Chapter 4

4 EXAMPLES IN LINEAR ELASTICITY

The objectives of this Chapter are to present the reader with a systematic method of first gaining an understanding of the problem on hand and, secondly, to establish the appropriate equations from which a solution *may* eventually be forthcoming. The solution, per say, is not important; the method of setting up the correct equations and correct boundary conditions, is important. In future cases where complicated problems are being tackled by the reader, this systematic approach leads to consistent sets of equations in which the Engineer may have confidence.

In all of the examples about to be discussed, the definitions of the stress vector, (**I**), the body force vector, (**II**), and the symmetry of the stress tensor, (**V**), are assumed to be applicable and will not be referred to further.

4.1 St. Venant's Principle

> ***If some distribution of forces acting on a portion of the surface of a body is replaced by a different distribution of forces acting on the same portion of the body, then the effects of the two different distributions on the parts of the body sufficiently far removed from the region of application of the forces are essentially the same provided the two distributions of forces are statically equivalent.***

4.2 The Deformation of a Long Rod Standing Vertically in a Gravitational Field

Although the following problem appears mundane, it has the redeeming feature that it illustrates nicely the power of tackling problems from the systematic point of view. Also, it does relate to a number of practical problems such as the design of a free standing tower and the laying of underwater communication cables.

4.2.1 PART A - The Set-up of the Problem

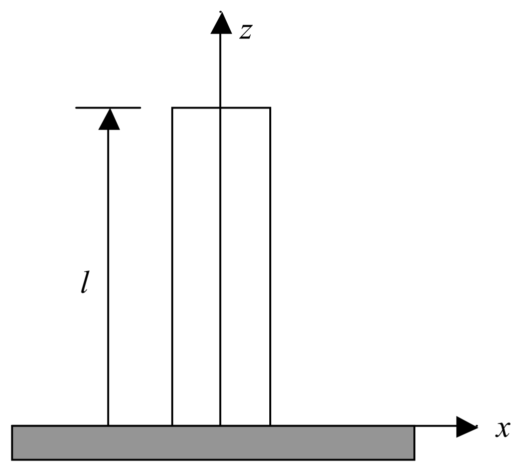

Figure 4.1 Long Prismatic Rod in a Gravitational Field

(III) The Stress Boundary Conditions; $T_i = \sigma_{ij} n_j$

1. <u>On all lateral surfaces</u> since there are no lateral forces (such as wind) and no taper in the rod:

$$T_i \sim (0 \quad 0 \quad 0) \quad ; \quad n_i \sim (n_1 \quad n_2 \quad 0) \;, \tag{4.1}$$

which, when taking into consideration (**III**), implies:

$$\begin{aligned} \sigma_{11} n_1 + \sigma_{21} n_2 &= 0 \;, \\ \sigma_{12} n_1 + \sigma_{22} n_2 &= 0 \;, \\ \sigma_{13} n_1 + \sigma_{23} n_2 &= 0 \;. \end{aligned} \tag{4.2}$$

Please note at this early stage that the stress field whatever it may eventually become ***must*** satisfy these boundary conditions. These conditions cannot be violated.

2. <u>On the top surface; *z = l*:</u>

$$T_i \sim (0 \quad 0 \quad 0) \quad ; \quad n_i \sim (0 \quad 0 \quad 1) \quad \Rightarrow \quad \sigma_{31} = \sigma_{32} = \sigma_{33} = 0 \;. \tag{4.3}$$

No bird or elephant sits on the top of the rod.

3. <u>On the bottom surface; *z = 0*:</u>

$$T_i \sim (0 \quad 0 \quad +\rho g l) \quad ; \quad n_i \sim (0 \quad 0 \quad -1) \quad \Rightarrow \quad \sigma_{31} = \sigma_{32} = 0 \;\; ; \;\; \sigma_{33} = -\rho g l \;. \tag{4.4}$$

The implication of these boundary conditions are again worth noting. They state that as the rod stands on a rigid foundation, there are no shear stresses induced at the bottom of the rod and that the normal stress is compressive and does not vary with the radius of the rod. This latter observation leads to anomalies in the about to be derived displacement field. Furthermore, by taking care in designating the direction of both the surface traction vector and the outward unit normal, the 'sense' of all stresses at this surface has been established beyond question. This is one of the plus points obtained by approaching the set-up of all problems in a systematic manner.

Finally, before moving onto the next basic equation, it is worthwhile making sure that ALL the load induced stress boundary conditions are deduced. In this case they are.

