ME 821 - Elasticity

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Grading

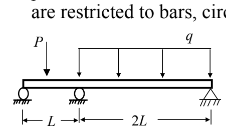
- A grade will be determined as follows:
  - Exam #1 25 percent
  - Exam #2 25 percent
  - Final 35 percent
  - Homework 15 percent
- Exams will be either in-class or take-home. HW will be from the book and/or from class hand-outs.
- If a project is undertaken in the class, the relative weighting above will be changed.

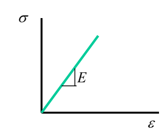
Course Objectives

- Comprehension of the structure of classic theory of elasticity
- Ability to recognize and formulate a well-posed problem
- Exposure to a variety of topics within the theory
- Presentation of basic theorems and solution techniques
- Illustration of alternate approaches to the same problem
- Ability to use the principle of superposition effectively

Preamble – Mech. Of Materials

In elementary mechanics of materials we study simple element's deformation under loading. Usually the problems solved with the mechanics of materials approach are restricted to bars, circular shafts, beams, and frames.



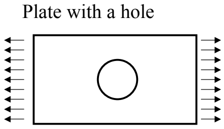


Generally, significant kinematic simplifying assumptions are made in obtaining mechanics of materials solutions.

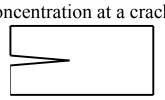
Preamble – Elasticity Solutions

- Elasticity with its general theory allow more difficult problems to be solved exactly such as:

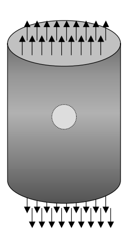
Plate with a hole



Stress concentration at a crack tip

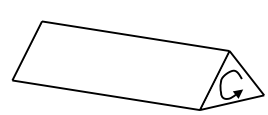


Solid with spherical inclusion

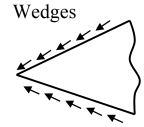


Preamble – Elasticity Solutions

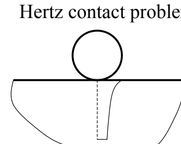
Torsion of noncircular cross section



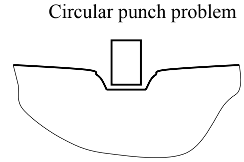
Wedges



Hertz contact problem



Circular punch problem



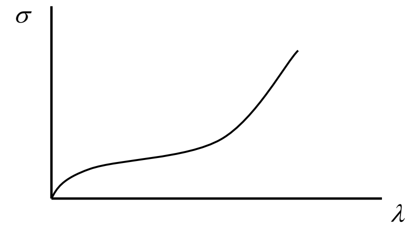
Preamble - Elastodynamics

Wave propagation is a topic which generally falls under the category of elasticity. Small amplitude stress waves travel through solids at speeds determined from the material's elastic properties. Applications include:

- measurement of elastic moduli
- measurement of residual stress
- measurement of texture

Preamble – Nonlinear Elasticity

Some materials don't exhibit linear stress-strain behavior to any extent



Preamble - Common Aspects

- For each of the topics discussed there are both undergraduate and graduate courses devoted to each. Yet there is quite a bit in common among them.
  - All of the previously mentioned topics have material occupying space.
  - Newton's Laws are valid for all topics
  - To solve specific problems in any discipline boundary conditions must be applied.

Preamble - Common Aspects

Continuum mechanics lays the foundation for the study of these problems.

- material particles
- description of position
- stress-strain relationships
- boundary conditions

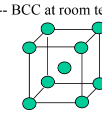
Preamble – Course Outline

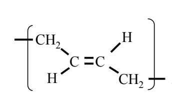
- Review of material assumed known
- Some basic theorems of elasticity
- St. Venant's problem and its relationship to mechanics of materials
- The plane problem
- Three dimensional elasticity
- Nonlinear elasticity
- Anisotropic elasticity (Stroh formalism)

Engineering Materials

- In mathematics a point has no dimension. Even in a metal a point within the boundary could be found which has no material occupying that position.
- A rubber, say *trans* 1,4 polybutadiene, is made up of carbon and hydrogen atoms connected in a repeating mer shown to the right.
- Structural foam
  - no Poisson's ratio

Iron -- BCC at room temperature





Continuum Concept

- ***Material properties are assumed to be the same no matter how many times we subdivide the region of interest.***
- By doing this, we can utilize mathematical tools involving continuous functions and obtain "exact" solutions.
- Before we tackle any problems, we must learn about tensors, indicial and direct notation, linear algebra, and vector calculus (using indicial notation!). This is similar to an undergraduate student mastering vector components in Statics and Dynamics.

