The H_∞ control problem:
a state space approach

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February 5, 2000

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Preface

The H_∞ norm as a measure has been thoroughly embedded in control theory during the last few years. Although the H_∞ norm has been used in control for a long time (a concept like bounded real is nothing else than an H_∞ norm bound), there was a gigantic surge in research efforts towards the minimization of the H_∞ norm of the closed loop system in the last 10 years. A fairly complete solution is now available.

This book tries to bring the reader up to date with respect to the, in our view, most elegant solution to the H_∞ control problem: the state space approach. This book contains numerous references towards other approaches to this theory which the reader can use as a starting point for further research.

We have tried to start this book at a basic level with the prerequisites being graduate level courses on linear algebra and state space systems. On the other hand, we included the most general solutions available at the time this book was written. At some times this presented a trade-off and we refer to the literature for the intuition behind some of the abstract concepts introduced in this book.

This book is the result of my work as a Ph.D. student at the Department of Mathematics of Eindhoven University of Technology in the Netherlands from October 1987 until October 1990. Numerous people contributed by assisting me whenever problems arose. In particular, I would like to thank Harry Trentelman, Malo Hautus and Ruth Curtain. Clearly they were not the only ones who offered their continuous support and assistance while preparing this book. The Dutch control community organized in the System and Control Theory Network consists of a great group of people. Whomever I asked for a favour or some assistance was always willing to help. I would like to thank them all.

Last but not least I would like to thank my family for their moral support during these years.

Ann Arbor   Anton Stoorvogel
July, 1991

Chapter 1

Introduction

In this book, we study the H_∞ control problem. This problem was originally formulated in [Za]. The motivation for this problem should become abundantly clear in this book. In this first chapter, we introduce a number of natural problems whose solutions depend on the results of H_∞ control. This chapter will be more or less a justification of the extensive research effort in the area of H_∞ control, and in particular, why we have written a book on this subject.

In section 1.1 we formulate and explain the importance of robustness and, in section 1.2, we sketch very briefly the basic H_∞ control problem. The section on robustness outlines the importance of the problems of uncertain systems and the gap metric, which are explained in reasonable detail in sections 1.3 and 1.4, respectively. These problems are formulated with the same objective of improving the robustness although from a completely different point of view and both have a solution which is intimately connected to H_∞ control. We shall also explain the mixed-sensitivity problem in this chapter. This is in essence already formulated as an H_∞ control problem and it is practically the only H_∞ control problem which has already been applied in practice. To conclude this chapter we shall outline the H_∞ control problems we shall treat in this book.

1.1   Robustness analysis

Control theory is concerned with the control of processes with inputs and outputs. We would like to know how we can achieve a desired goal we have for the output of our plant by choosing our inputs.

Example 1.1 : *Assume that we have a paper-making machine. This machine has certain inputs: wood-pulp, water, pressure and steam. The*

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wood-pulp is diluted with water. Then the fibres are separated from the water and a web is formed. Water is pressed out of the mixture and the paper is then dried on steam-heated cylinders (this is of course a very simplified view of the process). The product of the plant is the paper. More precisely we have two outputs: the thickness of the paper and the mass of fibres per unit area (expressing the quality of the paper). We would like both outputs to be equal to some desired value. That is, we have a process with a number of inputs and two goals: we would like to make the deviation from the desired values of the thickness and of the mass of fibres per unit area of the paper produced as small as possible. □

The first step is to find a mathematical model describing the behaviour of our plant. The second step is to use mathematical tools to find suitable inputs for our plant based on measurements we make of all, or of a subset, of our outputs. However, we apply these inputs to our plant and not to our model. Since our model should be simple enough for the mathematical tools of step 2 (for instance in this book we require that the model be linear) the model will not describe the plant exactly. Because we do not know how sensitive our inputs are with respect to the differences between model and plant, the obtained behaviour might differ significantly from the mathematically predicted behaviour. Hence our inputs will in general not be suitable for our plant and the behaviour we obtain can be completely surprising.

Therefore it is extremely important that, when we search for a control law for our model, we keep in mind that our model is far from perfect. This leads to the so-called robustness analysis of our plant and suggested controllers. Robustness of a system says nothing more than that the stability of the system (or another goal we have for the system) will stand against perturbation (structured or unstructured, depending on the circumstances).

The classic approach to this problem from the 1960s was the Linear Quadratic Gaussian (LQG) theory. In that approach the uncertainty is modelled as a white noise Gaussian process added as an extra (vector) input to the system. The major problem of this approach is that our uncertainty cannot always be modelled as white noise. While measurement noise can be quite well described by a random process, this is not the case with parameter uncertainty. If we model $a = 0.9$ instead of $a = 1$, then the error is not random but deterministic. The only problem is that the deterministic error is unknown. Another problem of main importance with parameter uncertainty is that uncertainty in the transfer from inputs to outputs cannot be modelled as state or output disturbances, i.e. extra

inputs. This is due to the fact that the size of the errors is relative to the size of the inputs and can hence only be modelled as an extra input in a *non-linear* framework.

In the last few years several approaches to robustness have been studied mainly for one goal: to obtain internal stability, where instead of trying to obtain this for one system, it is necessary for a class of systems simultaneously. It is then hoped that a controller which stabilizes all elements of this class of systems also stabilizes the plant itself.

In this chapter two approaches to this problem will be briefly discussed. Both approaches are in a linear, time-invariant setting and result in an H_∞ control problem.

1.2   The H_∞ control problem

We now state the H_∞ control problem. Assume that we have a system Σ:

$$
\begin{array}{c}
z \;\longleftarrow\; \boxed{\;\Sigma\;} \;\longleftarrow\; w \\[2pt]
y \;\longleftarrow\; \;\longleftarrow\; u
\end{array}
\tag{1.1}
$$

We assume Σ to be a linear time-invariant system either in continuous time or in discrete time. We note that Σ is a system with two kinds of inputs and two kinds of outputs. The input w is an exogenous input representing the disturbance acting on the system. The output z is an output of the system, whose dependence on the exogenous input w we want to minimize. The output y is a measurement we make on the system, which we shall use to choose our input u, which in turn is the tool we have to minimize the effect of w on z. A constraint we impose is that this mapping from y to u should be such that the closed-loop system is internally stable. This is quite natural since we do not want the states to become too large while we try to regulate our performance. The effect of w on z after closing the loop is measured in terms of the energy and the worst disturbance w. Our measure, which will turn out to be equal to the closed-loop H_∞ norm, is the supremum over all disturbances unequal to zero of the quotient of the energy flowing out of the system and the energy flowing into the system. A more precise definition is given in chapter 2.

Note that this problem formulation in itself does not have any connection with robustness.

Example 1.2 : *Assume that we have the following system:*

$$\Sigma: \begin{cases} \dot{x} = \quad - \ u + w, \\ y = x, \\ z = \qquad u. \end{cases}$$

It can be checked that a feedback law which minimizes the effect of w on z in the above sense is given by:

$$u = \varepsilon x$$

where ε is a very small positive number. On the other hand it is easily seen that a small perturbation of the system parameters might yield a closed-loop system which is unstable. Hence for this controller, the internal stability of the closed-loop system is certainly not robust with respect to perturbations of the state matrix. □

1.3   Stabilization of uncertain systems

As already mentioned, a method for handling the problem of robustness is to treat the uncertainty as additional input(s) to the system. The LQG design method treats these inputs as white noise and we noted that parameter uncertainty is not suitable for treatment as white noise. Also the idea of treating the error as extra inputs was not suitable because the size of the error might be relative to the size of the inputs. This yields an approach where parameter uncertainty is modelled as a disturbance *system* taking values in some range and modelled in a feedback setting (which allows us to incorporate the "relative" character of the error). We would like to know the effect with respect to stability of the "worst" disturbance in the prescribed parameter range (we want guaranteed performance so, even if the worst happens, it should still be acceptable). If this disturbance cannot destabilize the system, then we are certain (under the assumption that the plant is exactly described by a system we obtain for some value of the parameters in the prescribed range) that the plant is stabilized by our control law.

It is easy to verify that parameter uncertainty in a linear setting can

very often be modelled as:

$$
\begin{array}{c}
\boxed{\Sigma_K} \\
z \uparrow \qquad \downarrow w \\
y \leftarrow \boxed{\Sigma} \leftarrow u
\end{array}
\qquad (1.2)
$$

Here the system Σ_K represents the uncertainty and if the transfer matrix of Σ_K is zero, then we obtain our nominal model from u to y. The system Σ_K might contain uncertainty of parameters, ignored dynamics after model reduction or discarded non-linearities. The goal is to find a feedback law which stabilizes the model for a large range of systems Σ_K. In chapter 11 we shall give some examples of different kinds of uncertainties which can be modelled in the above sense. We shall also show what the results of this book applied to these problems look like. At this point, we only show by means of an example that a large class of parameter uncertainty can be considered as an interconnection of the form (1.2).

Example 1.3 : *Assume that we have a single-input, single-output system with two unknown parameters:*

$$
\Sigma_n : \left\{ \begin{array}{l} \dot{x} = -ax + bu, \\ y = \quad x. \end{array} \right.
$$

where a and b are parameters with values in the ranges $[a_0 - \varepsilon, a_0 + \varepsilon]$ and $[b_0 - \delta, b_0 + \delta]$ respectively. We can consider this system as an interconnection of the form (1.2) by choosing the system Σ to be equal to:

$$
\Sigma : \left\{ \begin{array}{l} \dot{x} = -a_0 x + \quad b_0 u + w, \\ y = \quad x, \\ z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u. \end{array} \right.
$$

and the system Σ_K to be the following static system:

$$
w = \left(\begin{array}{cc} a - a_0 & b - b_0 \end{array} \right) z.
$$

It is easily seen that by scaling we may assume that $\varepsilon = \delta = 1$. $\square$

If we want to use some special structure of the system Σ_K (e.g. that Σ_K is static and not dynamic as in the above example), then we have to resort to the so-called μ-synthesis of J. Doyle (see [Do2, Do3]) or to the (similar) theory of real stability radii (see [Hi]). μ-synthesis has the disadvantage that this method is so general that, at this moment, no reasonably efficient algorithms are available to improve robustness via μ-synthesis. The same is true for the related approach of working with real stability radii. The available technique for μ-synthesis is to use an H_∞ upper bound (depending on scales) for μ which is consequently minimized by choosing an appropriate controller. The result is again minimized over all possible choices for the scales.

On the other hand, if we want to find a controller from y to u such that the closed-loop system is stable for all *stable* systems Σ_K with H_∞ norm less than γ, then it has been shown (see [Hi]) that the problem is equivalent to the following problem: find a controller which is such that the closed-loop system (if the transfer matrix of Σ_K is zero) is internally stable and the H_∞ norm from w to z is strictly less than γ^{-1}. This problem can be solved using the techniques given in this book.

1.4   The graph topology

When we treat parameter uncertainty as we did in the previous section then we need to know explicitly how this uncertainty is structured in our system. In many practical cases we do not have such information and for these cases a different approach is needed.

We could formulate the problem abstractly as follows: assume that some nominal plant is given and some controller stabilizes this plant. Does the same controller also stabilize systems which are "close" to our nominal plant?

The problem in this formulation is how to define the concept of distance between two systems. Since systems are in fact nothing else than input–output operators a natural distance concept would be the induced operator norm. However, in this case an unstable system must be seen as an unbounded operator and hence this distance concept cannot be used to define distances between unstable systems. On the other hand, it is quite possible that our plant is unstable and hence we need a concept of distance which is still valid in the case of one or both of the systems being unstable. The graph topology turns out to yield a useful distance concept for these cases.

When do we call two systems close to each other? Firstly, if we have a controller which internally stabilizes one system, then it should also

be internally stabilizing for the other system. Secondly, if we apply the same stabilizing controller to each one of the systems, then the closed-loop systems should be close to each other, measured in the induced operator norm. Note that both closed-loop systems are stable and hence this operator norm is always finite. The first requirement is natural because we consider robustness of internal stability. The second requirement is added because we would like to prevent other performance criteria from being highly sensitive to perturbations even though stability is preserved.

Next, we shall formalize the above intuitive reasoning. We define the graph topology for the set of linear time-invariant and finite-dimensional systems. A sequence of systems $\{\Sigma_n\}$ is said to converge to Σ in the graph-topology if the following holds:

- For every controller Σ_F which stabilizes Σ, there exists a natural number N such that for all $n > N$, the controller Σ_F internally stabilizes Σ_n.

- For every controller Σ_F which stabilizes Σ denote the closed-loop operator by $\mathcal{G}_F$ and the closed-loop operator obtained by applying Σ_F to Σ_n by $\mathcal{G}_{F,n}$. Then

$$\|\mathcal{G}_F - \mathcal{G}_{F,n}\|_\infty \to 0 \ \ \text{as } n \to \infty.$$

Here $\| \cdot \|_\infty$ denotes the induced operator norm which will be formally defined in section 2.5.

The graph topology has been introduced in [Vi]. In [Zh] it was shown that this topology is always equal to the gap topology (in [Zh] this is also shown for more general classes of infinite-dimensional systems). The gap topology is yet another topology that can be applied to define convergence between possibly unstable plants. However, the gap topology is metrizable and hence we can now really discuss the distance between two plants. Note that the concept of coprime-factor perturbations as used in [Gl4, MF] again yields the same topology (see [Vi], in [GS] it was shown that the gap *metric* is related to so-called *normalized* coprime-factor perturbations).

Now, the problem of finding a maximally robust controller in this setting is defined as follows: for each internally stabilizing controller of the nominal plant we search for the distance to the nearest system in the gap metric which is not stabilized by our controller. This is called the stability margin of the controller. Finally we search for the controller with the largest stability margin. The above problem has been reduced in [GS, Hab, Zh] to an H_∞ control problem.

1.5   The mixed-sensitivity problem

The mixed-sensitivity problem is a special kind of H_∞ control problem. In the mixed-sensitivity problem it is assumed that the system under consideration can be written as the following interconnection where Σ_F is the controller which has to satisfy certain prerequisites.

$$(1.3)$$
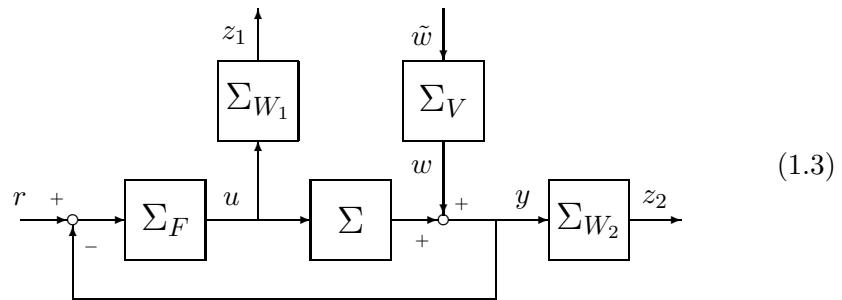
Many H_∞ control problems can be formulated in terms of an interconnection of the form (1.3). We shall show as an example how the tracking problem can be formulated in the setting described by the diagram (1.3). We first look at the following interconnection:

$$(1.4)$$
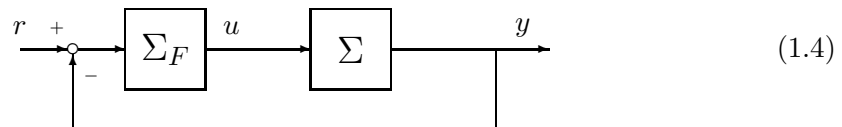
The problem is to regulate the output y of the system Σ to look like some given reference signal r by designing a precompensator Σ_F which has as its input the error signal, i.e. the input of the controller is the difference between the output y of Σ and the reference signal r. To prevent undesirable surprises we require internal stability. We could formulate the problem as "minimizing" the transfer function from r to $r - y$. As one might expect we shall minimize the H_∞ norm of this transfer function under the constraint of internal stability. The transfer matrix from r to u should also be under consideration. In practice the process inputs will often be restricted by physical constraints. This yields a bound on the transfer matrix from r to u. These transfer matrices from r to $r - y$ and

from r to u are given by:

$$
\begin{aligned}
S & := (I + GG_F)^{-1}, \\
T & := G_F (I + GG_F)^{-1},
\end{aligned}
$$

respectively, where G and G_F denote the transfer matrices of Σ and Σ_F. Here S is called the *sensitivity function* and T is called the *control sensitivity function*. A small function S expresses good tracking properties while a small function T expresses small inputs u. Note that there is a trade-off: making S smaller will in general make T larger. We add a signal w to the output y as in (1.3). Then the transfer matrix from w to y is equal to the sensitivity matrix S and the transfer matrix from w to u is equal to the control sensitivity matrix T.

As noted in section 1.2 the H_∞ norm can be viewed as the maximum amount of energy coming out of the system, subject to inputs with unit energy. However, if we apply the Laplace transform, then we obtain a frequency-domain characterization. For a single-input, single-output stable system the H_∞ norm is equal to the largest distance of a point on the Nyquist contour to the origin. Hence the H_∞ norm is a uniform bound over all frequencies on the transfer function. Although we assume the tracking signal to be, a priori, unknown, it might be that we know that our tracking signal will have a limited frequency spectrum. It is in general impossible to track signals of very high frequency reasonably well. On the other hand, since in general the model is only accurate up to a certain frequency, since we have bandwidth limitations on actuators and sensors and since we have problems implementing controllers with a large bandwidth, we only require the system to track signals of frequencies up to a certain bandwidth. In this situation straightforward application of H_∞ control might yield bad results because it only investigates a uniform bound over all frequencies. Also, certain frequencies may be more important than others for the error signal and the control input.

Thus in diagram (1.3), the systems Σ_{W_1}, Σ_{W_2} and Σ_V are weights which are chosen in such a way that we put more effort in regulating frequencies of interest than one uniform bound. For practical purposes the choice of these weights is extremely important. For single-input, single-output systems expressing performance criteria into requirements on the desired shape of the magnitude Bode diagram is well established (see, e.g. [FL, Hor]). This immediately translates into the appropriate choice for the weights. On the other hand, for multi-input, multi-output systems it is in general very hard to translate practical performance criteria into an appropriate choice for the weights. It should be noted that in practical circumstances it is often better to minimize the integrated tracking error

as we do in section 11.4 (this forces a zero steady-state tracking error). This can also be incorporated in the weights and is simply one way to emphasize our interest in tracking signals of low frequency.

In this way, we obtain the interconnection (1.3). Note that the transfer matrix from the disturbance $\tilde{w}$ to z_1 and z_2 is

$$
\begin{pmatrix} G_{W_1} T G_V \\ G_{W_2} S G_V \end{pmatrix} \tag{1.5}
$$

where G_{W_1}, G_{W_2} and G_V are the transfer matrices of Σ_{W_1}, Σ_{W_2} and Σ_V, respectively. Note that we can also use these weights to stress the relative importance of minimizing the sensitivity matrix S with respect to the importance of minimizing the control sensitivity matrix T by multiplying G_{W_1} by a scalar.

We want to find a controller which minimizes the H_∞ norm of the transfer matrix (1.5) and which yields internal stability. This problem can be solved using the techniques we present in this book.

1.6 Main items of this book

As already mentioned, this book will deal with several aspects of H_∞ control. In recent years many papers have been published on this subject and, in this section, we want to describe briefly the new contributions this book makes to the existing theory. We shall consider the time-domain approach to H_∞ control, which has received a large impulse from the paper [Do5]. This book is self-contained but we will put emphasis on the following aspects of the time-domain approach to H_∞ control:

- Singular systems

- Differential games

- The finite horizon H_∞ control problem

- The minimum entropy H_∞ control problem

- Discrete time systems.

First we introduce some notation and give some preliminary results in chapter 2. In chapter 3, we give some results on the state feedback H_∞ control problem of which the main results were already basically known in the literature. We have added this chapter for the sake of completeness and in order to have the results available for the rest of the book. In chapters 4 and 5 we extend the known H_∞ theory to so-called

singular systems. In chapter 6 we discuss the differential game and its relation to H_∞ control. Next, in chapter 7, we discuss the minimum entropy H_∞ control problem. In chapter 8 we study the finite horizon H_∞ control problem. Then, in chapters 9 and 10, we investigate the H_∞ control problem for discrete time systems. In chapter 11 we will apply the derived theory to find results for several robust stabilization problems. We will also give a worked-out design example of an inverted pendulum on a cart. Finally chapter 12 contains a number of concluding remarks. Appendix A gives the details of the state decomposition on which the proofs of our results for singular systems are based. Appendix B contains the proofs of two technical lemmas from chapter 5.

We shall discuss the main subjects of this book in some detail in the next five subsections.

1.6.1   Singular systems

In the paper [Do5] linear finite-dimensional time-invariant systems were considered which satisfy two kinds of essential assumptions:

- The subsystem from the control input to the output should not have invariant zeros on the imaginary axis and its direct-feedthrough matrix should be injective.

- The subsystem from the disturbance to the measurement should not have invariant zeros on the imaginary axis and its direct-feedthrough matrix should be surjective.

Invariant zeros are defined in chapter 2. For the moment it suffices to think of invariant zeros as points in the complex plane where the transfer matrix loses rank.

Throughout this book we shall make the same assumption with respect to invariant zeros, by excluding invariant zeros on the imaginary axis for both subsystems. A discussion of the difficulty of invariant zeros on the imaginary axis is given in chapter 12.

We define singular systems (contrary to regular systems) to be systems which do not satisfy at least one of the two above assumptions on the direct-feedthrough matrices. In chapters 4 and 5 we shall extend the results from [Do5] to the class of singular systems.

These singular systems are more difficult to analyse. In the case that the direct-feedthrough matrix from the control input to the output is not injective then either the system has an invariant zero at infinity or the subsystem from the control input to the output is not injective.

- **An invariant zero at infinity.** This is as difficult as invariant
  zeros on the imaginary axis. The problems with invariant zeros
  on the imaginary axis are explained in chapter 12. Basically the
  method of handling an invariant zero is to choose a controller which
  creates a pole in the same point. Then the invariant zero is cancelled
  via pole-zero cancellation. Because of our requirement of internal
  stability this is only possible for invariant zeros in the open left half
  plane. For invariant zeros in the open right half plane it is clearly
  not possible. In the case of an invariant zero on the imaginary axis
  we can achieve this cancellation *approximately* by creating a pole
  in the left half plane which is very close to the imaginary axis. A
  treatment like this for invariant zeros on the imaginary axis is given
  in [HSK]. At the moment we are only interested in invariant zeros
  at infinity. In general, the so-called central controller (as given
  in section 5.5) will be non-proper in this case. Hence we indeed
  have a pole in infinity which "cancels" our invariant zero at infinity.
  However, it will turn out that we can approximate this controller
  by a proper controller. It will be shown that the problem of this
  approximation can be reduced to the problem of almost disturbance
  decoupling. In this problem one is looking for conditions under
  which we can find internally stabilizing controllers which make the
  H_∞ norm arbitrarily small. Since this problem has been solved in
  [OSS1, OSS2, Tr, WW] we were able to use these results.

- **The system from the control input to the output is not
  injective.** This implies that there are several inputs which have
  the same effect on the output. Using a geometric approach this
  non-uniqueness can be filtered out. This is done explicitly in [S2].
  We do the same in this book but more implicitly because we handle
  invariant zeros at infinity at the same time.

On the other hand, the subsystem from the disturbance to the measure-
ment might not be surjective. The problems related to this fact play a
completely dual role and we shall tackle these problems by relating them
to the problems of a dual system.

   In [Do5] necessary and sufficient conditions are given for regular sys-
tems under which the existence of an internally stabilizing controller
which makes the H_∞ norm less than some, a priori given, number $\gamma > 0$
is guaranteed. These conditions are in terms of two algebraic Riccati
equations. For singular systems these Riccati equations will be replaced
by quadratic matrix inequalities. This is completely analogous to Linear
Quadratic (LQ) optimal control where for singular systems the role of the

algebraic Riccati equation is replaced by a *linear* matrix inequality.

Finally, a few words about the interest in singular systems. First of all, for mathematicians removing annoying assumptions is always of interest. Moreover, singular systems do arise in natural control problems, for example the problem of treating parameter uncertainty via the method discussed in section 1.3 will often yield H_∞ control problems for singular systems. This is worked out in more detail in chapter 12. Another example is the problem of Loop Transfer Recovery (see [Ni]) which also yields H_∞ control problems for singular systems.

Another reason for looking at singular systems is the omnipresent requirement of reduced-order controllers. By assuming that k states are observed without noise, it can be shown that the dynamic order of the controller can be reduced by k. However, to show this property we need to use singular systems (see [St13]).

Finally, a major reason for looking at singular systems is that quite often in applications direct-feedthrough matrices appear which are nearly singular. The results of [Do5] may still be applied but for deriving numerically reliable algorithms it is useful to know exactly what will happen if the direct-feedthrough matrices are no longer injective or surjective.

1.6.2 Differential game

As we mentioned in the previous subsection, the conditions under which we can make the H_∞ norm less than some, a priori given, number, are either in terms of the solutions of two algebraic Riccati equations or in terms of the solutions of two quadratic matrix inequalities. The solutions of these equations or inequalities have no direct meaning in H_∞ control. On the other hand, it would be good for the overall picture if we could understand the role of these solutions better.

We shall try to achieve this in chapter 6 for a special case. First of all we assume that we have a continuous time system. Secondly we only investigate the special case of state-feedback. In chapters 3 and 4 it will be shown that in this case we have only one Riccati equation or one quadratic matrix inequality.

It turns out that the theory of differential games yields the desired understanding of the role of the solution of either equation or inequality. The quadratic form associated with the solution of our Riccati equation turns out to be a Nash equilibrium for a differential game with a special cost criterion. The quadratic form associated with the solution of our quadratic matrix inequality turns out to be an *almost* Nash equilibrium for a differential game with the same cost criterion.

It is shown that being able to make the H_∞ norm strictly less than

our, a priori given, bound γ is a sufficient condition for the existence of an almost Nash equilibrium. On the other hand, being able to make the H_∞ norm less than *or equal to* γ is a necessary condition for the existence of an almost Nash equilibrium. Note that the cost criterion explicitly depends on our bound γ.

1.6.3   The minimum entropy H_∞ control problem

Besides robustness we often require controllers to yield a satisfactory performance. Therefore if we model uncertainty as we did in section 1.3 then we might have the following interconnection:

$$
\begin{array}{c}
\begin{array}{cc}
y & u \\
z_1 & w_1 \\
z_2 & w_2
\end{array} \\
\Sigma \\
\Sigma_K
\end{array}
\tag{1.6}
$$
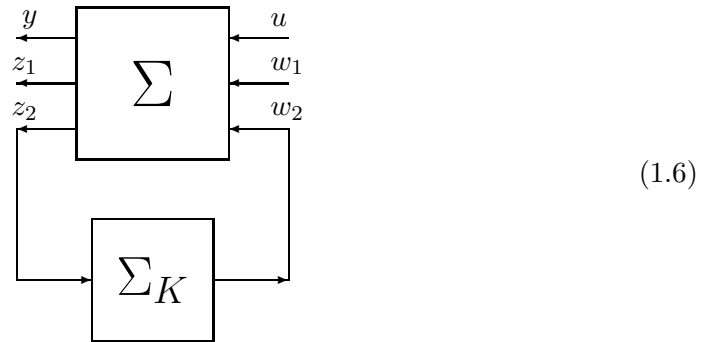
We assume that uncertainty is affecting the given system Σ via the unknown system Σ_K and we want to achieve a certain level of robustness with respect to internal stability and *simultanuously* attain a certain performance on the transfer matrix from w_1 to z_1. The latter performance requirement will be assumed to be the minimization of a quadratic performance criterion (which can be expressed in terms of the H_2 norm, defined in the next chapter, of the closed-loop transfer matrix by adding suitable weighting). Other criteria might be suitable and are interesting open problems for future research.

The above goal can be translated into the following objective: find a controller from y to u such that the closed-loop system is internally stable, the closed-loop transfer matrix from w_2 to z_2 has H_∞ norm less than some, a priori given, number γ and the H_2 norm of the closed-loop transfer matrix from w_1 to z_1 is minimized over all controllers which satisfy the first two conditions.

The above problem is still a completely unsolved problem. To simplify the problem the H_2 norm is replaced by an auxiliary cost function. This

auxiliary cost function is an upper bound for the H_2 norm and it is hoped that by minimizing this auxiliary cost function instead of the H_2 norm we still get a satisfactory performance.

The above problem has been investigated in [Mu] for regular systems under the assumption that $z_1 = z_2$ and $w_1 = w_2$. In general this auxiliary cost function is not very intuitive but in this special case we can replace it by an entropy function which yields more insight. In literature more general cases have been investigated (see [BH2, HB, HB2, RK]) and it remains an active research area. In this book we will investigate the special case $z_1 = z_2$ and $w_1 = w_2$ for singular systems. Our proofs are self-contained and in our opinion more straightforward then the proofs given in [Mu].

1.6.4   The finite horizon H_∞ control problem

As already remarked in section 1.2 the H_∞ control problem is concerned with minimizing the H_∞ norm. The H_∞ norm is equal to the maximum over all disturbances w of the quotient of the amount of energy going into the system and the amount of energy coming out of this system. In the standard H_∞ control problem, this energy is measured over an infinite time interval $[0, \infty)$. This might not always be realistic in practical applications. Therefore, in the finite horizon H_∞ control problem we minimize the same norm except that the energy is measured over a finite time interval $[0, T]$ for some given $T > 0$. This problem was solved for regular systems in [Li5]. We will extend these results to singular systems.

1.6.5   Discrete time systems

Early results for the H_∞ control problem were derived for the continuous time case. In the first chapters of this book we shall only concern ourselves with continuous time systems. However, in practical applications one is often concerned with discrete time systems.

One major reason is that to control a continuous time system one often applies a digital computer on which we can only implement a discrete time controller. One possible approach is to derive a continuous time H_∞ controller and then discretize the controller to be able to use your computer. This approach is followed in papers like [KA].

A direct digital design where a sample and hold function of given bandwidth are already incorporated into the plant (so-called sampled-data systems) and we design a discrete time controller for this system is a much better way to discover the limitations of our design. Although basically a much harder problem, major steps forward have been made

(see [Ch, Ch2, KH, SK]). In these papers the problem is reduced to a discrete time H_∞ control problem of which a solution is therefore needed.

Also, certain systems are in themselves inherently discrete and certainly for these systems it is useful to have results available for H_∞ control problems.

One approach to solve the discrete time H_∞ control problem, is to apply a transformation in the frequency-domain which transforms discrete time systems to continuous time systems. The transformation we have in mind is for instance discussed in [Gen, appendix 1]. With this transformation discrete time H_∞ functions are mapped isometrically onto continuous time H_∞ functions. One can then use the results available for continuous time systems and afterwards apply the inverse transformation on the controller thus obtained.

However, this transformation is not always attractive. It maps systems with a pole in 1 into non-proper systems. Also it clouds the understanding of specific features of discrete time H_∞ control because of its complexity. If it is possible to derive results for discrete time systems, why not apply these results directly instead of performing this unnatural transformation. Another problem is that the state feedback H_∞ control problem is transformed into a continuous time measurement feedback H_∞ control problem where indeed one might have problems observing the state. This prevents an understanding of discrete time H_∞ controllers as an interconnection of a state feedback and an observer since this distinction is scrambled by the transformation.

Therefore, in chapters 9 and 10 we shall derive results for the discrete time state feedback H_∞ control problem and the discrete time measurement feedback H_∞ control problem, respectively. We shall only consider discrete time analogues of regular systems, and hence it might not be too surprising that our conditions are formulated in terms of discrete time algebraic Riccati equations.

Chapter 2

Notation and basic properties

2.1 Introduction

In this chapter we introduce the notation and definitions we shall use throughout this book. Moreover, we give a number of basic properties we shall need further on. Most of these properties will not be proven here but we shall give appropriate references. In this book we deal with both discrete time systems as well as continuous time systems. Therefore, two sections of this chapter are split up into a discrete time part and a continuous time part in order to emphasize the differences. While reading this book, one should always keep in mind that we want to minimize the output with respect to the worst case disturbance. This goal is all we want to achieve and we shall try to achieve it under several different circumstances.

Let $\mathcal{R}$ denote the real numbers, $\mathcal{C}$ denote the complex numbers and let $\mathcal{N}$ denote the non-negative integers. Let $\mathcal{C}^+$ ($\mathcal{C}^0$, $\mathcal{C}^-$) denote the set of all $s \in \mathcal{C}$ such that Re $s > 0$ (Re $s = 0$, Re $s < 0$). Finally by $\mathcal{D}$ ($\delta\mathcal{D}$, $\mathcal{D}^+$) we denote the set of all $s \in \mathcal{C}$ such that $|s| < 1$ ($|s| = 1$, $|s| > 1$).

2.2 Linear systems

2.2.1 Continuous time

Except for chapters 9 and 10 we shall investigate systems with continuous time. These systems are described by a differential equation and two output equations.

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew, \\ y = C_1 x + D_{11}u + D_{12}w, \\ z = C_2 x + D_{21}u + D_{22}w. \end{cases} \qquad (2.1)$$

We shall always assume that x, u, w, y and z take values in finite-dimensional vector spaces: $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^m$, $w(t) \in \mathcal{R}^l$, $y(t) \in \mathcal{R}^q$ and $z(t) \in \mathcal{R}^p$. The system parameters A, B, E, C_1, C_2, D_{11}, D_{12}, D_{21} and D_{22} are matrices of appropriate dimensions. We assume that the system is time-invariant, i.e. the system parameters are independent of time. Except when stated explicitly, we shall always assume that the initial state is zero, i.e. $x(0) = 0$. The input w is the disturbance working on the system, whose effect on one of the outputs we want to minimize. The input u is the control which we use to achieve this goal. The output y is the measurement on the basis of which we choose our input u. The output z is the output we want to make small relative to the size of the disturbance w. More precisely, we are searching for a feedback from y to u, denoted by Σ_F, such that the closed-loop system $\Sigma \times \Sigma_F$ mapping w to z has a small induced norm. If all information on the system is available for feedback, i.e. $y = (x, w)$, or if the state is available for feedback, i.e. $y = x$, then we shall often delete the second equation in (2.1).

In several chapters the indices of the C- and D-matrices are different from the indices used in system (2.1). This is done to simplify the notation in the respective chapters. When comparing results from different chapters the reader should be careful whether these differences arise or not.

When we apply an input u and a disturbance w with initial condition $x(0) = \xi$ then we shall denote by $x_{u,w,\xi}$ and $z_{u,w,\xi}$ the state and the output of system (2.1), respectively. In the case that we have zero initial condition we shall write $x_{u,w}$ and $z_{u,w}$ instead of $x_{u,w,0}$ and $z_{u,w,0}$.

Quite often, we shall look at two special subsystems, one in which we restrict attention to the system from u to z:

$$\Sigma_{ci} : \begin{cases} \dot{x} = Ax + Bu, \\ z = C_2 x + D_{21} u, \end{cases} \tag{2.2}$$

and one in which we restrict attention to the system from w to y:

$$\Sigma_{di} : \begin{cases} \dot{x} = Ax + Ew, \\ y = C_1 x + D_{12} w. \end{cases} \tag{2.3}$$

If we only have one input and one output, as in the above two systems, then we can associate with a system the quadruple of the four system parameters. For the sake of simplicity we shall often write, for instance, the system (A, B, C_2, D_{21}) when, formally, we should write down the system equations (2.2).

The system equations will always be denoted by a Σ with some index to identify different systems. The input–output operator mapping the

inputs to the outputs with zero initial state will always be denoted by $\mathcal{G}$ with, again, some index to identify various operators. Finally, the transfer matrix of a system which, for instance for the system Σ_{ci}, is defined by

$$G_{ci}(s) := C_2 \left(sI - A\right)^{-1} B + D_{21},$$

will always be denoted by G with some index.

We shall investigate three kinds of feedback. In order of increasing generality: static state feedback, static feedback and dynamic output feedback. For the first two we shall always implicitly assume that the measurement is $y = x$ and $y = (x, w)$, respectively.

A dynamic output feedback is a system of the form

$$\Sigma_F : \begin{cases} \dot{p} = Kp + \ Ly, \\ u = Mp + Ny. \end{cases} \tag{2.4}$$

Except for chapter 8 we shall assume that the controller is time-invariant. If $D_{11} \neq 0$, then the equations (2.1) and (2.4) together might not have a unique solution for given w. This is clearly undesirable and related to the concept of well-posedness. Hence if $D_{11} \neq 0$, then we only consider controllers such that $I - D_{11}N$ is invertible (for all t if N is time-varying). This guarantees that the equations (2.1) and (2.4) together have a unique solution for given w. Moreover, it is a necessary and sufficient condition to guarantee that the closed-loop system is well-posed (for a detailed discussion of well-posedness see [Wi2]).

If the interconnection is well-posed, then the closed-loop system $\Sigma \times \Sigma_F$ will be of the form:

$$\Sigma_{cl} : \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = A_e \begin{pmatrix} x \\ p \end{pmatrix} + E_e w, \\ \ z \ = C_e \begin{pmatrix} x \\ p \end{pmatrix} + D_e w. \end{cases}$$

for certain matrices $A_e(t), E_e(t), C_e(t)$ and $D_e(t)$. Obviously, if we put $x(0) = 0$, $p(0) = 0$, then the closed-loop system Σ_{cl} defines a Volterra integral operator $\mathcal{G}_{cl} : \mathcal{L}_2^l \rightarrow \mathcal{L}_2^q$ given by

$$\left(\mathcal{G}_{cl}w\right)(t) = z(t) = \int_0^t C_e(t)\Phi_e(t, \tau)E_e(\tau)w(\tau)\, d\tau \ + D_e(t)w(t),$$

where $\Phi_e(t, \tau)$ is the transition matrix of $A_e(t)$. If the controller is time-invariant the matrices $A_e(t), E_e(t), C_e(t)$ and $D_e(t)$ are independent of time and $\Phi(t - \tau) = e^{A_e(t-\tau)}$.