(IV) The Equations of Equilibrium; $\sigma_{ij,j} + \chi_i = 0$:

By carefully writing out the component form of this equation in cartesian coordinates, it follows that:

$$\begin{aligned} \sigma_{11,1}+\sigma_{21,2}+\sigma_{31,3} \quad & = 0 \quad , \\ \sigma_{12,1}+\sigma_{22,2}+\sigma_{32,3} \quad & = 0 \quad , \\ \sigma_{13,3}+\sigma_{23,2}+\sigma_{33,3} - \rho g & = 0 \quad . \end{aligned} \tag{4.5}$$

These are the appropriate partial differential equations of equilibrium governing the entire stress field for this particular problem. Theoretically, by solving these equations and applying the stress boundary conditions listed above, the unique stress field throughout the rod may be determined. As noted in Appendix A, these equations coupled with (**V**) and the above boundary equations form a statically determined system.

(VI) The Strain-Displacement: $e_{ij} = 1/2\ (u_{i,j} + u_{j,i})$:

Certainly, these six strain-displacement equations may be written out at this time. They will, however, be left in this present form until the following "**B -** Analysis of the Problem" section.

At some point in the problem set-up portion **A**, the displacement boundary conditions must be listed. Since the displacements have been introduced by (**VI**), this is an appropriate place to consider these conditions.

A set of displacement boundary conditions consistent with the problem is as follows.

1. **For the two planes** $x_\alpha = 0$, the displacements $u_\alpha = 0$. (4.6)

   These two conditions simply state that whatever the displacement field turns out to be, the horizontal displacements of the centre of the rod must be zero. Other conditions could have been specified, the only thing to be made sure of being; that they are consistent with the problem being studied.

2. **At the single point** $x_i = 0$, i.e., the origin of the coordinates, $u_i = 0$. (4.7)

This condition rules out any possibility of a rigid body movement. In effect, the rod is being pinned at this single point and all displacements are relative to this point.

(VII) The Compatibility Equations

In this particular example the compatibility equations do not play a roll. It turns out, however, that both the resulting stress and displacement fields are compatible and do satisfy the St. Venant (**VII**)$_i$ and Beltrami-Michell (**VII**)$_{ii}$ equations, respectively.

(VIII)*i* The Constitutive Equations: $e_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}$.

In their expanded form they become:

$$e_{ij} \sim \begin{pmatrix} \frac{\sigma_{11}}{E} - \frac{\nu}{E}(\sigma_{22} + \sigma_{33}) & \frac{\sigma_{12}}{2G} & \frac{\sigma_{13}}{2G} \\ & \frac{\sigma_{22}}{E} - \frac{\nu}{E}(\sigma_{11} - \sigma_{33}) & \frac{\sigma_{23}}{2G} \\ \bullet & & \frac{\sigma_{33}}{E} - \frac{\nu}{E}(\sigma_{11} + \sigma_{22}) \end{pmatrix} \tag{4.8}$$

IX) The Navier Displacement Equations of Motion

These equations *can* be applied to this problem. However, since they are a 'nested' form of the even numbered equations: (**IV**), (**VI**) and (**VIII**), they duplicate the information already gathered from these particular equations. Considering the Navier equations *instead* of these three is often more attractive since only the three displacement components have to be determined.

Please note that all of the basic equations of linear isotropic elasticity have been investigated and their implications listed for the problem on hand. Other problems will have a completely different set of equations and boundary conditions, which result from an equally careful examination of the same basic equations, (**I**) through (**IX**). Mathematically, the number of equations equals the number of unknowns and a solution is therefore possible.

Attention will now be turned to part **B** where these equations along with the specified boundary conditions are analyzed.

4.2.2 PART B - Analysis of the Problem

By considering the equilibrium equations eqn(4.5) and the boundary conditions, eqns(4.1 – 4.4), it is easily verified that the following stress field satisfies all of these equations and conditions:

$$\sigma_{ij} \sim \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ \bullet & & -\rho g(l-z) \end{pmatrix} \tag{4.9}$$

Usually, this is the hardest part of a problem. In this case, where an effort is being made to illustrate the systematic approach, time will not be spent on 'solving' the equilibrium equations and then applying the boundary conditions. Instead, the reader should verify that the stress equations and boundary conditions are satisfied by this solution. Secondly and turning to the *design of this rod*, if the density and compressive strength of the rod's material and a suitable safety factor were to be specified, the maximum length of the rod could have been estimated. Concerning the related problem of the laying of undersea cables, the maximum allowable depth of ocean in which a cable could be suspended from the stern of a Cable-layer could be found in a similar fashion.