Review of material assumed known

a) Preliminaries
   i. Vectors, tensors, Cartesean tensors

b) Stress
   i. Definition, principal stress

c) Kinematics of deformation
   i. Position, displacement, strain
   ii. Infinitesimal linear displacements

d) Constitutive relationships
   i. elasticity

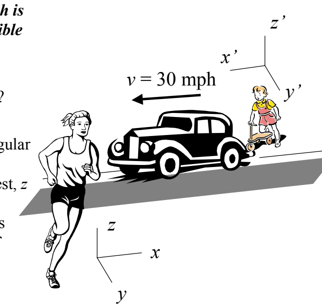
Tensor Basics

Tensor Order	Class	Examples
0^{th}	Scalar	temperature, density, speed
1^{st}	Vector (have directional properties)	velocity, force, acceleration
2^{nd}	Tensor	stress, strain, moment of inertia

There are 3rd, 4th, and higher order tensors which are common but have not been listed in this table.

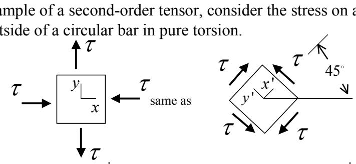
Tensor Definition

- ***A tensor is a quantity which is invariant under an admissible coordinate transformation.***
- What is meant by invariant?
- Say each of us has a rectangular coordinate system having x pointing away from our chest, z points up, and y is directed towards our left. Each of us stands on opposing sides of Third Ave. and observes a vehicle pass by.



Another Tensor Example

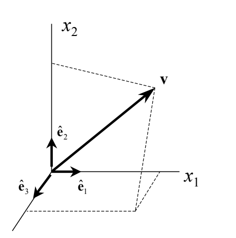
As an example of a second-order tensor, consider the stress on an element on the outside of a circular bar in pure torsion.



$$[\sigma'] = \begin{bmatrix} -\tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & 0 \end{bmatrix}_{xy} \quad \text{same as} \quad [\sigma] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{x'y'}$$

The admissible coordinate transformation is a 45° rotation about the z axis.

Vector Components



The "hat" designates a unit vector.

- Since we will be talking about coordinates which may be changing, a general notation must be used.
- Cartesian coordinates x_1, x_2, and x_3 will define three mutually orthogonal directions (these replace x, y, and z).
- $\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$ are unit vectors pointing in the x_1, x_2, and x_3 directions, respectively (these replace $\hat{\imath}, \hat{\jmath}$, and $\hat{k}$).

Any vector can be written in terms of components along these three mutually orthogonal directions.

$$\mathbf{v} = v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3 = \sum_{i=1}^{3} v_i\hat{\mathbf{e}}_i$$

We drop the summation sign and understand that indices will range from 1 to 3, and that indicies repeated once imply summation from 1 to 3.

$$\mathbf{v} = v_i\hat{\mathbf{e}}_i$$

This is called the ***summation convention***.

Compound Products of Vectors

- component form of triple product

$$\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = \mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = [\mathbf{u},\mathbf{v},\mathbf{w}]$$

- component form of triple product

$$\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = \varepsilon_{ijk}u_i v_j w_k$$

Permutation-Kronecker Eqs.