If the interconnection is well-posed, then it is called internally stable if, with $w = 0$, for every initial state of the system and every initial state of the controller the state of the system and the state of the controller in the interconnection converge to zero as $t \to \infty$. If the controller is given by (2.4) and the system is given by (2.1), then this is equivalent to the requirement that the matrix

$$
\begin{pmatrix}
A + BN(I - D_{11}N)^{-1}C_1 & B(I - ND_{11})^{-1}M \\
L(I - D_{11}N)^{-1}C_1 & K + L(I - D_{11}N)^{-1}D_{11}M
\end{pmatrix}
\tag{2.5}
$$

be asymptotically stable, i.e. all its eigenvalues lie in the open left half complex plane. For systems with discrete time, asymptotic stability will have a different meaning (see the next subsection). Moreover, note that if $D_{11} = 0$, then the interconnection is always well-posed. In that case the matrix (2.5) simplifies considerably and is equal to

$$
\begin{pmatrix}
A + BNC_1 & BM \\
LC_1 & K
\end{pmatrix}.
\tag{2.6}
$$

For a finite-dimensional system (A, B, C, D) we shall call the matrix A the *state matrix* of the system. Accordingly, the matrix (2.5) or (2.6) will be referred to as the *closed-loop state matrix*. We shall call B the *input matrix*, C the *output matrix* and D the *direct-feedthrough matrix*. Finally, the dimension of the state space is called the McMillan degree of the realization.

After applying the compensator Σ_F described by the static feedback law $u = F_1 x + F_2 w$ the closed-loop transfer matrix is given by

$$
G_F(s) := (C_2 + D_{21}F_1)(sI - A - BF_1)^{-1}(E + BF_2) + (D_{22} + D_{21}F_2).
\tag{2.7}
$$

Just as for dynamic compensators, the closed-loop system is called internally stable if, with $w = 0$ and for all initial states, the state converges to zero as $t \to \infty$. It is easy to check that the closed-loop system is internally stable if, and only if, the matrix $A + BF_1$ is asymptotically stable. This can also be derived from the dynamic feedback case discussed above by noting that

$$
y = \begin{pmatrix} I \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} w
$$

and that the matrix (2.6) then becomes equal to $A + BF_1$. Here K is a 0×0 matrix and hence disappears and $N = (\, F_1 \quad F_2 \,)$. A compensator

described by a static state feedback law $u = Fx$ can be considered as a special case of a static feedback law and the above definitions will be used correspondingly.

We shall say that matrices P and Q satisfy dual properties if P satisfies a certain property for the system (2.1) if, and only if, Q^{T} satisfies the same property for the dual system defined by

$$\Sigma^{\mathrm{T}} : \begin{cases} \dot{x} = A^{\mathrm{T}}x + \ C_1^{\mathrm{T}}u + \ C_2^{\mathrm{T}}w, \\ y = B^{\mathrm{T}}x + D_{11}^{\mathrm{T}}u + D_{21}^{\mathrm{T}}w, \\ z = E^{\mathrm{T}}x + D_{12}^{\mathrm{T}}u + D_{22}^{\mathrm{T}}w. \end{cases}$$

2.2.2   Discrete time

In chapters 9 and 10 we shall investigate systems with discrete time. These systems are described by a *difference* equation and two output equations.

$$\Sigma : \begin{cases} \sigma x = \ Ax + \ \ Bu + \ \ Ew, \\ y \ \ = C_1x + D_{11}u + D_{12}w, \\ z \ \ = C_2x + D_{21}u + D_{22}w. \end{cases} \tag{2.8}$$

where σ denotes the shift-operator, which is defined by

$$(\sigma x)(k) := x(k+1).$$

We shall only investigate shift-invariant systems, i.e. we assume throughout that the system parameters do not depend on time. All assumptions and definitions for continuous time as given in the previous subsection have an analogous meaning in discrete time. However, one point has to be discussed explicitly.

A dynamic output feedback is a system of the form

$$\Sigma_F : \begin{cases} \sigma p = \ Kp + \ Ly, \\ u \ \ = Mp + Ny. \end{cases} \tag{2.9}$$

As in the previous section we define the interconnection to be internally stable if the interconnection is well-posed and if, with $w = 0$, for every initial state of the system and for every initial state of the controller the state of the system and the state of the controller in the interconnection converge to zero as $t \to \infty$. If the controller is given by (2.9) and the system is given by (2.8) then this is equivalent to the requirement that the matrix (2.5) is asymptotically stable, i.e. all its eigenvalues lie in the open *unit disc.* Note that for systems with continuous time asymptotic

stability of a matrix has a different meaning. That is, we still have the same matrix as in the continuous time case only this time its eigenvalues should be in the open unit disc instead of the open left half plane.

Later on we shall also need a backwards difference equation of the form

$$\sigma^{-1}x = Ax + Bu.$$

A function x from $\mathcal{N} \cup \{-1\}$ to $\mathcal{R}^n$ is said to be a solution of this backwards difference equation if

$$x(k-1) = Ax(k) + Bu(k),$$

for all $k \in \mathcal{N}$.

2.3   Rational matrices

In this section we recall some basic notions on rational matrices which are either applied to system matrices (which will be defined on the next page) or transfer matrices.

Let $\mathcal{R}[s]$ denote the ring of polynomials with real coefficients. Let $\mathcal{R}^{n \times m}[s]$ be the set of all $n \times m$ matrices with coefficients in $\mathcal{R}[s]$. An element of $\mathcal{R}^{n \times m}[s]$ is called a polynomial matrix. Linear polynomial matrices of the form $sE - F$ are sometimes called a matrix pencil. $\mathcal{R}(s)$ denotes the field of rational functions with real coefficients, i.e. $\mathcal{R}(s)$ is the quotient field of $\mathcal{R}[s]$. Let $\mathcal{R}^{n \times m}(s)$ be the set of all $n \times m$ matrices with coefficients in $\mathcal{R}(s)$.

An element of $\mathcal{R}^{n \times m}(s)$ is called a rational matrix. A rational matrix G is called *stable* if G has no poles in the closed right half complex plane. G is called *proper* if $\lim_{s \to \infty} G(s)$ exists, and *strictly proper* if this limit is zero. Moreover, for a given proper rational matrix G, the matrix $\lim_{s \to \infty} G(s)$ is called the direct-feedthrough matrix. Note that the direct-feedthrough matrix of a system is equal to the direct-feedthrough matrix of its transfer matrix.

By $\text{rank}_{\mathcal{K}}$ we denote the rank of a matrix as a matrix with entries in the field $\mathcal{K}$. We shall often write only *rank* in the case that $\mathcal{K} = \mathcal{R}$ or $\mathcal{K} = \mathcal{C}$ (note that for a real matrix the rank over $\mathcal{C}$ always equals its rank over $\mathcal{R}$). Moreover, we often use the term *normal rank* for $\text{rank}_{\mathcal{K}}$ where $\mathcal{K} = \mathcal{R}(s)$.

We shall first discuss a number of properties of polynomial matrices. A square polynomial matrix is called unimodular if it is invertible *over the ring of polynomial matrices*. Two polynomial matrices P and Q are called unimodularly equivalent if unimodular matrices U and V exist

such that $Q = UPV$. In this book, we denote the fact that P and Q are unimodularly equivalent by $P \sim Q$. It is well known (see [Ga]) that for any $P \in \mathcal{R}^{n \times m}[s]$ there exists $\Psi \in \mathcal{R}^{n \times m}[s]$ of the form

$$
\Psi = \begin{pmatrix}
\psi_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & \vdots & & \vdots \\
\vdots & \ddots & \ddots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \psi_r & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
$$

such that $P \sim \Psi$. Here ψ_i are monic polynomials with the property that ψ_i divides ψ_{i+1} for $i = 1, \ldots, r-1$.

The polynomial matrix Ψ is called the *Smith form* of P (see [Ga]). The polynomials ψ_i are called the *invariant factors* of P. Their product $\psi = \psi_1 \psi_2 \cdots \psi_r$ is called the *zero polynomial* of P. The roots of ψ are called the *zeros* of P. The integer r is equal to the normal rank of P as defined before. If s is a complex number, then $P(s)$ is an element of $\mathcal{C}^{n \times m}$. It is easy to see that $\text{rank}_{\mathcal{R}(s)} P = \text{rank } P(s)$ for all $s \in \mathcal{C}$ if, and only if, P is unimodularly equivalent to the constant $n \times m$ matrix

$$
\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},
$$

where I_r is the $r \times r$ identity matrix.

Next we recall some important facts on the structure of a linear system $\Sigma_{ci} = (A, B, C, D)$. The *(Rosenbrock) system matrix* of Σ_{ci} is defined as the polynomial matrix

$$
P_{ci} := \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix}.
$$

The invariant factors of P_{ci} are called the *transmission polynomials* of Σ_{ci}. The transmission polynomials unequal to 1 are called the *non-trivial transmission polynomials* of P_{ci}. The zeros of P_{ci} are called the *invariant zeros* of Σ_{ci}. Clearly, $s \in \mathcal{C}$ is an invariant zero of Σ_{ci} if, and only if,

$$
\text{rank } P_{ci}(s) < \text{rank}_{\mathcal{R}(s)} P_{ci}.
$$

This concept of invariant zeros (see e.g. [Ro]) will play an important role in this book.

It is easy to see that if $F \in \mathcal{R}^{m \times n}$ and if $P_{ci,F}$ is the system matrix of $\Sigma_{ci,F} := (A + BF, B, C + DF, D)$, then $P_{ci} \sim P_{ci,F}$. In particular, this

implies that the transmission polynomials of Σ_{ci} and $\Sigma_{ci,F}$ coincide and, *a fortiori*, that the invariant zeros of Σ_{ci} and $\Sigma_{ci,F}$ coincide.

We can also associate with the system Σ_{ci} the controllability pencil

$$L(s) := \left(\begin{array}{cc} sI - A & -B \end{array} \right),$$

and the observability pencil

$$M(s) := \left(\begin{array}{c} sI - A \\ C \end{array} \right).$$

A special case which is often investigated in H_∞ control theory is the case that D is injective and $C^{\mathrm{T}} D = 0$. In which case the invariant zeros of Σ_{ci} are the zeros of M. That is, the invariant zeros of Σ_{ci} are the unobservable eigenvalues of (C, A).

A system $\Sigma_{ci} = (A, B, C, D)$ is called *left- (right-) invertible* if the transfer matrix of Σ_{ci} is left- (right-) invertible as a rational matrix. This is equivalent to the requirement that the system matrix of Σ_{ci} be left- (right-) invertible as a rational matrix. It can be shown that a system is left-invertible if, and only if, the input–output operator associated with this system is injective. However, it is in general not true that right-invertibility implies surjectivity of the input–output operator. We define the class of functions $\mathcal{C}_0^\infty$ as the class of infinitely often differentiable functions on $[0, \infty)$ such that all derivatives are equal to zero in zero. Then the continuous time system Σ_{ci} is right-invertible if, and only if, the input–output operator as a map from $\mathcal{C}_0^\infty$ to $\mathcal{C}_0^\infty$ is surjective. A discrete time system is right-invertible if all sequences f with $f(0) = f(1) = \ldots = f(n) = 0$ are contained in the image of the input–output operator. Here n is equal to the dimension of the state space.

2.4   Geometric theory

In this section we recall some basic notions from the geometric approach to linear system theory. We shall use the geometric approach only for continuous time systems and one should note that although we could use the geometric approach for discrete time systems too, its usefulness for our problems is mainly restricted to continuous time systems.

We first introduce the important concepts of controlled invariance and conditioned invariance (see [SH, Wo]).

Definition 2.1 : *A subspace $\mathcal{V}$ of $\mathcal{R}^n$ is said to be A-invariant if $A\mathcal{V} \subseteq \mathcal{V}$. A subspace $\mathcal{V}$ of $\mathcal{R}^n$ is said to be conditioned invariant (also called (C,A)-invariant) if a linear mapping G exists such that*

$(A + GC)\mathcal{V} \subseteq \mathcal{V}$.

A subspace $\mathcal{V}$ of $\mathcal{R}^n$ is said to be controlled invariant (also called (A,B)-invariant) if a linear mapping F exists such that

$(A + BF)\mathcal{V} \subseteq \mathcal{V}$. $\square$

Note that the alternative terminology of (C,A)-invariant and (A,B)-invariant is such that the only difference in the terminology for these two concepts is the order of the matrices. One has to remember that these two properties are in fact associated to the system $(A,B,C,0)$ where (A,B)-invariance means that a state feedback F exists which makes the subspace $\mathcal{V}$ invariant under the closed-loop state matrix $A + BF$ while (C,A)-invariance means that an output injection G exists which makes this subspace invariant under the state matrix $A+GC$. Finally, we would like to remark that these properties are dual to each other: $\mathcal{V}$ is (C,A)-invariant if, and only if, $\mathcal{V}^{\perp}$ is $(A^{\mathrm{T}}, C^{\mathrm{T}})$-invariant.

The following well-known lemma is a convenient tool for checking whether these properties hold for a certain subspace (see [SH, Wo]).

Lemma 2.2 : *A subspace $\mathcal{V}$ of $\mathcal{R}^n$ is (C,A)-invariant if, and only if,*

$A\,(\mathcal{V} \cap \ker C) \subseteq \mathcal{V}$.

A subspace $\mathcal{V}$ of $\mathcal{R}^n$ is (A,B)-invariant if, and only if,

$A\mathcal{V} \subseteq \mathcal{V} + \mathrm{im}\,B$. $\square$

We now define a number of particular linear subspaces of the state space, among which is the strongly controllable subspace. The latter will play a key role throughout this book.

Definition 2.3 : *Consider the system*

$$\Sigma_{ci} \;:\; \begin{cases} \dot{x} = Ax + Bu, \\ z = Cx + Du. \end{cases} \tag{2.10}$$

We define the strongly controllable subspace $\mathcal{T}(\Sigma_{ci})$ as the smallest subspace $\mathcal{T}$ of $\mathcal{R}^n$ for which a linear mapping G exists such that:

$$
\begin{align}
(A + GC)\mathcal{T} \quad &\subseteq \quad \mathcal{T}, \tag{2.11}\\
Im\,(B + GD) \quad &\subseteq \quad \mathcal{T}. \tag{2.12}
\end{align}
$$

We also define the detectable strongly controllable subspace $\mathcal{T}_g(\Sigma_{ci})$ as the smallest subspace $\mathcal{T}$ of $\mathcal{R}^n$ for which a linear mapping G exists such that (2.11) and (2.12) are satisfied and moreover $A + GC \,|\, \mathcal{R}^n/\mathcal{T}$ is asymptotically stable.

A system is called strongly controllable if its strongly controllable subspace is equal to the whole state space.   □

We also define the dual versions of these subspaces:

Definition 2.4 : *Consider the system (2.10). We define the weakly unobservable subspace $\mathcal{V}(\Sigma_{ci})$ as the largest subspace $\mathcal{V}$ of $\mathcal{R}^n$ for which a mapping F exists such that:*

$$
\begin{align}
(A + BF)\mathcal{V} \quad &\subseteq \quad \mathcal{V}, \tag{2.13}\\
(C + DF)\mathcal{V} \quad &= \quad \{0\}. \tag{2.14}
\end{align}
$$

We also define the stabilizable weakly unobservable subspace $\mathcal{V}_g(\Sigma_{ci})$ as the largest subspace $\mathcal{V}$ for which a mapping F exists such that (2.13) and (2.14) are satisfied and moreover $A + BF \,|\, \mathcal{V}$ is asymptotically stable.

A system is called strongly observable if its weakly unobservable subspace is equal to $\{0\}$.   □

We can give intuitive interpretations of these subspaces. $\mathcal{V}(\Sigma_{ci})$ is the subspace of all $x_0 \in \mathcal{R}^n$ such that for the system (2.10) with initial condition $x(0) = x_0$ there exists an input function u on $[0, \infty)$ such that the output function y of the system is identical zero on $[0, \infty)$. $\mathcal{V}_g(\Sigma_{ci})$ has the same interpretation but with the extra constraint on the input u that the resulting state trajectory $x(t)$ converges to 0 as $t \to \infty$.

On the other hand, $\mathcal{T}(\Sigma_{ci})$ consists of all $x_0 \in \mathcal{R}^n$ such that for the system (2.10) with initial condition $x(0) = x_0$ and for all $\varepsilon > 0$ there exist $T > 0$ and an input function u such that the resulting state satisfies $x(T) = 0$ while the $\mathcal{L}_1$-norm of the output y is less than ε, i.e. we can steer the initial state x_0 to 0 in finite time and, at the same time, we can make the $\mathcal{L}_1$-norm of the output y arbitrarily small. An interpretation

of $\mathcal{T}_g(\Sigma_{ci})$ can be given in terms of observers but is not very intuitive. Therefore, we shall not explain this interpretation in this book.

Note that $\mathcal{V}(\Sigma_{ci})$ and $\mathcal{T}(\Sigma_{ci})$ are dual subspaces, i.e. $\mathcal{V}(\Sigma_{ci})^{\perp} = \mathcal{T}(\Sigma_{ci}^{\mathrm{T}})$. Also $\mathcal{V}_g(\Sigma_{ci})$ and $\mathcal{T}_g(\Sigma_{ci})$ are dual subspaces. The following lemma gives explicit recursive algorithms to calculate these subspaces.

In this lemma we need the concept of *modal subspace* of a matrix A. Let some region $\mathcal{C}_d$ of the complex plane be given which is symmetric with respect to the real axis. The subspace $\mathcal{X}$ is called the *modal subspace* of A with respect to $\mathcal{C}_d$ if $\mathcal{X}$ is the largest A-invariant subspace of $\mathcal{R}^n$ such that if we restrict the mapping A to $\mathcal{X}$ then its spectrum is contained in $\mathcal{C}_d$. By $C^{-1}\mathcal{X}$ where $\mathcal{X}$ is some linear subspace, we shall denote $\{x \mid Cx \in \mathcal{X}\}$.

Lemma 2.5 : *The strongly controllable subspace $\mathcal{T}(\Sigma_{ci})$ is the limit of the sequence of subspaces $\{\mathcal{T}_i(\Sigma_{ci})\}$ generated by the recursive algorithm:*

$$\mathcal{T}_0(\Sigma_{ci}) := \{0\},$$
$$\mathcal{T}_{i+1}(\Sigma_{ci}) := \{x \in \mathcal{R}^n \mid \exists\, \tilde{x} \in \mathcal{T}_i(\Sigma_{ci}),\ u \in \mathcal{R}^m \text{ such that}$$
$$x = A\tilde{x} + Bu \text{ and } C\tilde{x} + Du = 0\} \quad (2.15)$$

$\mathcal{T}_i(\Sigma_{ci})$ $(i = 0, 1, \ldots)$ is a non-decreasing sequence of subspaces which attains its limit in a finite number of steps. In the same way $\mathcal{V}(\Sigma_{ci})$ equals the limit of the sequence of subspaces $\{\mathcal{V}_i(\Sigma_{ci})\}$ generated by:

$$\mathcal{V}_0(\Sigma_{ci}) := \mathcal{R}^n,$$
$$\mathcal{V}_{i+1}(\Sigma_{ci}) := \{x \in \mathcal{R}^n \mid \exists\, \tilde{u} \in \mathcal{R}^m,\ \text{such that}$$
$$Ax + B\tilde{u} \in \mathcal{V}_i(\Sigma_{ci}) \text{ and } Cx + D\tilde{u} = 0\} \quad (2.16)$$

$\mathcal{V}_i(\Sigma_{ci})$ $(i = 0, 1, \ldots)$ is a non-increasing sequence of subspaces which also attains its limit in a finite number of steps. Moreover, if G is a mapping such that (2.11) and (2.12) are satisfied for $\mathcal{T} = \mathcal{T}(\Sigma_{ci})$ and if F is a mapping such that (2.13) and (2.14) are satisfied for $\mathcal{V} = \mathcal{V}(\Sigma_{ci})$, then we have the following two equalities:

$$\mathcal{T}_g(\Sigma_{ci}) = [\mathcal{T}(\Sigma_{ci}) + \mathcal{X}_b(A + GC)] \cap\, < \mathcal{T}(\Sigma_{ci}) + C^{-1}\operatorname{im} D \mid A + GC > (2.17)$$

$$\mathcal{V}_g(\Sigma_{ci}) = \mathcal{V}(\Sigma_{ci}) \cap \mathcal{X}_g(A + BF) +\, < A + BF \mid \mathcal{V}(\Sigma_{ci}) \cap B \ker D > \quad (2.18)$$

Here $\mathcal{X}_b(A + GC)$ denotes the modal subspace of the matrix $A + GC$ with respect to the closed right half complex plane and $\mathcal{X}_g(A + BF)$ denotes the modal subspace of the matrix $A + BF$ with respect to the open left half complex plane. Moreover, $< A + BF \mid \mathcal{V}(\Sigma_{ci}) \cap B \ker D >$ denotes the smallest $A + BF$ invariant subspace containing $\mathcal{V}(\Sigma_{ci}) \cap B \ker D$ and finally, $< \mathcal{T}(\Sigma_{ci}) + C^{-1} \operatorname{im} D \mid A + GC >$ denotes the largest $A + GC$ invariant subspace contained in $\mathcal{T}(\Sigma_{ci}) + C^{-1} \operatorname{im} D$. □

Proof : This is all well known except for possibly (2.17) and (2.18) in the case that the D-matrix is unequal to zero. This can be proven by first showing that a G exists satisfying (2.11) and (2.12) for which (2.17) holds and after that, showing that the equality is independent of our particular choice of G satisfying (2.11) and (2.12). The same can be done for (2.18). Details are left to the reader.   ∎

We shall give some properties of the strongly controllable subspace at this point which will come in handy in the sequel (see [Ha, SH]). The following lemma can be easily checked using the properties (2.11) and (2.12) of the strongly controllable subspace.

Lemma 2.6 : *For all $F \in \mathcal{R}^{m \times n}$, the strongly controllable subspace $\mathcal{T}(\Sigma_{ci})$ is $(C + DF, A + BF)$-invariant.*   □

Lemma 2.7 : *Let F_0 be such that $D^{\mathrm{T}}(C + DF_0) = 0$. $\mathcal{T}(\Sigma_{ci})$ is the smallest $(C + DF_0, A + BF_0)$-invariant subspace containing $B \ker D$.* □

Proof : Let $\mathcal{T}$ be the smallest $(C + DF_0, A + BF_0)$-invariant subspace containing $B \ker D$. We know that $\mathcal{T}(\Sigma_{ci})$ is $(C+DF_0, A+BF_0)$-invariant by lemma 2.6. Moreover, in definition 2.3 we have $\mathcal{T}_1(\Sigma_{ci}) = B \ker D$. Since the $\mathcal{T}_i(\Sigma_{ci})$ are non-decreasing this implies that $\mathcal{T}(\Sigma_{ci}) \supseteq B \ker D$. Therefore we have $\mathcal{T} \subseteq \mathcal{T}(\Sigma_{ci})$.

Conversely we know:

$$\exists G_1 : \mathrm{im}\ (C + DF_0) \to \mathcal{R}^n \qquad [(A + BF_0) + G_1(C + DF_0)]\, \mathcal{T} \subseteq \mathcal{T},$$
$$\exists G_2 : \mathrm{im}\ D \to \mathcal{R}^n \qquad\qquad \mathrm{im}\ (B + G_0D) = B \ker D \subseteq \mathcal{T}.$$

Since $D^{\mathrm{T}}(C + DF_0) = 0$ the above two mappings G_1 and G_2 can be combined to one linear mapping G such that

$$G|_{\mathrm{im}\ (C+DF_0)} = G_1,$$
$$G|_{\mathrm{im}\ D} \qquad\ = G_2,$$

and hence we have found a G such that $(A + GC)\mathcal{T} \subseteq \mathcal{T}$ and $\mathrm{im}\ (B + GD) \subseteq \mathcal{T}$. Thus we find $\mathcal{T} \supseteq \mathcal{T}(\Sigma_{ci})$ and therefore $\mathcal{T} = \mathcal{T}(\Sigma_{ci})$.   ∎

We also have the following result available (see [Ha]).

Lemma 2.8 : *Assume that we have the system (2.10) with* $(C \quad D)$ *surjective. The system is strongly controllable if, and only if, the system matrix*

$$\begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix} \tag{2.19}$$

has full row rank for all $s \in \mathcal{C}$.   □

Note that this last lemma immediately implies that a strongly controllable system with $(C \quad D)$ surjective is controllable by applying the Popov–Belevitch–Hautus criterion (this result is true in general but it only follows from the above lemma if $(C \quad D)$ is surjective). A major reason why strongly controllable systems are interesting is the following corollary which can be derived after some effort from lemma 2.8:

Corollary 2.9 : *Assume that we have the system (2.10) with* $(C\ D)$ *surjective. Denote the transfer matrix of* Σ_{ci} *by* G_{ci}. *If the system is strongly controllable, then* G_{ci} *has a polynomial right inverse.*   □

Note that this implies that the mapping from u to z is surjective as a function from $\mathcal{C}_0^\infty$ to $\mathcal{C}_0^\infty$. Hence we can achieve any desired output in $\mathcal{C}_0^\infty$. The problem we shall encounter is that we cannot realize a polynomial part of a transfer matrix by a dynamic feedback of the form (2.4). It will turn out that it is possible to approximate the effect of the polynomial part of the desired controller arbitrarily well by the effect of a controller which is of the form (2.4). In other words, we shall not try to approximate the non-proper controller (which might be a legitimate alternative approach), but only the closed-loop transfer matrix this non-proper controller gives us.

2.5   The Hardy and Lebesgues spaces

2.5.1   Continuous time

We define the function space H_∞^1 as the set of all functions f on the open right half plane which are analytic and which satisfy

$$\|f\|_\infty := \sup_{s \in \mathcal{C}^+} |f(s)| < \infty. \tag{2.20}$$

This function space is a Banach space with respect to the norm $\| \cdot \|_\infty$. It can easily be seen that the set of all proper rational functions with no poles in the closed right half plane is contained in H_∞^1. Note that one can define such a Banach space of analytic functions on any simply connected region which is not equal to the whole complex plane. This is a direct consequence of Riemann's mapping theorem (see [Ru]). By H_∞ we denote all matrices with coefficients in H_∞^1. Note that the transfer matrix of an internally stable system with continuous time will be in H_∞ (hence it is not surprising that for discrete time systems we should replace the right half plane by the complement of the closed unit disc). On the space H_∞ we define the following norm:

$$\|G\|_\infty := \sup_{s \in \mathcal{C}^+} \sigma_1 \left[G(s) \right].$$

Here $\sigma_1[\cdot]$ denotes the largest singular value. Note that it is not a norm in the strict sense since H_∞ is not a vector space (we cannot add matrices with different dimensions). However, any subset of H_∞ of matrices with the same dimensions is a well-defined vector space on which $\| \cdot \|_\infty$ is a norm which makes this space into a Banach space. This norm will be referred to as the H_∞ norm.

We define H_2^1 as the set of all functions f on the open right half plane which are analytic and which satisfy

$$\|f\|_2 := \sup_{\alpha \in \mathcal{R}} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\alpha + i\omega) f^*(\alpha + i\omega) \, d\omega \right)^{1/2} < \infty.$$

By H_2 we denote the set of matrices with elements in H_2^1. On this space we define the following norm:

$$\|G\|_2 := \sup_{\alpha \in \mathcal{R}} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Trace } G(\alpha + i\omega) G^*(\alpha + i\omega) \, d\omega \right)^{1/2}.$$

Again, any subset of H_2 of matrices with the same dimensions is a well-defined vector space on which $\| \cdot \|_2$ is a norm which makes this space into a Banach space. Actually these subsets are Hilbert spaces since this norm is generated by an inner product. This norm will be referred to as the H_2 norm. Clearly any strictly proper stable rational matrix is in H_2. Both H_2 and H_∞ are special cases of the so-called Hardy spaces named after G.H. Hardy (1877–1947).

We define the space $\mathcal{L}_\infty$ as the space of essentially bounded measurable matrix valued functions on the imaginary axis. On this space we define the following norm

$$\|F\|_\infty := \sup_{\omega \in \mathcal{R}} \sigma_1 \left[F(i\omega) \right] < \infty. \tag{2.21}$$

With respect to this norm, subsets of all matrix valued functions in L_∞ with the same dimension form a Banach space. For general matrix valued functions in H_∞ we can identify a boundary function to the analytic function which is, a priori, only defined on the *open* left half plane. This boundary function is in L_∞ and its L_∞-norm is equal to the H_∞ norm of the original function (see [Yo]). Note that rational matrices in H_∞ do not have poles on the imaginary axis and hence this boundary function is simply the rational matrix itself evaluated on the imaginary axis. Since, in the above sense, we can embed H_∞ isometrically into L_∞ we shall use the same notation for the H_∞-norm and the L_∞-norm in this book.

Define $\mathcal{L}_2^n$ as the set of all Lebesgue measurable functions f from $\mathcal{R}$ to $\mathcal{R}^n$ for which

$$\|f\|_2 := \left(\int_0^\infty \|f(t)\|^2 \, dt \right)^{1/2} < \infty,$$

where $\|\cdot\|$ denotes the Euclidian norm. With respect to the norm $\|\cdot\|_2$ the function space $\mathcal{L}_2^n$ is a Banach space. It is even a Hilbert space since the $\mathcal{L}_2$-norm is induced by the following inner product

$$< f, g >_2 := \int_0^\infty < f(t), g(t) > \, dt,$$

where $< \cdot, \cdot >$ denotes the standard Euclidian inner product. By $\mathcal{L}_2$ we denote the set of all f for which an n exists such that $f \in \mathcal{L}_2^n$. Note that the inverse Laplace transform g of a vector-valued function $\bar{g} \in H_2$ is in $\mathcal{L}_2$ and by Parseval's theorem (see [Ru]) this is an isometrical mapping from H_2 to $\mathcal{L}_2$. Note that we can similarly define $\mathcal{L}_2(-i\infty, i\infty)$ of square integrable vector-valued functions on the imaginary axis. In that case we can embed a vector-valued function from H_2 into $\mathcal{L}_2(-i\infty, i\infty)$ and, via the Laplace transform, the latter is isometrically equivalent with $\mathcal{L}_2$. This embedding proceeds as in H_∞ by defining a boundary function.

In chapter 8 we shall discuss the function space $\mathcal{L}_2$ over a finite interval $[0, T]$ for some $T > 0$. We define $\mathcal{L}_2^n[0, T]$ as the set of all Lebesgue measurable functions f from $[0, T]$ to $\mathcal{R}^n$ for which

$$\|f\|_2 := \left(\int_0^T \|f(t)\|^2 \, dt \right)^{1/2} < \infty.$$

The context will always make clear which $\|\cdot\|_2$ is intended. The H_2 norm will only be applied to transfer matrices while the $\mathcal{L}_2$ norm will only be used for input–output signals. The use of the $\mathcal{L}_2[0, T]$ norm will be restricted to chapter 8.

For a more detailed expose of the spaces introduced above we refer to [Ho, RR, Ru, Yo].

In this book we shall frequently use a time-domain characterization of the H_∞ norm. Let $\Sigma \times \Sigma_F$ be the closed-loop system when we apply a controller Σ_F of the form (2.4) to the system Σ given by (2.1). If the closed-loop system is internally stable, then the closed-loop transfer matrix G_F is in H_∞. Denote by $\mathcal{G}_{ci}$ the closed-loop operator mapping w to z. The H_∞ norm is equal to the $\mathcal{L}_2$-induced operator norm of the closed-loop operator, i.e.

$$\|G_F\|_\infty = \|\mathcal{G}_F\|_\infty := \sup_w \left\{ \left. \frac{\|\mathcal{G}_{ci}w\|_2}{\|w\|_2} \right| w \in \mathcal{L}_2^l,\ w \neq 0 \right\}. \quad (2.22)$$

Because of the above equality we shall often refer to the $\mathcal{L}_2$-induced operator norm of the closed-loop operator $\mathcal{G}_F$ as the H_∞ norm of $\mathcal{G}_F$. By the H_∞ norm of a stable system we denote the H_∞ norm of the corresponding transfer matrix.

In chapter 8 we shall search for a controller which minimizes the $\mathcal{L}_2[0,T]$-induced operator norm. This norm will also be referred to as the H_∞ norm because of its relation with (2.22) and is defined by:

$$\|\mathcal{G}\|_\infty := \sup \left\{ \frac{\|\mathcal{G}w\|_2}{\|w\|_2} \mid 0 \neq w \in \mathcal{L}_2^l[0,T] \right\}.$$

Note that for the finite horizon H_∞ control problem we admit time-varying controllers. This implies that the closed-loop input–output operator $\mathcal{G}$ is a Volterra integral operator whose Laplace transform will, in general, not exist.

A system (A, B, C, D) is called inner if the system is internally stable and the input–output operator $\mathcal{G}$ is unitary, i.e. $\mathcal{G}$ maps $\mathcal{L}_2^m$ into itself and $\mathcal{G}$ is such that for all $f \in \mathcal{L}_2^m$ we have

$$\|\mathcal{G}f\|_2 = \|f\|_2.$$

Often inner is defined as a property of the transfer matrix but in our setting this is a more natural definition. It can be shown that $\mathcal{G}$ is unitary if, and only if, the transfer matrix of the system, denoted by G, satisfies:

$$G^{\mathrm{T}}(-z)G(z) = G(z)G^{\mathrm{T}}(-z) = I, \quad (2.23)$$

for all $z \notin \sigma(A)$ where $\sigma(\cdot)$ denotes the spectrum. A transfer matrix G satisfying (2.23) is called unitary. Note that if G is unitary then G will not be automatically stable. If the transfer matrix is unitary and stable we call it inner. In general, for an operator from $\mathcal{L}_2^m$ to $\mathcal{L}_2^p$, in literature two concepts, inner and co-inner, are defined which coincide if $m = p$. Since we only need the latter case of $p = m$, we shall not make this distinction.

We now formulate a result from [Gl].

Lemma 2.10 : *Assume that we have a system Σ_{ci} described by (2.10) where u and z take values in the same vector space $\mathcal{R}^k$ and where A is asymptotically stable. The system Σ_{ci} is inner if a matrix X exists satisfying:*

  (i) $A^{\mathrm{T}}X + XA + C^{\mathrm{T}}C = 0,$

  (ii) $D^{\mathrm{T}}C + B^{\mathrm{T}}X = 0,$

  (iii) $D^{\mathrm{T}}D = I.$ □

Remarks :

  (i) If (A, B) is controllable the reverse of the above implication is also true. However, in general, the reverse does not hold. A simple counter example is given by $\Sigma_{ci} := (-1, 0, 1, 1)$ which is inner but for which (ii) does not hold for any choice of X.

  (ii) Note that since A is asymptotically stable, the (unique) matrix X satisfying part (i) of lemma 2.10 is equal to the observability gramian of (C, A). We know, for instance, that $X > 0$ if, and only if, (C, A) is observable. In general we only have $X \geq 0$.

Inner systems are often used in H_∞ control. Lemma 2.12 is one of the main reasons why this class of systems is interesting. However, we first give a preliminary lemma needed to prove the lemma in which we are mainly interested.

Lemma 2.11 : *Let $K \in \mathcal{R}^{n \times n}(s)$ be such that*

  • $K \in \mathcal{L}_\infty$ *and* $\|K\|_\infty < 1,$

  • $(I - K)^{-1} \in H_\infty .$

Then we have $K \in H_\infty .$ □

Proof : We know that $\|K(s)\| < 1$ for all s on the imaginary axis. Therefore $\det(I - \alpha K(s)) \neq 0$ for all $s \in \mathcal{C}^0$ and for all $\alpha \in [0,1]$. The Nyquist contour of $\det I - \alpha K$, which is defined as the contour $\{\det(I - \alpha K(s)) \,|\, s \in \mathcal{C}^0 \}$, therefore has the same winding number around zero for all $\alpha \in [0,1]$. Since the winding number for $\alpha = 0$ is zero we find that the winding number for $\alpha = 1$ is also zero. However plus or minus the winding number is equal to the number of poles minus the number of zeros of $(I - K)^{-1}$ in the right half plane (depending on the direction one follows on the contour, see [Ru]). Since $I - K$ has no zeros in the right half plane (its inverse is in H_∞) it does not have poles in the right half plane either and therefore its inverse $I - K$ is stable. ∎

We now give the result which shows the importance of inner systems to H_∞ control. The original result was obtained in [Re]. Our adapted version however stems from [Do5]:

Lemma 2.12 : *Suppose that two systems Σ_1 and Σ_2, both described by some state space representation, are interconnected in the following way:*

$$(2.24)$$

Assume that Σ_1 is inner and that its transfer matrix G has the following decomposition:

$$G \begin{pmatrix} w \\ u \end{pmatrix} =: \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} z \\ y \end{pmatrix} \qquad (2.25)$$

which is compatible with the sizes of w, u, z and y, such that $G_{21}^{-1} \in H_\infty$ and G_{22} is strictly proper.

   Under the above assumptions the following two statements are equivalent:

   (i) The closed-loop system (2.24) is internally stable and its closed-loop transfer matrix has H_∞ norm less than 1.

   (ii) The system Σ_2 is internally stable and its transfer matrix has H_∞ norm less than 1. □

Proof : Denote the transfer matrix of Σ_2 by G_2 and denote the closed-loop transfer matrix of the interconnection (2.24) by G_{cl}. By $\mathcal{G}$ with some index we denote the input–output operator associated with the transfer matrix with the same index.

(i) $\Rightarrow$ (ii): Note that if the closed-loop system (2.24) is internally stable, then Σ_2 is stabilizable and detectable. We have

$$G_{cl} = G_{11} + G_{12}G_2 \left(I - G_{22}G_2\right)^{-1} G_{21}.$$

Note that $I - G_{22}G_2$ is invertible as a rational matrix since G_{22} is strictly proper. Assume that G_2 has a pole on the imaginary axis or has $\mathcal{L}_\infty$ norm larger than or equal to one. Then, an s_0 on the imaginary axis exists such that s_0 is not a pole of G_2 and s_0 is such that there exists an $0 \neq y \in \mathcal{C}^q$ for which $\|G_2(s_0)y\| \geq \|y\|$ (in the case that we have to choose $s_0 = \infty$ we simply replace all transfer matrices evaluated in s_0 by their respective direct-feedthrough matrices). Define

$$
\begin{aligned}
u &= G_2(s_0)y, \\
w &= G_{21}^{-1}(s_0)(I - G_{22}(s_0)G_2(s_0))y, \\
z &= G_{11}(s_0)w + G_{12}(s_0)u.
\end{aligned}
$$

Note that

$$\begin{pmatrix} z \\ y \end{pmatrix} = G(s_0) \begin{pmatrix} w \\ u \end{pmatrix}.$$

Hence, since $G(s_0)$ is a unitary matrix, we have $\|z\|^2 + \|y\|^2 = \|w\|^2 + \|u\|^2$. We already know that $\|y\| \leq \|u\|$ which implies $\|z\| \geq \|w\|$. However, $z = G_{cl}(s_0)w$ and since $\|G_{cl}\|_\infty < 1$ this yields a contradiction. Hence G_2 is in $\mathcal{L}_\infty$ and has $\mathcal{L}_\infty$ norm strictly less than 1. The closed-loop system is internally stable which implies that $(I - G_{22}G_2)^{-1}$ and $G_2(I - G_{22}G_2)^{-1}$ are in H_∞. Moreover $G_{22}G_2$ is in $\mathcal{L}_\infty$ and has $\mathcal{L}_\infty$ norm strictly less than 1. By applying lemma 2.11 this implies that $I - G_{22}G_2$ is in H_∞. Using that $G_2(I - G_{22}G_2)^{-1}$ is in H_∞ we find that G_2 is in H_∞ and hence stable. Combined with the detectability and stabilizability of the realization of Σ_2 this yields the desired result.