Substituting this stress field eqn(4.9), into the constitutive equations, eqn(4.8), immediately results in the strain field:

$$e_{ij} \sim \begin{pmatrix} \frac{\nu\rho g}{E}(l-z) & 0 & 0 \\ & \frac{\nu\rho g}{E}(l-z) & 0 \\ \bullet & & -\frac{\rho g}{E}(l-z) \end{pmatrix}. \tag{4.10}$$

Substituting this strain field, into the strain-displacement relations, (**VI**), gives:

$$e_{ij} \sim \begin{pmatrix} u_{1,1} = \frac{\nu\rho g}{E}(l-z) & u_{1,2}+u_{2,1}=0 & u_{1,3}+u_{3,1}=0 \\ & u_{2,2} = \frac{\nu\rho g}{E}(l-z) & u_{2,3}+u_{3,2}=0 \\ \bullet & & u_{3,3} = -\frac{\rho g}{E}(l-z) \end{pmatrix}. \tag{4.11}$$

Integrating the diagonal terms of e_{ij} and recalling from Appendix A that a "comma" indicates *partial differentiation*, results in general expressions for the displacement field, i.e.:

$$u_1 = \frac{\nu\rho g}{E}(l - z)x + f_1(y,z) \ ,$$

$$u_2 = \frac{\nu\rho g}{E}(l - z)y + f_2(x,z) \ ,$$

$$u_3 = -\frac{\rho g}{E}\left(lz - \frac{z^2}{2}\right) + f_3(x,y) \ .$$

(4.12)

We now apply the above determined boundary conditions to these displacement expressions.

1. The boundary condition: on the plane $x = 0$, $u_1 = 0$, implies, **for all applicable values of *y* and *z***, that:

$$f_1(y,z) = 0 \ .$$

(4.13)

2. The boundary condition: on the plane $y = 0$, $u_2 = 0$, implies, **for all applicable values of *x* and *z***, that:

$$f_2(x,z) = 0 \ .$$

(4.14)

3. Considering the shear strain-displacement conditions expressed in eqn(4.11), it now immediately follows that $u_{1,2} + u_{2,1} = 0$ is identically satisfied.

4. However, the implication of the off-diagonal relation $u_{1,3} + u_{3,1} = 0$ of eqn(4.11) has to be determined differently. Applying the observation eqn(4.13) and taking the partial differentiation of eqn$(4.12)_1$ with respect to z gives:

$$u_{1,3} = -\frac{\nu\rho g}{E}x \ ,$$

$(4.14)_1$

while the partial derivative of eqn$(4.12)_3$ with respect to x results in:

$$u_{3,1} = f_{3,1} \ .$$

$(4.14)_2$

Hence, from $u_{1,3} + u_{3,1} = 0$ it follows that:

$$f_{3,1} = \frac{\nu\rho g}{E}x \quad \Rightarrow \quad f_3 = \frac{\nu\rho g}{2E}x^2 + h_1(y) \ .$$

(4.15)

5. In a similar manner, the implication of the off-diagonal equation $u_{2,3} + u_{3,2} = 0$ is that:

$$f_{3,2} = \frac{\nu\rho g}{E} y \quad \Rightarrow \quad f_3 = \frac{\nu\rho g}{2E} y^2 + h_2(x) \quad . \tag{4.16}$$

6. Comparing the expressions for f_3 in eqns(4.15 & 4.16) and taking note of h_1 and h_2, it follows that a general expression for f_3 may be written as:

$$f_3 = \frac{\nu\rho g}{2E}\left(x^2 + y^2\right) + C \quad . \tag{4.17}$$

7. Finally, by substituting this eqn(4.17) into eqn(4.12)$_3$ and applying the displacement boundary condition, eqn(4.7), it follows that:

$$C = 0 \quad . \tag{4.18}$$

Hence eqn(4.12) finally becomes:

$$u_1 = \frac{\nu\rho g}{E}(l - z)x \quad ,$$

$$u_2 = \frac{\nu\rho g}{\mathrm{E}}(l - z)y \quad ,$$

$$u_3 = -\frac{\rho g}{2E}\left[2lz - z^2 - \nu\left(x^2 + y^2\right)\right] \quad . \tag{4.19}$$

Now, the last thing to be undertaken in a systematic approach to any problem is to check if the answers makes sense. For example, the above expressions, which describe the lateral displacements appear correct in that they adhere to the applied boundary conditions and they indicate a general expansion of the rod as it approaches the base. Secondly, the vertical displacement is negative as one would expect.