The Kronecker delta and permutation symbol have the following useful identities

$$\varepsilon_{miq}\varepsilon_{jkq} = \delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}$$

$$\varepsilon_{mkq}\varepsilon_{jkq} = \delta_{jm}$$

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6$$

Tensor Product of Two Vectors

The tensor product of two vectors results in a **dyad**.

$$\mathbf{a}\otimes\mathbf{b} = a_i\hat{\mathbf{e}}_i \otimes b_j\hat{\mathbf{e}}_j = a_i b_j \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$$

Tensor products are defined through their action on an arbitrary vector **v**. The following relation must hold:

$$\mathbf{a}\otimes\mathbf{b}\cdot\mathbf{v} = \mathbf{a}(\mathbf{b}\cdot\mathbf{v})$$

If vectors **a** and **b** form a tensor **T** through the tensor product, then

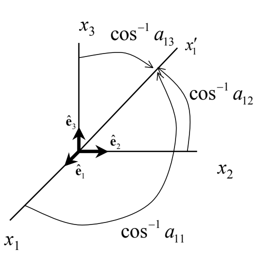
$$\mathbf{T} = \mathbf{a}\otimes\mathbf{b} \text{ then } \mathbf{T}\cdot\mathbf{v} = \mathbf{a}(\mathbf{b}\cdot\mathbf{v}) \text{ for all } \mathbf{v}$$

This relationship will be used later when stress is studied.

Transformation of Cartesian Tensors

It is often necessary to write the components of a tensor in terms of another reference frame. Such is the case in elementary mechanics of solids when Mohr's circle is used.

Consider an unprimed reference frame $\hat{e}_i$ and a primed reference frame $\hat{e}_i'$.



Directly from the sketch on previous page, the unit vectors are related to each other as follows:

$$\hat{\mathbf{e}}_1' = a_{11}\hat{\mathbf{e}}_1 + a_{12}\hat{\mathbf{e}}_2 + a_{13}\hat{\mathbf{e}}_3$$
$$\hat{\mathbf{e}}_2' = a_{21}\hat{\mathbf{e}}_1 + a_{22}\hat{\mathbf{e}}_2 + a_{23}\hat{\mathbf{e}}_3$$
$$\hat{\mathbf{e}}_3' = a_{31}\hat{\mathbf{e}}_1 + a_{32}\hat{\mathbf{e}}_2 + a_{33}\hat{\mathbf{e}}_3$$

Or, more compactly $\hat{\mathbf{e}}_i' = a_{ij}\hat{\mathbf{e}}_j$

where $a_{ij} = \hat{\mathbf{e}}_i'\cdot\hat{\mathbf{e}}_j$

A convenient way to display the components of a_{ij}, the **transformation matrix**, is by a transformation table.

Transformation Table

	$\hat{\mathbf{e}}_1$	$\hat{\mathbf{e}}_2$	$\hat{\mathbf{e}}_3$
$\hat{\mathbf{e}}'_1$	a_{11}	a_{12}	a_{13}
$\hat{\mathbf{e}}'_2$	a_{21}	a_{22}	a_{23}
$\hat{\mathbf{e}}'_3$	a_{31}	a_{32}	a_{11}

Transformation of a Vector

To transform an arbitrary vector, simply transform the unit vectors and let the components tag along.

$$v'_j = a_{ji} v_i \quad \text{or} \quad \mathbf{v}' = \mathbf{A}\mathbf{v} = \mathbf{v}\mathbf{A}^T$$

This gives the components of **v** in the primed system in terms of the components in the unprimed system (a transformation). It can be shown that

$$a_{ij} a_{ik} = \delta_{jk} \quad \text{or} \quad \mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$$

which means that the transformation matrix is an orthogonal matrix.

Transformation of a Vector cont.

Using the orthogonality of the transformation matrix allows us to derive an inverse between v_i and v_i'.

$$v'_j = a_{ji} v_i$$

$$a_{jk} v'_j = a_{jk} a_{ji} v_i = \delta_{ki} v_i = v_k$$

$$v_k = a_{jk} v'_j \quad \text{or} \quad \mathbf{v} = \mathbf{A}^T \mathbf{v}'$$

(Notice the difference in the indice's order in the two transformation equations. This difference determines whether to use A or A^T in the direct, or symbolic, notation.)