(ii) $\Rightarrow$ (i) First note that, since both Σ_1 and Σ_2 are internally stable, the interconnection is internally stable if, and only if, $I - G_{22}G_2$ has an inverse in H_∞. We have $\|G_2\|_\infty < 1$ and G_{22}, as a submatrix of a unitary matrix, satisfies $\|G_{22}\|_\infty \leq 1$. Hence, using a small gain argument, it can be shown that $I - G_{22}G_2$ has an inverse in H_∞.

That $\mathcal{G}_{cl}$ has H_∞ norm less than 1 remains to be proved. Since $\mathcal{G}_{21}$ is invertible over H_∞ there exists $\mu > 0$ such that

$$\mu\|\mathcal{G}_{21}^{-1}\left(I - \mathcal{G}_{22}\mathcal{G}_2\right)\|_\infty < 1.$$

Let $w \in \mathcal{L}_2^l$ be given. We have

$$y = (I - \mathcal{G}_{22}\mathcal{G}_2)^{-1} \mathcal{G}_{21}w$$

and thus we get $\mu\|w\|_2 \leq \|y\|_2$. Let $u = G_2y$. Since $\mathcal{G}$ in unitary and $\|\mathcal{G}_2\|_\infty < 1$ we find

$$
\begin{aligned}
\|z\|_2^2 &= \|w\|_2^2 + \|u\|_2^2 - \|y\|_2^2 \\
&\leq \|w\|_2^2 + \left(\|\mathcal{G}_2\|_\infty^2 - 1\right)\|y\|_2^2 \\
&\leq \|w\|_2^2 + \mu^2\left(\|\mathcal{G}_2\|_\infty^2 - 1\right)\|w\|_2^2 \\
&= \left(1 + \mu^2\left(\|\mathcal{G}_2\|_\infty^2 - 1\right)\right)\|w\|_2^2.
\end{aligned}
$$

Because $1 + \mu^2(\|\mathcal{G}_2\|_\infty^2 - 1) < 1$ we find that $\|\mathcal{G}_{cl}\|_\infty < 1$. $\blacksquare$

Note that the above theorem can be extended to time-varying systems (see [RNK]).

2.5.2 Discrete time

In chapters 9 and 10 we shall discuss systems with discrete time. In this subsection we shall repeat the definitions and results from the previous subsection but adapted to the discrete time case.

For the discrete time case we define the function space H_∞^1 as the set of all functions f which are analytic *outside* the closed unit disc and which satisfy

$$\|f\|_\infty := \sup_{z \in \mathcal{D}^+} |f(z)| < \infty.$$

This function space is a Banach space with respect to the norm $\|\cdot\|_\infty$. It can easily be seen that the set of rational functions with all poles inside the open unit disc is contained in H_∞^1. By H_∞ we denote all matrices with coefficients in H_∞^1. On this space we define the following norm:

$$\|G\|_\infty := \sup_{z \in \mathcal{D}^+} \sigma_1\left[G(z)\right].$$

Note that any subset of H_∞ of matrices with the same dimensions is a Banach space with respect to this norm. This norm will be referred to as the H_∞ norm. As in the continuous time case for rational matrices G we have the following equality:

$$\|G\|_\infty = \sup_{z \in \delta\mathcal{D}} \sigma_1\left[G(z)\right].$$

Note that we use the same notation as in the continuous time case although in discrete time the role of the right half plane is replaced by the complement of the closed unit disc. Since it will always be clear from the context whether we have discrete time or continuous time this will cause no confusion.

Define ℓ_2^n as the set of all Lebesgue measurable functions f from $\mathcal{N}$ to $\mathcal{R}^n$ for which

$$\|f\|_2 := \left(\sum_{k=0}^{\infty} \|f(k)\|^2 \right)^{1/2} < \infty.$$

With respect to the norm $\|\cdot\|_2$ it can be shown that ℓ_2^n is a Banach space. It is even a Hilbert space since this ℓ_2-norm is induced by the following inner product:

$$< f, g >_2 := \sum_{k=0}^{\infty} < f(k), g(k) >,$$

where $< \cdot, \cdot >$ denotes the standard Euclidian inner product. By ℓ_2 we denote the set of all f for which an n exists such that $f \in \ell_2^n$.

As in the continuous time case we shall use a time-domain characterization of the H_∞ norm. Let $\Sigma \times \Sigma_F$ be the closed-loop system when we apply a controller Σ_F to the system Σ. If the closed-loop system is internally stable (as a discrete time system!), then the closed-loop transfer matrix G_F is in H_∞. Denote the closed-loop operator mapping w to z with zero initial conditions by $\mathcal{G}_F$. The H_∞ norm is equal to the ℓ_2-induced operator norm of the closed-loop operator, i.e.

$$\|G_F\|_\infty = \|\mathcal{G}_F\|_\infty := \sup_w \left\{ \frac{\|\mathcal{G}_F w\|_2}{\|w\|_2} \;\middle|\; w \in \ell_2^l, \; w \neq 0 \right\}. \quad (2.26)$$

Because of the above equality, we often refer to the ℓ_2- induced operator norm of $\mathcal{G}_F$ as the H_∞ norm of $\mathcal{G}_F$ (like the continuous time case).

As in the continuous time case a system (A, B, C, D) is called inner if it is internally stable and the input–output operator is a unitary operator from ℓ_2^k into itself. The input–output operator is unitary if, and only if, the transfer matrix of the system, denoted by G, satisfies:

$$G^{\mathrm{T}}(z^{-1})G(z) = G(z)G^{\mathrm{T}}(z^{-1}) = I. \quad (2.27)$$

We now formulate a result from [St6] which itself is a generalization of a result from [Gu]. A proof can be given by simply writing out (2.27).

Lemma 2.13 : *Assume that we have a system*

$$\Sigma_{ci} : \begin{cases} \sigma x = Ax + Bu, \\ z = Cx + Du, \end{cases}$$

where A is asymptotically stable and where u and z both take values in the same vector space $\mathcal{R}^m$. The system Σ_{ci} is inner if a matrix X exists satisfying:

(i) $X = A^{\mathrm{T}} X A + C^{\mathrm{T}} C,$

(ii) $D^{\mathrm{T}} C + B^{\mathrm{T}} X A = 0,$

(iii) $D^{\mathrm{T}} D + B^{\mathrm{T}} X B = I.$ □

Remark : As in the continuous time case it can be shown that the reverse implication holds if (A, B) is controllable. A counter example for the general case is this time given by $\Sigma_{ci} := (0.5, 0, 1, 1)$.

We shall now rephrase lemma 2.12 for discrete time systems. We shall not prove this result since the proof of lemma 2.12 needs only minor adjustments to yield a proof of our discrete time version.

Lemma 2.14 : *Suppose that two systems Σ_1 and Σ_2, both described by some state space representation, are interconnected in the following way:*

                                    (2.28)

Assume that Σ_1 is inner. Moreover, assume that if we decompose the transfer matrix G_1 of Σ_1:

$$G_1 \begin{pmatrix} w \\ u \end{pmatrix} =: \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} z \\ y \end{pmatrix} \qquad (2.29)$$

compatible with the sizes of w, u, z and y, then we have $G_{21}^{-1} \in H_\infty$ and G_{22} is strictly proper.
The following two statements are equivalent:

(*i*) *The closed-loop system (2.28) is internally stable and its closed-loop transfer matrix has H_∞ norm less than 1.*

(*ii*) *The system Σ_2 is internally stable and its transfer matrix has H_∞ norm less than 1.* □

2.6 (Almost) disturbance decoupling problems

In this section we shall discuss the problems of (almost) disturbance decoupling for continuous time systems. The disturbance decoupling problems for discrete time systems are defined in a similar way. We discuss both the case of state feedback as well as the case of dynamic measurement feedback. In this book our central objective is minimizing the H_∞ norm of the closed-loop system over all internally stabilizing controllers. This section discusses the special case that this infimum is 0 (almost disturbance decoupling) and under which conditions the infimum is attained (disturbance decoupling). Several people contributed to this area. For the problem of disturbance decoupling excellent references are [SH, Wo]. For the problem of almost disturbance decoupling prime references are [OSS1, OSS2, Tr, WW, Wi5, Wi6].

We start by defining the several problems and after that we state several known solutions to the respective problems and some extensions. These extensions are concerned with the inclusion of direct-feedthrough matrices which were in general excluded in the literature. We shall mainly present results we need and not the most general results available.

Definition 2.15 : *Consider the system*

$$\Sigma : \begin{cases} \dot{x} = Ax + \ Bu + Ew, \\ z = Cx + Du. \end{cases} \tag{2.30}$$

We say that the Disturbance Decoupling Problem with internal Stability (denoted by DDPS) is solvable if a state feedback $u = Fx$ exists for Σ such that the closed-loop system Σ_{cl} is internally stable, i.e. $A + BF$ is asymptotically stable, and such that Σ_{cl} has a transfer matrix which is equal to 0.

We say that the Almost Disturbance Decoupling Problem with internal Stability(ADDPS) is solvable if for all $\varepsilon > 0$ a state feedback $u = Fx$ exists for Σ such that the closed-loop system Σ_{cl} is internally stable and Σ_{cl} has H_∞ norm less than ε. □

We also define related problems for the case of dynamic measurement feedback:

Definition 2.16 : *Consider the system (2.1). We say that the Disturbance Decoupling Problem with Measurement feedback and internal Stability (DDPMS) is solvable if a controller Σ_F of the form (2.4) exists such that the closed-loop system $\Sigma \times \Sigma_F$ is well-posed, internally stable and has a transfer matrix which is equal to 0.*

We say that the Almost Disturbance Decoupling Problem with Measurement feedback and internal Stability (ADDPMS) is solvable if for all $\varepsilon > 0$ a controller Σ_F of the form (2.4) exists such that the closed-loop system $\Sigma \times \Sigma_F$ is well-posed, internally stable and has H_∞ norm less than ε. □

It was shown in [Tr, Wo, Wi5] that necessary and sufficient conditions for the solvability of DDPS and ADDPS can be stated in terms of the strongly controllable subspace, $\mathcal{T}(\Sigma_{ci})$, and the stabilizable weakly unobservable subspace, $\mathcal{V}_g(\Sigma_{ci})$, associated with the system $\Sigma_{ci} = (A, B, C, D)$. (In [Tr, Wi5] the subspace $\mathcal{T}(\Sigma_{ci})$ is denoted by $\mathcal{R}_b^*(\ker C)$ while D is assumed to be equal to 0.) The exact result is as follows:

Theorem 2.17 : *Consider the system (2.30) with zero initial condition. Let $\mathcal{T}(\Sigma_{ci})$ and $\mathcal{V}_g(\Sigma_{ci})$ denote the strongly controllable subspace and the stabilizable weakly unobservable subspace, respectively, associated with the system $\Sigma_{ci} = (A, B, C, D)$.*

(i) The disturbance decoupling problem is solvable if, and only if,

$$\mathrm{Im} E \subseteq \mathcal{V}_g(\Sigma_{ci}).$$

(ii) Assume that Σ_{ci} is such that

$$\mathcal{V}_g(\Sigma_{ci}) + \mathcal{T}(\Sigma_{ci}) = \mathcal{R}^n.$$

In this case a sequence of matrices F_n exists such that the controllers $\Sigma_{F,n}$ of the form $u = F_n x$, when applied to Σ, yield closed-loop systems which are internally stable for all n and such that the closed-loop transfer matrices $G_{cl,n}$ are strictly proper and satisfy:

$$\|G_{cl,n}\|_\infty \;\to\; 0 \qquad as\ n \to \infty \qquad\qquad (2.31)$$

$$\|G_{cl,n}\|_2 \;\to\; 0 \qquad as\ n \to \infty \qquad\qquad (2.32)$$

where $\|\cdot\|_2$ denotes the H_2 norm. In particular the almost disturbance decoupling problem with internal stability is solvable. □

Proof : The first part of this theorem is proven in [St11]. Therefore, we shall only prove the second part of this theorem since this is the result we shall explicitly use in this book. It is an extension of [OSS2] in the sense of including (2.32). On the other hand, it is an extension of [Tr] since it includes direct-feedthrough matrices.

By definition 2.4 we know a mapping $\tilde{F}$ exists such that

$$(A + B\tilde{F})\mathcal{V}_g(\Sigma_{ci}) \quad \subset \quad \mathcal{V}_g(\Sigma_{ci}) \tag{2.33}$$

$$(C_2 + D_2\tilde{F})\mathcal{V}_g(\Sigma_{ci}) \quad = \quad \{0\} \tag{2.34}$$

and such that $A + B\tilde{F} \,|\, \mathcal{V}_g(\Sigma_{ci})$ is asymptotically stable. Let Π denote the canonical projection $\mathcal{R}^n \to \mathcal{R}^n/\mathcal{V}_g(\Sigma)$. By (2.33) and (2.34) linear mappings $\bar{A}$, $\bar{B}$ and $\bar{C}$ exist such that

$$\bar{A}\Pi \quad = \quad \Pi(A + B\tilde{F}), \tag{2.35}$$

$$\bar{B} \quad = \quad \Pi B, \tag{2.36}$$

$$\bar{C}\Pi \quad = \quad C_2 + D_2\tilde{F}. \tag{2.37}$$

We then define the system:

$$\Sigma_{fs} : \left\{ \begin{array}{l} \dot{p} = \bar{A}p + \bar{B}u, \\ z = \bar{C}p + D_2u. \end{array} \right. \tag{2.38}$$

It can easily be shown by induction using algorithm 2.5 that for $i = 0, 1, \ldots$ we have $\mathcal{T}_i(\Sigma_{fs}) = \Pi\mathcal{T}_i(\Sigma_{ci})$. Hence we have:

$$\mathcal{T}(\Sigma_{fs}) = \Pi\mathcal{T}(\Sigma_{ci}) = \Pi\{\mathcal{T}(\Sigma_{ci}) + \mathcal{V}_g(\Sigma_{ci})\} = \Pi\mathcal{R}^n = \mathcal{R}^n/\mathcal{V}_g(\Sigma_{ci}).$$

This implies that the system (2.38) is strongly controllable.

Let F_0 and M be such that

$$(\bar{C} + D_2F_0)^{\mathrm{T}} D_2 = 0, \tag{2.39}$$

$$\ker D_2 = \operatorname{im} M. \tag{2.40}$$

Using lemma 2.7, it is straightforward to check that

$$\mathcal{T}(\Sigma_{fs}) = \mathcal{T}(\bar{A} + \bar{B}F_0, \bar{B}M, \bar{C} + D_2F_0, 0).$$

Hence by [Tr, Theorem 3.25] we know that for all $\varepsilon > 0$ an $\bar{F}$ exists such that:

$$\| (\bar{C} + D_2F_0) \, e^{((\bar{A}+\bar{B}F_0)+\bar{B}M\bar{F})t} \,\|_1 \quad < \quad \varepsilon \tag{2.41}$$

$$\| (\bar{C} + D_2F_0) \, e^{((\bar{A}+\bar{B}F_0)+\bar{B}M\bar{F})t} \,\|_2 \quad < \quad \varepsilon \tag{2.42}$$

and such that $\bar{A} + \bar{B}F_0 + \bar{B}M\bar{F}$ is asymptotically stable. Here $\|\cdot\|_1$ denotes the $\mathcal{L}_1$ norm which is defined by

$$\|M\|_1 := \int_0^\infty \|M(t)\| \, dt.$$

Define $F := \tilde{F} + (F_0 + M\bar{F})\Pi$ then

$$
\begin{aligned}
A + BF \,|\, \mathcal{V}_g(\Sigma_{ci}) &= A + B\tilde{F} \,|\, \mathcal{V}_g(\Sigma_{ci}), \\
\Pi(A + BF) &= (\bar{A} + \bar{B}F_0 + \bar{B}M\bar{F})\,\Pi.
\end{aligned}
$$

It is easy to show that this implies that $A + BF$ is asymptotically stable. Moreover we have:

$$(C_2 + D_2 F)\, e^{(A+BF)t} = (\bar{C} + D_2 F_0)\, e^{((\bar{A}+\bar{B}F_0)+\bar{B}M\bar{F})t}\Pi \qquad (2.43)$$

for all $t \geq 0$. Using (2.43) we find for all $s \in i\mathcal{R}$ (Use that $|\,e^{st}\,| = 1$):

$$
\begin{aligned}
\|\,(C_2 + D_2 F)\,(sI - (A+BF))^{-1}\,\| \\
&= \|\int_0^\infty (C_2 + D_2 F)\, e^{((A+BF)-sI)t}\, dt\| \\
&\leq \int_0^\infty \|\,(C_2 + D_2 F)\, e^{((A+BF)-sI)t}\|\, dt \\
&= \|\,(C_2 + D_2 F)\, e^{(A+BF)t}\|_1 \\
&= \|\,(\bar{C} + D_2 F_0)\, e^{((\bar{A}+\bar{B}F_0)+\bar{B}M\bar{F})t}\Pi\|_1 \\
&\leq \varepsilon.
\end{aligned}
$$

This implies that the closed-loop transfer matrix G_{cl} has H_∞ norm less than ε. By applying Parseval's theorem (see [Ru]) it can be shown that (2.42) implies that G_{cl} also has H_2 norm less than ε.

Since ε was arbitrary, this implies the existence of the desired sequence F_n. ∎

As an immediate consequence of the above we obtain the following fact: if $\Sigma_{ci} = (A, B, C, D)$ is strongly controllable, then for all $\varepsilon > 0$ a static state feedback $u = Fx$ exists such that the closed-loop system has H_∞ norm less than ε and such that $A + BF$ is asymptotically stable.

In chapter 6 we shall need a different version of theorem 2.17 which includes initial states:

Lemma 2.18 : *Assume that the system $(A, B, C, 0)$ is strongly control-lable. Then for all bounded sets $\mathcal{V} \subset \mathcal{R}^n$ and all $\varepsilon > 0$ there exists $F \in \mathcal{R}^{m \times n}$ such that*

(i) $A + BF$ is asymptotically stable.

(ii) For all $w \in \mathcal{L}_2^l$ and all $\xi \in \mathcal{V}$ we have $\|z\|_2 \le \varepsilon\,(\|w\|_2 + 1)$, where z is given by

$$\dot{x} = (A + BF)\,x + Ew, \qquad x(0) = \xi \in \mathcal{V},$$
$$z = Cx. \hspace{5cm} \square$$

Proof : Let M_1 be such that for all $\xi \in \mathcal{V}$ we have $\|\xi\| < M_1$. Apply theorem 2.17 to (2.30) with $E = I$ and $D = 0$. We find that for all $\varepsilon > 0$ a matrix F exists such that $A + BF$ is asymptotically stable and

$$\|C(sI - A - BF)^{-1})\|_2 \le \frac{\varepsilon}{M_1}, \quad \|C(sI - A - BF)^{-1}\|_\infty \le \frac{\varepsilon}{\|E\| + 1}.$$

The last inequality clearly implies:

$$\|C\,(sI - A - BF)^{-1}\,E\|_\infty < \varepsilon.$$

Using the above it can be shown straightforwardly that F satisfies (ii) for this choice of ε and $\mathcal{V}$. $\blacksquare$

We shall now discuss the (almost) disturbance decoupling problems with measurement feedback and internal stability. For the disturbance decoupling problem with measurement feedback and internal stability we have the following result from [SH, St11] available:

Theorem 2.19 : *Let the following system Σ be given:*

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew, \\ y = C_1 x + D_1 w, \\ z = C_2 x + D_2 u. \end{cases} \tag{2.44}$$

The following conditions are equivalent:

(i) (A, B) *is stabilizable,* (C_2, A) *is detectable and*

    (a) $\mathrm{Im}\,E \subseteq \mathcal{V}_g(\Sigma_{ci}) + B\,\mathrm{Ker}\,D_2$,

    (b) $\mathrm{Ker}\,C_2 \supseteq \mathcal{T}_g(\Sigma_{di}) \cap C_1^{-1}\,\mathrm{Im}\,D_1$,

    (c) $\mathcal{T}_g(\Sigma_{di}) \subseteq \mathcal{V}_g(\Sigma_{ci})$.

    where $\Sigma_{ci} = (A, B, C_2, D_2)$ *and* $\Sigma_{di} = (A, E, C_1, D_1)$.

(*ii*) *The disturbance decoupling problem with measurement feedback and internal stability is solvable, i.e. a controller of the form (2.4) for Σ exists such that the closed-loop system is internally stable and has a transfer matrix which is equal to 0.*   □

For ADDPMS we shall only present sufficient conditions for solvability. These conditions are all we need in this book. General necessary and sufficient conditions for solvability are, as far as we know, not available.

Theorem 2.20 : *Let the system (2.44) be given. Assume that Σ satisfies the following two rank conditions:*

$$rank \begin{pmatrix} sI - A & -B \\ C_2 & D_2 \end{pmatrix} = n + rank \begin{pmatrix} C_2 & D_2 \end{pmatrix} \quad \forall \, s \in \mathcal{C}^0 \cup \mathcal{C}^+, \quad (2.45)$$

and

$$rank \begin{pmatrix} sI - A & -E \\ C_1 & D_1 \end{pmatrix} = n + rank \begin{pmatrix} E \\ D_1 \end{pmatrix} \quad \forall \, s \in \mathcal{C}^0 \cup \mathcal{C}^+. \quad (2.46)$$

Under the above assumptions a sequence of controllers $\Sigma_{F,n}$ of the form (2.4) exists such that the closed-loop systems are internally stable for all n and the closed-loop transfer matrices $G_{cl,n}$ are strictly proper and satisfy:

$$\|G_{cl,n}\|_\infty \;\to\; 0 \qquad \text{as } n \to \infty \tag{2.47}$$

$$\|G_{cl,n}\|_2 \;\to\; 0 \qquad \text{as } n \to \infty \tag{2.48}$$

In particular the almost disturbance decoupling problem with measurement feedback and internal stability is solvable.   □

Remark: in fact in [St13] it is shown that we can find a sequence of controllers of McMillan degree $n - \text{rank} \begin{pmatrix} C_1 & D_1 \end{pmatrix} + \text{rank} \, D_1$ satisfying (2.47) and (2.48). In other words, if we observe k states directly we only have to build an observer for $n - k$ states which then gives us a controller of McMillan degree $n - k$. In our proof we shall restrict attention to controllers of McMillan degree n.

Before we can prove this result we have to do some preparatory work. A system (A, B, C_2, D_2) satisfying (2.45) has all its invariant zeros in $\mathcal{C}^-$ and is right-invertible. The property that a system has all its invariant

zeros in the open left half complex plane is called minimum-phase. In the same way a system (A, E, C_1, D_1) satisfying (2.46) is left-invertible and minimum-phase. We can express the rank conditions (2.45) and (2.46) in terms of the subspaces introduced in section 2.4 (see [Fr, SH3]):

Lemma 2.21 : *The rank condition (2.45) is satisfied if, and only if,*

$$\mathcal{V}_g(\Sigma_{ci}) + \mathcal{T}(\Sigma_{ci}) = \mathcal{R}^n,$$

where $\Sigma_{ci} = (A, B, C_2, D_2)$. *The rank condition (2.46) is satisfied if, and only if,*

$$\mathcal{V}(\Sigma_{di}) \cap \mathcal{T}_g(\Sigma_{di}) = \{0\},$$

where $\Sigma_{di} = (A, E, C_1, D_1)$. $\qquad\qquad\square$

Proof of theorem 2.20 : Let $\varepsilon > 0$. We first choose a mapping F such that:

$$\| \, (C_2 + D_2 F)\, (sI - A - BF)^{-1} \, \|_\infty < \frac{\varepsilon}{3\|E\| + 1}, \qquad (2.49)$$

$$\| \, (C_2 + D_2 F)\, (sI - A - BF)^{-1} \, \|_2 < \frac{\varepsilon}{3\|E\| + 1}, \qquad (2.50)$$

and such that $A + BF$ is asymptotically stable. This can be done according to lemmas 2.17 and 2.21. Next choose a mapping G such that:

$$\| \, (sI - A - GC_1)^{-1}\, (E + GD_1) \, \|_\infty$$
$$< \quad \min\left\{ \frac{\varepsilon}{3\|D_2 F\| + 1} \, , \, \frac{\|E\|}{\|BF\| + 1} \right\}, \quad (2.51)$$

$$\| \, (sI - A - GC_1)^{-1}\, (E + GD_1) \, \|_2 \quad < \frac{\|E\|}{\|BF\| + 1}, \qquad (2.52)$$

and such that $A + GC_1$ is asymptotically stable. The dual version of theorem 2.17 guarantees the existence of such a G. We apply the following feedback compensator to the system (2.44):

$$\Sigma_{F,G} : \begin{cases} \dot{p} = Ap \ + Bu + G\,(C_1 p - y), \\ u = Fp. \end{cases} \qquad (2.53)$$

The closed-loop system is given by (where e:=x-p):

$$\Sigma_{cl} : \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A + BF & -BF \\ 0 & A + GC_1 \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} E \\ E + GD_1 \end{pmatrix} w, \\ \\ z \ = \begin{pmatrix} C_2 + D_2 F & -D_2 F \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}. \end{cases}$$

It is clear that this is an internally stabilizing feedback. The transfer matrix from w to z can be shown to be equal to:

$$\left(C_2 + D_2 F\right)\left(sI - A - BF\right)^{-1} E$$

$$- \left(C_2 + D_2 F\right)\left(sI - A - BF\right)^{-1} BF \left(sI - A - GC_1\right)^{-1} \left(E + GD_1\right)$$

$$- D_2 F \left(sI - A - GC_1\right)^{-1} \left(E + GD_1\right).$$

Using (2.49), (2.50), (2.51) and (2.52) it can be easily shown that this closed-loop transfer matrix has H_∞ and H_2 norm less than ε. ∎

We conclude this section by noting that we can extend our results to the more general system (2.1) by some standard loop shifting arguments as discussed in sections 4.6 and 5.6. However, we should be careful. As discussed in example 4.4, we should make a careful distinction between proper and non-proper controllers.

Chapter 3

The regular full-information H_∞ control problem

3.1 Introduction

Both in this chapter and in the next one, we discuss the H_∞ control problem where all states are available for feedback. We also discuss the alternative where all states <u>and</u> the disturbance are available for feedback.

The state feedback H_∞ control problem was the first to be solved using time-domain techniques. The main ideas stem from the stabilization of uncertain systems where one of the problems is to find a controller which maximizes the complex stability radius of some system with structured uncertainty (see [Kh5, Pe, Pe2, Pe4, RK2, ZK]). This is closely related to the H_∞ control problem since this problem can be reduced to the problem of finding a controller which minimizes the H_∞ norm of some related system. Later these ideas were applied directly to H_∞ control theory (see [Kh2, Kh3, Pe3, Pe5, ZK2]). The main result of these papers is the following: there exists an internally stabilizing state feedback which makes the H_∞ norm less than some a priori given bound $\gamma > 0$ if, and only if, a positive definite matrix P and a positive constant ε exist such that P is a stabilizing solution of an algebraic Riccati equation parametrized by ε. A main drawback of this result is that the Riccati equation is parametrized. However, under certain assumptions it turned out that this parameter could be removed (see [Do5, St, Ta]). The assumptions are twofold. Firstly the direct-feedthrough matrix from control input to output should be injective. Secondly a certain subsystem should have no invariant zeros on the imaginary axis. In this chapter we shall give necessary and sufficient conditions for the existence of a suitable controller under these assumptions. In the next chapter we shall then remove the first assumption.

We shall generalize the results in [Do5, St, Ta] since we shall not assume that the direct-feedthrough matrix from control input to output is zero. Moreover we do not need any extra assumptions besides the two assumptions mentioned above.

It will be shown that a "suitable" controller (i.e. an internally stabilizing controller such that the closed-loop system has H_∞ norm less than some a priori given bound $\gamma > 0$) exists if, and only if, the following condition is satisfied: a positive semi-definite stabilizing solution of a certain algebraic Riccati equation exists and, moreover, a given matrix, explicitly specified in terms of the system parameters, is positive definite. This Riccati equation has an indefinite quadratic term. Riccati equations of this type first appeared in the theory of differential games (see [Ban, Ma, We]) and have different properties from the Riccati equation used in linear quadratic control and Kalman filtering which all have a definite quadratic term (see, e.g. [Wi]). Our proof will have a strong relation to differential games. Differential games are discussed in more detail in chapter 6.

We shall show in the present chapter that, if an arbitrary "suitable" compensator exists, then a "suitable" static feedback also exists. In the next chapter we shall show that this property is not necessarily true if we remove our assumption on the direct-feedthrough matrix. We shall give an explicit formula for one "suitable" static feedback. It will be shown that if we only allow for static <u>state</u> feedback, then we have to make an extra assumption to guarantee existence of a suitable controller.

The main ideas for this chapter stem from [PM, Ta].

The outline of this chapter is the following. In section 3.2 we shall give the problem formulation and our main results. In section 3.3 we give an intuitive proof of the necessity part. This is done because the formal proof is rather technical so it appeared to be a good idea to first explain how we obtained the necessary intuition for the formal proof. In section 3.4 we shall then prove necessity in a formal way and in section 3.5 we shall prove sufficiency.

3.2   Problem formulation and main results

We consider the linear, time-invariant, finite-dimensional system:

$$\Sigma : \begin{cases} \dot{x} = Ax + \ Bu + \ Ew, \\ z = Cx + D_1u + D_2w, \end{cases} \tag{3.1}$$

where for each t we have that $x(t) \in \mathcal{R}^n$ is the state, $u(t) \in \mathcal{R}^m$ the control input, $w(t) \in \mathcal{R}^l$ the disturbance and $z(t) \in \mathcal{R}^q$ the output to be

controlled. A, B, E, C, D_1 and D_2 are matrices of appropriate dimensions. We want to minimize the effect of the disturbance w on the output z by finding an appropriate control input u. More precisely, we seek a compensator Σ_F described by a static feedback law $u = F_1 x + F_2 w$ such that after applying this feedback law to the system (3.1), the resulting closed-loop system $\Sigma \times \Sigma_F$ is internally stable and its transfer matrix, denoted by G_F, has minimal H_∞ norm.

Although minimizing the H_∞ norm is always our ultimate goal, in this chapter as well as in the rest of this book, we shall only derive necessary and sufficient conditions under which we can find an internally stabilizing compensator which makes the resulting H_∞ norm of the closed-loop system strictly less than some a priori given bound γ. In principle one can then obtain the infimum of the closed-loop H_∞ norm over all internally stabilizing compensators via a search procedure (a simple binary search procedure is straightforward, for more advanced, quadratically convergent, algorithms see [S, S2]).

We are now in the position to formulate our main result.

Theorem 3.1 : *Consider the system (3.1) and let $\gamma > 0$. Assume that the system (A, B, C, D_1) has no invariant zeros on the imaginary axis and D_1 is injective. Then the following three statements are equivalent:*

(i) *A static feedback law $u = F_1 x + F_2 w$ exists such that after applying this compensator to the system (3.1) the resulting closed-loop system is internally stable and the closed-loop operator $\mathcal{G}_F$ has H_∞ norm less than γ, i.e. $\|\mathcal{G}_F\|_\infty < \gamma$.*

(ii) *(A, B) is stabilizable and a $\delta < \gamma$ exists such that for all $w \in \mathcal{L}_2^l$ an $u \in \mathcal{L}_2^m$ exists such that $x_{u,w} \in \mathcal{L}_2^n$ and $\|z_{u,w}\|_2 \leq \delta \|w\|_2$.*

(iii) *We have*

$$D_2^{\mathrm{T}} \left(I - D_1 \left(D_1^{\mathrm{T}} D_1 \right)^{-1} D_1^{\mathrm{T}} \right) D_2 < \gamma^2 I. \tag{3.2}$$

Moreover, there is a positive semi-definite solution P of the algebraic Riccati equation

$$0 = A^{\mathrm{T}} P + PA + C^{\mathrm{T}} C$$
$$- \begin{pmatrix} B^{\mathrm{T}} P + D_1^{\mathrm{T}} C \\ E^{\mathrm{T}} P + D_2^{\mathrm{T}} C \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} D_1^{\mathrm{T}} D_1 & D_1^{\mathrm{T}} D_2 \\ D_2^{\mathrm{T}} D_1 & D_2^{\mathrm{T}} D_2 - \gamma^2 I \end{pmatrix}^{-1} \begin{pmatrix} B^{\mathrm{T}} P + D_1^{\mathrm{T}} C \\ E^{\mathrm{T}} P + D_2^{\mathrm{T}} C \end{pmatrix}$$

such that A_{cl} is asymptotically stable where:

$$A_{cl} := A - \begin{pmatrix} B & E \end{pmatrix} \begin{pmatrix} D_1^{\mathrm{T}} D_1 & D_1^{\mathrm{T}} D_2 \\ D_2^{\mathrm{T}} D_1 & D_2^{\mathrm{T}} D_2 - \gamma^2 I \end{pmatrix}^{-1} \begin{pmatrix} B^{\mathrm{T}} P + D_1^{\mathrm{T}} C \\ E^{\mathrm{T}} P + D_2^{\mathrm{T}} C \end{pmatrix}.$$

If P satisfies the conditions in part (iii), then a controller satisfying the conditions in part (i) is given by:

$$
\begin{aligned}
F_1 &:= -\left(D_1^{\mathrm{T}} D_1\right)^{-1}\left(D_1^{\mathrm{T}} C + B^{\mathrm{T}} P\right), \tag{3.3}\\
F_2 &:= -\left(D_1^{\mathrm{T}} D_1\right)^{-1} D_1^{\mathrm{T}} D_2. \tag{3.4}
\end{aligned}
$$

$\square$

Remarks :

(i) Note that our assumption that D_1 is injective together with (3.2) guarantees the existence of the inverse in the algebraic Riccati equation and in the definition of A_{cl}.

(ii) The implication (i) $\Rightarrow$ (ii) is trivial. Hence we only need to prove (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). The first step will be done in section 3.4 and in the section thereafter we shall complete the proof. We shall only prove this theorem for the case that $D_2 = 0$. The proof for the general case can be found in [St9] and is a straightforward extension although it requires some extra algebraic manipulations.

(iii) The case that either one of the assumptions made in the theorem above is not satisfied will be discussed in the next chapter. The extension to the case that D_1 is not necessarily injective is in the author's opinion completely satisfactory. However, an elegant extension of the above result to the case that invariant zeros on the imaginary axis are allowed is still an open problem.

In a large part of the literature one is searching for an internally stabilizing static <u>state</u> feedback which makes the H_∞ norm less than γ, rather than a feedback that is also allowed to depend on w. This is clearly more acceptable from a practical point of view. In practice one might be able to measure the state in some special cases but one does not apply a feedback depending on a measurement of the disturbance. From the above theorem it is immediately obvious that if $D_2 = 0$ then we can not do better by general static feedback than we can by static state feedback. The next theorem gives necessary and sufficient conditions for the existence of a suitable static state feedback.

Theorem 3.2 : *Consider the system (3.1). Let $\gamma > 0$. Assume that the system (A, B, C, D_1) has no invariant zero on the imaginary axis and D_1 is injective. Then the following statements are equivalent:*

(*i*) *A static state feedback law $u = Fx$ exists such that after applying this compensator to the system (3.1) the resulting closed-loop system is internally stable and the closed-loop operator $\mathcal{G}_F$ has H_∞ norm less than γ, i.e. $\|\mathcal{G}_F\|_\infty < \gamma$.*

(*ii*) *We have*

$$D_2^{\mathrm{T}} D_2 \; < \; \gamma^2 I.$$

Moreover, there is a positive semi-definite solution P of the algebraic Riccati equation

$$0 = A^{\mathrm{T}} P + PA + C^{\mathrm{T}} C$$

$$- \begin{pmatrix} B^{\mathrm{T}} P + D_1^{\mathrm{T}} C \\ E^{\mathrm{T}} P + D_2^{\mathrm{T}} C \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} D_1^{\mathrm{T}} D_1 & D_1^{\mathrm{T}} D_2 \\ D_2^{\mathrm{T}} D_1 & D_2^{\mathrm{T}} D_2 - \gamma^2 I \end{pmatrix}^{-1} \begin{pmatrix} B^{\mathrm{T}} P + D_1^{\mathrm{T}} C \\ E^{\mathrm{T}} P + D_2^{\mathrm{T}} C \end{pmatrix}$$

such that A_{cl} is asymptotically stable where:

$$A_{cl} := A - \begin{pmatrix} B & E \end{pmatrix} \begin{pmatrix} D_1^{\mathrm{T}} D_1 & D_1^{\mathrm{T}} D_2 \\ D_2^{\mathrm{T}} D_1 & D_2^{\mathrm{T}} D_2 - \gamma^2 I \end{pmatrix}^{-1} \begin{pmatrix} B^{\mathrm{T}} P + D_1^{\mathrm{T}} C \\ E^{\mathrm{T}} P + D_2^{\mathrm{T}} C \end{pmatrix}.$$

If P satisfies the conditions in part (ii), then a controller satisfying the conditions in part (i) is defined by:

$$F := - \left(D_1^{\mathrm{T}} \left(I - \gamma^{-2} D_2 D_2^{\mathrm{T}} \right)^{-1} D_1 \right)^{-1} \times$$

$$\left[D_1^{\mathrm{T}} C + B^{\mathrm{T}} P + D_1^{\mathrm{T}} D_2 \left(\gamma^2 I - D_2^{\mathrm{T}} D_2 \right)^{-1} \left(D_2^{\mathrm{T}} C + E^{\mathrm{T}} P \right) \right] \quad (3.5)$$

$$\square$$

Remarks :

(*i*) Note that the only difference between part (iii) of theorem 3.1 and part (ii) of this theorem is that we already have $D_2^{\mathrm{T}} D_2 < I$ instead of having the chance of obtaining this condition after a preliminary disturbance feedforward. The condition $D_2^{\mathrm{T}} D_2 < I$ is clearly necessary for the existence of a "suitable" state feedback, since if we are only allowed to apply state feedback then we cannot change D_2. The surprising part is that it is also sufficient: if $D_2^{\mathrm{T}} D_2 < I$ and if a "suitable" feedback exists, then there is also a "suitable" static state feedback.