However, there appears to be a problem at the base where $z = 0$. On this surface, as x or y increases, there is a positive u_3 displacement. This anomaly is as a result of the chosen stress boundary assumption that the vertical reaction load on the surface $z = 0$ be uniform. In practice it is not and this approximation caused the anomaly noted in the u_3 expression.

4.3 Thin Rotating Discs

Now that an understanding of the systematic approach to the solution of problems has been gained, the actual order in which the set-up and analysis is carried out is no longer of importance. What is important is to make sure that all parts of the process are addressed at some point or other.

In this case of thin rotating discs, for example, a detailed review of the stress boundary conditions is delayed until the end. Also, the Navier displacement equations of motion (**IX**) could have been implemented effectively, rather than the (**IV**), (**VI**) & (**VIII**) approach illustrated here. Finally, this example illustrates the use of the cylindrical coordinates detailed in Appendix B.

The stress distribution in rotating circular discs is of considerable practical importance, for example, in the design of turbines and spray driers. If the thickness of the disc is small in comparison to its radius, the variation of the radial and tangential stresses over the thickness, as a first approximation, may be neglected. As will be seen in the next chapter, this is equivalent to a plane stress assumption.

4.3.1 PART A - The Set-up of the Problem

(**II**) **The Body Force Vector: χ.** For a disc rotating at a constant angular velocity ω, the centrifugal force may be considered as a body force acting radially outwards at each point of the disc. Using polar cylindrical coordinates, this may be expressed as:

$$\chi_r = \rho\omega^2 r \qquad ; \qquad \chi_\theta = 0 \qquad ; \qquad \chi_z = 0 \quad . \tag{4.21}$$

(**IV**) **The Equilibrium Equations:** $\sigma_{ij,j} + \chi_i = 0$

From a consultation of the equations eqn(B.1) in Appendix B and acknowledging that this is a strictly axisymmetric problem, these equations reduce to the single equation:

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr}}{r} - \frac{\sigma_{\theta\theta}}{r} + \rho\omega^2 r = 0 \qquad \text{or} \qquad \frac{dr\sigma_{rr}}{dr} - \sigma_{\theta\theta} + \rho\omega^2 r^2 = 0 \quad . \tag{4.22}$$

(**VI**) **The strain-displacement relations**: $e_{ij} = \frac{1}{2}\left(u_i,_j + u_j,_i\right)$

Again consulting equations presented in the Appendix B we see that equation (B.3) is applicable in this case, i.e.:

$$e_{ij} \sim \begin{pmatrix} \frac{du_r}{dr} & 0 & 0 \\ & \frac{u_r}{r} & 0 \\ & & 0 \end{pmatrix} \quad .$$

(4.23)

(**VIII**)i **The Constitutive Equations:** $e_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{(1-\nu)}{E}\sigma_{ij}$

Since these equations do not involve partial differentiation, signified by the "comma", they may simply be written down in terms of the stress and strain components that appear to be non-zero in the above equilibrium and strain-displacement equations, i.e.:

$$e_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}) \quad ; \quad e_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \sigma_{rr}) \quad .$$

(4.24)

Noting that both the stress boundary equations (**III**) and the compatibility equations will be discussed subsequently, this completes the set-up of the spinning disc problem.

4.3.2 PART B - The Analysis of the Problem

From here on, every problem is different. In this particular case, a review of the equilibrium equation eqn$(4.22)_2$ reveals that it is identically satisfied by setting:

$$r\sigma_{rr} = F(r) \quad \text{and} \quad \sigma_{\theta\theta} = \frac{dF(r)}{dr} + \rho\omega^2 r^2 \quad .$$

(4.25)

This is the first example of many where solutions may be found by introducing a ***STRESS FUNCTION***, namely, *F(r)*. This is a relatively common way of solving problems and more will be said about this technique in the next chapter. For now, just notice that these expressions for the stresses satisfy the equilibrium equation (**IV**). The emphasis will now be concentrated on finding the governing equation for *F(r)*. Once that equation is solved, it may be substituted into eqn(4.25) to deduce the stresses, from whence the strains and eventually the displacements can be found.