Transformation of a Tensor

For a second-order tensor the transformation matrices are applied twice: once for each free indice.

$$T'_{ij} = a_{iq} a_{jm} T_{qm} \quad \text{or} \quad \mathbf{T}' = \mathbf{A}\mathbf{T}\mathbf{A}^T$$

$$T_{ij} = a_{iq} a_{mj} T'_{qm} \quad \text{or} \quad \mathbf{T} = \mathbf{A}^T \mathbf{T}' \mathbf{A}$$

In general, for an arbitrary tensor of any rank, the transformation equation can be written as

$$R'_{ij\cdots k} = a_{iq} a_{jm} \cdots a_{kp} R_{qm\cdots p}$$

Invariants of a Tensor

Three quantities composed from a tensor's components are invariant under an admissible coordinate transformation. These quantities are called the **invariants**.

$$I = T_{ii} \quad \text{or} \quad I = tr\,\mathbf{T}$$

$$II = \frac{1}{2}\left(T_{ii}T_{jj} - T_{ij}T_{ij}\right) \quad \text{or} \quad II = \frac{1}{2}\left[\left(tr\,\mathbf{T}\right)^2 - tr\left(\mathbf{T}^2\right)\right]$$

$$III = \varepsilon_{ijk} T_{1i} T_{2j} T_{3k} \quad \text{or} \quad III = \det \mathbf{T}$$

Note: Often $I = I_1$, $II = I_2$, $III = I_3$

Linear Algebra Review

In general, a second order tensor **T** operating on a vector **v** results in another vector **w**. Vectors **v** and **w** are, in general, completely different vectors.

$$\mathbf{T} \cdot \mathbf{v} = \mathbf{w} \quad \text{or} \quad T_{ij} v_j = w_i \qquad (1)$$

However, if a nonzero vector **v** is such that when **T** operates on it the resulting vector **w** is parallel to **v**, then **v** is called an eigenvector of the tensor **T**.

$$\mathbf{T} \cdot \mathbf{v} = \beta \mathbf{v} \quad \text{or} \quad T_{ij} v_j = \beta v_i \qquad (2)$$

β is a scalar called an eigenvalue

From this definition,

$$T_{ij} v_j = \beta v_i = \beta v_j \delta_{ji} \Rightarrow \left(T_{ij} - \beta \delta_{ij}\right) v_j = 0 \qquad (3)$$

Eq. (3) represents three equations which must be satisfied by components v_i in order that **v** be an eigenvector of **T**. Furthermore, this system of equations has nontrivial solutions iff the determinant of the coefficients vanish, i. e.,

$$\det\left(T_{ij} - \beta\delta_{ij}\right) = 0 \qquad (4)$$

$$\phi\left(\beta\right) = -\beta^3 + I_1\beta^2 - I_2\beta + I_3 = 0 \qquad (5)$$

where I_1, I_2, and I_3 are invarients of **T**. In general, Eq. (5) has three roots of which some could be imaginary.

Linear Algebra Theorem 1: If **T** is symmetric, all of the roots of Eq. (5) are real.

Suppose the roots of Eq. (5) are not real. From a theorem of algebra the complex roots occur in conjugate pairs. Thus, two of the roots would be of the form:

$$\beta^{(1)} = \mu + i\gamma \quad \beta^{(2)} = \mu - i\gamma \text{ where } i^2 = -1 \qquad (6)$$

Corresponding eigenvectors would be of the form:

$$v_j^{(1)} = \alpha_j + i\delta_j \qquad v_j^{(2)} = \alpha_j - i\delta_j \qquad (7)$$

Eq. $(2)_2$ must hold for all eigenvectors and their corresponding eigenvalues, hence

$$T_{ij}v_j^{(1)} = \beta^{(1)}v_i^{(1)} \text{ and } T_{ij}v_j^{(2)} = \beta^{(2)}v_i^{(2)} \qquad (8)$$

Multiply $(8)_1$ by $v_i^{(2)}$ and $(8)_2$ by $v_i^{(1)}$ to get

$$\begin{aligned}\beta^{(1)}v_i^{(1)}v_i^{(2)} &= T_{ij}v_j^{(1)}v_i^{(2)} \\ \beta^{(2)}v_i^{(2)}v_i^{(1)} &= T_{ij}v_j^{(2)}v_i^{(1)} = T_{ji}v_j^{(2)}v_i^{(1)} = T_{ij}v_j^{(1)}v_i^{(2)}\end{aligned} \qquad (9)$$

Subtraction of these expressions gives

$$\left(\beta^{(1)} - \beta^{(2)}\right)v_i^{(1)}v_i^{(2)} = 0 \qquad (10)$$

Using Eq. (6)

$$0 = 2i\gamma\left[\left(\alpha_i + i\delta_i\right)\left(\alpha_i - i\delta_i\right)\right] = 2i\gamma\left(\alpha_i\alpha_i - \delta_i\delta_i\right) \Rightarrow \gamma = 0 \qquad (11)$$

Thus, $\beta^{(1)}$, $\beta^{(2)}$ are real.