(ii) If $D_2^{\mathrm{T}} D_1 = 0$, then the compensator suggested is the same as the compensator suggested in theorem 3.1. Hence only if $D_2^{\mathrm{T}} D_1 \neq 0$ we can possibly do better by allowing disturbance feedforward.

In this book we only prove the above results for the case that $D_2 = 0$. In that case we can restrict attention to static state feedbacks and the results of theorems 3.1 and 3.2 melt together. The general case can be proven along the same lines as we are following in this chapter. The difference being that we have fewer algebraic manipulations and some technicalities become straightforward. We feel that this allows the reader to get a better understanding of the main core of the proof.

3.3   Intuition for the formal proof

Since the approach proving the result of this chapter is crucial for this book and is used again in chapter 9 for discrete-time systems, we first explain and prove the results <u>intuitively</u>. In the next section we give a formal proof.

Clearly making the H_∞ norm as small as possible is directly related to the following "sup-inf" problem:

$$\inf_{f} \sup_{w \neq 0} \left\{ \left. \frac{\|z_{u,w}\|_2}{\|w\|_2} \ \right| \ w \in \mathcal{L}_2^l,\ f : \mathcal{L}_2^m \to \mathcal{L}_2^n \text{ causal and } u = f(w) \right\}$$

(3.6)

for the initial condition $x(0) = 0$. We know from more classical results on "inf-sup" problems that when the criterion function is not quadratic then it is nearly impossible to solve this problem explicitly. Therefore we note that if the number defined by (3.6) is smaller than some bound $\gamma > 0$ then clearly

$$\inf_{f} \sup_{w} \left\{ \|z_{u,w}\|_2^2 - \gamma^2 \|w\|_2^2 \ \middle| \ w \in \mathcal{L}_2^l, f \text{ causal and } u = f(w) \right\} \leq 0$$

(3.7)

for initial condition $x(0) = 0$. However, for H_∞ control we have an extra constraint since we require our controller to be internally stabilizing. Internal stability is a property which is related to <u>all</u> initial conditions of the system. Thus, in order to be able to build this constraint into our "inf-sup" problem, we investigate (3.7) for an arbitrary initial condition $x(0) = \xi$. We define

$$\mathcal{C}(u, w, \xi) := \|z_{u,w,\xi}\|_2^2 - \gamma^2 \|w\|_2^2$$

and we investigate

$$\bar{\mathcal{C}}^*(\xi) := \inf_f \sup_w \left\{ \mathcal{C}(u,w,\xi) \mid f \text{ causal}, u = f(w), w \in \mathcal{L}_2^l \right.$$
$$\left. \text{such that } x_{u,w,\xi} \in \mathcal{L}_2^n \right\}$$

for arbitrary initial state $x(0) = \xi$. A related problem is the following:

$$\mathcal{C}^*(\xi) := \sup_w \inf_u \left\{ \mathcal{C}(u,w,\xi) \mid u \in \mathcal{L}_2^m, w \in \mathcal{L}_2^l \text{ such that } x_{u,w,\xi} \in \mathcal{L}_2^n \right\}$$

for arbitrary initial state $x(0) = \xi$. It can be easily shown that, without the extra assumption that f should be causal, the above two problems are equivalent. The surprising fact is that even with the extra assumption of causality these problems are equal, i.e.

$$\mathcal{C}^*(\xi) = \bar{\mathcal{C}}^*(\xi) \quad \forall \xi \in \mathcal{R}^n. \tag{3.8}$$

The main difference with our original problem formulation is that we do not require causality, i.e. $u(t)$ may depend on the future of w. This was excluded in our original formulation because we only look at controllers of the form (2.4) which are automatically causal. The surprising fact is that our formal proof will show that we cannot do better by allowing for a non-causal dependence on w (note that this is *not* true for discrete time systems as shown in chapter 9). From now on we only investigate $\mathcal{C}^*(\xi)$. It can be shown that $\mathcal{C}^*(\xi) < \infty$ for all $\xi \in \mathcal{R}^n$ if the number defined by (3.6) is strictly less than γ.

Next by using a method from [Mo] it can be shown that a matrix P exists such that $\mathcal{C}^*(\xi) = \xi^{\mathrm{T}} P \xi$ for all $\xi \in \mathcal{R}^n$. (The same method can be used to show that for a large class of optimization problems with a quadratic cost criterion the optimal cost is also quadratic.)

We denote by $x_{u,w,\xi,t}(\tau)$ and $z_{u,w,\xi,t}(\tau)$ respectively the state and the output of our system at time $\tau > t$ if we apply inputs u and w and $x(t) = \xi$. We define:

$$\mathcal{C}(u,w,\xi,t) := \int_t^\infty \|z_{u,w,\xi,t}(\tau)\|^2 - \gamma^2 \|w(\tau)\|^2 \, d\tau.$$

Our system is time-invariant and therefore we find

$$\mathcal{C}^*(\xi,t) := \sup_w \inf_u \left\{ \mathcal{C}(u,w,\xi,t) \mid u \in \mathcal{L}_2^m, w \in \mathcal{L}_2^l \right.$$
$$\left. \text{such that } x_{u,w,\xi,t} \in \mathcal{L}_2^n \right\}$$
$$= \xi^{\mathrm{T}} P \xi.$$

Now we make the, illegal, move to assume a priori that u is indeed a causal function of w (which, at this point, we justify by the equality (3.8)). Then the last "sup-inf" problem can be rewritten and we find

$$0 = \sup_{w|_{[0,t]}} \inf_{u|_{[0,t]}} \sup_{w|_{(t,\infty)}} \inf_{u|_{(t,\infty)}} \int_0^\infty \|z_{u,w,\xi}(\tau)\|^2 - \gamma^2\|w(\tau)\|^2 \, d\tau \; - \xi^{\mathrm{T}}P\xi$$

$$= \sup_{w|_{[0,t]}} \inf_{u|_{[0,t]}} \int_0^t \|z_{u,w,\xi}(\tau)\|^2 - \gamma^2\|w(\tau)\|^2 d\tau + x^{\mathrm{T}}(t)Px(t) - \xi^{\mathrm{T}}P\xi$$

where $u \in \mathcal{L}_2^n$ and $w \in \mathcal{L}_2^l$ should be such that $x_{u,w,\xi} \in \mathcal{L}_2^n$. We differentiate this expression with respect to t and take the derivative at $t = 0$, acting as if this expression were differentiable and as if we were allowed to interchange the supremum and infimum with the differentiation operator. We then obtain

$$0 = \sup_{w(0)} \inf_{u(0)} \; \|z(0)\|^2 - \gamma^2\|w(0)\|^2 + \frac{d}{dt}x^{\mathrm{T}}(t)Px(t)\Big|_{t=0}$$

We can now investigate this latter expression and note that we thus obtained a static "sup-inf" problem which we can solve explicitly. Writing this out in terms of the system parameters of our system Σ (remember that we have $D_2 = 0$) as defined by (3.1) then yields:

$$0 = \sup_{w(0)} \inf_{u(0)} \begin{pmatrix} \xi \\ u(0) \\ w(0) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C & B^{\mathrm{T}}P + C^{\mathrm{T}}D_1 & E^{\mathrm{T}}P \\ PB + D_1^{\mathrm{T}}C & D_1^{\mathrm{T}}D_1 & 0 \\ PE & 0 & -\gamma^2 I \end{pmatrix} \begin{pmatrix} \xi \\ u(0) \\ w(0) \end{pmatrix}$$

Next we define

$$q \;\; := \;\; u + (D_1^{\mathrm{T}}D_1)^{-1}(B^{\mathrm{T}}P + D_1^{\mathrm{T}}C)\xi,$$
$$p \;\; := \;\; w - \gamma^{-2}E^{\mathrm{T}}P\xi,$$

where we assume that W is invertible. Finally, we denote by $R(P)$ the right hand side of the algebraic Riccati equation in theorem 3.1. By using Schur complements we can then rephrase our previous "sup-inf" problem in the form

$$0 = \sup_p \inf_q \begin{pmatrix} \xi \\ q \\ p \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} R(P) & 0 & 0 \\ 0 & D_1^{\mathrm{T}}D_1 & 0 \\ 0 & 0 & -\gamma^2 I \end{pmatrix} \begin{pmatrix} \xi \\ q \\ p \end{pmatrix}.$$

The above equality should be true for all initial conditions $\xi \in \mathcal{R}^n$. This implies $R(P) = 0$. This is one of the two conditions given in part (iii) of theorem 3.1 (note that the first condition is superfluous since $D_2 = 0$). The optimal p and q are zero, i.e.

$$w^*(0) \;\; = \;\; \gamma^{-2}E^{\mathrm{T}}P\xi,$$
$$u^*(0) \;\; = \;\; -(D_1^{\mathrm{T}}D_1)^{-1}(B^{\mathrm{T}}P + D_1^{\mathrm{T}}C)\xi.$$

However, we can show in the same way that these equalities are satisfied by the optimal u^* and w^* for all t, i.e.

$$
\begin{aligned}
w^*(t) &= \gamma^{-2} E^{\mathrm{T}} P x^*(t), \\
u^*(t) &= -\left(D_1^{\mathrm{T}} D_1\right)^{-1}\left(B^{\mathrm{T}} P + D_1^{\mathrm{T}} C\right) x^*(t),
\end{aligned}
$$

which implies that

$$
\dot{x}^* = A_{cl} x^*.
$$

Since $x^* \in \mathcal{L}_2^n$ for all initial conditions $\xi \in \mathcal{R}^n$ this implies that A_{cl} is asymptotically stable. This is the last condition in part (iii) of theorem 3.1. We shall formalize the above reasoning in the next section to yield a rigorous and reasonably elegant proof.

3.4 Solvability of the Riccati equation

Throughout this section we assume that there exists a $\delta < \gamma$ such that condition (ii) of theorem 3.1 is satisfied. Note that, as mentioned before, we only give a proof for the case $D_2 = 0$. Hence we shall assume throughout the next two sections that $D_2 = 0$. We show that this implies that there exists a matrix P satisfying condition (iii) of theorem 3.1. For the time being we assume that

$$
D_1^{\mathrm{T}} C = 0 \tag{3.9}
$$

and $\gamma = 1$. We derive the more general statement at the end of this section. We first define the following function

$$
\mathcal{C}(u, w, \xi) := \|z_{u,w,\xi}\|_2^2 - \|w\|_2^2.
$$

In order to prove the existence of the desired P we shall investigate the following "sup-inf" problem:

$$
\mathcal{C}^*(\xi) := \sup_w \inf_u \left\{ \mathcal{C}(u, w, \xi) \mid u \in \mathcal{L}_2^m, w \in \mathcal{L}_2^l \text{ such that } x_{u,w,\xi} \in \mathcal{L}_2^n \right\}
$$

for arbitrary initial state ξ. Since condition (ii) of theorem 3.1 holds, it will turn out that the "sup-inf" is finite for all initial states. Moreover we shall show that there exists a $P \geq 0$ such that $\mathcal{C}^*(\xi) = \xi^{\mathrm{T}} P \xi$. It will be proven that this matrix P exactly satisfies condition (iii) of theorem 3.1.

For given $w \in \mathcal{L}_2^l$ and $\xi \in \mathcal{R}^n$, we shall first infimize the function $\mathcal{C}(u, w, \xi)$ over all $u \in \mathcal{L}_2^m$ for which $x_{u,w,\xi} \in \mathcal{L}_2^n$. After that we shall maximize over $w \in \mathcal{L}_2^l$.

As a tool we shall use Pontryagin's maximum principle. This only gives necessary conditions for optimality. However in [LM, Section 5.2]

a sufficient condition for optimality is derived over a finite horizon. The stability requirement $x_{u,w,\xi} \in \mathcal{L}_2^n$ allows us to adapt the proof to the infinite horizon case. We start by constructing a solution to the Hamilton–Jacobi–Bellman boundary value problem associated with this optimization problem.

Let L be the positive semi-definite solution of the following algebraic Riccati equation:

$$A^\mathrm{T} L + LA + C^\mathrm{T} C - LB \left(D_1^\mathrm{T} D_1 \right)^{-1} B^\mathrm{T} L = 0 \tag{3.10}$$

for which

$$A_L := A - B \left(D_1^\mathrm{T} D_1 \right)^{-1} B^\mathrm{T} L \tag{3.11}$$

is asymptotically stable. The existence and uniqueness of such L is guaranteed under the assumptions that $(A,\ B,\ C,\ D_1)$ has no invariant zeros on the imaginary axis, D_1 is injective and (A, B) is stabilizable (see [Wi7]). Let $w \in \mathcal{L}_2^l$ be given. We define

$$r(t) := - \int_t^\infty e^{A_L^\mathrm{T}(\tau - t)} LEw(\tau) \, d\tau$$

$(t \in [0, \infty))$. Note that r is well defined since A_L is asymptotically stable. Next we define x_+ and η by the equations:

$$\dot{x}_+ = A_L x_+ + B \left(D_1^\mathrm{T} D_1 \right)^{-1} B^\mathrm{T} r + Ew \tag{3.12}$$
$$\eta = -Lx_+ + r \tag{3.13}$$

where $x_+(0) = \xi$. It can be easily checked that r, x_+ and η are all $\mathcal{L}_2$ functions. Moreover we have

$$\lim_{t \to \infty} r(t) = \lim_{t \to \infty} x_+(t) = \lim_{t \to \infty} \eta(t) = 0. \tag{3.14}$$

After some calculations, we find the following lemma:

Lemma 3.3 : *Let $\xi \in \mathcal{R}^n$ and $w \in \mathcal{L}_2^l$ be given. The function η as defined by (3.13) satisfies:*

$$\dot{\eta} = -A^\mathrm{T} \eta + C^\mathrm{T} C x_+. \tag{3.15}$$

$\square$

In the statement of Pontryagin's Maximum Principle this equation is the so-called "adjoint equation" and η is called the "adjoint state variable". We have constructed a solution to this equation and we shall show that this η indeed yields a minimizing u. The proof is adapted from [LM, Theorem 5.5]:

Lemma 3.4 : *Let the system (3.1) be given. Moreover let w and $x(0) = \xi$ be fixed. Then*

$$
\begin{aligned}
u_+ &:= \left(D_1^{\mathrm{T}} D_1\right)^{-1} B^{\mathrm{T}} \eta \\
&= \arg \inf_u \left\{ \, \mathcal{C}(u, w, \xi) \mid u \in \mathcal{L}_2^m \text{ such that } x_{u,w,\xi} \in \mathcal{L}_2^n \, \right\}. \quad \square
\end{aligned}
$$

Proof : Since w is fixed it is sufficient to minimize

$$
\|z_{u,w,\xi}\|_2^2 = \mathcal{C}(u, w, \xi) + \|w\|_2^2.
$$

It can be easily checked that $x_+ = x_{u_+,w,\xi}$. Let $u \in \mathcal{L}_2^m$ be an arbitrary control input such that $x_{u,w,\xi} \in \mathcal{L}_2^n$. Since $x_{u,w,\xi} \in \mathcal{L}_2^n$ and $\dot{x}_{u,w,\xi} \in \mathcal{L}_2^n$ we have

$$
\lim_{t \to \infty} x_{u,w,\xi}(t) = 0. \tag{3.16}
$$

For all $t \in \mathcal{R}^+$ we find

$$
\begin{aligned}
\|z_{u,w,\xi}(t)\|^2 - 2\frac{d}{dt}\eta^{\mathrm{T}}(t)x(t) &= \|Cx(t)\|^2 - 2x^{\mathrm{T}}(t)C^{\mathrm{T}}Cx_+(t) \\
&\quad - 2\eta^{\mathrm{T}}(t)Ew(t) + \left[D_1^{\mathrm{T}}D_1 u(t) - 2B^{\mathrm{T}}\eta(t)\right]^{\mathrm{T}} u(t)
\end{aligned} \tag{3.17}
$$

and

$$
\begin{aligned}
\|z_{u_+,w,\xi}\|^2 - 2\frac{d}{dt}\eta^{\mathrm{T}}(t)x_+(t) &= -\|Cx_+(t)\|^2 - 2\eta^{\mathrm{T}}(t)Ew(t) \\
&\quad + \left[D_1^{\mathrm{T}}D_1 u_+(t) - 2B^{\mathrm{T}}\eta(t)\right]^{\mathrm{T}} u_+(t).
\end{aligned} \tag{3.18}
$$

If we integrate (3.17) and (3.18) from zero to infinity, subtract from each other and use (3.14), (3.16) we find (note that $x(0) = x_+(0) = \xi$)

$$
\|z_{u_+,w,\xi}\|_2^2 - \|z_{u,w,\xi}\|_2^2 = -\int_0^\infty \|C\left[x(t) - x_+(t)\right]\|^2 \, dt +
$$

$$
\int_0^\infty \left[D_1^{\mathrm{T}}D_1 u_+(t) - 2B^{\mathrm{T}}\eta(t)\right]^{\mathrm{T}} u_+(t) - \left[D_1^{\mathrm{T}}D_1 u(t) - 2B^{\mathrm{T}}\eta(t)\right]^{\mathrm{T}} u(t) \, dt
$$

Using the definition of u_+ we also find for each fixed $t \in \mathcal{R}^+$ that

$$
\left[D_1^{\mathrm{T}}D_1 u_+(t) - 2B^{\mathrm{T}}\eta(t)\right]^{\mathrm{T}} u_+(t) = \inf_v \left[D_1^{\mathrm{T}}D_1 v - 2B^{\mathrm{T}}\eta(t)\right]^{\mathrm{T}} v. \tag{3.19}
$$

Combining the last two equations then yields

$$\|z_{u_+,w,\xi}\|_2^2 \;\leq\; \|z_{u,w,\xi}\|_2^2. \tag{3.20}$$

which is exactly what we had to prove. Since D_1 is injective it is straightforward to show the minimizing v in (3.19) is unique and hence the minimizing u is unique. ∎

We are now going to maximize over $w \in \mathcal{L}_2^l$. This will then yield $\mathcal{C}^*(\xi)$. Define $\mathcal{F}(\xi,w) := (x_+, u_+, \eta)$ and $\mathcal{G}(\xi,w) := z_{u_+,w,\xi}$. It is clear from the previous lemma that $\mathcal{F}$ and $\mathcal{G}$ are bounded linear operators (linear in (ξ,w) not in ξ and w separately). Define

$$\mathcal{C}(\xi,w) \;:=\; \|\mathcal{G}(\xi,w)\|_2^2 - \|w\|_2^2, \tag{3.21}$$

$$\|w\|_C \;:=\; \left(-\mathcal{C}(0,w)\right)^{1/2}. \tag{3.22}$$

It can easily be shown, using condition (ii) of theorem 3.1 with $\gamma = 1$, that $\|.\|_C$ defines a norm on $\mathcal{L}_2^l$. Moreover we find

$$\|w\|_2 \geq \|w\|_C \geq \rho\|w\|_2 \tag{3.23}$$

where $\rho > 0$ is such that $\rho^2 = 1 - \delta^2$ and δ is such that condition (ii) of theorem 3.1 with $\gamma = 1$ is satisfied. Hence $\|.\|_C$ and $\|.\|_2$ are equivalent norms.

Note that lemma 3.4 still holds if condition (ii) of theorem 3.1 does not hold. However the result that $\|.\|_C$ is a norm and that even $\|.\|_C$ and $\|.\|_2$ are equivalent norms is the essential property which is implied by condition (ii) of theorem 3.1 and which is the key to our derivation. We have

$$\mathcal{C}^*(\xi) = \sup_{w \in \mathcal{L}_2^l} \; \mathcal{C}(\xi,w). \tag{3.24}$$

We can derive the following properties of $\mathcal{C}^*$:

Lemma 3.5 :

(i) *For all $\xi \in \mathcal{R}^n$ we have*

$$0 \leq \xi^{\mathrm{T}} L\xi \leq \mathcal{C}^*(\xi) \leq \frac{\xi^{\mathrm{T}} L\xi}{1 - \delta^2}, \tag{3.25}$$

where L as defined by (3.10) and the stability of (3.11) and where δ is such that condition (ii) of theorem 3.1 is satisfied.

(*ii*) *For all* $\xi \in \mathcal{R}^n$ *there exists an unique* $w_* \in \mathcal{L}_2^l$ *such that* $\mathcal{C}^*(\xi) = \mathcal{C}(\xi, w_*)$. $\quad\square$

Proof : Part (i): It is well known that L, as the stabilizing solution of the algebraic Riccati equation (3.10), yields the optimal cost of the linear quadratic problem with internal stability (see [Wi7]). Hence $\|\mathcal{G}(\xi, 0)\|_2^2 = \mathcal{C}(\xi, 0) = \xi^{\mathrm{T}} L \xi$. Therefore we have $0 \le \xi^{\mathrm{T}} L \xi \le \mathcal{C}^*(\xi)$. Moreover

$$
\begin{aligned}
\mathcal{C}(\xi, w) &= \|\mathcal{G}(\xi, w)\|_2^2 - \|w\|_2^2 \\
&\le (\|\mathcal{G}(\xi, 0)\|_2 + \|\mathcal{G}(0, w)\|_2)^2 - \|w\|_2^2 \\
&\le \left(\sqrt{\xi^{\mathrm{T}} L \xi} + \delta \|w\|_2\right)^2 - \|w\|_2^2 \\
&\le \frac{\xi^{\mathrm{T}} L \xi}{1 - \delta^2}.
\end{aligned}
$$

Part (ii) can be proven in the same way as in [Ta]. First it is shown that $\|.\|_C$ satisfies:

$$
-\|w_\alpha - w_\beta\|_C^2 = 2\mathcal{C}(\xi, w_\alpha) + 2\mathcal{C}(\xi, w_\beta) - 4\mathcal{C}\left(\xi, 1/2\left(w_\alpha + w_\beta\right)\right) \quad (3.26)
$$

for arbitrary $\xi \in \mathcal{R}^n$. Then it can be shown that a maximizing sequence of $\mathcal{C}(\xi, w)$ is a Cauchy sequence with respect to the $\|.\|_C$-norm and hence, since $\|.\|_C$ and $\|.\|_2$ are equivalent norms, there exists a maximizing $\mathcal{L}_2$ function w_*. Using (3.26) it is straightforward to show uniqueness. $\quad\blacksquare$

Define $\mathcal{H} : \mathcal{R}^n \to \mathcal{L}_2^l$ by $\mathcal{H}\xi := w_*$.

Lemma 3.6 : *Let* $\xi \in \mathcal{R}^n$ *be given.* $w_* = \mathcal{H}\xi$ *is the unique* $\mathcal{L}_2$-*function* w *satisfying*

$$
w = -E^{\mathrm{T}} \eta, \quad\quad\quad (3.27)
$$

where $(x, u, \eta) = \mathcal{F}(\xi, w)$. $\quad\square$

Proof : Define $(x_*, u_*, \eta_*) = \mathcal{F}(\xi, w_*)$. Moreover, we define

$$
w_0 := -E^{\mathrm{T}} \eta_*
$$

and $(x_0, u_0, \eta_0) := \mathcal{F}(\xi, w_0)$. We find

$$\|z_{u_0,w_0,\xi}(t)\|^2 - \|w_0(t)\|^2 - 2\frac{d}{dt}\,\eta_*^{\mathrm{T}}(t)x_0(t) =$$

$$\|w_0(t)\|^2 - \|z_{u_*,w_*,\xi}(t)\|^2 + \|z_{u_*,w_*,\xi}(t) - z_{u_0,w_0,\xi}(t)\|^2.$$

We also find:

$$\|z_{u_*,w_*,\xi}(t)\|^2 - \|w_*(t)\|^2 - 2\frac{d}{dt}\,\eta_*^{\mathrm{T}}(t)x_*(t) =$$

$$\|w_0(t)\|^2 - \|z_{u_*,w_*,\xi}(t)\|^2 - \|w_*(t) - w_0(t)\|^2. \qquad (3.28)$$

Integrating the last two equations from zero to infinity and subtracting from each other gives us

$$\mathcal{C}(\xi, w_*) = \mathcal{C}(\xi, w_0) - \|w_0 - w_*\|_2^2 - \|z_{u_0,w_0,\xi} - z_{u_*,w_*,\xi}\|_2^2.$$

Since w_* maximizes $\mathcal{C}(\xi, w)$ over all w, this implies that $w_0 = w_*$. Thus we find that w_* satisfies (3.27).

That w_* is the only solution of the equation (3.27) can be shown in a similar way. Assume that, apart from w_*, also w_1 satisfies (3.27). Let $(x_1, u_1, \eta_1) := \mathcal{F}(\xi, w_1)$. From (3.28) we find

$$\|z_{u_*,w_*,\xi}(t)\|^2 - \|w_*(t)\|^2 - 2\frac{d}{dt}\,\eta_*^{\mathrm{T}}(t)x_*(t) = \|w_*(t)\|^2 - \|z_{u_*,w_*,\xi}(t)\|^2.$$

$$(3.29)$$

We also find:

$$\|z_{u_1,w_1,\xi}(t)\|^2 - \|w_1(t)\|^2 - 2\frac{d}{dt}\,\eta_*^{\mathrm{T}}(t)x_1(t) = \|z_{u_1,w_1,\xi}(t)\|^2 - \|w_1(t)\|^2$$

$$+2w_*^{\mathrm{T}}(t)w_1(t) - 2z_{u_*,w_*,\xi}^{\mathrm{T}}(t)z_{u_1,w_1,\xi}(t).$$

Hence, if we integrate the last two equations from 0 to ∞ and subtract from each other we get

$$\mathcal{C}(\xi, w_*) = \mathcal{C}(\xi, w_1) - \|w_* - w_1\|_C^2. \qquad (3.30)$$

Since w_* was maximizing we find $\|w_* - w_1\|_C = 0$ and hence $w_* = w_1$. ∎

Next, we show that $\mathcal{C}^*(\xi) = \xi^{\mathrm{T}} P \xi$ for some matrix P. In order to do that we first show that η_* is a linear function of x_*:

Lemma 3.7 : *There exists a constant matrix P such that*

$$\eta_* = -Px_*. \tag{3.31}$$

<div style="text-align: right">□</div>

Proof : We shall first look at time 0. From (3.13) it is straightforward that $\eta_*(0)$ depends linearly on (ξ, w_*). By lemma 3.6 we know w_* is uniquely defined by equation (3.27). If we investigate the initial condition $\xi = \alpha\xi_1 + \beta\xi_2$ and if we define $w_p = \alpha\mathcal{H}\xi_1 + \beta\mathcal{H}\xi_2$ then it is easy to check that w_p, ξ satisfies equation (3.27) and hence $w_p = \mathcal{H}\xi$. In this way we have shown that $\mathcal{H} : \xi \to w_*$ is linear. Since w_* depends linearly on ξ, this implies that $\eta_*(0)$ depends linearly on ξ and hence there exists a matrix P such that $\eta_*(0) = -P\xi$.

We shall now look at time t. The sup-inf problem starting at time t with initial state $x(t)$ can now be solved. Due to time-invariance we see that w_* restricted to $[t, \infty)$ satisfies (3.27) and hence, for this problem, the optimal x and η are x_* and η_*. But since t is the initial time for this optimization problem, which is exactly equal to the original optimization problem, we find equation (3.31) at time t with the same matrix P as at time 0. Since t was arbitrary this completes the proof. ■

Lemma 3.8 : *The matrix P defined by lemma 3.7 satisfies*

$$\mathcal{C}^*(\xi) = \xi^{\mathrm{T}}P\xi. \tag{3.32}$$

<div style="text-align: right">□</div>

Proof : We integrate equation (3.29) from zero to infinity. By (3.14) we find

$$\mathcal{C}(\xi, w_*) + 2\eta_*^{\mathrm{T}}(0)\xi = -\mathcal{C}(\xi, w_*).$$

Since $\mathcal{C}(\xi, w_*) = \mathcal{C}^*(\xi)$ and $\eta_*(0) = -P\xi$ we find (3.32). ■

Using the above it will be shown that this matrix P satisfies condition (iii) of theorem 3.1.

Lemma 3.9 : *Assume that the system (A, B, C, D_1) has no invariant zeros on the imaginary axis. Moreover assume that D_1 is injective. Finally assume that $D_1^{\mathrm{T}} C = 0$ and $\gamma = 1$. If the statement in part (ii) of theorem 3.1 is satisfied, then there exists a symmetric matrix P satisfying part (iii) of theorem 3.1.* □

Proof : By lemma 3.7 we have $\eta_* = -P x_*$. Using this we find that:

$$
\begin{aligned}
w_* &= E^{\mathrm{T}} P x_*, \\
u_* &= -\left(D_1^{\mathrm{T}} D_1\right)^{-1} B^{\mathrm{T}} P x_*.
\end{aligned}
\tag{3.33}
$$

Thus we get

$$
\dot{x}_* = A_{cl} x_*
$$

where A_{cl} as defined in theorem 3.1. Since $x_* \in \mathcal{L}_2^n$ for every initial state ξ we know that A_{cl} is asymptotically stable. Next we show that P satisfies the algebraic Riccati equation as given in theorem 3.1. From (3.15) and (3.31) combined with (3.33) we find

$$
-P A_{cl} = A^{\mathrm{T}} P + C^{\mathrm{T}} C.
$$

Using the definition of A_{cl} this equation turns out to be equivalent to the algebraic Riccati equation. Next we show that P is symmetric. Note that both P and P^{T} satisfy the algebraic Riccati equation. Using this we find that

$$
A_{cl}^{\mathrm{T}} \left(P - P^{\mathrm{T}}\right) + \left(P - P^{\mathrm{T}}\right) A_{cl} = 0.
$$

Since A_{cl} is asymptotically stable this implies that $P = P^{\mathrm{T}}$. P can be shown to be positive semi-definite by combining lemma 3.5 and (3.32). ∎

From lemma 3.9 we can derive the implication (ii) $\Rightarrow$ (iii) in theorem 3.1 without the assumption $\gamma = 1$ and without the assumption $D_1^{\mathrm{T}} C = 0$:

Corollary 3.10 : *Assume that (A, B, C, D_1) has no invariant zeros on the imaginary axis. Moreover assume that D_1 is injective. If part (ii) of theorem 3.1 is satisfied, then there exists a symmetric matrix $P \geq 0$ satisfying part (iii) of theorem 3.1.* □

Proof : First we scale the γ to 1, i.e. we set $E_{\text{new}} = E/\gamma$. The rest of the system parameters do not change. Then we apply a preliminary feedback $u = \tilde{F}x + v$ such that $D_1^{\mathrm{T}}(C + D_1\tilde{F}) = 0$. Denote the new A, C and E_{new} by $\tilde{A}, \tilde{C}$ and $\tilde{E}$. For this new system part (ii) of theorem 3.1 is satisfied for $\gamma = 1$. We also know that by applying a preliminary state feedback the invariant zeros of a system do not change. Therefore our new system does not have invariant zeros on the imaginary axis. Hence, since for this new system we also know that $D_1^{\mathrm{T}}\tilde{C} = 0$, we may apply lemma 3.9. Thus we find conditions in terms of the new parameters. Rewriting in terms of the original parameters gives the desired conditions in part (iii) of theorem 3.1. ∎

3.5 Existence of a suitable controller

In this section we shall show that if there exists a matrix P satisfying the conditions in part (iii) of theorem 3.1, then the feedback as suggested by theorem 3.1 satisfies condition (i). To this end we shall assume throughout this section that there is a matrix P satisfying the conditions in part (iii) of theorem 3.1 (which are the same as the conditions on P in part (ii) of theorem 3.2). Moreover we assume that (3.2) is satisfied and we set $\gamma = 1$. The implications will only be proven under the assumption $\gamma = 1$. Just as in the proof of corollary 3.10 the general case can be reduced to the special case $\gamma = 1$ by scaling.

We define the following system:

$$\Sigma_U : \begin{cases} \dot{x}_U = A_U x_U + B_U u_U + Ew, \\ y_U = C_{1,U} x_U + \quad\quad + w, \\ z_U = C_{2,U} x_U + D_{21,U} u_U, \end{cases} \tag{3.34}$$

where

$$\begin{aligned}
A_U &:= A - B \left(D_1^{\mathrm{T}} D_1 \right)^{-1} \left(D_1^{\mathrm{T}} C + B^{\mathrm{T}} P \right), \\
B_U &:= B \left(D_1^{\mathrm{T}} D_1 \right)^{-1/2}, \\
C_{1,U} &:= -E^{\mathrm{T}} P, \\
C_{2,U} &:= C - D_1 \left(D_1^{\mathrm{T}} D_1 \right)^{-1} \left(D_1^{\mathrm{T}} C + B^{\mathrm{T}} P \right), \\
D_{21,U} &:= D_1 \left(D_1^{\mathrm{T}} D_1 \right)^{-1/2}.
\end{aligned}$$

Lemma 3.11 : *The system Σ_U as defined by (3.34) is inner. Denote the transfer matrix of Σ_U by G_U. We decompose G_U:*

$$G_U \begin{pmatrix} w \\ u_U \end{pmatrix} =: \begin{pmatrix} G_{11,U} & G_{12,U} \\ G_{21,U} & G_{22,U} \end{pmatrix} \begin{pmatrix} w \\ u_U \end{pmatrix} = \begin{pmatrix} z_U \\ y_U \end{pmatrix}$$

compatible with the sizes of w, u_U, z_U and y_U. Then $G_{21,U}$ is invertible as a rational matrix and its inverse is in H_∞. Moreover $G_{22,U}$ is strictly proper. □

Proof : Since P satisfies the conditions of part (iii) of theorem 3.1 it is easy to check that $X = P$ satisfies the equations (i) and (ii) of lemma 2.10 for the system Σ_U with inputs w, u_U and outputs z_U, y_U. Indeed, the algebraic Riccati equation in theorem 3.1 is equivalent to the equation (i) of lemma 2.10. Also, by simply writing out the equations in the original system parameters of system (3.1) we find conditions (ii) and (iii) of lemma 2.10.

Moreover, we know that $P \geq 0$ and the equation (i) of lemma 2.10 can be rewritten as

$$PA_U + A_U^{\mathrm{T}} P + \begin{pmatrix} C_{1,U}^{\mathrm{T}} & C_{2,U}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} C_{1,U} \\ C_{2,U} \end{pmatrix} = 0.$$

Note that $A_{cl} = A_U - E_U C_{1,U}$. Therefore, since A_{cl} is asymptotically stable we find that $(C_{1,U}, A_U)$ is detectable. Using standard Lyapunov theory it can then be shown that A_U is asymptotically stable. (A more precise proof can be based on the proof of corollary A.8.) By applying lemma 2.10 we thus find that Σ_U is inner.

To show that $G_{21,U}^{-1}$ is an H_∞ function we can write down a realization for $G_{21,U}^{-1}$ and use the expression for A_{cl} as given above. ∎

Lemma 3.12 : *Assume that the conditions in part (iii) of theorem 3.1 are satisfied. In that case the compensator Σ_F described by the feedback law $u = F_1 x$, where F_1 is given by (3.3) satisfies condition (i) of theorem 3.1.* □

Proof : First note that for this particular feedback the transfer matrix G_F as given by (2.7) is equal to $G_{11,U}$ and, moreover, $A + BF_1$ is equal to

A_U. This implies that the compensator Σ_F is internally stabilizing. Because $G_{21,U}$ is invertible over H_∞ we can derive the following inequalities:

$$\|G_{11,U}\|^2_\infty + \|G_{21,U}^{-1}\|^{-2}_\infty \leq \| \begin{pmatrix} G_{11,U} \\ G_{21,U} \end{pmatrix} \|^2_\infty \leq 1, \tag{3.35}$$

which shows that we have $\|G_F\|_\infty < 1$. The second inequality in (3.35) follows because a submatrix of an inner matrix has H_∞ norm less than or equal to 1.   ∎

Note that theorem 3.1 for $\gamma = 1$ is simply a combination of corollary 3.10 and lemma 3.12. The general result (for $D_2 = 0$) can then be obtained by scaling. The case that $D_2 \neq 0$ needs a similar proof but it takes some extra algebra.

Chapter 4

The general full-information H_∞ control problem

4.1 Introduction

As we already mentioned in section 3.1 the full-information H_∞ control problem was discussed in many papers. We showed that if we make two assumptions on the system parameters, then we could derive elegant, necessary and sufficient conditions under which an internally stabilizing controller exists which makes the H_∞ norm of the closed-loop system less than some, a priori given, bound. Moreover, under these assumptions, it was possible to derive an explicit formula for one "suitable" controller. These assumptions were:

- The direct-feedthrough matrix from the control input to the output is injective.

- The subsystem from the control input to the output has no invariant zeros on the imaginary axis.

In this chapter we shall discuss the general full-information H_∞ problem where these conditions are not necessarily satisfied. We shall remove the first assumption in an, in our opinion, elegant way. The second assumption is removed in [S3] but in this reference the first assumption is still made. At this moment no satisfactory method is available which removes both assumptions at the same time. Some methods to remove the second assumption will be briefly discussed in section 4.7.

In [Kh2, Kh3, Pe3, Pe5, ZK2] conditions were derived for the existence of suitable controllers without these assumptions. However, these conditions are in terms of a *family* of algebraic Riccati equations parametrized by a positive constant ε. In the previous chapter we derived conditions which were expressed in terms of one *single* Riccati equation. The

problem is that this Riccati equation no longer exists when the direct-feedthrough matrix from control input to output is not injective. In this chapter we shall discuss the method from [St2]. We shall express our conditions in terms of a *quadratic matrix inequality* and a number of rank conditions. In the case that the direct-feedthrough matrix is injective it will be shown that the quadratic matrix inequality together with one rank condition reduces to the algebraic Riccati equation from the previous chapter. This quadratic matrix inequality is reminiscent of the dissipation inequality appearing in linear quadratic optimal control (see [Gee, SH2, Tr, Wi7]). The existence of solutions satisfying the rank conditions can be checked by solving a reduced order Riccati equation. However, in contrast with the previous chapter, we are not able to derive an explicit formula for one controller, if one exists, which is internally stabilizing and which makes the H_∞ norm of the closed-loop system less than some, a priori given, bound γ.

