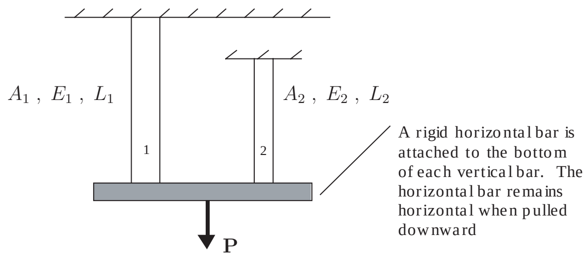


Figure 10.2:

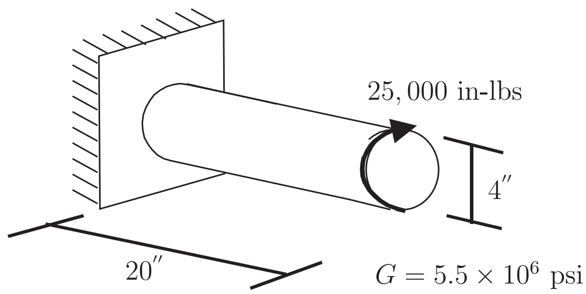


Figure 10.3:

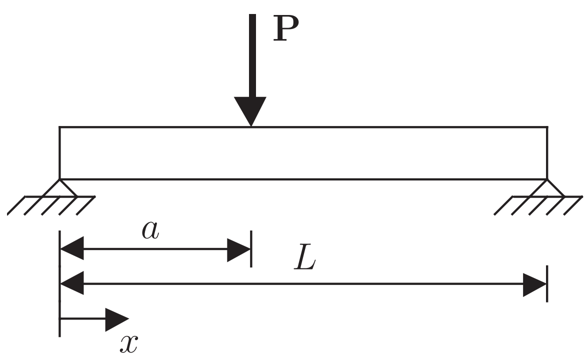


Figure 10.4:

10.4 General Solution Procedure

The governing equations are for an elastic solid body are in general a system of coupled partial differential equations that must be solved simultaneously. As in the solution of any differential equation, *boundary conditions must be specified to solve this boundary value problem.* These boundary conditions must be *either* displacements or tractions on *every* point of the boundary. Consequently, for every problem we must satisfy the following four (4) sets of equations:

Four Sets of Equations to be Satisfied for Any Solid Deformable Body

	3-D	*2-D*	*1-D*
Static Equilibrium Equation (from COLM):	(10.1)	(10.5)	(10.9)
Constitutive Equation (Stress-Strain):	(10.3)	(10.7)*	(10.10)
Kinematics (strain-displacement):	(10.4)	(10.8)	(10.11)
Boundary Conditions:	Depends on	problem geometry	and loading
		* for plane stress	condition

Example 10-1

Consider a uniaxial bar that experiences a **simple extension** in the x direction under a uniform loading $\sigma = \sigma_0$ on the end surfaces as shown below:

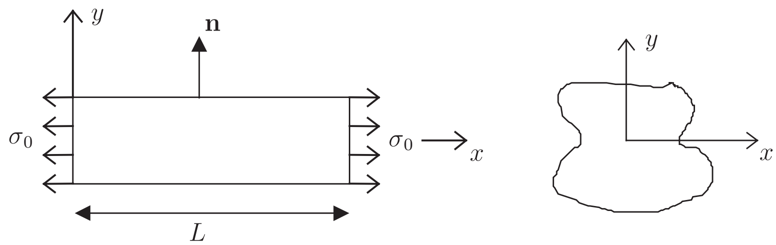


Figure 10.5:

Try $\sigma_{xx} = \sigma_0$, $\sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$

- Equilibrium equations:
  identically satisfied

- Constitutive Equations

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] = \frac{\sigma_0}{E} \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} - \sigma_{zz})\right] = -\nu\frac{\sigma_0}{E} \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{yy} + \sigma_{xx})\right] = -\nu\frac{\sigma_0}{E} \\
\varepsilon_{xy} &= 0 \\
\varepsilon_{xz} &= 0 \\
\varepsilon_{yz} &= 0
\end{aligned} \tag{10.12}$$

- Kinematics

$$u_x = \left(\frac{\sigma_0}{E}\right) x_x, \quad u_y = -\left(\frac{\nu\sigma_0}{E}\right) x_y, \quad u_z = -\left(\frac{\nu\sigma_0}{E}\right) x_z \tag{10.13}$$