The first step in finding the governing equation for *F(r)* is performed by considering the strain-displacement relations, eqn(4.23). It is noticed, by taking the derivative of $e_{\theta\theta}$ with respect to *r*, that the resulting expression will contain the e_{rr} term. Hence:

$$
\begin{aligned}
\frac{de_{\theta\theta}}{dr} &= \frac{d}{dr}\left(\frac{u_r}{r}\right) \\
&= \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} \\
&= \frac{1}{r}\left(\frac{du_r}{dr} - \frac{u_r}{r}\right) \\
&= \frac{1}{r}\left(e_{rr} - e_{\theta\theta}\right) \quad .
\end{aligned}
\tag{4.26}
$$

This equation eqn(4.26) takes the place of the **compatibility equations**, (**VII**)$_i$, discussed in Chapter 3, in that it may be viewed as an integration of the single compatibility equation that applies in this situation.

The strains expressed in terms stresses, namely eqn(4.24), may now be substituted into this 'pseudo' compatibility relation eqn(4.26), to generate:

$$
\frac{d}{dr}\left[\frac{1}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right)\right] = \frac{1}{r}\left[\frac{1}{E}\left(\sigma_{rr} - \nu\sigma_{\theta\theta} - \sigma_{\theta\theta} + \nu\sigma_{rr}\right)\right]
$$

which, upon rearranging gives:

$$
\frac{d\sigma_{\theta\theta}}{dr} - \nu\frac{d\sigma_{rr}}{dr} = \frac{1}{r}(1-\nu)(\sigma_{rr} - \sigma_{\theta\theta}) \quad .
\tag{4.27}
$$

This in turn may be viewed as an integrated form of the Beltrami Michell compatibility equations (**VII**)$_{ii}$.

Finally, the stresses in terms of the stress function, *F(r)*, may now be substituted into this compatibility equation to give, after some manipulation, the required governing equation for the stress function *F(r)*:

$$
r^2\frac{d^2F(r)}{dr^2} + r\frac{dF(r)}{dr} - F(r) = -(3+\nu)\rho\omega^2 r^3 \quad .
\tag{4.28}
$$

This is a non-homogeneous, 2nd order, linear, ordinary differential equation with variable coefficients, which may be solved using standard solution techniques. For example, the solution of the homogeneous part of eqn(4.28) may be found prior to finding its particular integral. Such a process eventually results in the general expression for the stress function *F(r)*, namely:

$$
F(r) = A_1 r + A_2\frac{1}{r} + \frac{(3+\nu)\rho\omega^2}{8}r^3 \quad .
\tag{4.29}
$$

At this point it is worth noting that this general expression could not have been found without employing all of the basic linear isotropic elastic relations, including the compatibility equations. With the determination of the stress function *F(r)*, it becomes a relatively simple process to find all the other variables, starting with the stresses.

Substituting for *F(r)* in eqn(4.25) the stress field becomes:

$$\sigma_{rr} = A_1 + \frac{A_2}{r^2} - \frac{(3+\nu)\rho\omega^2}{8} r^2 \quad ,$$

$$\sigma_{\theta\theta} = A_1 - \frac{A_2}{r^2} - \frac{(1+3\nu)\rho\omega^2}{8} r^2 \quad . \tag{4.30}$$

It is important to review the form of these equations. First, consider these expressions when there is no rotational effect. The third term of both becomes zero and exceptionally nice forms of the stresses ***for all generally axisymmetric problems*** results. For any thin circular disc where the loads do not vary with θ all that is required is to find the two constants of integration from the applied boundary conditions.

Having obtained these general expressions the **boundary conditions** (**III**) may now be specified in order to complete the problem. Among a variety of boundary conditions, consider the following two boundary value problems.

(A) The Solid Disc:

For this case, the first thing that is noticed is that since σ_{rr} must remain finite at the center of the disc:

$$A_2 = 0 \quad , \tag{4.31}$$

otherwise the stresses at the centre would tend to infinity. Secondly, the stress boundary condition at $r = a$, the outside radius of the disc, becomes from (**III**):

$$T_r = \sigma_{rr} n_r + \sigma_{\theta r} n_\theta + \sigma_{zr} n_z = 0 \qquad \text{with} \qquad n_i \sim (1 \quad 0 \quad 0) \quad , \tag{4.32}$$

such that:

$$\sigma_{rr} = 0 \qquad \text{at} \qquad r = a \quad . \tag{4.33}$$

Evaluating eqn$(4.30)_1$ at $r = a$ dictates that:

$$A_1 = \frac{(3+\nu)\rho\omega^2}{8} a^2 \quad , \tag{4.34}$$

from which it follows, in the case of a solid disc, that the stresses are:

$$\sigma_{rr} = \frac{(3+\nu)\rho\omega^2}{8}\left(a^2 - r^2\right) \quad ,$$

$$\sigma_{\theta\theta} = \frac{(3+\nu)\rho\omega^2}{8}a^2 - \frac{(1+3\nu)\rho\omega^2}{8}r^2 \quad .$$