We shall first assume that the direct-feedthrough matrix from disturbance to output is zero and we shall show at the end of this chapter how the general result can be obtained. This is done to prevent the complexity of the formulas from clouding the comprehension of the reader.

The outline of this chapter is as follows. In section 4.2 we briefly recall the problem to be studied and give a statement of our main result. We shall also show how the results of this chapter reduce to the results from chapter 3 in the case that the direct-feedthrough matrix from control input to output is injective. In section 4.3 we shall prove necessity and in section 4.4 we shall show sufficiency. These sections will strongly depend on the decompositions as introduced in appendix A. In section 4.5 we shall discuss techniques to construct a suitable controller. In section 4.6 we shall give a method to solve the full-information H_∞ control problem in case the direct-feedthrough matrix from disturbance to output is unequal to zero. Finally, in section 4.7, we shall give methods for handling invariant zeros on the imaginary axis. None of these methods for handling invariant zeros on the imaginary axis is completely satisfactory, but combining the results of this chapter with the results from [S3] will probably yield elegant conditions. However, this still has to be investigated.

4.2   Problem formulation and main results

We consider the finite-dimensional, linear, time-invariant system

$$\Sigma : \begin{cases} \dot{x} = Ax + \quad Bu + Ew, \\ z = Cx + D_1 u, \end{cases} \tag{4.1}$$

where, for each t, $x(t) \in \mathcal{R}^n$ is the state, $u(t) \in \mathcal{R}^m$ is the control input, $w(t) \in \mathcal{R}^l$ is the disturbance and $z(t) \in \mathcal{R}^q$ is the output to be controlled. A, B, E, C and D_1 are matrices of appropriate dimensions. As in the previous chapter we would like to minimize the effect of the disturbance w on the output z by finding an appropriate control input u. More precisely, we seek a compensator Σ_F described by a static state feedback law $u = Fx$ such that after applying this feedback law in the system (4.1), the resulting closed-loop system $\Sigma \times \Sigma_F$ is internally stable and its transfer matrix, denoted by G_F, has H_∞ norm strictly less than some, a priori given, bound γ. In principle one can then obtain, via a search procedure, the infimum over all internally stabilizing compensators of the H_∞ norm of the closed-loop operator (see e.g. [S, S2]).

For any real number $\gamma > 0$ and matrix $P \in \mathcal{R}^{n \times n}$ we define a matrix $F_\gamma(P) \in \mathcal{R}^{(n+m) \times (n+m)}$ by

$$F_\gamma(P) := \begin{pmatrix} PA + A^\mathrm{T}P + \gamma^{-2}PEE^\mathrm{T}P + C^\mathrm{T}C & PB + C^\mathrm{T}D_1 \\ B^\mathrm{T}P + D_1^\mathrm{T}C & D_1^\mathrm{T}D_1 \end{pmatrix}. \quad (4.2)$$

If $\gamma = 1$ we shall simply write $F(P)$ instead of $F_1(P)$. Clearly if P is symmetric, then $F_\gamma(P)$ is symmetric as well. If $F_\gamma(P) \geq 0$, then we shall say that P is a solution of the quadratic matrix inequality at γ. Note the close relationship of this quadratic matrix inequality with the dissipation inequality used in [Gee, Wi7]. The difference being the extra term $\gamma^{-2}PEE^\mathrm{T}P$ in the upper left corner. If P satisfies the dissipation inequality then the system (with $w = 0$) is dissipative with supply rate $z^\mathrm{T}z$ and storage function $-x^\mathrm{T}Px$ (for a definition of these concepts we refer to [WW2, Wi3, Wi4]). On the other hand, if P satisfies the quadratic matrix inequality then there is a feedback $w = Fx$ such that the closed-loop system is dissipative with supply rate $z^\mathrm{T}z - \gamma^2 w^\mathrm{T}w$ and storage function $-x^\mathrm{T}Px$. Note that we are clearly not interested in feedbacks of the disturbance. However, the disturbance w wants to make the quotient $z^\mathrm{T}z - \gamma^2 w^\mathrm{T}w$ large. It turns out that by making the system dissipative with this storage function and supply rate the disturbance is doing this in some way. The above is only meant to give some intuition behind the introduction of this quadratic matrix inequality.

In addition to (4.2), for any $\gamma > 0$ and $P \in \mathcal{R}^{n \times n}$, we define a $n \times (n+m)$ matrix pencil $L_\gamma(P, s)$ by

$$L_\gamma(P, s) := \begin{pmatrix} sI - A - \gamma^{-2}EE^\mathrm{T}P & -B \end{pmatrix}. \quad (4.3)$$

Again if $\gamma = 1$ we shall write $L(P, s)$ instead of $L_1(P, s)$. We note that $L_\gamma(P, s)$ is the controllability pencil associated with the system

$$\dot{x} = \left(A + \gamma^{-2}EE^\mathrm{T}P\right)x + Bu.$$

G_{ci} will denote the transfer matrix of the system $\Sigma_{ci} := (A, B, C, D_1)$ which is a subsystem of Σ as described by (4.1). We are now in the position to formulate the main result of this chapter:

Theorem 4.1 : *Consider the system (4.1). Let $\gamma > 0$. Assume that the system (A, B, C, D_1) has no invariant zeros on the imaginary axis. Then the following three statements are equivalent:*

   (i) *A static feedback law $u = Fx$ exists such that after applying this compensator to the system (4.1) the resulting closed-loop system is internally stable and the closed-loop operator $\mathcal{G}_F$ has H_∞ norm less than γ, i.e. $\|\mathcal{G}_F\|_\infty < \gamma$.*

  (ii) *(A, B) is stabilizable and for the system (4.1) a $\delta < \gamma$ exists such that for all $w \in \mathcal{L}_2^l$ there exists an $u \in \mathcal{L}_2^m$ such that $x_{u,w} \in \mathcal{L}_2^n$ and $\|z_{u,w}\|_2 \leq \delta\|w\|_2$.*

 (iii) *A real symmetric solution $P \geq 0$ to the quadratic matrix inequality $F_\gamma(P) \geq 0$ exists such that*

$$rank\, F_\gamma(P) = rank_{\mathcal{R}(s)}\, G_{ci} \tag{4.4}$$

and

$$rank \begin{pmatrix} L_\gamma(P, s) \\ F_\gamma(P) \end{pmatrix} = n + rank_{\mathcal{R}(s)}\, G_{ci} \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+. \tag{4.5}$$

$\square$

Remarks :

   (i) The implication (i) $\Rightarrow$ (ii) is trivial. Hence we only need to prove (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). The first step will be done in the next section and in the section after that we shall complete the proof. We shall only prove this result for $\gamma = 1$, the general result can be easily obtained by scaling.

  (ii) The case that the assumption concerning the invariant zeros on the imaginary axis is not satisfied will be discussed in section 4.7. The extension to the more general system (3.1), where D_2 is arbitrary, will be discussed in section 4.6.

Before embarking on a proof of this theorem we would like to point out how the results from the previous chapter for $D_2 = 0$ and D_1 injective can be obtained from our theorem as a special case. First note that in this case we have

$$\text{rank}_{\mathcal{R}(s)} G_{ci} = \text{rank } D_1 = m$$

Define

$$R_\gamma(P) := PA + A^{\mathrm{T}}P + \gamma^{-2}PEE^{\mathrm{T}}P + C^{\mathrm{T}}C$$

$$- (PB + C^{\mathrm{T}}D_1)(D_1^{\mathrm{T}}D_1)^{-1}(B^{\mathrm{T}}P + D_1^{\mathrm{T}}C).$$

Furthermore, define a real $(n+m) \times (n+m)$ matrix by

$$S(P) := \begin{pmatrix} I & -(PB + C^{\mathrm{T}}D_1)(D_1^{\mathrm{T}}D_1)^{-1} \\ 0 & I \end{pmatrix}.$$

Then clearly we have

$$S(P)F_\gamma(P)S(P)^{\mathrm{T}} = \begin{pmatrix} R_\gamma(P) & 0 \\ 0 & D_1^{\mathrm{T}}D_1 \end{pmatrix}.$$

From this we can see that the conditions $F_\gamma(P) \geq 0$ and $\text{rank}_{\mathcal{R}(s)} F_\gamma(P) = m$ are equivalent to the single condition $R_\gamma(P) = 0$. We now analyse the second rank condition appearing in our theorem. It is easy to verify that for all $s \in \mathcal{C}$ we have

$$\begin{pmatrix} I & 0 & B(D_1^{\mathrm{T}}D_1)^{-1} \\ 0 & I & -(PB + C^{\mathrm{T}}D_1)(D_1^{\mathrm{T}}D_1)^{-1} \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} L_\gamma(P,s) \\ F_\gamma(P) \end{pmatrix} =$$

$$\begin{pmatrix} sI - A - \gamma^{-2}EE^{\mathrm{T}}P + B(D_1^{\mathrm{T}}D_1)^{-1}(B^{\mathrm{T}}P + D_1^{\mathrm{T}}C) & 0 \\ R_\gamma(P) & 0 \\ B^{\mathrm{T}}P + D_1^{\mathrm{T}}C & D_1^{\mathrm{T}}D_1 \end{pmatrix}.$$

Consequently, if $R_\gamma(P) = 0$, then for each $s \in \mathcal{C}$ the condition

$$\text{rank} \begin{pmatrix} L_\gamma(P,s) \\ F_\gamma(P) \end{pmatrix} = n + \text{rank}_{\mathcal{R}(s)} G_{ci}. \tag{4.6}$$

is equivalent to

$$\text{rank}\left(sI - A - \gamma^{-2}EE^{\mathrm{T}}P + B(D_1^{\mathrm{T}}D_1)^{-1}(B^{\mathrm{T}}P + D_1^{\mathrm{T}}C)\right) = n$$

and hence, since (4.6) should hold for all $s \in \mathcal{C}^0 \cup \mathcal{C}^+$, we find that this is equivalent to the requirement that the matrix

$$A + \gamma^{-2} E E^{\mathrm{T}} P - B \left(D_1^{\mathrm{T}} D_1\right)^{-1} \left(B^{\mathrm{T}} P + D_1^{\mathrm{T}} C\right)$$

is asymptotically stable. Theorem 3.1 for the case that $D_2 = 0$ is then obtained immediately.

4.3 Solvability of the quadratic matrix inequality

In this section we shall establish a proof of the implication (ii) $\Rightarrow$ (iii) in theorem 4.1: assuming that the condition in part (ii) is satisfied we shall show that there is a solution of the quadratic matrix inequality that satisfies the two rank conditions. As announced just after theorem 4.1 we shall only prove this result for $\gamma = 1$. Therefore throughout this section we assume that $\gamma = 1$.

Consider our system (4.1). For the special case that D_1 is injective, we already have theorem 3.1 available. Our proof will use this result.

This time we do not make assumptions on the matrix D_1. Choose bases in the state space, the input space and the output space as in appendix A (where D is replaced by D_1) and apply the preliminary feedback $u = F_0 x + v$ where F_0 is given by (A.1). We obtain the system

$$\hat{\Sigma} : \left\{ \begin{array}{ll} \dot{x} = & (A + B F_0)x + \quad Bv + Ew, \\ z = & (C + D_1 F_0)x + D_1 v \end{array} \right. \tag{4.7}$$

After this transformation, in terms of our decomposition, we have the equations (A.5)–(A.7). We can consider our control system as the interconnection of the subsystem $\tilde{\Sigma}$ given by the equations (A.5) and (A.7) and the subsystem Σ_0 given by (A.6). This interconnection is depicted in picture (A.8). The idea we want to pursue is the following. We know condition (ii) of theorem 4.1 is still satisfied after the preliminary feedback, i.e. there exists a $\delta < 1$ such that for all $w \in \mathcal{L}_2^l$ there exists a $v \in \mathcal{L}_2^m$ such that

- $\|z_{v,w}\|_2 \leq \delta \|w\|_2$,
- $x_{v,w} \in \mathcal{L}_2^n$. $\tag{4.8}$

Then, for a given w, let v be such that these conditions are satisfied. Define v_1 as the first component of v and take x_3 as the third component

of $x_{v,w}$. Interpret (v_1, x_3) as an input for the subsystem $\tilde{\Sigma}$ defined by the equations (A.5) and (A.7). It then follows from (4.8) that

- $\|z_1\|_2^2 + \|z_2\|_2^2 \leq \delta^2 \|w\|_2^2$,

- $x_1 \in \mathcal{L}_2^n$.

Moreover note that the fictitious input $x_3 \in \mathcal{L}_2$. Since $w \in \mathcal{L}_2$ was arbitrary we note that $\tilde{\Sigma}$ satisfies condition (ii) of theorem 4.1 with "inputs" (v_1, x_3), disturbance w and output (z_1, z_2). The crucial observation is now that the direct-feedthrough matrix of $\tilde{\Sigma}$ is *injective* (see lemma A.2). Thus we can apply theorem 3.1 to the system $\tilde{\Sigma}$. Before doing this we should make sure that $(A, (B_{11}, A_{13}))$ is stabilizable and that $\tilde{\Sigma}_{ci}$ given by (A.12) has no invariant zeros on the imaginary axis. It is easy to see that if (A, B) is stabilizable, then $(A, (B_{11}, A_{13}))$ is also stabilizable. Furthermore if $\Sigma_{ci} = (A, B, C, D_1)$ has no invariant zeros on the imaginary axis, then the same holds for $\tilde{\Sigma}_{ci}$ (see lemma A.3). Consequently, we may apply theorem 3.1 to $\tilde{\Sigma}$ and we obtain:

Corollary 4.2 : *Consider the system (4.1). Assume that (A, B, C, D_1) has no invariant zeros on the imaginary axis. Moreover, assume that part (ii) of theorem 4.1 is satisfied. Then a real symmetric solution $P_1 \geq 0$ to the algebraic Riccati equation $R(P_1) = 0$ exists, where R is defined by (A.17), such that $Z(P_1)$ is asymptoticaly stable, where Z is defined by (A.18).* □

The basic idea pursued in section A.2 is that there is a one–one relation between solutions of the algebraic Riccati equation $R(P_1) = 0$ and solutions of the quadratic matrix inequality $F_\gamma(P) \geq 0$ which satisfy the first rank condition (4.4). This is formalized in theorem A.6. The implication (ii) $\Rightarrow$ (iii) is then obtained directly by combining corollary 4.2 with theorem A.6.

4.4 Existence of state feedback laws

In this section we give a proof of the implication (iii) $\Rightarrow$ (i) in theorem 4.1. We shall first explain the idea of the proof. Again, we consider the parameters of our control system with respect to the bases of appendix A and after applying the preliminary feedback $u = F_0 x + v$ where F_0 is defined by (A.1). In this way we obtain the system $\hat{\Sigma}$ defined by (4.7). Also, as in the previous section, we consider our control system as the

interconnection of the subsystem $\tilde{\Sigma}$ given by the equations (A.5) and (A.7) and the subsystem Σ_0 given by (A.6). Suppose that the quadratic matrix inequality has a positive semi-definite solution at $\gamma = 1$ such that the rank conditions (4.4) and (4.5) hold. Then, according to theorem A.6, the algebraic Riccati equation associated with the subsystem $\tilde{\Sigma}$ (with "inputs" v_1 and x_3) has a positive semi-definite solution P_1 such that the corresponding matrix $Z(P_1)$, defined by (A.18), is asymptotically stable. Thus, by applying theorem 3.1 to the subsystem $\tilde{\Sigma}$, we find that the "feedback law"

$$v_1 \;\; = \;\; -\left(\hat{D}^\mathrm{T}\hat{D}\right)^{-1} B_{11}^\mathrm{T} P_1 x_1, \tag{4.9}$$

$$x_3 \;\; = \;\; -\left(C_{23}^\mathrm{T} C_{23}\right)^{-1} \left(A_{13}^\mathrm{T} P_1 + C_{23}^\mathrm{T} C_{21}\right) x_1, \tag{4.10}$$

applied to the system $\tilde{\Sigma}$ yields an internally stable system and the closed-loop transfer matrix has H_∞ norm smaller than 1. Now what we shall do is the following: we shall construct a state feedback law for the *original* system (4.1) in such a way that in the subsystem $\tilde{\Sigma}$ the equality (4.10) holds *approximately* and the equality (4.9) holds exactly. The closed-loop transfer matrix of the original system will then be approximately equal to that of the subsystem $\tilde{\Sigma}$ and will therefore also have H_∞ norm smaller than 1.

In our proof an important role will be played by a result in the context of the problem of *almost disturbance decoupling* as studied in [Tr] and [Wi5]. The results we need are recapitulated in section 2.6. We shall now formulate and prove the converse of corollary 4.2:

Theorem 4.3 : *Consider the system (4.1). Assume that (A, B, C, D_1) has no invariant zeros on the imaginary axis. Assume that a real symmetric solution $P_1 \geq 0$ to the algebraic Riccati equation $R(P_1) = 0$ exists, where R is defined by (A.17), such that $Z(P_1)$, defined by (A.18), is asymptotically stable. Then $F \in \mathcal{R}^{m \times n}$ exists such that $A + BF$ is asymptotically stable and such that after applying the state feedback law $u = Fx$ to Σ we have $\|G_F\|_\infty < 1$.* ☐

Proof : Clearly it is sufficient to prove the existence of such state feedback law $v = Fx$ for the system (4.7). Let this system be decomposed according to (A.5)–(A.7). Choose

$$v_1 = -\left(\hat{D}^\mathrm{T}\hat{D}\right)^{-1} B_{11}^\mathrm{T} P_1 x_1$$

and introduce a new state variable q_3 by

$$q_3 := x_3 + (C_{23}^\mathrm{T} C_{23})^{-1} (A_{13}^\mathrm{T} P_1 + C_{23}^\mathrm{T} C_{21}) x_1$$

Then the equations (A.5)–(A.7) can be rewritten as

$$\dot{x}_1 \; = \tilde{A}_{11} x_1 + A_{13} q_3 + E_1 w, \tag{4.11}$$

$$\begin{pmatrix} \dot{x}_2 \\ \dot{q}_3 \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & \tilde{A}_{33} \end{pmatrix} \begin{pmatrix} x_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix} v_2 + \begin{pmatrix} \tilde{A}_{21} \\ \tilde{A}_{31} \end{pmatrix} x_1 + \begin{pmatrix} E_2 \\ \tilde{E}_3 \end{pmatrix} w, \tag{4.12}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ C_{23} \end{pmatrix} q_3. \tag{4.13}$$

Here we used the following definitions:

$$
\begin{aligned}
\tilde{A}_{11} &:= A_{11} - A_{13}(C_{23}^\mathrm{T} C_{23})^{-1}(A_{13}^\mathrm{T} P_1 + C_{23}^\mathrm{T} C_{21}) \\
&\qquad\qquad\qquad - B_{11}(\hat{D}^\mathrm{T}\hat{D})^{-1} B_{11}^\mathrm{T} P_1, \\
\tilde{A}_{21} &:= A_{21} - A_{23}(C_{23}^\mathrm{T} C_{23})^{-1}(A_{13}^\mathrm{T} P_1 + C_{23}^\mathrm{T} C_{21}) \\
&\qquad\qquad\qquad - B_{21}(\hat{D}^\mathrm{T}\hat{D})^{-1} B_{11}^\mathrm{T} P_1, \\
\tilde{A}_{31} &:= A_{31} - A_{33}(C_{23}^\mathrm{T} C_{23})^{-1}(A_{13}^\mathrm{T} P_1 + C_{23}^\mathrm{T} C_{21}) \\
&\qquad\qquad\qquad - B_{31}(\hat{D}^\mathrm{T}\hat{D})^{-1} B_{11}^\mathrm{T} P_1 \\
&\qquad\qquad\qquad + (C_{23}^\mathrm{T} C_{23})^{-1}(A_{13}^\mathrm{T} P_1 + C_{23}^\mathrm{T} C_{21})\tilde{A}_{11}, \\
\tilde{A}_{33} &:= A_{33} + (C_{23}^\mathrm{T} C_{23})^{-1}(A_{13}^\mathrm{T} P_1 + C_{23}^\mathrm{T} C_{21})A_{13}, \\
\tilde{C}_1 &:= -\hat{D}(\hat{D}^\mathrm{T}\hat{D})^{-1} B_{11}^\mathrm{T} P_1, \\
\tilde{C}_2 &:= C_{21} - C_{23}(C_{23}^\mathrm{T} C_{23})^{-1}(A_{13}^\mathrm{T} P_1 + C_{23}^\mathrm{T} C_{21}), \\
\tilde{E}_3 &:= E_3 + (C_{23}^\mathrm{T} C_{23})^{-1}(A_{13}^\mathrm{T} P_1 + C_{23}^\mathrm{T} C_{21})E_1.
\end{aligned}
$$

By corollary (A.8), we know that $\tilde{A}_{11}$ is asymptotically stable. According to theorem 3.1, if in the subsystem formed by the equations (4.11) and (4.13) we have $q_3 = 0$, then its transfer matrix from w to z has H_∞ norm smaller than 1. Moreover, by corollary (A.8), we know that $\tilde{A}_{11}$ is asymptotically stable. Hence, there exist $M > 0$ and $\rho > 0$ such that for all w and q_3 in $\mathcal{L}_2$ we have

$$\|z\|_2 < (1-\rho)\|w\|_2 + M\|q_3\|_2 \tag{4.14}$$

Also, since $\tilde{A}_{11}$ is asymptotically stable, $M_1, M_2 > 0$ exist such that for all w and q_3 in $\mathcal{L}_2$ we have

$$\|x_1\|_2 < M_1\|w\|_2 + M_2\|q_3\|_2. \tag{4.15}$$

We claim that the following system is strongly controllable:

$$
\left[\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & \tilde{A}_{33} \end{pmatrix}, \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix}, \begin{pmatrix} 0 & I \end{pmatrix}, 0 \right] \tag{4.16}
$$

This can be seen by the following transformation:

$$
\begin{pmatrix} I & 0 & 0 \\ 0 & I & A_{33} - \tilde{A}_{33} \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} sI - A_{22} & -A_{23} & -B_{22} \\ -A_{32} & sI - A_{33} & -B_{32} \\ 0 & I & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} sI - A_{22} & -A_{23} & -B_{22} \\ -A_{32} & sI - \tilde{A}_{33} & -B_{32} \\ 0 & I & 0 \end{pmatrix}.
$$

Since the first matrix on the left is unimodular and the second matrix has full row rank for all $s \in \mathcal{C}$ (combine lemmas 2.8 and A.2), the matrix on the right has full row rank for all $s \in \mathcal{C}$. Hence the system (4.16) is strongly controllable by lemma 2.8.

Consider now the almost disturbance decoupling problem for the system (4.12) with output q_3 and "disturbance" (x_1, w). According to lemma 2.17 the strong controllability of (4.16) implies that there exists a feedback law

$$
v_2 = F_1 \begin{pmatrix} x_2 \\ q_3 \end{pmatrix}
$$

such that in (4.12) we have

$$
\|q_3\|_2 < \frac{\rho}{2} \left(M + M_1 M + \rho M_2 \right)^{-1} \{\|w\|_2 + \|x_1\|_2\} \tag{4.17}
$$

for all w and x_1 in $\mathcal{L}_2$, and such that the matrix

$$
\tilde{A} := \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & \tilde{A}_{33} \end{pmatrix} + \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix} F_1
$$

is asymptotically stable. Combining (4.14), (4.15) and (4.17) gives us

$$
\|z\|_2 < \left(1 - \frac{\rho}{2} \right) \|w\|_2
$$

for all w in $\mathcal{L}_2$. Summarizing, we have shown that if in our original system (4.7) we apply the state feedback law

$$
v_1 = - \left(\hat{D}^{\mathrm{T}} \hat{D} \right)^{-1} B_{11}^{\mathrm{T}} P_1 x_1
$$

$$
v_2 = F_1 \begin{pmatrix} x_2 \\ x_3 + (C_{23}^{\mathrm{T}} C_{23})^{-1} (A_{13}^{\mathrm{T}} P_1 + C_{23}^{\mathrm{T}} C_{21}) x_1 \end{pmatrix}, \tag{4.18}
$$

then for all $w \in \mathcal{L}_2^l$ we have $\|z\|_2 < \delta\|w\|_2$ with $\delta < 1$. Thus, the H_∞ norm of the resulting closed-loop transfer matrix is smaller than 1.

It remains to be shown that the closed-loop system is internally stable. We know that

$$\|\left(sI - \tilde{A}_{11}\right)^{-1} A_{13}\|_\infty \leq M_2 \tag{4.19}$$

$$\|\left(\begin{array}{cc} 0 & I \end{array}\right)\left(sI - \tilde{A}\right)^{-1} \left(\begin{array}{c} \tilde{A}_{21} \\ \tilde{A}_{31} \end{array}\right)\|_\infty \;\; \leq \;\; \frac{\rho}{2}\left(M + M_1 M + \rho M_2\right)^{-1}$$

$$\leq \;\; \frac{1}{2M_2} \tag{4.20}$$

The closed-loop state matrix resulting from the feedback (4.18) is given by

$$A_{cl} := \left(\begin{array}{cc} \tilde{A}_{11} & \left(\begin{array}{cc} 0 & A_{13} \end{array}\right) \\ \left(\begin{array}{c} \tilde{A}_{21} \\ \tilde{A}_{31} \end{array}\right) & \tilde{A} \end{array}\right)$$

Assume that $(x^{\mathrm{T}}, y^{\mathrm{T}}, z^{\mathrm{T}})^{\mathrm{T}}$ is an eigenvalue of A_{cl} with eigenvalue λ with $\mathrm{Re}\,\lambda \geq 0$. It can be seen that

$$x = \left(\lambda I - \tilde{A}_{11}\right)^{-1} A_{13}z, \tag{4.21}$$

$$z = \left(\begin{array}{cc} 0 & I \end{array}\right)\left(\lambda I - \tilde{A}\right)^{-1} \left(\begin{array}{c} \tilde{A}_{21} \\ \tilde{A}_{31} \end{array}\right) x. \tag{4.22}$$

(Note that the inverses exist due to the fact that $\tilde{A}_{11}$ and $\tilde{A}$ are stable matrices.) Combining (4.19) and (4.21) we find $\|x\| \leq M_2\|z\|$ and combining (4.20) and (4.22) yields $\|x\| \geq 2M_2\|z\|$. Hence $x = z = 0$. This would however imply that $(\begin{array}{cc} y^{\mathrm{T}} & 0 \end{array})^{\mathrm{T}}$ is an unstable eigenvector of $\tilde{A}$. Since $\tilde{A}$ is asymptotically stable this yields a contradiction. This proves that the closed-loop system is internally stable. ∎

Again using the one–one relation between solutions of the Riccati equation and solutions of the quadratic matrix inequality as given in theorem A.6, the implication (iii) $\Rightarrow$ (i) in theorem 4.1 is now obtained by combining theorem A.6 and theorem 4.3.

Remarks : In the regular case (i.e. D_1 injective) it is quite easy to give an explicit expression for a suitable state feedback law. This was

done in theorem 3.1. In the singular case (i.e. D_1 is not injective) a
suitable state feedback law is given by $u = F_0 x + v$. Here, F_0 is given by
(A.1) and $v = (\ v_1^{\mathrm{T}} \quad v_2^{\mathrm{T}}\ )^{\mathrm{T}}$ is given by (4.18). The matrix P_1 is obtained
by solving the quadratic matrix inequality or, equivalently, by solving
the reduced order Riccati equation $R(P_1) = 0$ where R is defined by
(A.17). The matrix F_1 is a "state feedback" for the strongly controllable
auxiliary system (4.12). This state feedback achieves almost disturbance
decoupling between the "disturbance" (x_1, w) and the "output" q_3. The
required accuracy of decoupling is expressed by (4.17). A conceptual
algorithm to construct F_1 is given in section 4.5. In this section we
also discuss the numerical reliability of several algorithms available in the
literature.

4.5   The design of a suitable compensator

Several algorithms are available in the literature for the construction of a
suitable compensator for singular H_∞ control problems. The best-known
algorithm is a small perturbation of the system parameters such that
the perturbed system becomes regular. We can then apply the results of
the previous chapter where a suitable controller is given explicitly. This
method is outlined in subsection 4.7.2.

   An alternative approach is to use the steps outlined in this chapter.
In other words, bring the system in the form of the basis of appendix
A. Then a suitable controller is given by (4.18) where F_1 should be such
that (4.17) is satisfied. Hence the problem is reduced to finding a state
feedback for the strongly controllable system defined by the quadruple in
(4.16) such that the closed-loop H_∞ norm is less than:

$$\delta := \frac{\rho}{2}\left(M + M_1 M + \rho M_2\right)^{-1}\left\|\left(\begin{array}{cc} \tilde{A}_{21} & E_2 \\ \tilde{A}_{31} & \tilde{E}_3 \end{array}\right)\right\|$$

where all matrices are as defined in the previous section. We shall de-
scribe a method which is an adapted version of the one given in [Tr]. An
alternative method is described in [OSS1].

   We denote the first three matrices of the quadruple in (4.16) by $\tilde{A}$,
$\tilde{B}$ and $\tilde{C}$ respectively.

(i) We construct a new basis for the state space. We shall construct it
   by induction. Choose $0 \neq x_1 \in \ker \tilde{C} \cap \operatorname{im} \tilde{B}$ and, if such x_1 exists,
   choose v_1 such that $Bv_1 = x_1$. If x_1 does not exist then set $i = 0$
   and $\mathcal{S}_i = \{0\}$ and go to item (ii).

Assume that $\{x_1, \ldots, x_i\}$ and $\{v_1, \ldots, v_i\}$ are given. Denote by $\mathcal{S}_i$ the linear span of $\{x_1, \ldots, x_i\}$. If $(\tilde{A}x_i + \text{ im } \tilde{B}) \cap \text{ ker } \tilde{C} \subset \mathcal{S}_i$ and $\text{im } \tilde{B} \cap \text{ ker } \tilde{C} \subset \mathcal{S}_i$ then go to step (ii). Otherwise, if $(\tilde{A}x_i + \text{ im } \tilde{B}) \cap \text{ ker } \tilde{C} \not\subset \mathcal{S}_i$ then choose v such that $\tilde{A}x_i + \tilde{B}v \in \text{ ker } \tilde{C}$ and $\tilde{A}x_i + \tilde{B}v \notin \mathcal{S}_i$. Set $x_{i+1} = \tilde{A}x_i + \tilde{B}v$ and $v_{i+1} = v$. If $(\tilde{A}x_i + \text{ im } \tilde{B}) \cap \text{ ker } \tilde{C} \subset \mathcal{S}_i$, then choose v such that $\tilde{B}v \in \text{ ker } \tilde{C}$ and $\tilde{B}v \notin \mathcal{S}_i$. Set $x_{i+1} = \tilde{B}v$ and $v_{i+1} = v$. Set $i := i + 1$ and repeat this paragraph again.

(ii) Define $\mathcal{R}_a^*(\text{ ker } \tilde{C}) = \mathcal{S}_i$. Define a linear mapping F such that $Fx_j = v_j$, $j = 1, \ldots, i$ and extend it arbitrarily to the whole state space. In [Tr] it has been shown that $\tilde{A}\mathcal{R}_a^*(\text{ ker } \tilde{C}) + \text{im } \tilde{B} = \mathcal{T}(\Sigma_{ht}) = \mathcal{R}^n$. Therefore we can extend the basis of $\mathcal{S}_i$, $\{x_1, \ldots, x_i\}$ to a basis of $\mathcal{R}^n$ which can be written as

$$
\begin{aligned}
&\tilde{B}v_1, \ A_F\tilde{B}v_1, \ \ldots \ , A_F^{r_1}\tilde{B}v_1, \\
&\tilde{B}v_2, \ A_F\tilde{B}v_2, \ \ldots \ , A_F^{r_2}\tilde{B}v_2, \\
&\quad \vdots \qquad \vdots \qquad\qquad \vdots \\
&\tilde{B}v_i, \ A_F\tilde{B}v_i, \ \ldots \ , A_F^{r_i}\tilde{B}v_i, \\
&\tilde{B}v_{i+1}, \ldots, \tilde{B}v_k,
\end{aligned}
$$

where $A_F = \tilde{A} + \tilde{B}F$ and where we may have to extend our set $\{v_1, \ldots, v_i\}$ with some vectors v_j, $j = i + 1, \ldots, k$ in order to obtain a basis of $\mathcal{R}^n$. Moreover, for $j = 1, \ldots, i$ we should have

$$
\tilde{B}v_j, A_F\tilde{B}v_j, \ldots, A_F^{r_j - 1}\tilde{B}v_j \in \text{ ker } \tilde{C}.
$$

We define $r_j = 0$ for $j = i + 1, \ldots, k$.

(iii) We define the following sequence of vectors. For $j = 1, \ldots, k$ we define:

$$
\begin{aligned}
x_{j,1}(n) \ &:= \ \left(I + \tfrac{1}{n}A_F\right)^{-1} \tilde{B}v_j \\
x_{j,2}(n) \ &:= \ \left(I + \tfrac{1}{n}A_F\right)^{-1} A_F x_{j,1}(n) \\
\vdots \qquad &\qquad \vdots \qquad\qquad \vdots \\
x_{j,r_j+1}(n) \ &:= \ \left(I + \tfrac{1}{n}A_F\right)^{-1} A_F x_{j,r_j}(n)
\end{aligned}
$$

It is easily shown that $x_{j,h}(n) \to A_F^{h-1}\tilde{B}v_j$ as $n \to \infty$ for $j = 1, \ldots, k$ and $h = 1, \ldots, r_j + 1$. Therefore, for n sufficiently large, the vectors

$$
\{x_{j,h}(n), \ j = 1, \ldots, k; \ h = 1, \ldots, r_j + 1\}
$$

are linearly independent and hence form a basis of $\mathcal{R}^n$ again. Let N be such that for all $n > N$ these vectors indeed form a basis.

(iv) For all $n > N$ define a linear mapping $\bar{F}_n$ by

$$
\begin{aligned}
\bar{F}_n x_{j,1}(n) &:= -nv_j \\
\bar{F}_n x_{j,2}(n) &:= -n^2 v_j \\
\vdots & \qquad \vdots \\
\bar{F}_n x_{j,r_j+1}(n) &:= -n^{r_j+1} v_j
\end{aligned}
$$

for $j = 1, \ldots, k$. This determines $\bar{F}_n$ uniquely. Define $F_n := F + \bar{F}_n$. It is shown in [Tr] that the spectrum of $\tilde{A} + \tilde{B}F_n$ is the set $\{-n\}$. Moreover, we have

$$
\lim_{n \to \infty} \|\tilde{C} e^{(A+BF_n)t}\|_1 = 0
$$

Choose n such that the impulse response has $\mathcal{L}_1$ norm smaller than δ.

We would like to conclude this section by comparing the three methods mentioned in this section. Several numerical problems arise:

(i) In order to transform the system in the bases of appendix A we need the subspace $\mathcal{T}(\Sigma_{ci})$. The construction of this subspace as given in lemma 2.5 depends strongly on the determination of kernels and images of matrices. However, a rank evaluation is needed to determine, e.g. the kernel, and this is known to be a numerically ill-posed problem. To determine this subspace we can also use the algorithm as described in [SS]. This has the same problem but this algorithm has been thoroughly tested via examples and seems to work well.

(ii) The method described in this section and also the method described in subsection 4.7.2 involves a parameter (ε and $1/n$ respectively) which goes to infinity. This also results in numerical problems.

Fortunately there is an easy and numerically reliable test whether the resulting feedback is stabilizing and satisfies the H_∞ norm bound for the closed-loop system. A good algorithm to calculate the H_∞ norm can be found in [BBK, BS]. In other words, we can check whether the numerical problems resulted in a bad design.

The algorithm in [OSS1, OSS2] also involves rank evaluations. However, by using the infinite zero structure and especially the different

timescales of eigenvalues that go to infinity, this algorithm results in feedbacks with lower gain to achieve the same level of disturbance decoupling. Ozcetin et al.'s work shows that combining the results in this book on the singular H_∞ control problem with their algorithm yielded better results (i.e. lower feedback gain) in all the examples they tried when compared to the often proposed technique of subsection 4.7.2.

4.6 A direct-feedthrough matrix from disturbance to output

In this section we shall show how to handle the state feedback case for the more general system

$$\Sigma : \begin{cases} \dot{x} = Ax + \ Bu + \ Ew, \\ z = Cx + D_1u + D_2w. \end{cases} \tag{4.23}$$

(i.e. we allow for $D_2 \neq 0$). We shall use a technique from [Gl6] to reduce this problem to the problem studied in section 4.2. Throughout this section we shall assume that $\gamma = 1$. The more general result can be easily derived by scaling. There is one notable difference between the regular case and the singular case if the extra direct-feedthrough matrix is present. In the regular case conditions (i) and (ii) of theorem 3.1 are equivalent. This is *not* true in the singular case. An easy counter example is the following:

Example 4.4 : *Let the following system be given:*

$$\Sigma : \begin{cases} \dot{x} = \quad\quad u, \\ z = x + \quad 100w. \end{cases}$$

After applying an arbitrary static feedback of the form $u = f_1 x + f_2 w$ the closed-loop transfer matrix will be given by

$$g_f(s) = \frac{f_2}{s - f_1} + 100.$$

Clearly $\|g_f\|_\infty \geq 100$. However, if we set $u = -100\dot{w}$, then $z = 0$ for all w which are differentiable and satisfy w(0)=0 (if w is not differentiable or if $w(0) \neq 0$, then we can approximate w arbitrarily by a differentiable function w_1 which satisfies $w_1(0) = 0$). This implies that condition (ii) of theorem 3.1 is satisfied but on the other hand condition (i) of theorem 3.1 is certainly not satisfied. This yields the desired contradiction. □

In general the controllers we would need in the case that condition (ii) is satisfied while condition (i) is not satisfied will be non-proper, i.e. they will include differentiations of the disturbance. Since this is very undesirable we shall focus all our attention on controllers of the form (2.4). We shall restrict ourselves to the case of state feedback, i.e. $y = x$. The result for the full-information case, i.e. $y = (x, w)$, can be obtained in a similar way.