Linear Algebra Theorem 2: The eigenvectors of a symmetric tensor corresponding to distinct eigenvalues are orthogongal.

Suppose $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are eigenvectors of **T** corresponding to eigenvalues $\beta^{(1)}$ and $\beta^{(2)}$, respectively. ($\beta^{(1)}$<>$\beta^{(2)}$) Again, Eq. (2) must hold for each eigenvector:

$$T_{ij}v_j^{(1)} = \beta^{(1)}v_i^{(1)} \text{ and } T_{ij}v_j^{(2)} = \beta^{(2)}v_i^{(2)} \qquad (12)$$

Multiply $(12)_1$ by $v_i^{(2)}$ and $(12)_2$ by $v_i^{(1)}$ to obtain

$$\begin{aligned}\beta^{(1)}v_i^{(1)}v_i^{(2)} &= T_{ij}v_j^{(1)}v_i^{(2)} \\ \beta^{(2)}v_i^{(2)}v_i^{(1)} &= T_{ij}v_j^{(2)}v_i^{(1)} = T_{ji}v_j^{(1)}v_i^{(2)}\end{aligned} \qquad (13)$$

Subtracting Eqs. (13)

$$\left(\beta^{(1)} - \beta^{(2)}\right)v_i^{(1)}v_i^{(2)} = \left(T_{ij} - T_{ji}\right)v_i^{(1)}v_j^{(2)} = 0 \qquad (14)$$

Also

$$\begin{aligned}&\beta^{(1)} \neq \beta^{(2)} \Rightarrow v_i^{(1)}v_i^{(2)} = 0 \\ &\text{or } \mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)} = 0 \Rightarrow \mathbf{v}^{(1)} \perp \mathbf{v}^{(2)}\end{aligned} \qquad (15)$$

That is, eigenvectors are perpendicular.

Linear Algebra Theorem 3: Every symmetric second order tensor has three linearly independent principal directions.

See **Advanced Engineering Mathematics**, Wylie, p. 542.

Extremal Properties of Quadratic Forms

Associated with a symmetric tensor **T** is a scalar valued function, quadratic in the direction cosines of a unit vector m_i

$$Q(\hat{\mathbf{m}}) = T_{ij} m_i m_j \tag{16}$$

We want to know the extreme values of $Q(m_i)$ subject to the constraint that m_i is a unit vector. Using the method of Lagrange multipliers, the condition for extremum of (16) is

$$\frac{\partial}{\partial m_k}\left\{T_{ij} m_i m_j - \beta\left(m_i m_j - 1\right)\right\} = 0 \tag{17}$$

$$T_{ij}\delta_{ik} m_j + T_{ij} m_i \delta_{jk} - 2\beta\delta_{ik} m_i = 0$$

$$T_{kj} m_j + T_{ik} m_i - 2\beta m_k = 0 \tag{18}$$

Since $T_{kj} = T_{jk}$,

$$2\left(T_{kj} m_j - \beta m_k\right) = 0 \tag{19}$$

$$T_{kj} m_j = \beta m_k = \beta\delta_{kj} m_j \tag{20}$$

Hence, $Q(m_i)$ attains extremum values when m_i is an eigenvector of **T**. We also note that is m_i is an eigenvector of **T** with corresponding eigenvalue β, then

$$Q(\hat{\mathbf{m}}) = T_{ij} m_i m_j = \beta m_i m_i = \beta \tag{21}$$

From the previous development, **T** has three real eigenvalues and three orthogonal eigenvectors. Referred to the basis $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ the tensor **T** is represented in diagonal form

$$\mathbf{T} = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix} \tag{22}$$

Eigenvectors of a tensor make the associated quadratic form attain extreme values which are associated eigenvalues.