(4.35)

Notice, that the stresses are the greatest at the disc center *r = 0* with:

$$\sigma_{rr} = \sigma_{\theta\theta} = \frac{(3+\nu)\rho\omega^2}{8}a^2 \quad .$$

(4.36)

The displacement field may be simply determined by substituting these expressions for the stresses back into a combination of eqn(4.23)and eqn(4.24) to obtain:

$$u_r = re_{\theta\theta} = \frac{r}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right) = \frac{r\rho\omega^2}{8E}\left[\left(3 - 2\nu - \nu^2\right)a^2 + \left(\nu^2 - 1\right)r^2\right] \ .$$

(4.37)

Again it is worth noting that the displacement tends to zero as the radius tends to zero.

With a knowledge of these stresses and displacements, coupled with the mechanical properties of the material from which the disc is to be manufactured, the safe design of the disc may be completed. Attention will now turn to another problem that results from a choice of different boundary conditions.

(B) A Disc with a Stress Free Circular Hole:

The Stress Boundary conditions, (**III**) , in this case may be formulated as follows.

1. On the outside radius $r = a$ of the disc:

$$T_i \sim (0 \quad 0 \quad 0) \quad , \quad n_i \sim (1 \quad 0 \quad 0) \quad \Rightarrow \quad \sigma_{rr} = 0 \quad .$$

(4.38)

Hence from $\text{eqn}(4.30)_1$:

$$A_1 = \frac{A_2}{a^2} + \frac{(3+\nu)\rho\omega^2}{8}a^2 \quad .$$

(4.39)

2. On the inside radius $r = b$ of the disc:

$$T_i \sim (0 \quad 0 \quad 0) \quad , \quad n_i \sim (-1 \quad 0 \quad 0) \quad \Rightarrow \quad \sigma_{rr} = 0 \quad , \tag{4.40}$$

which, in turn implies:

$$A_1 = -\frac{A_2}{b^2} + \frac{(3+\nu)\rho\omega^2}{8} b^2 \quad . \tag{4.41}$$

By solving for A_1 and A_2 from eqns(4.39 & 4.40), it may be shown that:

$$A_1 = \frac{(3+\nu)\rho\omega^2}{8}\left(a^2 + b^2\right) \quad ; \quad A_2 = -\frac{(3+\nu)\rho\omega^2}{8} a^2 b^2 \quad . \tag{4.42}$$

Hence, in the case of a spinning disc with a stress free hole, the stresses given in eqn(4.30) become:

$$\begin{aligned} \sigma_{rr} &= \frac{(3+\nu)\rho\omega^2}{8}\left(a^2 + b^2 - \frac{a^2 b^2}{r^2} - r^2\right) \quad , \\ \sigma_{\theta\theta} &= \frac{(3+\nu)\rho\omega^2}{8}\left[a^2 + b^2 + \frac{a^2 b^2}{r^2} - \frac{(1+3\nu)}{(3+\nu)} r^2\right] \quad . \end{aligned} \tag{4.44}$$

The position of maximum "radial" stress may be shown to be at $r = \sqrt{ab}$, while the maximum "hoop" stress is at $r = b$. Furthermore, the stresses at these positions are:

$$\begin{aligned} \sigma_{rr\,max} &= \frac{(3+\nu)\rho\omega^2}{8}(a-b)^2 \quad , \\ \sigma_{\theta\theta\,max} &= \frac{(3+\nu)\rho\omega^2}{8}\left[2a^2 + \frac{2(1-\nu)}{(3+\nu)} b^2\right] \quad . \end{aligned} \tag{4.45}$$

Now, as $b \to 0$, it may be observed by comparing the maximum "hoop" stresses described by eqn$(4.36)_2$ and eqn$(4.45)_2$ that:

$$\sigma_{\theta\theta}\,|\text{hole} \quad = \quad 2\sigma_{\theta\theta}\,|\text{solid} \quad . \tag{4.46}$$

The value of 2 is the ***stress concentration factor*** for this particular problem. It is of considerable importance in the design of spinning discs.