It is easily seen that when there exists a dynamic controller of the form (2.4) where $y = x$ which makes the H_∞ norm less than 1 then $\|D_2\| < 1$. We therefore assume that $\|D_2\| < 1$. We define the following matrices
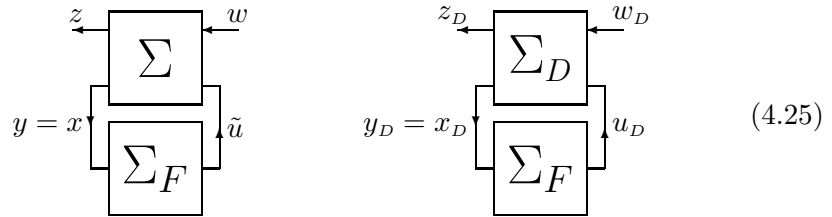
$$
\begin{aligned}
A_D &:= A + E\,(I - D_2^{\mathrm{T}}D_2)^{-1}\,D_2^{\mathrm{T}}C, \\
B_D &:= B + E\,(I - D_2^{\mathrm{T}}D_2)^{-1}\,D_2^{\mathrm{T}}D_1, \\
E_D &:= -E\,(I - D_2^{\mathrm{T}}D_2)^{-1/2}, \\
C_D &:= (I - D_2 D_2^{\mathrm{T}})^{-1/2}\,C, \\
D_{1,D} &:= (I - D_2 D_2^{\mathrm{T}})^{-1/2}\,D_1.
\end{aligned}
$$

Using these matrices we define the following system:

$$
\Sigma_D : \quad \left\{
\begin{aligned}
\dot{x}_D &= A_D x_D + B_D u_D + E_D w_D, \\
z_D &= C_D x_D + D_{1,D} u_D.
\end{aligned}
\right. \tag{4.24}
$$

We have the following lemma connecting Σ_D and Σ:

Lemma 4.5 : *Let Σ_F be a dynamic controller of the form (2.4). Consider the following two systems:*

$$\tag{4.25}$$

The system on the left is the interconnection of Σ described by (4.23) and Σ_F described by (2.4). The system on the right is the interconnection of Σ_D described by (4.24) and the same feedback compensator Σ_F. The following two conditions are equivalent
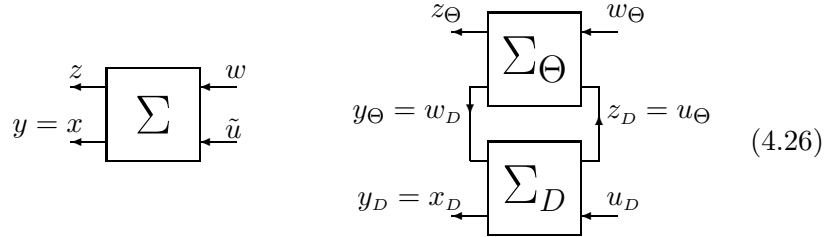
  (i) The system on the left is internally stable and the closed-loop transfer matrix from w to z has H_∞ norm less than 1.

(*ii*) *The system on the right is internally stable and the closed-loop transfer matrix from w_D to z_D has H_∞ norm less than 1.* □
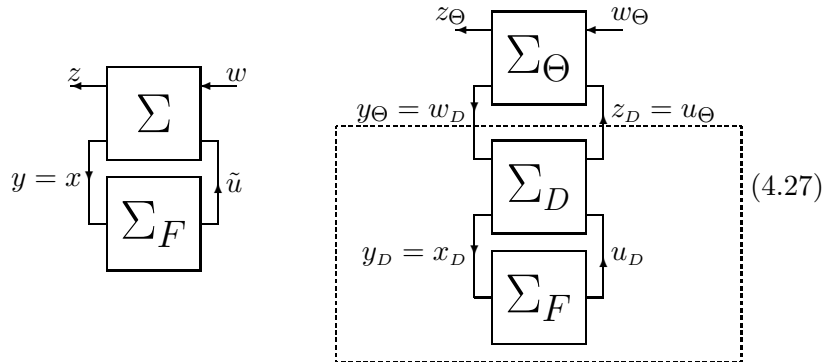
Proof : We define the following static system

$$\Sigma_\Theta : \begin{pmatrix} z_\Theta \\ y_\Theta \end{pmatrix} = \begin{pmatrix} D_2 & (I - D_2 D_2^{\mathrm{T}})^{1/2} \\ -(I - D_2^{\mathrm{T}} D_2)^{1/2} & D_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} w_\Theta \\ u_\Theta \end{pmatrix}.$$

It is straightforward to check that Σ_Θ is inner. We investigate the following two interconnections:

(4.26)

Then it is easily shown that these systems have identical realizations. Next, we investigate the following interconnections:

(4.27)

Because for the system on the left and the system on the right in (4.26) we have identical realizations it is immediate that in (4.27) the system on the left is internally stable and has H_∞ norm less than 1 if, and only if, the system on the right is internally stable and has H_∞ norm less than 1.

We would now like to apply lemma 2.12 to the interconnection on the right where $\Sigma_1 = \Sigma_\Theta$ and Σ_2 is the dashed system. As required, Σ_Θ is inner and the 2,1 block of its transfer matrix is invertible in H_∞.

However, the 2,2 block of its transfer matrix, G_{22}, is not strictly proper. On the other hand, if one checks the proof of lemma 2.12 it is easily seen that the requirement that G_{22} is strictly proper is only needed to show that $I - G_{22}G_2$ is invertible as a rational matrix, where G_2 is the transfer matrix of the dashed system. It is easy to check that $I - G_{22}G_2$ evaluated at infinity is equal to $I - D_2^{\mathrm{T}}D_2$. Hence, since $I - D_2^{\mathrm{T}}D_2$ is invertible, the rational matrix $I - G_{22}G_2$ is invertible. This implies that the result of lemma 2.12 is still valid. Therefore in (4.27) the system on the right is internally stable and has H_∞ norm less than 1 if, and only if, the dashed system is internally stable and has H_∞ norm less than 1.

Combining, we find that in (4.27) the system on the left is internally stable and has H_∞ norm less than 1 if, and only if, the dashed system is internally stable and has H_∞ norm less than 1. By noting that the system on the left in (4.27) is equal to the system on the left in (4.25) and the dashed system in (4.27) is equal to the system on the right in (4.25) this completes the proof. ∎

We can also derive the following result:

Lemma 4.6 : *Assume that the system Σ and the system Σ_D, described by (4.23) and (4.24), respectively, are given. The invariant zeros of (A, B, C, D_1) are equal to the invariant zeros of $(A_D, B_D, C_D, D_{1,D})$.* □

Proof : We have the following relationship between the system matrices of the systems (A, B, C, D_1) and $(A_D, B_D, C_D, D_{1,D})$ respectively:

$$
\begin{pmatrix} I & ED_2^{\mathrm{T}}(I - D_2 D_2^{\mathrm{T}})^{-1/2} \\ 0 & (I - D_2 D_2^{\mathrm{T}})^{1/2} \end{pmatrix} \begin{pmatrix} sI - A_D & -B_D \\ C_D & D_{1,D} \end{pmatrix} = \begin{pmatrix} sI - A & -B \\ C & D_1 \end{pmatrix}.
$$

Since the first matrix on the left is unimodular, the above equality immediately gives the desired result. ∎

We shall now state and prove the results for the more general system (4.23). The following theorem generalizes theorem 3.2.

Theorem 4.7 : *Let the system Σ described by (4.23) be given. Assume that (A, B, C, D_1) has no invariant zeros on the imaginary axis. Then the following three statements are equivalent:*

(*i*) *A compensator Σ_F described by the static state feedback law $u = Fx$ exists such that the closed-loop system $\Sigma \times \Sigma_F$ is internally stable and has H_∞ norm less than 1.*

(*ii*) *A compensator Σ_F of the form (2.4) with $y=x$ exists such that the closed-loop system $\Sigma \times \Sigma_F$ is internally stable and has H_∞ norm less than 1.*

(*iii*) *We have $\|D_2\| < 1$ and Σ_D defined by (4.24) satisfies condition (iii) of theorem 4.1 with $\gamma = 1$.* □

Proof : The implication (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii): Note that if there is a dynamic state feedback which makes the H_∞ norm less than 1, then $\|D_2\| < 1$. By lemma 4.5 if there is an internally stabilizing controller Σ_F which makes the H_∞ norm less than 1, then the same controller is internally stabilizing for the system Σ_D defined by (4.24) and this controller also makes the H_∞ norm of the closed-loop system $\Sigma_D \times \Sigma_F$ less than 1. By lemma 4.6 Σ_D satisfies the assumptions of theorem 4.1 and we already know that Σ_D satisfies condition (ii) of theorem 4.1. Therefore condition (iii) of theorem 4.1 is satisfied for the system Σ_D.

(iii) $\Rightarrow$ (i): By lemma 4.6 the system Σ_D satisfies the assumptions of theorem 4.1 and moreover condition (iii) of theorem 4.1 is satisfied. Therefore, the system Σ_D satisfies condition (i) of theorem 4.1. This implies that for the system Σ_D there is an internally stabilizing static state feedback which makes the closed-loop H_∞ norm less than 1. Finally by lemma 4.5 this state feedback satisfies condition (i) of theorem 4.7 for the system Σ. ∎

4.7 Invariant zeros on the imaginary axis

In this section we shall discuss two methods which can be used to solve the H_∞ problem with zeros on the imaginary axis. A method we shall not discuss and which is given in [S3] will probably lead to much more elegant conditions but these conditions are too complex to explain here in detail (some discussions on this alternative method are given in subsection 12.2.1).

4.7.1   Frequency-domain loop shifting

The basic method (see [Kh2, Li3, Sa]) here is based on applying a transformation in the frequency-domain. This method is applicable for the full-information feedback case. On the other hand, for the state feedback case we have, after transformation, a problem with measurement feedback which we can only solve using techniques which will be given in the next chapter. Therefore we assume that $y = (x, w)$.

For all $\varepsilon > 0$ we define the following transformation:

$$G(s) \longrightarrow \tilde{G}(s) := G\left(\frac{\varepsilon + s}{1 + \varepsilon s}\right)$$

and instead of minimizing the H_∞ norm of a system described by the transfer matrix G we minimize the H_∞ norm of a system described by the transfer matrix $\tilde{G}$. If $I - \varepsilon A$ is invertible a state space realization of $\tilde{G}$ is given by:

$$\tilde{\Sigma}(\varepsilon) : \begin{cases} \dot{x} = \tilde{A}x + \tilde{B}u + \tilde{E}w, \\ \tilde{y} = \tilde{C}_1 x + \tilde{D}_{11}u + \tilde{D}_{12}w, \\ z = \tilde{C}_2 x + \tilde{D}_{21}u + \tilde{D}_{22}w, \end{cases} \tag{4.28}$$

where

$$\begin{aligned} \tilde{A} &:= (A - \varepsilon I)(I - \varepsilon A)^{-1}, \\ \tilde{B} &:= (1 - \varepsilon^2)(I - \varepsilon A)^{-1}B, \\ \tilde{E} &:= (1 - \varepsilon^2)(I - \varepsilon A)^{-1}E, \\ \tilde{C}_1 &:= \begin{pmatrix} (I - \varepsilon A)^{-1} \\ 0 \end{pmatrix}, \\ \tilde{C}_2 &:= C(I - \varepsilon A)^{-1}, \\ \tilde{D}_{11} &:= \begin{pmatrix} \varepsilon(I - \varepsilon A)^{-1}B \\ 0 \end{pmatrix}, \\ \tilde{D}_{12} &:= \begin{pmatrix} \varepsilon(I - \varepsilon A)^{-1}E \\ I \end{pmatrix}, \\ \tilde{D}_{21} &:= D_1 + \varepsilon C(I - \varepsilon A)^{-1}B, \\ \tilde{D}_{22} &:= D_2 + \varepsilon C(I - \varepsilon A)^{-1}E. \end{aligned}$$

Assume that a compensator $\tilde{\Sigma}_F$ described by $u = \tilde{F}_1 x + \tilde{F}_2 w$ for $\tilde{\Sigma}(\varepsilon)$ exists such that the closed-loop system is internally stable, the closed-loop transfer matrix $\tilde{G}_{cl}$ has H_∞ norm less than 1 and, moreover, the

matrix $I - \varepsilon B F_1$ is invertible (the latter we can always achieve by an arbitrarily small perturbation). We define

$$F := \begin{pmatrix} \tilde{F}_1(I + \varepsilon B \tilde{F}_1) & \tilde{F}_2 \end{pmatrix} \begin{pmatrix} I - \varepsilon A & -\varepsilon(E - B\tilde{F}_2) \\ 0 & I \end{pmatrix}. \qquad (4.29)$$

The feedback $u = F\tilde{y}$ applied to the same system $\tilde{\Sigma}(\varepsilon)$ but with $\tilde{y}$ instead of $y = (x, w)$ yields the same closed-loop system and is hence internally stabilizing and the closed-loop transfer matrix is $\tilde{G}_{cl}$.

If we apply the feedback Σ_F described by $u = Fy$ to our original system, then the closed-loop system $\Sigma_F \times \Sigma$, with transfer matrix G_{cl}, is related to the closed-loop system $\tilde{\Sigma}(\varepsilon) \times \tilde{\Sigma}_F$, with transfer matrix $\tilde{G}_{cl}$, via the above transformation, i.e.

$$\tilde{G}_{cl}(s) = G_{cl}\left(\frac{\varepsilon + s}{1 + \varepsilon s}\right).$$

Moreover, it can be shown that the state matrix of the closed-loop system $\Sigma \times \Sigma_F$ has all its eigenvalues inside a circle $\tilde{\mathcal{D}}$ which is symmetric with respect to the real axis and lies in the left half plane between $-\varepsilon$ and $-1/\varepsilon$. Hence the closed-loop system is certainly internally stable. Denote the set of all $s \in \mathcal{C}$ outside that circle by $\tilde{\mathcal{D}}^+$. We have

$$
\begin{aligned}
1 \;>\; & \|\tilde{G}_F\|_\infty = \sup_{s \in \mathcal{C}^+} \|\tilde{G}_F(s)\| = \sup_{s \in \tilde{\mathcal{D}}^+} \|G_F(s)\| \\
\geq\; & \sup_{s \in \mathcal{C}^+} \|G_F(s)\| = \|G_F\|_\infty.
\end{aligned}
$$

Hence Σ_F makes the H_∞ norm of the closed-loop transfer matrix G_F strictly less than 1.

On the other hand, if, for the system Σ, there is a stabilizing static compensator Σ_F with $y = (x, w)$ which makes the H_∞ norm of the closed-loop system strictly less than 1 then it can be shown that $\varepsilon_1 > 0$ exists such that for all $0 < \varepsilon \leq \varepsilon_1$ the transformed system $\tilde{\Sigma}(\varepsilon)$ with the same compensator $\tilde{\Sigma}_F$ but with measurement $\tilde{y}$ is internally stable and the H_∞ norm of the closed-loop system is strictly less than 1.

Now for all but finitely many $\varepsilon > 0$ the system $\tilde{\Sigma}(\varepsilon)$, described by (4.28) is such that $(\tilde{A}, \tilde{B}, \tilde{C}_2, \tilde{D}_{21})$ has no invariant zeros on the imaginary axis. Therefore to check if for $\tilde{\Sigma}(\varepsilon)$ there is an internally stabilizing feedback which makes the H_∞ norm of the closed-loop system less than 1 we can use the results of the previous section.

We can formalize the above intuition in the following theorem. We shall not formally prove this theorem.

Theorem 4.8 : *Let a system Σ be given described by (4.23). For all $\varepsilon > 0$ such that $I - \varepsilon A$ is invertible we define $\tilde{\Sigma}(\varepsilon)$ by (4.28). The following two statements are equivalent:*

(*i*) *For the system Σ there is a static feedback Σ_F described by $u = F_1 x + F_2 w$ which is internally stabilizing and which makes the H_∞ norm of the closed-loop system less than 1.*

(*ii*) *$\varepsilon_1 > 0$ exists such that for all $0 < \varepsilon < \varepsilon_1$ the matrix $I - \varepsilon A$ is invertible and the subsystem $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}_1)$ of $\tilde{\Sigma}(\varepsilon)$ has no invariant zeros on the imaginary axis. Moreover, ε_1 can be chosen such that for all $0 < \varepsilon < \varepsilon_1$ a feedback $\tilde{\Sigma}_F$ exists for $\tilde{\Sigma}(\varepsilon)$ of the form $u = \tilde{F}_1 x + \tilde{F}_2 w$ which is internally stabilizing and makes the H_∞ norm of the closed-loop system less than 1.* $\qquad\square$

Remarks

(*i*) Note that we can check part (ii) for some $\varepsilon > 0$ using the methods from the previous section. The problem is that we do not know how small we must make ε. If for some $\varepsilon > 0$ part (ii) is not satisfied, then either part (i) is not true or part (ii) is true for some smaller ε. Therefore we are never sure.

(*ii*) A controller satisfying part (ii) such that $I - \varepsilon B \tilde{F}_1$ is invertible can be transformed into a controller $u = Fy$ which satisfies part (i) where F is defined by (4.29). This condition is a well-posedness condition we need since the system $\tilde{\Sigma}(\varepsilon)$ with measurement $\tilde{y}$ has $\tilde{D}_{11} \neq 0$.

(*iii*) The advantage of this method is that controllers for Σ found in this way will yield closed-loop systems which have all poles inside $\tilde{\mathcal{D}}$ and hence inside the left half plane. Since poles close to the imaginary axis with high imaginary part are outside $\tilde{\mathcal{D}}$ they can never be eigenvalues of the closed-loop state matrix. Since this kind of ill-damped, high-frequency poles is highly undesirable from a practical point of view this is a nice property.

(*iv*) We think that this transformation should make it clear to the reader that transformations in the frequency-domain are not well-suited for the state feedback case or for the full-information case. Often these problems are transformed into systems with measurement feedback where we might really have trouble observing the state. This is the reason why most of the approaches we shall discuss in section 5.1

for the H_∞ control problem with measurement feedback are often not very suitable to treat the special cases of state feedback and full-information feedback.

4.7.2 Cheap control

In this subsection we shall briefly describe how the method used in [Kh3, Kh2, Pe3, Pe5, ZK2] still gives necessary and sufficient conditions for the existence of internally stabilizing controllers which make the H_∞ norm of the closed-loop system less than 1 even when we have invariant zeros on the imaginary axis. This subsection is worked out in more detail in [St].

We assume that we have a system Σ of the form (4.23). For each $\varepsilon > 0$ we define the following system:

$$\tilde{\Sigma}(\varepsilon) : \begin{cases} \dot{x} = Ax + \ Bu + \ Ew, \\ z = \tilde{C}x + \tilde{D}_1 u + \tilde{D}_2 w, \end{cases} \tag{4.30}$$

where

$$\tilde{C} = \begin{pmatrix} C \\ \varepsilon I \\ 0 \end{pmatrix}, \quad \tilde{D}_1 = \begin{pmatrix} D_1 \\ 0 \\ \varepsilon I \end{pmatrix}, \quad \tilde{D}_2 = \begin{pmatrix} D_2 \\ 0 \\ 0 \end{pmatrix}.$$

The structure of the perturbations on the matrices is such that it is easy to show that any controller Σ_F of the form (2.4) with $y = x$ has the property that Σ_F is internally stabilizing when applied to Σ if, and only if, the same controller Σ_F is internally stabilizing when applied to $\tilde{\Sigma}(\varepsilon)$ (for $y = (x, w)$, the same is true).

Let Σ_F be internally stabilizing when applied to Σ. Denote the closed-loop operator by $\mathcal{G}_F$. Moreover denote the closed-loop operator when the controller is applied to $\tilde{\Sigma}(\varepsilon)$ by $\tilde{\mathcal{G}}_F(\varepsilon)$. Then we have

$$\|\mathcal{G}_F\|_\infty \leq \|\tilde{\mathcal{G}}_F(\varepsilon_1)\|_\infty \leq \|\tilde{\mathcal{G}}_F(\varepsilon_2)\|_\infty$$

for all $0 \leq \varepsilon_1 \leq \varepsilon_2$. Hence if we have an internally stabilizing controller which makes the H_∞ norm of the closed-loop system less than 1 when applied to the system $\tilde{\mathcal{G}}_F(\varepsilon)$, then the same controller is internally stabilizing and makes the H_∞ norm of the closed-loop system less than 1 when applied to the original system. Thus we can obtain the following result:

Theorem 4.9 : *Let a system Σ be given by (4.23). For all $\varepsilon > 0$ define $\tilde{\Sigma}(\varepsilon)$ by (4.28). The following two statements are equivalent:*

(i) *For the system Σ there is a feedback Σ_F of the form (2.4) with $y = x$ which is internally stabilizing and which makes the H_∞ norm of the closed-loop system less than 1.*

(ii) *$\varepsilon_1 > 0$ exists such that for all $0 < \varepsilon < \varepsilon_1$ there is a feedback Σ_F of the form (2.4) for $\tilde{\Sigma}(\varepsilon)$ with $y = x$ which is internally stabilizing and which makes the H_∞ norm of the closed-loop system less than 1.*

Any controller satisfying part (ii) for some $\varepsilon > 0$ also satisfies part (i).

$\square$

On the other hand for the system $\tilde{\Sigma}(\varepsilon)$ the subsystem $(A, B, \tilde{C}, \tilde{D}_1)$ does not have any invariant zeros and hence certainly no invariant zeros on the imaginary axis. Moreover, $\tilde{D}_1$ is injective. Hence we may apply the results of chapter 3 to the system $\tilde{\Sigma}(\varepsilon)$ to obtain necessary and sufficient conditions for the existence of internally stabilizing controllers which make the H_∞ norm of the closed-loop system less than 1.

If we compare the method of the previous subsection with the method of the current subsection, then we see that the method of this subsection is much easier. First of all because we do not have to apply transformations on the controller. Secondly because for all $\varepsilon > 0$ the system (4.30) is well defined and satisfies the assumptions of chapter 3. In contrast the system (4.28) is not always well defined and for certain values of $\varepsilon > 0$ there may be invariant zeros on the imaginary axis. Moreover, in general it will only satisfy the conditions of chapter 4 and at this point the reader is probably aware of the fact that the necessary and sufficient conditions of chapter 4 are more difficult to check than the necessary and sufficient conditions of chapter 3.

Both methods have the disadvantage that the conditions cannot actually be checked since both condition (ii) of theorem 4.8 and condition (ii) of theorem 4.9 have to be checked for an infinite number of $\varepsilon > 0$.

One of the main reasons for using the method of the previous subsection is because all closed-loop poles will be placed inside a circle in the open left half plane and therefore the closed-loop system will not have ill-damped high-frequency poles. Hence the method has advantages from an engineering point of view.

4.8 Conclusion

In this chapter we have completed the results on the full-information H_∞ control problem. It turns out that as long as we do not have invariant zeros on the imaginary axis, we find nice, checkable, necessary

and sufficient conditions under which there exist an internally stabilizing controller which makes the H_∞ norm of the closed-loop system less than some, a priori given, bound γ. In the case that we have a regular problem we have one Riccati equation and, if one exists, an explicit formula for one suitable controller. On the other hand, for the singular problem we have a quadratic matrix inequality and a couple of rank conditions. Using a state space decomposition, the latter conditions can be shown to be equivalent to a reduced order Riccati equation (see appendix A). A problem is that we do not have an explicit formula for the desired controller and there may be numerical problems when trying to find such a controller.

When there are invariant zeros on the imaginary axis the methods of section 4.7 can be used. However, these methods are not really satisfactory and one would have to investigate the technique used in [S3] better to obtain nicer conditions for the general case (the conditions we have in mind are suggested in subsection 12.2.1).

Chapter 5

The H_∞ control problem with measurement feedback

5.1 Introduction

In the previous two chapters we investigated two special cases of the H_∞ control problem with measurement feedback, namely $y = x$ and $y = (x, w)$. In this chapter we shall investigate the more general system (2.1).

Around 1984 practically all the work on H_∞ control theory with measurement feedback was done with a mixture of time-domain and frequency-domain techniques (see [Do3, Fr2, Gl, Gl6]). The main drawback of these methods was that it yielded high order controllers. In [Li, Li2] it was shown that the order of the controller could be reduced considerably: it was proven that for the one block H_∞ control problem (the matrices D_{12} and D_{21} appearing in (2.1) are both square matrices) and for the two block H_∞ control problem (either the matrix D_{12} or the matrix D_{21} in (2.1) is a square matrix) there are always "suitable" controllers with an order no greater than the order of the plant.

During the last few years the H_∞ control problem with measurement feedback was investigated via several new methods:

- **The interpolation approach**. In fact, the interpolation approach already has quite a history and several authors have worked on this problem (see e.g. [Gr, Kh, Li3, Za2]). However, it can only treat the special case of a one-block H_∞ control problem. One should note that the classic interpolation techniques were used for discrete time systems. At first the continuous time case was treated via a frequency-domain transformation to the discrete time case (a linear fractional transformation as given in [Gen, appendix 1]).

- **The time-domain approach** (see [Do5, Kh3, Kh4, Pe, Ta]). This

method was the first to suggest the use of Riccati equations in H_∞ control, which was an important breakthrough. The detailed formulations of the results obtained via this method will be given in this chapter.

- **The polynomial approach** (see [Bo, Kw, Kw2]). This method starts with a polynomial left (or right) coprime factorization of the transfer matrix of the system. Then it is shown that a controller which minimizes the H_∞ norm of the closed-loop system is a so-called equalizing solution of a certain minimization problem. Conditions for obtaining an internally stabilizing controller which minimizes the H_∞ norm are then given in terms of diophantine equations. The current research in this area is to investigate the relation between these diophantine equations and the conditions found in other methods.

- **The J-spectral factorization approach** (see [Gr2, HSK, Ki]). This method is strongly based on the classic frequency-domain approach. The conditions for the existence of an internally stabilizing controller which makes the H_∞ norm of the closed-loop system less than some, a priori given, bound are given in terms of a rational matrix which should have a so-called J-spectral factorization. Whether such a factorization exists can be checked via the solvability of certain algebraic Riccati equations.

All of these methods show that if suitable controllers exist, then a suitable controller can be found of the same complexity as the original plant. A second feature that all of these methods have in common is a number of basic assumptions they all have to make. The assumptions are twofold. Firstly two direct-feedthrough matrices should be injective and surjective, respectively. Secondly, two given subsystems should not have invariant zeros on the imaginary axis. These assumptions exclude, for instance, the special cases of state feedback and full-information feedback.

A special feature of the time-domain approach we are using throughout this book is that it first solves the full-information feedback case as we did in the previous chapters and then uses these results to obtain the general result.

Under the basic assumptions mentioned above and using the time-domain approach, the necessary and sufficient conditions for the existence of internally stabilizing controllers which make the H_∞ norm of the closed-loop system less than some, a priori given, bound γ are the following: two given Riccati equations should have positive semi-definite solutions and the product of these two matrices should have spectral radius less than

γ^2. These two Riccati equations are not coupled and in fact one of them
is the same as the Riccati equation from chapter 3. The other Riccati
equation is dual to the first one and is related to the problem of state
estimation. The coupling condition that the product of the solutions of
these Riccati equations should have spectral radius less than γ^2 is very
hard to explain intuitively. It is a kind of test whether state estimation
and state feedback combined in some way yield the desired result: an
internally stabilizing feedback which makes the H_∞ norm less than γ.
The most general result under these assumptions is stated without proof
in [Gl3].

In the literature two methods have been proposed to solve the H_∞ pro-
blem without assumptions on the direct-feedthrough matrices and with-
out assumptions on the invariant zeros. For the special case of full-
information feedback, these methods are discussed in section 4.7. Using
the first method discussed in subsection 4.7.1, which applies a transforma-
tion in the complex plane combined with the results in [Do5, Kh3, Pe, Ta],
we still have to make assumptions: two given subsystems should be left
and right invertible respectively. The second method combined with the
methods of the latter papers can tackle the most general case. These
methods, however, have the drawback that the conditions are in terms of
Riccati equations which are parametrized by some parameter $\varepsilon > 0$.

In this chapter we shall present a method which, independently of the
latter papers, solves the measurement feedback case using the results of
our previous chapter. In contrast to [Do5, Kh3, Pe, Ta] we shall impose no
assumptions on the direct-feedthrough matrices. However, we still have
to exclude invariant zeros on the imaginary axis. Our method will not
have the above-mentioned drawback of a parametrized Riccati equation.
Also our results reduce to the known results in [Do5, Ta] when these
singularities of the direct-feedthrough matrices do not occur.

Another advantage of our (weaker) assumptions will be that the spe-
cial cases of state feedback and full-information feedback fall within the
framework of the general problem formulation.

The necessary and sufficient conditions under which there exists an
internally stabilizing dynamic compensator which makes the H_∞ norm
strictly less than some a priori given bound γ are formulated differently
from recent publications [Do5, Ta]. As mentioned above, in these papers
the conditions are formulated in terms of two given Riccati equations.
However, when there are singularities of the direct-feedthrough matrices
these Riccati equations do not exist. To replace the role of these Riccati
equations we have two quadratic matrix inequalities. The solution of each
of these quadratic matrix inequalities has to satisfy two rank conditions.

Moreover, we have a condition which couples these two matrix inequalities. The spectral radius of the product of the two solutions of these matrix inequalities should be smaller than γ^2. In the regular case the quadratic matrix inequality together with the corresponding first rank condition reduces to a Riccati equation and the second rank condition guarantees that it is a stabilizing solution of the Riccati equation. As for the regular case, the first matrix inequality with the two corresponding rank conditions exactly forms the necessary and sufficient conditions for the existence of a "suitable" state feedback as derived in chapter 4. The second quadratic matrix inequality with the remaining two rank conditions are, again as in the regular case, dual to the first matrix inequality and the first two rank conditions.

Our proof will use ideas given in [Do5] to solve the regular H_∞ problem with measurement feedback but is independent of the results in [Do5] and is entirely self-contained. Most of the results of this chapter already appeared in [St4].

The outline of this chapter is as follows: In section 5.2 we formulate the problem and present the main result for the case when two given direct-feedthrough matrices are zero. Moreover, we show that in the regular case and the state feedback case this result reduces to the known results in [Do5] and chapter 4, respectively. In section 5.3 it is shown that the conditions for the existence of a suitable compensator as given in our main theorem are necessary and sufficient. It is shown that the problem of finding such a compensator is equivalent to finding such a compensator for a certain transformed system, i.e. we prove that any compensator which internally stabilizes this new system and makes the H_∞ norm of the closed-loop system less than γ has the same properties when applied to the original system and vice versa. This new system has some desirable properties and, using the results from section 2.6, it is shown that for this new system we can even make the H_∞ norm of the closed-loop system arbitrarily small. In section 5.4 a method for finding the desired compensator is discussed. In section 5.5 we discuss the characterization of all stable closed-loop systems with H_∞ norm less than γ which we can obtain via a suitable feedback and the related question of characterization of all controllers which achieve this H_∞ norm bound for the closed-loop system. In section 5.6 we shall briefly discuss how to extend the results to the case that all direct-feedthrough matrices are allowed to be unequal to zero. We conclude in section 5.7 with some final remarks. The proofs of section 5.3 depend upon the basis transformations of appendix A and are given in appendix B since they are rather technical and detract from the main ideas of the proof.

5.2   Problem formulation and main results

We consider the linear, time-invariant, finite-dimensional system:

$$
\Sigma : \left\{
\begin{array}{l}
\dot{x} = Ax + Bu + Ew, \\
y = C_1 x + D_1 w, \\
z = C_2 x + D_2 u,
\end{array}
\right.
\tag{5.1}
$$

where for all t we have that $x(t) \in \mathcal{R}^n$ is the state, $u(t) \in \mathcal{R}^m$ is the control input, $w(t) \in \mathcal{R}^l$ is the unknown disturbance, $y(t) \in \mathcal{R}^p$ is the measured output and $z(t) \in \mathcal{R}^q$ is the unknown output to be controlled. A, B, E, C_1, C_2, D_1, and D_2 are matrices of appropriate dimensions. Note that compared with system (2.1) we assume that two direct-feedthrough matrices are zero. In section 5.6 it will be shown how this assumption can be removed. As in the previous chapters we would like to minimize the effect of the disturbance w on the output z by finding an appropriate control input u. This time, however, the measured output y is not necessarily (x, w) or x but is a more general linear function of state and disturbance. In general the controller has less information and hence the necessary and sufficient conditions for the existence of internally stabilizing controllers which make the H_∞ norm of the closed-loop system less than some given bound will be stronger. It will turn out that we need an extra quadratic matrix inequality which tests how well we can observe the state and the disturbance.

More precisely, we seek a *dynamic* compensator Σ_F described by (2.4) such that after applying the feedback Σ_F to the system (5.1), the resulting closed-loop system is internally stable and has H_∞ norm strictly less than some a priori given bound γ. We shall derive necessary and sufficient conditions under which such a compensator exists.

A central role in our study of the above problem will be played by the quadratic matrix inequality. For $\gamma > 0$ and $P \in \mathcal{R}^{n \times n}$ we again consider the following matrix:

$$
F_\gamma(P) := \left(
\begin{array}{cc}
A^{\mathrm{T}} P + PA + C_2^{\mathrm{T}} C_2 + \gamma^{-2} PEE^{\mathrm{T}} P & PB + C_2^{\mathrm{T}} D_2 \\
B^{\mathrm{T}} P + D_2^{\mathrm{T}} C_2 & D_2^{\mathrm{T}} D_2
\end{array}
\right).
$$

Recall that if $F_\gamma(P) \geq 0$, we say that P is a solution of the quadratic matrix inequality at γ. Note that this is the same quadratic matrix inequality as the one we used in the previous chapter (see (4.2), where C and D_1 are replaced by C_2 and D_2 respectively).

We also define a dual version of this quadratic matrix inequality. For any $\gamma > 0$ and matrix $Q \in \mathcal{R}^{n \times n}$ we define the following matrix:

$$G_\gamma(Q) := \begin{pmatrix} AQ + QA^{\mathrm{T}} + EE^{\mathrm{T}} + \gamma^{-2}QC_2^{\mathrm{T}}C_2Q & QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}} \\ C_1Q + D_1E^{\mathrm{T}} & D_1D_1^{\mathrm{T}} \end{pmatrix}.$$

If $G_\gamma(Q) \geq 0$, we say that Q is a solution of the dual quadratic matrix inequality at γ. In addition to these two matrices, we define two matrices pencils, which play dual roles:

$$L_\gamma(P, s) := \begin{pmatrix} sI - A - \gamma^{-2}EE^{\mathrm{T}}P & -B \end{pmatrix},$$

$$M_\gamma(Q, s) := \begin{pmatrix} sI - A - \gamma^{-2}QC_2^{\mathrm{T}}C_2 \\ -C_1 \end{pmatrix}.$$

We note that $L_\gamma(P, s)$ is the controllability pencil associated with the system:

$$\dot{x} = (A + \gamma^{-2}EE^{\mathrm{T}}P)\,x + Bu.$$

Note that L_γ is the same controllability pencil as the one we used in the previous chapter (see (4.3)). On the other hand $M_\gamma(Q, s)$ is the observability pencil associated with the system:

$$\begin{cases} \dot{x} = (A + \gamma^{-2}QC_2^{\mathrm{T}}C_2)\,x, \\ y = \quad\quad\quad -C_1x. \end{cases}$$

Finally, we define the following two transfer matrices:

$$\begin{aligned} G_{ci}(s) &:= C_2\,(sI - A)^{-1}\,B + D_2, \\ G_{di}(s) &:= C_1\,(sI - A)^{-1}\,E + D_1, \end{aligned}$$

where G_{ci} is the same transfer matrix as the one used in the previous chapter with C and D_1 replaced by C_2 and D_2 respectively. Let $\rho(\cdot)$ denote the spectral radius. We are now in a position to formulate the main result of this chapter.

Theorem 5.1 : *Consider the system (5.1). Assume that both the system* (A, B, C_2, D_2) *as well as the system* (A, E, C_1, D_1) *have no invariant zeros on the imaginary axis. Then the following two statements are equivalent:*

(*i*)   *For the system (5.1) a time-invariant, finite-dimensional dynamic compensator Σ_F of the form (2.4) exists such that the resulting closed-loop system, with transfer matrix G_F, is internally stable and has H_∞ norm less than γ, i.e. $\|G_F\|_\infty < \gamma$.*

(*ii*)   *There are positive semi-definite solutions P, Q of the quadratic matrix inequalities $F_\gamma(P) \geq 0$ and $G_\gamma(Q) \geq 0$ satisfying $\rho(PQ) < \gamma^2$, such that the following rank conditions are satisfied*

   (*a*)   *rank $F_\gamma(P) = rank_{\mathcal{R}(s)} G_{ci}$,*

   (*b*)   *rank $G_\gamma(Q) = rank_{\mathcal{R}(s)} G_{di}$,*

   (*c*)   *rank $\begin{pmatrix} L_\gamma(P,s) \\ F_\gamma(P) \end{pmatrix} = n + rank_{\mathcal{R}(s)} G_{ci} \qquad \forall\, s \in \mathcal{C}^0 \cup \mathcal{C}^+$,*

   (*d*)   *rank $\begin{pmatrix} M_\gamma(Q,s) & G_\gamma(Q) \end{pmatrix} = n + rank_{\mathcal{R}(s)} G_{di} \quad \forall\, s \in \mathcal{C}^0 \cup \mathcal{C}^+$.*

$\square$

Remarks :

(*i*)   Note that the conditions on P in part (iii) of theorem 4.1 are exactly the same as the conditions on P in part (ii) of the above theorem. Hence the conditions on P are related to the full-information H_∞ control problem. The conditions on Q are exactly dual to the conditions on P. It can be shown that the existence of Q is related to the question how well we are able to estimate the state x on the basis of our observations y. For instance, if $y = x$ it will be shown that $Q = 0$. The test of whether we are able to estimate and control simultaneously with the desired effect is expressed in the coupling condition $\rho(PQ) < \gamma^2$.