Substitute $\sigma_0 = \frac{P}{A}$ into the equation for u:

$$u_x = \frac{Px}{EA} \tag{10.14}$$

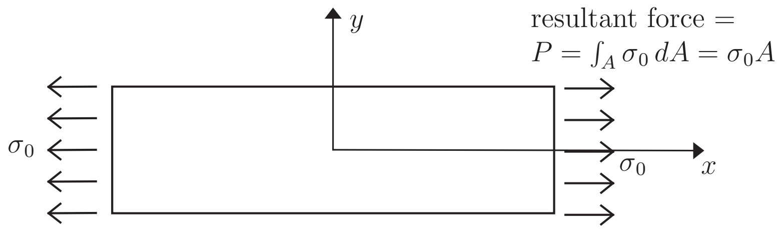


Figure 10.6:

- Boundary Conditions:

$$\left[\mathbf{t}_{(\mathbf{n})}\right] = [\mathbf{n}] \cdot [\boldsymbol{\sigma}]$$

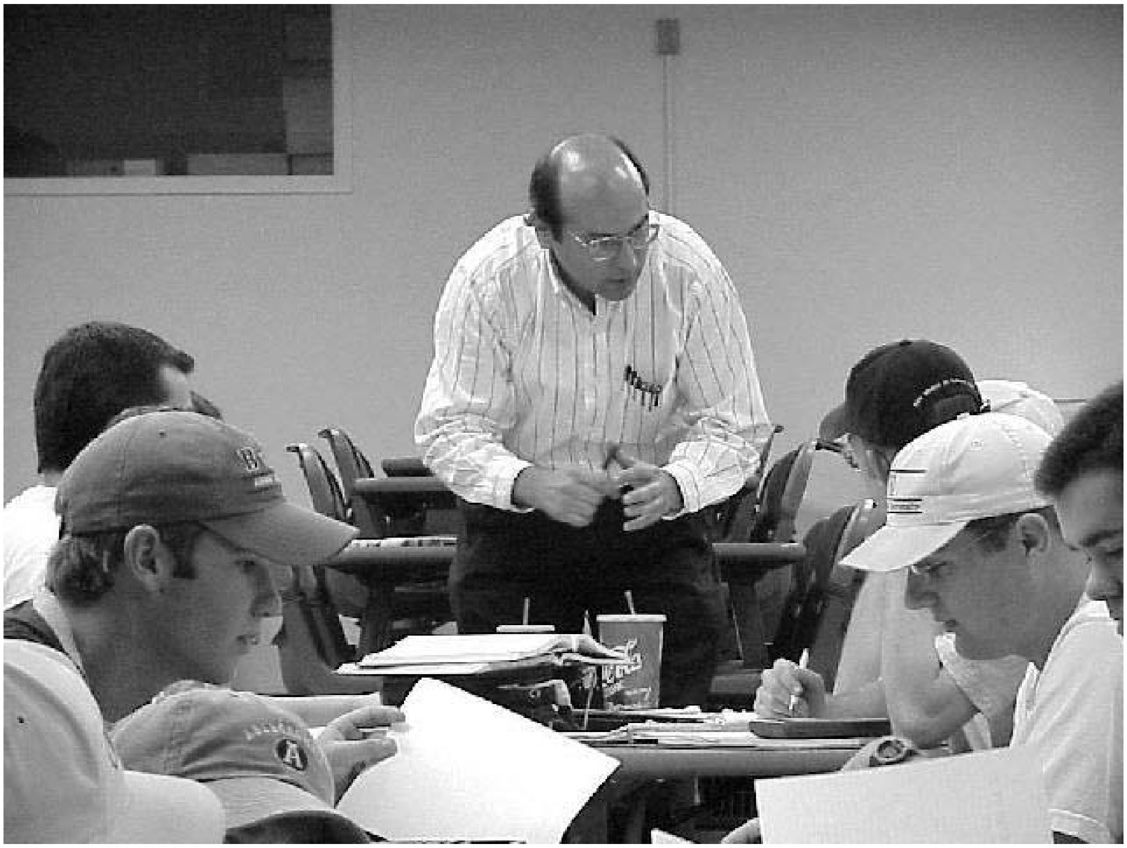
$$\text{lateral surface}: \ \left[\mathbf{t}_{(\mathbf{n})}\right] = \begin{bmatrix} 0 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\text{end surfaces}: \ \left[\mathbf{t}_{(\mathbf{n})}\right] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_0 & 0 & 0 \end{bmatrix}$$

$$x = L$$

$$x = 0 \rightarrow \left[\mathbf{t}_{(\mathbf{n})}\right] = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\sigma_0 & 0 & 0 \end{bmatrix}$$

Deep Thought



Collaborative learning brings you one step closer to the solution steps of BVP in continuum mechanics.