(*ii*)   The construction of a dynamic compensator satisfying (i) can be done using a method that will be described in section 5.4. It turns out that it is always possible to find a compensator of the same dynamic order as the original plant. In fact, in [St13] it has been shown that we can find a suitable controller of McMillan degree $n - \text{rank} (\ C_1 \quad D_1\ ) + \text{rank}\, D_1$.

The case where D_1 is surjective and D_2 is injective is called the regular H_∞ control problem (or regular case) as opposed to the singular H_∞ control problem (or singular case) where either one of these assumptions is not satisfied. In the regular case an explicit formula for one controller satisfying part (i) can be given (see [Gl3]).

That is, if P and Q exist satisfying the conditions of part (ii) of theorem 5.1, then a controller satisfying part (i) is given by:

$$\Sigma_F : \left\{ \begin{array}{l} \dot{p} = K_{P,Q}p + L_{P,Q}y, \\ u = M_{P,Q}p, \end{array} \right.$$

where

$$
\begin{aligned}
M_{P,Q} &:= -\left(D_2^{\mathrm{T}}D_2\right)^{-1}\left(D_2^{\mathrm{T}}C_2 + B^{\mathrm{T}}P\right), \\
L_{P,Q} &:= \left[ED_1^{\mathrm{T}} + \left(I - \gamma^{-2}QP\right)^{-1}Q\left(C_1^{\mathrm{T}} + \gamma^{-2}PED_1^{\mathrm{T}}\right)\right]\left(D_1D_1^{\mathrm{T}}\right)^{-1}, \\
K_{P,Q} &:= A + \gamma^{-2}EE^{\mathrm{T}}P + BM_{P,Q} - L_{P,Q}\left(C_1 + \gamma^{-2}D_1E^{\mathrm{T}}P\right).
\end{aligned}
$$

For the regular H_∞ control problem, it is possible to parametrize all suitable controllers. This will be discussed in section 5.5.

(*iii*) By corollary A.7 we know that a solution P of the quadratic matrix inequality $F_\gamma(P) \geq 0$ satisfying (a) and (c) is unique. By dualizing corollary A.7 it can also be shown that a solution Q of the dual quadratic matrix inequality $G_\gamma(Q) \geq 0$ satisfying (b) and (d) is unique. The existence of P and Q can be checked via state space transformations and investigating reduced order Riccati equations.

(*iv*) We shall prove this theorem only for the case $\gamma = 1$. The general result can then be easily obtained by scaling.

Before we prove this result we shall look more closely to the result for two special cases:

State feedback: $C_1 = I, \; D_1 = 0.$
In this case we have $y = x$, i.e. we know the state of the system. The first matrix inequality $F_\gamma(P) \geq 0$ together with rank conditions (a) and (b) does not depend on C_1 or D_1 so we cannot expect a simplification there. However, $G_\gamma(Q)$ does get a special form:

$$G_\gamma(Q) = \left(\begin{array}{cc} AQ + QA^{\mathrm{T}} + EE^{\mathrm{T}} + \gamma^{-2}QC_2^{\mathrm{T}}C_2Q & Q \\ Q & 0 \end{array} \right).$$

Using this special form it can be easily seen that $G_\gamma(Q) \geq 0$ if, and only if, $Q = 0$. For the rank conditions it is interesting to investigate the normal rank of G_{di}. We have:

$$\text{rank}_{\mathcal{R}(s)} G_{di} = \text{rank}_{\mathcal{R}(s)} (sI - A)^{-1} E = \text{rank } E. \qquad (5.2)$$

By using the equality (5.2), it can be easily checked that $Q = 0$ satisfies the rank conditions (b) and (d). The condition $\rho(PQ) < \gamma^2$ is trivially satisfied if $Q = 0$. We find that in this case condition (ii) of theorem 5.1 becomes:

> There exists a positive semi-definite solution P of the quadratic matrix inequality $F_\gamma(P) \geq 0$ such that the following two rank conditions are satisfied:
>
> (i) rank $F_\gamma(P) = \text{rank}_{\mathcal{R}(s)} G_{ci}$,
>
> (ii) rank $\begin{pmatrix} L_\gamma(P, s) \\ F_\gamma(P) \end{pmatrix} = n + \text{rank}_{\mathcal{R}(s)} G_{ci}, \qquad \forall\, s \in \mathcal{C}^0 \cup \mathcal{C}^+$

which is exactly the result obtained in the previous chapter (see theorem 4.1).

Regular case: D_1 surjective and D_2 injective.
In this case it can be shown in the same way as in chapter 4 that $F_\gamma(P) \geq 0$ together with rank condition (a) is equivalent to the condition:

$$A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 + \gamma^{-2}PEE^{\mathrm{T}}P$$
$$- (PB + C_2^{\mathrm{T}}D_2)(D_2^{\mathrm{T}}D_2)^{-1}(B^{\mathrm{T}}P + D_2^{\mathrm{T}}C_2) = 0.$$

The dual version of this proof can be applied to the dual matrix inequality $G_\gamma(Q) \geq 0$ together with rank condition (b). These conditions turn out to be equivalent to the condition:

$$AQ + QA^{\mathrm{T}} + EE^{\mathrm{T}} + \gamma^{-2}QC_2^{\mathrm{T}}C_2Q$$
$$- (QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}})(D_1D_1^{\mathrm{T}})^{-1}(C_1Q + D_1E^{\mathrm{T}}) = 0.$$

The two remaining rank conditions (c) and (d) turn out to be equivalent to the requirement that the following two matrices be asymptotically stable:

$$A + \gamma^{-2}EE^{\mathrm{T}}P - B(D_2^{\mathrm{T}}D_2)^{-1}(B^{\mathrm{T}}P + D_2^{\mathrm{T}}C_2),$$

$$A + \gamma^{-2}QC_2^{\mathrm{T}}C_2 - (QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}})(D_1D_1^{\mathrm{T}})^{-1}C_1.$$

Together with the remaining condition $\rho(PQ) < \gamma^2$, we thus obtain exactly the same conditions as [Do5, Gl3].

5.3   Reduction of the original problem to an almost disturbance decoupling problem

In this section theorem 5.1 will be proven. We shall show that the problem of finding a suitable compensator Σ_F for the system (5.1) is equivalent to finding a suitable compensator Σ_F for a *new system* which has some very nice structural properties. On the basis of the results in section 2.6 we can show that for this new system the (almost) disturbance decoupling problem with measurement feedback and stability is satisfied. Clearly, this implies that we can find for this new system, and hence also for our original system, a suitable compensator. We recall that in the remainder of this chapter we assume that $\gamma = 1$. Define $F(P)$, $G(Q)$, $L(P,s)$ and $M(Q,S)$ to be equal to $F_1(P)$, $G_1(Q)$, $L_1(P,s)$ and $M_1(Q,s)$, respectively.

Lemma 5.2 : *Assume that the systems (A, B, C_2, D_2) and (A, E, C_1, D_1) have no invariant zeros on the imaginary axis. If a dynamic compensator Σ_F exists such that the resulting closed-loop system is internally stable and has H_∞ norm less than 1, then the following two conditions are satisfied:*

(i) *There is a symmetric solution $P \geq 0$ of the quadratic matrix inequality $F(P) \geq 0$ satisfying the following two rank conditions:*

   (a) *$rank\, F(P) = rank_{\mathcal{R}(s)} G_{ci}$,*

   (b) *$rank \begin{pmatrix} L(P,s) \\ F(P) \end{pmatrix} = n + rank_{\mathcal{R}(s)} G_{ci}, \qquad \forall\, s \in \mathcal{C}^0 \cup \mathcal{C}^+.$*

(ii) *There is a symmetric solution $Q \geq 0$ of the dual quadratic matrix inequality $G(Q) \geq 0$ satisfying the following two rank conditions:*

   (a) *$rank\, G(Q) = rank_{\mathcal{R}(s)} G_{di}$,*

   (b) *$rank \begin{pmatrix} M(Q,s) & G(Q) \end{pmatrix} = n + rank_{\mathcal{R}(s)} G_{di}, \qquad \forall\, s \in \mathcal{C}^0 \cup \mathcal{C}^+.$*
   $\square$

Proof : Since there is an internally stabilizing feedback which makes the H_∞ norm of the closed-loop system less than 1 for the problem with measurement feedback, it is easy to check that condition (ii) of theorem 4.1 is satisfied. This implies, according to theorem 4.1, the existence

of a matrix P satisfying the conditions in part (i) of the above lemma. By dualization it can be easily shown that there also exists a matrix Q satisfying the conditions in part (ii) of the above lemma. ∎

Assume that there exist matrices P and Q satisfying conditions (i) and (ii) in lemma 5.2. We make the following factorization of $F(P)$:

$$F(P) = \begin{pmatrix} C_{2,P}^{\mathrm{T}} \\ D_P^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} C_{2,P} & D_P \end{pmatrix} \tag{5.3}$$

where $C_{2,P}$ and D_P are matrices of suitable dimensions. This can be done since $F(P) \geq 0$. We define the following system:

$$\Sigma_P : \begin{cases} \dot{x}_P = A_P x_P + B u_P + E w_P, \\ y_P = C_{1,P} x_P + D_1 w_P, \\ z_P = C_{2,P} x_P + D_P u_P, \end{cases} \tag{5.4}$$

where $A_P := (A + E E^{\mathrm{T}} P)$ and $C_{1,P} := (C_1 + D_1 E^{\mathrm{T}} P)$. We first derive the following lemma

Lemma 5.3 : *Assume that the systems (A, B, C_2, D_2) and (A, E, C_1, D_1) have no invariant zeros on the imaginary axis. In that case the systems $(A_P, B, C_{2,P}, D_P)$ and $(A_P, E, C_{1,P}, D_1)$ have no invariant zeros on the imaginary axis either.* ☐

Proof : Note that the system $(A_P, E, C_{1,P}, D_1)$ can be obtained from the system (A, E, C_1, D_1) by applying the preliminary feedback $u = E^{\mathrm{T}} P x + v$. Therefore, the invariant zeros of the two systems coincide. Hence the system $(A_P, E, C_{1,P}, D_1)$ has no invariant zeros on the imaginary axis. The rank condition (b) of lemma 5.2 can be reformulated as

$$\mathrm{rank} \begin{pmatrix} sI - A_P & -B \\ C_{2,P} & D_P \end{pmatrix} = n + \mathrm{rank} \begin{pmatrix} C_{2,P} & D_P \end{pmatrix}, \ \ \forall\, s \in \mathcal{C}^0 \cup \mathcal{C}^+.$$

This immediately yields that $(A_P, B, C_{2,P}, D_P)$ has no invariant zeros on the imaginary axis. ∎

Next, we derive the following relation between the systems Σ and Σ_P:

Lemma 5.4 : *Let P satisfy the conditions (i) of lemma 5.2. Moreover, let an arbitrary dynamic compensator Σ_F be given, described by (2.4). Consider the following two systems, where the system on the left is the interconnection of (5.1) and (2.4) and the system on the right is the interconnection of (5.4) and (2.4):*
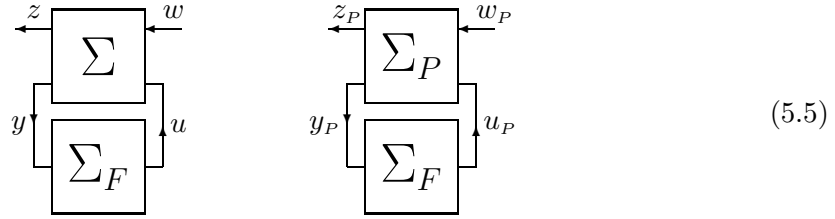
$$(5.5)$$

Then the following statements are equivalent :

(i) *The system on the left is internally stable and its transfer matrix from w to z has H_∞ norm less than 1.*

(ii) *The system on the right is internally stable and its transfer matrix from w_P to z_P has H_∞ norm less than 1.*   □

Proof : see appendix B for the proof.   ■

We assumed that for the original system (5.1) there exists an internally stabilizing dynamic compensator such that the resulting closed-loop matrix has H_∞ norm less than 1. Hence, by applying lemma 5.4, we know that the *same* compensator is internally stabilizing for the new system (5.4) and yields a closed-loop transfer matrix with H_∞ norm less than 1. Moreover we know by lemma 5.3 that Σ_F satisfies the assumptions on the invariant zeros needed to apply lemma 5.2. Therefore, if we consider for this new system the two quadratic matrix inequalities we know that there are positive semi-definite solutions to these inequalities satisfying a number of rank conditions. We shall now formalize this in the following lemma.

For arbitrary X and Y in $\mathcal{R}^{n \times n}$ we define the following matrices:

$$\bar{F}(X) := \begin{pmatrix} A_P^{\mathrm{T}} X + X A_P + C_{2,P}^{\mathrm{T}} C_{2,P} + X E E^{\mathrm{T}} X & X B + C_{2,P}^{\mathrm{T}} D_P \\ B^{\mathrm{T}} X + D_P^{\mathrm{T}} C_{2,P} & D_P^{\mathrm{T}} D_P \end{pmatrix},$$

$$\bar{G}(Y) := \begin{pmatrix} A_P Y + Y A_P^{\mathrm{T}} + E E^{\mathrm{T}} + Y C_{2,P}^{\mathrm{T}} C_{2,P} Y & Y C_{1,P}^{\mathrm{T}} + E D_1^{\mathrm{T}} \\ C_{1,P} Y + D_1 E^{\mathrm{T}} & D_1 D_1^{\mathrm{T}} \end{pmatrix},$$

$$\bar{L}(X, s) := \left(\begin{array}{cc} sI - A_P - EE^{\mathrm{T}}X & -B \end{array} \right),$$

$$\bar{M}(Y, s) := \left(\begin{array}{c} sI - A_P - YC_{2,P}^{\mathrm{T}}C_{2,P} \\ -C_{1,P} \end{array} \right).$$

Moreover we define two new transfer matrices:

$$\begin{array}{rcl} \bar{G}_{ci}(s) & := & C_{2,P}\left(sI - A_P\right)^{-1}B + D_P, \\ \bar{G}_{di}(s) & := & C_{1,P}\left(sI - A_P\right)^{-1}E + D_1. \end{array}$$

Lemma 5.5 : *Let P and Q satisfy conditions (i) and (ii) in lemma 5.2, respectively. Assume that the systems (A, B, C_2, D_2) and (A, E, C_1, D_1) have no invariant zeros on the imaginary axis. Then we have the following two results:*

(i) $X := 0$ *is a solution of the quadratic matrix inequality $\bar{F}(X) \geq 0$ and the following two rank conditions are satisfied:*

   (a) $\operatorname{rank} \bar{F}(X) = \operatorname{rank}_{\mathcal{R}(s)} \bar{G}_{ci},$

   (b) $\operatorname{rank} \left(\begin{array}{c} \bar{L}(X, s) \\ \bar{F}(X) \end{array} \right) = n + \operatorname{rank}_{\mathcal{R}(s)} \bar{G}_{ci}, \qquad \forall\, s \in \mathcal{C}^0 \cup \mathcal{C}^+.$

(ii) *A symmetric matrix Y exists which satisfies the quadratic matrix inequality $\bar{G}(Y) \geq 0$ together with the following two rank conditions:*

   (a) $\operatorname{rank} \bar{G}(Y) = \operatorname{rank}_{\mathcal{R}(s)} \bar{G}_{di},$

   (b) $\operatorname{rank} \left(\begin{array}{cc} \bar{M}(Y, s) & \bar{G}(Y) \end{array} \right) = n + \operatorname{rank}_{\mathcal{R}(s)} \bar{G}_{di}, \qquad \forall\, s \in \mathcal{C}^0 \cup \mathcal{C}^+,$

   if, and only if, $I - QP$ is invertible. Moreover, in this case there is a unique solution $Y := (I - QP)^{-1}Q$. This matrix Y is positive semi-definite if, and only if,

$$\rho\left(PQ\right) < 1. \tag{5.6}$$

$\square$

Proof : see appendix B for the proof. ∎

Proof of (i) $\Rightarrow$ (ii) in theorem 5.1 : The existence of P and Q satisfying the quadratic matrix inequalities and the corresponding four rank

conditions can be obtained directly from lemma 5.2. We know by lemma
5.3 that the two subsystems $(A_P, B, C_{2,P}, D_P)$ and $(A_P, E, C_{1,P}, D_1)$ have
no invariant zeros on the imaginary axis. We also know by lemma 5.4
that a dynamic compensator exists for the transformed system Σ_P which
internally stabilizes the system and makes the H_∞ norm of the closed-
loop system less than 1. By applying lemma 5.2 to this new system we
find that a matrix $Y \geq 0$ exists satisfying part (ii) of lemma 5.5. Hence
by lemma 5.5 we have (5.6) and therefore all the conditions in theorem
5.1, part (ii) are satisfied.   ∎

We are now going to prove the reverse implication (ii) $\Rightarrow$ (i) in theorem
5.1. Therefore we no longer assume that there exists an internally sta-
bilizing compensator which makes the H_∞ norm less than 1. Instead we
assume that matrices P and Q exist satisfying part (ii) of theorem 5.1.

In order to prove the implication (ii) $\Rightarrow$ (i) in theorem 5.1 we trans-
form the system (5.4) once again, this time however using the dual-
ized version of the original transformation. By lemma 5.5 we know
$Y = (I - QP)^{-1} Q \geq 0$ satisfies $\bar{G}(Y) \geq 0$. We factorize $\bar{G}(Y)$:

$$\bar{G}(Y) = \begin{pmatrix} E_{P,Q} \\ D_{P,Q} \end{pmatrix} \begin{pmatrix} E_{P,Q}^{\mathrm{T}} & D_{P,Q}^{\mathrm{T}} \end{pmatrix}. \tag{5.7}$$

where $E_{P,Q}$ and $D_{P,Q}$ are matrices of suitable dimensions. We define the
following system:

$$\Sigma_{P,Q} : \begin{cases} \dot{x}_{P,Q} = A_{P,Q} x_{P,Q} + B_{P,Q} u_{P,Q} + E_{P,Q} w, \\ y_{P,Q} = C_{1,P} x_{P,Q} \qquad\qquad\;\; + D_{P,Q} w, \\ z_{P,Q} = C_{2,P} x_{P,Q} + \; D_P u_{P,Q}, \end{cases} \tag{5.8}$$

where

$$\begin{aligned} A_{P,Q} &:= A_P + Y C_{2,P}^{\mathrm{T}} C_{2,P}, \\ B_{P,Q} &:= B + Y C_{2,P}^{\mathrm{T}} D_P. \end{aligned}$$

Note that the subsystems $(A_P, B, C_{2,P}, D_P)$ and $(A_P, E, C_{1,P}, D_1)$ of Σ_P
have no invariant zeros on the imaginary axis by lemma 5.3. By applying
the dual version of lemma 5.5 (i.e. the combination of lemmas B.3 and
B.4) to the system $\Sigma_{P,Q}$ with the corresponding matrix inequalities we
observe that $X_{P,Q} := 0$ and $Y_{P,Q} := 0$ satisfy the matrix inequalities and
the corresponding rank conditions for this new system. It is easy to show
that this implies:

$$\mathrm{rank} \begin{pmatrix} sI - A_{P,Q} & -B_{P,Q} \\ C_{2,P} & D_P \end{pmatrix} = n + \mathrm{rank} \begin{pmatrix} C_{2,P} & D_P \end{pmatrix}, \;\; \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+ \tag{5.9}$$

and

$$\text{rank} \begin{pmatrix} sI - A_{P,Q} & -E_{P,Q} \\ C_{1,P} & D_{P,Q} \end{pmatrix} = n + \text{rank} \begin{pmatrix} E_{P,Q} \\ D_{P,Q} \end{pmatrix}, \ \ \forall \, s \in \mathcal{C}^0 \cup \mathcal{C}^+. \quad (5.10)$$

Equation (5.9) implies that the subsystem from $u_{P,Q}$ to $z_{P,Q}$ is right-invertible and minimum-phase. On the other hand, equation (5.10) implies that the subsystem from w to $y_{P,Q}$ is left-invertible and minimum-phase. By applying lemma 5.4 and its dualized version the following corollary can be derived:

Corollary 5.6 : *Let an arbitrary compensator Σ_F of the form (2.4) be given. The following two statements are equivalent:*

  (i) *The compensator Σ_F applied to the system Σ described by (5.1), is internally stabilizing and the resulting closed-loop transfer matrix has H_∞ norm less than 1.*

  (ii) *The compensator Σ_F applied to the system $\Sigma_{P,Q}$ described by (5.8), is internally stabilizing and the resulting closed-loop transfer matrix has H_∞ norm less than 1.*   □

Remark :   We note that even if, for this new system, we can make the H_∞ norm arbitrarily small, for the original system we are only sure that the H_∞ norm will be less than 1. It is possible that a compensator for the new system yields an H_∞ norm of say 0.0001 while the same compensator makes the H_∞ norm of the original plant only 0.9999.

In section 2.6 we have shown how to solve the H_∞ problem for a system satisfying the extra conditions (5.9) and (5.10). It turned out that for such a system we can even make the H_∞ norm arbitrarily small. We are now able to complete the proof of theorem 5.1:

Proof of the implication (ii) $\Rightarrow$ (i) of theorem 5.1:   Since we can transform the original system into a system satisfying (5.9) and (5.10) we know by lemma 2.20 that we can find an internally stabilizing dynamic compensator for this new system such that the closed-loop transfer matrix has H_∞ norm less than 1. By applying corollary 5.6 we know that this compensator Σ_F satisfies the requirements in theorem 5.1, part (i).   ■

5.4   The design of a suitable compensator

In this section we shall give a method to calculate a dynamic compensator Σ_F such that the closed-loop system is internally stable and, moreover, the closed-loop transfer matrix has H_∞ norm less than 1. We shall derive this Σ_F step by step, using the following conceptual algorithm:

(i)   Calculate P and Q satisfying part (ii) of theorem 5.1. This can, for instance, be done using lemma A.6. If they do not exist or if $\rho(PQ) \geq 1$ then a feedback satisfying part (i) of theorem 5.1 does not exist and we stop.

(ii)   Perform the factorizations (5.3) and (5.7). We can now construct the system $\Sigma_{P,Q}$ as given by (5.8).

We now start solving the almost disturbance decoupling problem with measurement feedback for the system (5.8) we obtained in step (ii). We shall rename our variables and assume that we have a system in the form (5.1) which satisfies (2.45) and (2.46). We use the results as derived in section 2.6. We have to construct matrices F and G such that (2.49) and (2.51) are satisfied and, moreover, such that $A + BF$ and $A + GC_1$ are asymptotically stable. We shall only discuss the construction of F. The construction of G can be obtained by dualization.

(iii)   Construct $\mathcal{V}_g(\Sigma_{ci})$ by using lemma 2.5.

(iv)   Construct an $\tilde{F}$ such that (2.33) and (2.34) are satisfied and, moreover, such that $A + B\tilde{F} \,|\, \mathcal{V}_g(\Sigma_{ci})$ is asymptotically stable.

(v)   Let Π be the canonical projection $\mathcal{R}^n \to \mathcal{R}^n/\mathcal{V}_g(\Sigma)$ and let $\bar{A}$, $\bar{B}$ and $\bar{C}$ be such that (2.35)–(2.37) are satisfied. Construct the system Σ_{fs} as given by (2.38).

(vi)   Construct F_0 and M such that (2.39) and (2.40) are satisfied. Define the following matrices:

(a) $\tilde{A} := \bar{A} + \bar{B}F_0$,

(b) $\tilde{B} := \bar{B}M$,

(c) $\tilde{C} := \bar{C} + D_2 F_0$.

and the system

$$\Sigma_{ht} : \begin{cases} \dot{x} = \tilde{A}x + \tilde{B}u \\ z = \tilde{C}x \end{cases} \tag{5.11}$$

In this way we obtain a strongly controllable system (5.11), for which we have to find a static state feedback law $u = \bar{F}x$ such that the closed-loop system is internally stable and such that the closed-loop impulse response has H_∞ norm smaller than $\varepsilon/(3\|E\| + 1)$. This can be done via the technique as outlined in section 4.5.

We construct G by dualizing the construction of F and the required dynamic compensator is finally given by (2.53).

Note that for the regular case we can stop after step (iv) since in this case $\mathcal{V}_g(\Sigma_{ci}) = \mathcal{R}^n$ and hence Π defined in step (v) equals 0. Moreover, the technique presented will clearly yield high-gain feedback. This is generally necessary, but for the singular case it might be possible to prevent it. The latter problem, when and how to prevent high-gain feedback, is still open and will need more research.

Note that the numerical problems mentioned in section 4.5 also arise in this construction. The alternative methods mentioned in that section can also be used for the general problem with measurement feedback.

## 5.5   Characterization of achievable closed-loop systems

A characterization of all controllers which achieve the required H_∞ norm bound has been given in [Do5, Ta]. This was a kind of ball around the so-called central controller (which is given in remark (ii) after theorem 5.1). This central controller has the alternative interpretation as the controller which minimizes the entropy function defined in chapter 7. However, this characterization of all suitable controllers is only done for the regular H_∞ control problem. The question we would like to address in this section is whether similar characterizations can be given for the singular H_∞ control problem.

In our opinion it is not possible to obtain a characterization similar to the one obtained in [Do5]. *This is due to the fact that the so-called central controller might be non-proper.* But it turns out that it is much easier to characterize the closed-loop systems with H_∞ norm less than 1 we can obtain via a suitable dynamic compensator.

Let Σ, Σ_P and $\Sigma_{P,Q}$ be defined by (5.1), (5.4) and (5.8) respectively. In appendix B we construct a system Σ_U which satisfies all conditions of lemma 2.12. Moreover, for all controllers Σ_F of the form (2.4) the system on the left and the system on the right in (B.7) have the same transfer matrix. Compared to the system on the left the system on the right has some extra uncontrollable stable dynamics. Similarly we can construct a system Σ_V such that $\Sigma_{P,Q}$ and Σ_P are related via Σ_V in the same way as

Σ_P and Σ are related via Σ_U. This construction goes via dualization and therefore Σ_V^{T} (instead of Σ_V) will satisfy the conditions of lemma 2.12. To conclude this argument we look at the following interconnections:

(5.12)
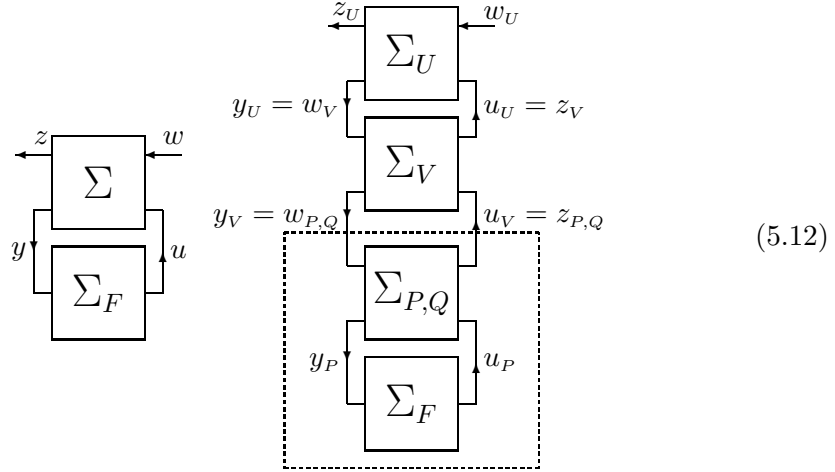
By the arguments above, the system on the left and the system on the right have the same realization except for some stable uncontrollable or unobservable dynamics on the right hand side. We denote the system inside the dashed box by X. We also denote the orthogonal projectors on the image of $(C_{2,P} \quad D_P)$ and $(E_{P,Q}^{\mathrm{T}} \quad D_{P,Q}^{\mathrm{T}})$ by Π_1 and Π_2 respectively. Finally, we denote the interconnection of Σ_U and Σ_V by $\tilde{\Sigma}_C$. The system $\tilde{\Sigma}_C$ is a system with inputs (w_U, u_V) and output (z_U, y_V). By setting $u_V = \Pi_1 \tilde{u}_V$ and $\tilde{y}_V = \Pi_2 y_V$ we define a new system from $(w_U, \tilde{u}_V)$ to $(z_U, \tilde{y}_V)$ which we denote by Σ_C. Finally by definition of Π_1 and Π_2 we can find a system Q such that $X = \Pi_1 Q \Pi_2$ and $\|X\|_\infty = \|Q\|_\infty$. We can then simplify the picture (5.12):

(5.13)
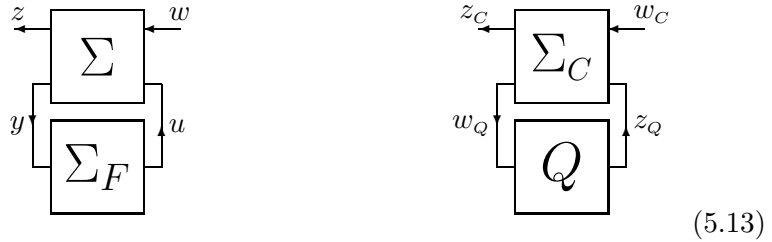
We can now derive the following theorem:

Theorem 5.7 : *Let Σ and Σ_C be as defined before. We have the following relationships between the interconnections in picture 5.13.*

(i) *For any controller Σ_F of the form (2.4) which internally stabilizes the interconnection on the left and makes the the transfer matrix from w to z strictly proper and its H_∞ norm less than 1, we can find a stable, linear, time-invariant, finite-dimensional and strictly proper system Q with H_∞ norm less than 1, such that the interconnections on the left and the right have the same closed-loop transfer matrix. Moreover, the system on the right is internally stable.*

(ii) *Conversely, for any stable, linear, time-invariant, finite-dimensional and strictly proper system Q with H_∞ norm less than 1, the interconnection on the right is internally stable. Moreover, a compensator Σ_F exists which internally stabilizes the interconnection on the left and the difference between the transfer matrices of the two interconnections has an arbitrarily small H_∞ norm.*   □

Proof : Part (i) For any compensator Σ_F which satisfies these conditions we define X as the system described by the dashed box in (5.12). It is immediate that by definition of Π_1 and Π_2 that a system Q exists such that $X = \Pi_1 Q \Pi_2$ and $\|Q\|_\infty = \|X\|_\infty$. We know that Σ_U satisfies the conditions of lemma 2.12 and we know that Σ_F makes the H_∞ norm of the interconnection on the right in (5.12) less than 1. Lemma 2.12 then yields that the interconnection of the bottom three blocks on the right in (5.12) has H_∞ norm less than 1 and is internally stable. We know that Σ_V^{T} also satisfies the conditions of lemma 2.12. Therefore, application of the dual version of this lemma yields that the dashed box in (5.12), which is by definition equal to X, is stable and has H_∞ norm less than 1. By construction Q then satisfies the same properties.

Part (ii) Assume a system Q which satisfies the conditions is given. Let $\varepsilon > 0$ be given. We first show that we can find a compensator Σ_F such that the dashed box in (5.12) with transfer matrix $G_{F,d}$ is internally stable and satisfies $\|G_F - \Pi_1 Q \Pi_2\|_\infty < \varepsilon$. We define the following system:

$$
\Sigma_n : \begin{cases}
\dot{x}_n = \begin{pmatrix} A_{P,Q} & 0 \\ 0 & F \end{pmatrix} x_n + \begin{pmatrix} B_{P,Q} \\ 0 \end{pmatrix} u + \begin{pmatrix} E_{P,Q} \\ G \end{pmatrix} w, \\
y = \begin{pmatrix} C_{1,P} & 0 \end{pmatrix} x_n \qquad\qquad + \quad D_{P,Q} w, \\
z = \begin{pmatrix} C_{2,P} & -H \end{pmatrix} x_n + \quad D_P u,
\end{cases}
$$

where $(F, G, H, 0)$ is a realization of $\Pi_1 Q \Pi_2$ with F asymptotically stable. The latter can always be found since Q is stable and strictly proper. Note that if we apply Σ_F from y to u in Σ_n then the transfer matrix from w to z is equal to $G_{F,d} - \Pi_1 Q \Pi_2$. Therefore, we have to check whether we can

find a stabilizing controller Σ_F for Σ_n which makes the H_∞ norm of the closed-loop system arbitrarily small. We already noted that $\Sigma_{P,Q}$ is such that equations (5.9) and (5.10) are satisfied. It is then easily checked that Σ_n satisfies the conditions of theorem 2.20 and hence the result follows by applying theorem 2.20. It remains to be shown that this Σ_F also makes the transfer matrices of the interconnections in picture (5.13) arbitrarily close in H_∞ norm. Note that Σ_C is asymptotically stable. Let its transfer matrix be decomposed in four (stable) components:

$$\begin{pmatrix} z_C \\ w_X \end{pmatrix} = G_C \begin{pmatrix} w_C \\ z_X \end{pmatrix} = \begin{pmatrix} G_{11,C} & G_{12,C} \\ G_{21,C} & G_{22,C} \end{pmatrix} \begin{pmatrix} w_C \\ z_X \end{pmatrix}$$

Using the fact that the closed-loop transfer matrices of the two interconnections in picture (5.12) are equal we find that the closed-loop transfer matrix of the interconnection on the left in picture (5.13) is equal to

$$G_{11,C} + G_{12,C} G_{F,d} \left(I - G_{22,C} G_{F,d} \right)^{-1} G_{21,C} \tag{5.14}$$

It is straightforward that the interconnection on the right in (5.13) is equal to:

$$G_{11,C} + G_{12,C} \Pi_1 Q \Pi_2 \left(I - G_{22,C} \Pi_1 Q \Pi_2 \right)^{-1} G_{21,C} \tag{5.15}$$

Therefore, since $\|G_{F,d} - \Pi_1 Q \Pi_2\|_\infty$ can be made arbitrarily small, since $\|G_{22,C} G_{F,d}\|_\infty < 1$ and since all terms are stable, it is immediate that the difference between the transfer matrices in (5.14) and (5.15) can be made arbitrarily small in H_∞ norm.   ∎

Remarks:

(i) The above theorem states that the interconnection on the right in (5.13) where Q is an arbitrary stable, linear, time-invariant, finite-dimensional system with H_∞ norm less than 1 parametrizes a class of stable systems with H_∞ norm strictly less than 1 which contains all systems we can obtain via a suitable controller applied to Σ. The remaining systems can be arbitrarily well approximated by a closed-loop system obtained by applying a suitable controller to Σ. In some way our class is the closure of all attainable stable closed-loop systems with H_∞ norm less than 1 (not formally, since our class is not closed).

(ii) The above characterization can be extended to include non-strictly proper closed-loop plants. However, in that case we have to constrain Q in the sense that $\Pi_1 Q(\infty) \Pi_2 = D_2 N D_1$ for some matrix N.

(*iii*) This theorem is not valid to characterize time-varying closed-loop systems. We apply the dual version of lemma 2.12. Lemma 2.12 is still true for time-varying systems (see [RNK]). However Σ_V does not satisfy the conditions of lemma 2.12. Instead Σ_V^{T} satisfies the conditions of this lemma. Therefore, in the above proof we apply the dual version of lemma 2.12. However, this dualization argument is not valid for time-varying systems. Specifically, if we require in lemma 2.12 that $G_{12}^{-1} \in H_\infty$ instead of $G_{21}^{-1} \in H_\infty$ then the lemma is still valid for time-invariant systems but not for time-varying systems.

We would like to complete this section by showing how the above is simplified if we have a regular problem, i.e. if D_1 is surjective and D_2 is injective. First of all, we have an explicit expression for Σ_P in (5.4) with C_P and D_P given by (B.3) and (B.4) without the need for the special basis of appendix A. Hence we have an explicit expression for Σ_U in (B.6). Some straightforward calculations then yield the following realization for Σ_C:

$$\Sigma_C : \begin{cases} \dot{x}_U = A x_U + B \left[M_{P,Q} e + (D_2^{\mathrm{T}} D_2)^{-1} D_2^{\mathrm{T}} z_Q \right] + E w_C \\ \dot{e} = K_{P,Q} e + L_{P,Q}(C_1 x_U + D_1 w) + S_{P,Q} z_Q \\ w_Q = D_1^{\mathrm{T}}(D_1^{\mathrm{T}} D_1)^{-1} \left[C_1 x_U + D_1 w - (C_1 + D_1 E^{\mathrm{T}} P)e \right], \\ z_U = C_2 x_U + D_2 M_{P,Q} e + z_Q, \end{cases} \quad (5.16)$$

where

$$\begin{aligned} M_{P,Q} &:= -(D_2^{\mathrm{T}} D_2)^{-1}(D_2^{\mathrm{T}} C_2 + B^{\mathrm{T}} P), \\ L_{P,Q} &:= [E D_1^{\mathrm{T}} + (I - QP)^{-1} Q(C_1^{\mathrm{T}} + P E D_1^{\mathrm{T}})](D_1 D_1^{\mathrm{T}})^{-1}, \\ K_{P,Q} &:= A + E E^{\mathrm{T}} P + B M_{P,Q} - L_{P,Q}(C_1 + D_1 E^{\mathrm{T}} P), \\ S_{P,Q} &:= [B + Y(C_2^{\mathrm{T}} D_2 + PB)](D_2^{\mathrm{T}} D_2)^{-1} D_2^{\mathrm{T}}, \end{aligned}$$

Moreover, in this special case where D_1 is surjective and D_2 is injective, for every stable system Q with H_∞ norm strictly less than 1 we can find a (unique) system Σ_F such that the interconnections in (5.13) are both internally stable and have the same transfer matrix. Here the closed-loop system is not necessarily strictly proper. This Σ_F is equal to the following

interconnection:

$$(5.17)$$
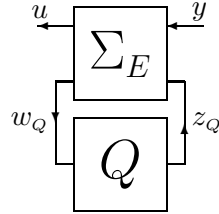
where

$$\Sigma_E : \left\{ \begin{array}{l} \dot{p} \ = K_{P,Q}p + L_{P,Q}y + S_{P,Q}z_Q, \\ w_Q = D_1^{\mathrm{T}}(D_1^{\mathrm{T}}D_1)^{-1}[y - (C_1 + D_1E^{\mathrm{T}}P)p], \\ u \ = M_{P,Q}p - (D_2^{\mathrm{T}}D_2)^{-1}D_2^{\mathrm{T}}z_Q, \end{array} \right.$$

Therefore, (5.17) where Q is stable and has H_∞ norm strictly less than 1, is a characterization of all controllers which yield a stable closed-loop system with H_∞ norm strictly less than 1. A proof of this can be derived by finding an explicit expression for $\Sigma_{P,Q}$ and then noting that Σ_F should be such that the dashed box in (5.12) is equal to $\Pi_1 Q \Pi_2$. The special properties of $\Sigma_{P,Q}$ then allow us to express Σ_F uniquely as a function of Q.

Note that because we investigate the regular case (D_1 injective and D_2 injective), we have an exact characterization of all suitable closed-loop systems and of all suitable controllers. In the singular case we can only characterize a kind of closure of the set of all suitable closed-loop systems and we *cannot* characterize all suitable controllers.

It should be noted that if we set $Q = 0$ then we obtain the central controller (as given in remark (ii) after theorem 5.1) and the corresponding closed-loop system. Note that the central controller can be given an interpretation as minimizing the entropy (see chapter 7) and as the optimal controller for the linear exponential Gaussian stochastic control problem (see [BvS, Wh]).

For the regular case this characterization is completely similar to the characterization given in [Do5, Ta].

5.6 No assumptions on any direct-feedthrough matrix

In this section we shall briefly discuss how we can extend our result in theorem 5.1 to the more general system (2.1). We set $\gamma = 1$ but the general result can be easily obtained by scaling. For this system we still

have to assume that both the system (A, B, C_2, D_{21}) as well as the system (A, E, C_1, D_{12}) have no invariant zeros on the imaginary axis. We first tackle the extra direct-feedthrough matrix D_{22} and after that the extra direct-feedthrough matrix D_{11}.

5.6.1   An extra direct-feedthrough matrix from disturbance to output

The method we use for the case $D_{22} \neq 0$ is as the method presented in section 4.6 and stems from [Gl6]. If there is an internally stabilizing feedback of the form (2.4) that makes the H_∞ norm less than 1 for the system Σ described by (2.1), then it is easy to check that a matrix S must exist such that $I - SD_{11}$ is invertible and

$$\|D_{22} + D_{21} (I - SD_{11})^{-1} SD_{12}\| < 1. \tag{5.18}$$

We assume that an S satisfying (5.18) exists (note that in section 4.6 we assume that $S = 0$ satisfies (5.18), this section extends this result). We define the following system:

$$\Sigma_S : \begin{cases} \dot{x}_S = A_S x_S + B_S u_S + E_S w_S, \\ y_S = C_{1,S} x_S + D_{11,S} u_S + D_{12,S} w_S, \\ z_S = C_{2,S} x_S + D_{21,S} u_S. \end{cases} \tag{5.19}$$

where

$$
\begin{aligned}
\tilde{D}_{22} &:= D_{22} + D_{21} (I - SD_{11})^{-1} SD_{12}, \\
E_S &:= -\left(E + B (I - SD_{11})^{-1} SD_{12}\right) \left(I - \tilde{D}_{22}^{\mathrm{T}} \tilde{D}_{22}\right)^{-1/2}, \\
C_{2,S} &:= \left(I - \tilde{D}_{22} \tilde{D}_{22}^{\mathrm{T}}\right)^{-1/2} \left(C_2 + D_{21} (I - SD_{11})^{-1} SC_1\right), \\
D_{12,S} &:= (I - D_{11} S)^{-1} D_{12} \left(I - \tilde{D}_{22}^{\mathrm{T}} \tilde{D}_{22}\right)^{-1/2}, \\
D_{21,S} &:= \left(I - \tilde{D}_{22} \tilde{D}_{22}^{\mathrm{T}}\right)^{-1/2} D_{21} (I - SD_{11})^{-1}, \\
A_S &:= \left(A + B (I - SD_{11})^{-1} SC_1\right) - E_S \tilde{D}_{22}^{\mathrm{T}} C_{2,S}, \\
B_S &:= B (I - SD_{11})^{-1} - E_S \tilde{D}_{22}^{\mathrm{T}} D_{21,S}, \\
C_{1,S} &:= (I - D_{11} S)^{-1} C_1 - D_{12,S} \tilde{D}_{22}^{\mathrm{T}} C_{2,S}, \\
D_{11,S} &:= (I - D_{11} S)^{-1} D_{11} - D_{12,S} \tilde{D}_{22}^{\mathrm{T}} D_{21,S}.
\end{aligned}
$$

We can then derive the following lemma:

Lemma 5.8 : *Let Σ_F be a dynamic controller of the form (2.4) defining an operator $\mathcal{G}_F$ from y to u. Consider the following two systems:*

Here $\tilde{\Sigma}_F$ is a compensator defined by the operator $u = (\mathcal{G}_F + S)y$. The system on the left is the interconnection of Σ described by (2.1) and $\tilde{\Sigma}_F$. The system on the right is the interconnection of Σ_S described by (5.19) and the feedback compensator Σ_F. The following two conditions are equivalent:

- *(i) The system on the left is well-posed, internally stable and the closed-loop transfer matrix from w to z has H_∞ norm less than 1.*

- *(ii) The system on the right is well-posed, internally stable and the closed-loop transfer matrix from w to z has H_∞ norm less than 1.*

$\square$

Remark : Note that we have to assume that the interconnection is well-posed. For the interconnection of Σ and Σ_F described by (2.1) and (2.4), this requirement is equivalent to the condition that the matrix $I - D_{11}N$ is invertible.

Proof : We define the following static system

$$
\Sigma_\Theta : \quad \begin{pmatrix} z_\Theta \\ y_\Theta \end{pmatrix} = \begin{pmatrix} \tilde{D}_{22} & \left(I - \tilde{D}_{22}\tilde{D}_{22}^{\mathrm{T}}\right)^{1/2} \\ -\left(I - \tilde{D}_{22}^{\mathrm{T}}\tilde{D}_{22}\right)^{1/2} & \tilde{D}_{22}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} w_\Theta \\ u_\Theta \end{pmatrix} .
$$

Let $\tilde{\Sigma}$ be the system we obtain by applying to Σ the preliminary static output feedback $u = Sy + v$, where S is such that (5.18) is satisfied. Then

we note that in the picture
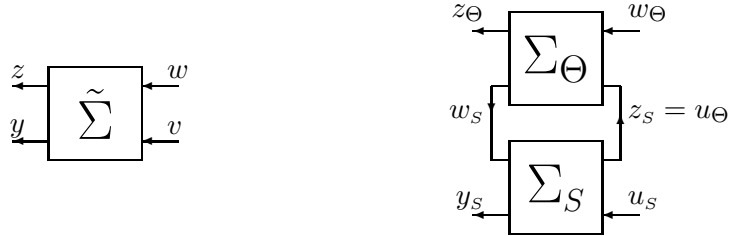
the system on the left and the interconnection on the right have identical realizations. Moreover, note that the interconnection on the right is always well-posed. The rest of the proof is as the proof of lemma 4.5. ■

In the same way as lemma 4.6 we are able to derive the following lemma:

Lemma 5.9 : *Let S be such that (5.18) is satisfied. Assume that Σ and Σ_S are described by (2.1) and (5.19) respectively.*

The systems (A, B, C_2, D_{21}) and $(A_S, B_S, C_{2,S}, D_{21,S})$ have the same invariant zeros. Also, the invariant zeros of the systems (A, E, C_1, D_{12}) and $(A_S, E_S, C_{1,S}, D_{12,S})$ are equal.   □

Thus we are able to obtain the following theorem:

Theorem 5.10 : *Consider the system Σ described by (2.1). Assume that the systems (A, B, C_2, D_{21}) and (A, E, C_1, D_{12}) have no invariant zeros on the imaginary axis. Then the following statements are equivalent:*

(i) *A compensator $\tilde{\Sigma}_F$ of the form (2.4) exists such that the closed-loop system $\Sigma \times \tilde{\Sigma}_F$ is well-posed, internally stable and has H_∞ norm less than 1.*

(ii) *An S exists such that (5.18) is satisfied and if, for this S, we define the system Σ_S by (5.19), then for Σ_S a controller Σ_F of the form (2.4) exists such that the interconnection $\Sigma_S \times \Sigma_F$ is well-posed, internally stable and has H_∞ norm less than 1. Moreover the subsystems $(A_S, B_S, C_{2,S}, D_{21,S})$ and $(A_S, E_S, C_{1,S}, D_{12,S})$ have no invariant zeros on the imaginary axis.*   □

Remarks :

(i) Note that if we find a controller $\tilde{\Sigma}_F$ with associated input–output operator $\tilde{\mathcal{G}}_F$ satisfying part (ii), then we know that a controller with input–output operator $\tilde{\mathcal{G}}_F + S$ satisfies part (i). Conversely, if we find a controller Σ_F with associated input–output operator $\mathcal{G}_F$ satisfying part (i) then the controller with associated input–output operator $\mathcal{G}_F - S$ satisfies part (i).

(ii) Remember that the existence of suitable controllers for the system Σ_S is independent of our specific choice for the matrix S satisfying (5.18).

(iii) In this way we have reduced the problem of finding necessary and sufficient conditions under which there exists an internally stabilizing controller which makes the H_∞ norm of the closed-loop system less than 1 to the same problem for the system Σ_S. However, for the latter system we have $D_{22} = 0$. In the next subsection we show how we can reduce it to the same problem for a system which has both $D_{11} = 0$ and $D_{22} = 0$. We can show that after these two steps for the system obtained in this way the two subsystems do not have invariant zeros on the imaginary axis if, and only if, the two subsystems for the original system (2.1) do not have invariant zeros on the imaginary axis. Hence we may apply theorem 5.1 to this new system.

5.6.2 An extra direct-feedthrough matrix from control to measurement

In this section we investigate a system Σ, described by (2.1) with $D_{22} = 0$. It has been shown in the previous subsection how we can reduce the problem of finding internally stabilizing controllers which make the closed-loop H_∞ norm less than 1 to the same problem for a system with $D_{22} = 0$. In this subsection we shall show how we can reduce it even further to the case that both $D_{11} = 0$ as well as $D_{22} = 0$.

We define the following system:

$$\tilde{\Sigma} : \begin{cases} \dot{x} = Ax + Bu + Ew, \\ y = C_1 x \qquad\quad + D_{12}w, \\ z = C_2 x + D_{21}u. \end{cases} \tag{5.20}$$

Assume that for the system Σ we have a controller Σ_F of the form (2.4) such that the interconnection $\Sigma \times \Sigma_F$ is well-posed, i.e. $I - D_{11}N$ is

invertible. Moreover, assume that this controller is internally stabilizing and makes the H_∞ norm less than 1. We define the following controller

$$
\tilde{\Sigma}_F : \quad \begin{cases} \dot{p} = \tilde{K}p + \tilde{L}y, \\ y = \tilde{M}p + \tilde{N}y, \end{cases} \tag{5.21}
$$

where

$$
\begin{aligned}
\tilde{K} &:= K + L\left(I - D_{11}N\right)^{-1} D_{11}M, \\
\tilde{L} &:= L\left(I - D_{11}N\right)^{-1}, \\
\tilde{M} &:= \left(I - ND_{11}\right)^{-1} M, \\
\tilde{N} &:= N\left(I - D_{11}N\right)^{-1}.
\end{aligned}
$$

Then it is easily shown that $\tilde{\Sigma}_F$ is an internally stabilizing feedback for the system $\tilde{\Sigma}$. Moreover, the interconnection $\tilde{\Sigma} \times \tilde{\Sigma}_F$ and the interconnection $\Sigma \times \Sigma_F$ have identical realizations and hence the interconnection $\tilde{\Sigma} \times \tilde{\Sigma}_F$ has H_∞ norm less than 1.

On the other hand, assume that we have a controller $\tilde{\Sigma}_F$ of the form (5.21) for $\tilde{\Sigma}$ which is internally stabilizing and makes the H_∞ norm strictly less than 1. Note that this interconnection is always well-posed. We know that for all $\varepsilon > 0$ there exists a matrix Z with $\|Z\| < \varepsilon$ such that

$$
I + \left(\tilde{N} + Z\right) D_{11}
$$

is invertible. Denote the controller we obtain by replacing $\tilde{N}$ by $\tilde{N} + Z$ by $\hat{\Sigma}_F(Z)$. It is easy to see that for sufficiently small ε (and hence sufficiently small Z) the closed-loop system, after applying the compensator $\hat{\Sigma}_F(Z)$ to $\tilde{\Sigma}$, is still internally stable and still has H_∞ norm less than 1. Assume that Z is chosen like this then the controller Σ_F of the form (2.4) and defined by

$$
\begin{aligned}
K &:= \tilde{K} - \tilde{L}\left(I + D_{11}\left(\tilde{N} + Z\right)\right) D_{11}\tilde{M}, \\
L &:= \tilde{L}\left(I + D_{11}\left(\tilde{N} + Z\right)\right)^{-1}, \\
M &:= \left(I + \left(\tilde{N} + Z\right) D_{11}\right)^{-1} \tilde{M}, \\
N &:= \left(I + \left(\tilde{N} + Z\right) D_{11}\right)^{-1} \left(\tilde{N} + Z\right),
\end{aligned}
$$

is internally stabilizing for the system Σ and the closed-loop system $\Sigma \times \Sigma_F$ has H_∞ norm less than 1.

In the above we derived the following theorem:

Theorem 5.11 : *Let the system* Σ *described by (2.1) with* $D_{22} = 0$ *be given. Then the following statements are equivalent:*

(*i*) *A compensator* Σ_F *of the form (2.4) exists such that the closed-loop system* $\Sigma \times \Sigma_F$ *is well-posed, internally stable and has* H_∞ *norm less than* 1.

(*ii*) *For the system (5.20) a controller of the form (5.21) exists which is internally stabilizing and makes the* H_∞ *norm less than* 1. $\qquad\Box$

Remarks : Thus we see that we can reduce the problem of finding a controller which is internally stabilizing and which makes the H_∞ norm less than 1 (a suitable controller) for the system Σ described by (2.1) with $D_{22} = 0$ to the problem of finding a suitable controller for the system we obtain by setting $D_{11} = 0$ in (2.1). The system we thus obtain has $D_{11} = 0$ as well as $D_{22} = 0$. In the previous subsection we showed how we could reduce the problem of finding a suitable controller for the general system (2.1) to the problem of finding a suitable controller for a system of the same form (2.1) but with $D_{22} = 0$. In both steps the property that the two subsystems have no invariant zeros on the imaginary axis is preserved. Hence, if for some general system Σ of the form (2.1) we need to find a suitable controller and if the subsystems (A, B, C_2, D_{21}) and (A, E, C_1, D_{12}) have no invariant zeros on the imaginary axis, then we can reduce this problem to the problem of finding a suitable controller for a system on which we may apply the main result of this chapter : theorem 5.1.

5.7 Conclusion

In this chapter we have given a complete treatment of the H_∞ problem with measurement feedback without restrictions on the direct-feedthrough matrices. However, how we can treat invariant zeros on the imaginary axis remains an open problem. This problem is studied in [S4]. Other open problems are the determination of the minimally required dynamic order of the controller and the characterization of the behaviour of the feedbacks and closed-loop system if we tighten the bound γ. The latter problem has already been investigated. It is possible that the infimum over all stabilizing controllers of the closed-loop H_∞ norm is never attained, attained by a non-proper controller or attained by a proper controller (see [Fr2]). Using the ideas of this chapter it might be possible to characterize whether we can attain the infimum by a proper controller.

Finally, it would be interesting to characterize all controllers which achieve the H_∞ norm bound. For the singular problem a major step has been made in section 5.5. However, this mainly focuses on achievable closed-loop systems instead of suitable controllers.

In our opinion this chapter again gives extra support to our claim that the approach to solve the H_∞ problem in the time-domain is a much more intuitive and appealing approach than the other methods used in recent papers. Only when using frequency-related weighting do we lose the intuition of the meaning of the filters. Hence, when discussing choices of the filters it might be preferable to work in the frequency-domain. However, for intuition when actually solving the H_∞ control problem, one is really better off in the time-domain.

Chapter 6

The singular zero-sum differential game with stability

6.1 Introduction

In this chapter we shall consider the zero-sum linear quadratic finite-dimensional differential game. This is an area of research which was rather popular during the seventies (see e.g. [Ban, Ma, BO, Sw]).

In the last few years, the solution of the regular H_∞ control problem (see chapter 3 and [Do5, Kh3, Pe]) turned out to contain the same kind of algebraic Riccati equation as the one appearing in the solution of the zero-sum differential game (see [Ban, Ma, Sw]). This Riccati equation has the special property that the quadratic term is, in general, indefinite, in contrast to, for instance, the equation appearing in linear quadratic optimal control theory (see [Wi7]), where the quadratic term in the Riccati equation is always definite.

Since in H_∞ control theory the solution of the algebraic Riccati equation has no meaning in itself it is interesting to give a more intuitive characterization such as a Nash equilibrium in the theory of differential games. Recently, a number of papers appeared which studied a zero-sum differential game with the goal of obtaining such a characterization (see [Pe2, Pe3, We]). A very recent book on the relation between H_∞ and differential games is [BB].

In chapter 4 it has been shown that if the direct-feedthrough matrix from the control input to the output is not injective then, instead of an algebraic Riccati equation, we get a quadratic matrix inequality. A similar phenomenon also occurs in linear quadratic optimal control theory, although in that case we get a *linear* matrix inequality (see [Wi7]).

This chapter is concerned with the zero-sum differential game in the case that the direct-feedthrough matrix is not injective. It will be

shown that, as expected, existence of equilibria is related to solutions of a quadratic matrix inequality. By using results from H_∞ control theory, we are able to derive necessary conditions for the existence of an equilibrium, which, to our knowledge, has not been done in previous papers. We shall study the differential game with certain *stability* requirements since it turns out to give results which indeed centre around the same solution of the quadratic matrix inequality as the one which appears in H_∞ control. If we assume detectability, then the problems with and without stability turn out to be equivalent.

We give this treatment of the differential game in this book although the formal proofs of the results on H_∞ control theory do not depend on results derived in this chapter. The reason for nevertheless including this chapter lies in the fact that the intuition, which yielded the results on H_∞ control, mostly stems from this chapter. The results of this chapter have already appeared in [St3].

The outline of this chapter is as follows: in section 6.2 we formulate the problem and give our main results. In section 6.3 we prove the existence of an equilibrium under certain sufficient conditions. After that, in section 6.4 we derive necessary conditions for the existence of equilibria. The proofs in these two sections are very much concerned with the specific choice of bases and other results of appendix A. In section 6.5 we show that if the direct-feedthrough matrix from control input to output is injective, then the necessary conditions of section 6.4 are also sufficient. We conclude in section 6.6 with some remarks.

6.2   Problem formulation and main results

We shall consider the zero-sum, infinite horizon, linear quadratic differential game with cost criterion

$$\mathcal{J}(u,w) = \int_0^\infty z^{\mathrm{T}}(t)z(t) - w^{\mathrm{T}}(t)w(t)dt, \tag{6.1}$$

and dynamics given by the following linear and finite-dimensional system

$$\Sigma \ : \ \begin{cases} \dot{x} = Ax + Bu + Ew, \qquad x(0) = \xi, \\ z = Cx + Du. \end{cases} \tag{6.2}$$

Here, for all t we have $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^m$, $w(t) \in \mathcal{R}^l$ and $z(t) \in \mathcal{R}^p$. A,B,C,D and E are matrices of appropriate dimensions. Note that contrary to the other chapters we shall in general consider systems with

initial state unequal to zero. We assume that (A, B) is stabilizable. We define the following class of functions:

$$\mathcal{U}_{fb}^k = \left\{ v : \mathcal{R}^n \times \mathcal{R} \to \mathcal{R}^k \mid \forall x \in \mathcal{L}_2^n \quad v(x(.), .) \in \mathcal{L}_2^k \right\}.$$

Note that we can consider $\mathcal{L}_2^k$ as a subset of $\mathcal{U}_{fb}^k$ by identifying with each function $v \in \mathcal{L}_2^k$ a function $\tilde{v} \in \mathcal{U}_{fb}^k$ as follows:

$$\tilde{v}(x, t) := v(t) \ \forall x \in \mathcal{R}^n, t \in \mathcal{R}^+.$$

We also define the following class of functions:

$$\mathcal{U}_{st}^m(\xi) := \left\{ u \in \mathcal{U}_{fb}^m \mid \forall \, w \in \mathcal{L}_2^l : \ (u, w) \text{ is an admissible pair} \right.$$

$$\left. \text{for initial value } \xi \right\}.$$

We call $(\tilde{u}, \tilde{w}) \in \mathcal{U}_{st}^m \times \mathcal{U}_{fb}^l$ an *admissible pair* for initial value ξ if by setting $u(.) = \tilde{u}(x(.), .)$ and $w(.) = \tilde{w}(x(.), .)$ the differential equation (6.2) <u>has a unique solution</u> $x_{u,w,\xi}$ on $[0, \infty)$ which satisfies $x_{u,w,\xi} \in \mathcal{L}_2^n$. Hence by the definition of $\mathcal{U}_{fb}$ we have $u \in \mathcal{L}_2^m$ and $w \in \mathcal{L}_2^l$. This implies that the resulting output trajectory satisfies $z_{u,w,\xi} \in \mathcal{L}_2^p$ and hence $\mathcal{J}(u, w)$ as defined in (6.1) is well-defined. We shall only consider inputs of this form. Since (A, B) is stabilizable the class of admissible pairs is non-empty for every initial state ξ.

We shall call u a *minimizing player* and its goal is to minimize the cost criterion $\mathcal{J}(u, w)$. In the same way we shall call w a *maximizing player* which would like to maximize the cost criterion $\mathcal{J}(u, w)$.

Definition 6.1 : *The system (6.2) with criterion function (6.1) is said to have an equilibrium if, for every initial value ξ, $u_0 \in \mathcal{U}_{st}^m(\xi)$ and $w_0 \in \mathcal{U}_{fb}^l$ exist such that (u_0, w_0) is an admissible pair and, moreover,*

$$\mathcal{J}(u_0, w) \leq \mathcal{J}(u_0, w_0) \leq \mathcal{J}(u, w_0),$$

for all $u \in \mathcal{U}_{st}^m(\xi)$ and $w \in \mathcal{U}_{fb}^l$ such that (u, w_0) and (u_0, w) are admissible pairs. □

Note that in the theory of differential games, this is often called a Nash equilibrium. The existence of an equilibrium is, in general, too strong a condition. We shall define a weaker version.

Definition 6.2 : *The system (6.2) with cost criterion (6.1) is said to have an almost equilibrium if a function $\mathcal{J}^* : \mathcal{R}^n \to \mathcal{R}$ exists such that for all $\varepsilon > 0$ and for every state $\xi \in \mathcal{R}^n$ there are $u_0 \in \mathcal{U}^m_{st}(\xi)$ and $w_0 \in \mathcal{U}^l_{fb}$ such that (u_0, w_0) is admissible and, moreover,*

$$
\begin{aligned}
\mathcal{J}(u_0, w) &\leq \mathcal{J}^*(\xi) + \varepsilon, \\
\mathcal{J}(u, w_0) &\geq \mathcal{J}^*(\xi) - \varepsilon,
\end{aligned}
\tag{6.3}
$$

for all $u \in \mathcal{U}^m_{st}(\xi)$ and $w \in \mathcal{U}^l_{fb}$ such that (u, w_0) and (u_0, w) are admissible pairs. $\qquad\square$

Remarks :

(i) Note that an equilibrium certainly defines an almost equilibrium since, in this case, we can find u_0 and w_0 such that $\mathcal{J}^*(\xi) := \mathcal{J}(u_0, w_0)$ satisfies (6.3) for $\varepsilon = 0$.

(ii) Note that if either u_0 or w_0 is fixed, then choosing the other input in $\mathcal{U}_{fb}$ such that we have an admissible pair, results in well-defined functions in $\mathcal{L}_2$ for the state, the minimizing player and the maximizing player. Hence in definition 6.2 and also in definition 6.1 we can, without loss of generality, assume that u and w are in $\mathcal{L}_2$ instead of $\mathcal{U}_{fb}$. In this case there are no longer any restrictions on w because (u_0, w) is admissible for all $w \in \mathcal{L}^l_2$ since $u_0 \in \mathcal{U}^m_{st}(\xi)$. This condition is rather unusual in zero-sum linear quadratic differential games. Intuitively it means that we hand over the responsibility of the condition $x \in \mathcal{L}^n_2$ to the minimizing player. Without this assumption it is possible that u_0 and w_0 exist such that

$$
\begin{aligned}
\mathcal{J}(u_0, w) &\leq M_1 \\
\mathcal{J}(u, w_0) &\geq M_2
\end{aligned}
$$

for all u and w such that (u_0, w) and (u, w_0) are admissible pairs and $M_2 > M_1$. Clearly (u_0, w_0) is not admissible but neither u_0 nor w_0 will change since that would be contrary to their objectives of minimizing and maximizing the cost criterion respectively. To prevent such a deadlock we hand over the responsibility for $x \in \mathcal{L}^n_2$ to one of the players. An example of this phenomenon is given by the system

$$
\left\{
\begin{aligned}
\dot{x} &= x + u + w, \\
z &= \phantom{x +{}} u,
\end{aligned}
\right.
$$

with $u_0 = w_0 = 0$.

We shall derive conditions for the existence of an almost equilibrium. Since we do not assume that the D matrix is injective it is not surprising that, as in the singular LQ problem, we find a matrix inequality instead of a Riccati equation. We repeat a number of definitions already used in chapter 4:

$$F(P) := \left(\begin{array}{cc} A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C + PEE^{\mathrm{T}}P & PB + C^{\mathrm{T}}D \\ B^{\mathrm{T}}P + D^{\mathrm{T}}C & D^{\mathrm{T}}D \end{array} \right).$$

We call a symmetric matrix $P \in \mathcal{R}^{n \times n}$ a solution of the quadratic matrix inequality if $F(P) \geq 0$. Furthermore, we define

$$L(P, s) := \left(\begin{array}{cc} sI - A - EE^{\mathrm{T}}P & -B \end{array} \right),$$

and finally we define $G_{ci}(s) := C(sI - A)^{-1}B + D$. We shall now present the main results of this chapter:

Theorem 6.3 : *Consider the system (6.2) with cost criterion (6.1) and assume that (A, B) is stabilizable. An almost equilibrium exists if the following condition is satisfied:*

A positive semi-definite solution P of $F(P) \geq 0$ exists such that

    (i) $\operatorname{rank} F(P) = \operatorname{rank}_{\mathcal{R}(s)} G_{ci}$,

    (ii) $\operatorname{rank} \left(\begin{array}{c} L(P, s) \\ F(P) \end{array} \right) = n + \operatorname{rank}_{\mathcal{R}(s)} G_{ci}, \quad \forall s \in \mathcal{C}^+ \cup \mathcal{C}^0.$

Moreover $\mathcal{J}^*(\xi) = \xi^{\mathrm{T}} P \xi$ defines an almost equilibrium and for each bounded set of initial values and for each $\varepsilon > 0$ we can find static state feedbacks F_u, F_w such that $u_0 = F_u x$ and $w_0 = F_w x$ satisfy (6.3) for all initial values in that set. $\qquad\square$

Remarks :

    (i) If the matrix D is injective, the existence of P satisfying the conditions of theorem 6.3 guarantees the existence of an (exact) equilibrium.

    (ii) We shall show that P also satisfies the following equality:

$$\xi^{\mathrm{T}} P \xi = \inf_{u \in \mathcal{U}_{st}^m(\xi)} \sup_{w \in \mathcal{L}_2^l} \mathcal{J}(u, w).$$

(*iii*) Note that these sufficient conditions for the existence of an almost equilibrium are nothing other than the requirement that the H_∞ norm of the closed-loop system can be made less than 1 by a suitable state feedback $u = Fx$.

The next theorem shows the necessity of the above conditions:

Theorem 6.4 : *Consider the system (6.2) with cost criterion (6.1). Assume that (A, B) is stabilizable and assume that (A, B, C, D) has no invariant zeros on the imaginary axis. If an almost equilibrium exists, then the following condition is satisfied:*

A positive semi-definite solution P of $F(P) \geq 0$ exists such that

(*i*) $rank\ F(P) = rank_{\mathcal{R}(s)}\ G_{ci}$,

(*ii*) $rank \begin{pmatrix} L(P, s) \\ F(P) \end{pmatrix} = n + rank_{\mathcal{R}(s)}\ G_{ci}, \quad \forall s \in \mathcal{C}^+.$

Moreover, in the case that D is injective, the above condition is also sufficient for the existence of an almost equilibrium.   □

Remarks : Although in the case that D is injective we can prove the existence of an almost equilibrium under the conditions of theorem 6.4, we find time-varying state feedback laws for u_0 and w_0 contrary to the conditions of theorem 6.3 under which we could find time-invariant feedback laws. Moreover, the conditions of theorem 6.3 guaranteed the existence of an (exact) equilibrium when D is injective. This time we only find an almost equilibrium.

6.3 Existence of almost equilibria

In this section we assume that a matrix P satisfying the conditions of theorem 6.3 exists. We shall show that there exists an *almost equilibrium*. Moreover, if D is injective, we show that an (exact) equilibrium exists. The proof will be strongly related to the proof given in section 4.4. We first apply the preliminary feedback as defined in appendix A, i.e. $u = F_0 x + v$ where F_0 is defined by (A.1). We obtain the system

$$\hat{\Sigma} : \begin{cases} \dot{x} = (A + BF_0)x + Bv + Ew, \\ z = (C + DF_0)x + Dv. \end{cases}$$

For this system we define the cost criterion:

$$\hat{\mathcal{J}}(v,w) = \int_0^\infty z^{\mathrm{T}}(t)z(t) - w^{\mathrm{T}}(t)w(t)dt. \qquad (6.4)$$

Clearly this system has an (almost) equilibrium if, and only if, the original system Σ has an (almost) equilibrium. Moreover, for initial condition ξ, (u,w) is an admissible pair if, and only if, $(v,w) = (u - F_0 x, w)$ is an admissible pair. Finally, if $v = u - F_0 x$, then we have $\hat{\mathcal{J}}(v,w) = \mathcal{J}(u,w)$. Therefore in the remainder of this section we may investigate the system $\hat{\Sigma}$, instead of Σ, with cost criterion (6.4).

We shall use the following lemma which will give theorem 6.3 as an almost direct result.

Lemma 6.5 : *Let P be given such that $F(P) \geq 0$. Choose the bases of appendix A so that P has the form (A.15). Then for all admissible pairs (v,w) we have:*

$$
\begin{aligned}
\hat{\mathcal{J}}(v,w) &= \|z\|_2^2 - \|w\|_2^2 \\
&= \xi^{\mathrm{T}} P \xi + \int_0^\infty x_1(\tau)^{\mathrm{T}} R(P_1)x_1(\tau)d\tau + \|C_{23}q_3\|_2^2 + \|\hat{D}\tilde{v}_1\|_2^2 - \|\tilde{w}\|_2^2.
\end{aligned}
$$
$$(6.5)$$

where $R(P_1)$ is defined by (A.17) and

$$
\begin{aligned}
q_3 &:= x_3 + (C_{23}^{\mathrm{T}} C_{23})^{-1} \left(A_{13}^{\mathrm{T}} P_1 + C_{23}^{\mathrm{T}} C_{21}\right) x_1, & (6.6) \\
\tilde{v}_1 &:= v_1 + \left(\hat{D}^{\mathrm{T}} \hat{D}\right)^{-1} B_{11}^{\mathrm{T}} P_1 x_1, & (6.7) \\
\tilde{w} &:= w - E^{\mathrm{T}} P x. & (6.8)
\end{aligned}
$$

Moreover the dynamics in these new coordinates are given by

$$\dot{x}_1 = \bar{A}_{11}x_1 + A_{13}q_3 + B_{11}\tilde{v}_1 + E_1\tilde{w} \qquad (6.9)$$

$$
\begin{pmatrix} \dot{x}_2 \\ \dot{q}_3 \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & \tilde{A}_{33} \end{pmatrix} \begin{pmatrix} x_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix} v_2 + \begin{pmatrix} \bar{A}_{21} \\ \bar{A}_{31} \end{pmatrix} x_1 + \begin{pmatrix} B_{21} \\ B_{31} \end{pmatrix} \tilde{v}_1 + \begin{pmatrix} E_2 \\ \tilde{E}_3 \end{pmatrix} \tilde{w}
$$
$$(6.10)$$

$$
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_3 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ C_{23} \end{pmatrix} q_3 \qquad (6.11)
$$

Here we used the definitions given just after (4.13) in chapter 4. Moreover we used the following three definitions

$$
\begin{aligned}
\bar{A}_{11} &:= \tilde{A}_{11} + E_1 E_1^{\mathrm{T}} P_1, \\
\bar{A}_{21} &:= \tilde{A}_{21} + E_2 E_1^{\mathrm{T}} P_1, \\
\bar{A}_{31} &:= \tilde{A}_{31} + E_3 E_1^{\mathrm{T}} P_1.
\end{aligned} \qquad \Box
$$

Remarks :

(i) Note that the system dynamics given here are almost the same as the system dynamics given in (4.11)– (4.13). Only in chapter 4 we set $\tilde{v}_1 = 0$ and we did not make the transformation from w to $\tilde{w}$.

(ii) Note that $\bar{A}_{11}$ and $\tilde{A}_{11}$ are both asymptotically stable by theorem A.6 and corollary A.8 respectively.

(iii) When D is injective the above lemma simplifies considerably. In that case, we find

$$
\begin{aligned}
\hat{\mathcal{J}}(v,w) &= \|z\|_2^2 - \|w\|_2^2 \\
&= \xi^{\mathrm{T}} P \xi + \int_0^\infty x_1(\tau)^{\mathrm{T}} R(P) x_1(\tau) d\tau + \|Dv\|_2^2 - \|\tilde{w}\|_2^2.
\end{aligned}
$$

where $R(P)$ is defined by

$$
R(P) = A^{\mathrm{T}} P + P A + C^{\mathrm{T}} C + P E E^{\mathrm{T}} P
$$

$$
- (B^{\mathrm{T}} P + D^{\mathrm{T}} C) (D^{\mathrm{T}} D)^{-1} (B^{\mathrm{T}} P + D_1^{\mathrm{T}} C)
$$

Moreover,

$$
\begin{aligned}
\tilde{v} &:= v + (D^{\mathrm{T}} D)^{-1} B^{\mathrm{T}} P x, \\
\tilde{w} &:= w - E^{\mathrm{T}} P x,
\end{aligned}
$$

and the dynamics are governed by:

$$
\begin{aligned}
\dot{x} &= \bar{A} x + B \tilde{v} + E \tilde{w}, \\
z &= \bar{C} x + D \tilde{v},
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{A} &:= A - B (D^{\mathrm{T}} D)^{-1} (B^{\mathrm{T}} P + C^{\mathrm{T}} D) + E E^{\mathrm{T}} P, \\
\bar{C} &:= C - D (D^{\mathrm{T}} D)^{-1} (B^{\mathrm{T}} P + C^{\mathrm{T}} D).
\end{aligned}
$$

Proof : By using the system equations (6.2) we find

$$\frac{d}{dt}\left[x^{\mathrm{T}}(t)Px(t) - \xi^{\mathrm{T}}P\xi + \int_0^t z(\tau)^{\mathrm{T}}z(\tau) - w(\tau)^{\mathrm{T}}w(\tau)\ d\tau\right] \qquad (6.12)$$

$$= \begin{pmatrix} x \\ v \\ w \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} A_{F_0}^{\mathrm{T}}P + PA_{F_0} + C_{F_0}^{\mathrm{T}}C_{F_0} & PB & PE \\ B^{\mathrm{T}}P & D^{\mathrm{T}}D & 0 \\ E^{\mathrm{T}}P & 0 & -I \end{pmatrix}\begin{pmatrix} x \\ v \\ w \end{pmatrix}$$

$$= \begin{pmatrix} x \\ v \\ \tilde{w} \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} A_{F_0}^{\mathrm{T}}P + PA_{F_0} + C_{F_0}^{\mathrm{T}}C_{F_0} + PEE^{\mathrm{T}}P & PB & 0 \\ B^{\mathrm{T}}P & D^{\mathrm{T}}D & 0 \\ 0 & 0 & -I \end{pmatrix}\begin{pmatrix} x \\ v \\ \tilde{w} \end{pmatrix}. \qquad (6.13)$$

where we have

$$\begin{aligned} A_{F_0} &:= A + BF_0, \\ C_{F_0} &:= C + DF_0, \end{aligned}$$

and $\tilde{w}$ as given by (6.8) and F_0 as defined by (A.1). We can now use the decomposition as defined in (A.2) and we find that (6.13) is equal to:

$$\begin{pmatrix} x_1 \\ x_3 \\ v_1 \\ \tilde{w} \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} P_1A_{11} + A_{11}^{\mathrm{T}}P_1 + C_{21}^{\mathrm{T}}C_{21} + P_1E_1E_1^{\mathrm{T}}P_1 & P_1A_{13} + C_{21}^{\mathrm{T}}C_{23} & P_1B_{11} & 0 \\ A_{13}^{\mathrm{T}}P_1 + C_{23}^{\mathrm{T}}C_{21} & C_{23}^{\mathrm{T}}C_{23} & 0 & 0 \\ B_{11}^{\mathrm{T}}P_1 & 0 & \hat{D}^{\mathrm{T}}\hat{D} & 0 \\ 0 & 0 & 0 & -I \end{pmatrix}\begin{pmatrix} x_1 \\ x_3 \\ v_1 \\ \tilde{w} \end{pmatrix}$$

When we finally use the definitions (6.6) and (6.7) and integrate the equation (6.12) from 0 to ∞ we find the equation (6.5). Here we used that

$$\lim_{t\to\infty} x^{\mathrm{T}}(t)Px(t) = 0,$$

since the pair (v, w) is admissible and hence $x \in \mathcal{L}_2^n$ and $\dot{x} \in \mathcal{L}_2^n$. Moreover (v, w) admissible implies that the integral in (6.5) is well-defined. ∎

Assume that we have a matrix P such that $F(P) \geq 0$ and such that rank $F(P) = \text{rank}_{\mathcal{R}(s)}G_{ci}$. Then in the bases of appendix A the matrix P can be written in the form (A.15) with $R(P_1) = 0$. The minimizing player can control v_1 and v_2. As we did in chapter 4 we assume for the moment that the minimizing player can control v_1 and x_3 and hence also $\tilde{v}_1$ and q_3. The maximizing player can control $\tilde{w}$. From the previous lemma it is then intuitively clear that the minimizing player should set $\tilde{v}_1 = 0$ and $q_3 = 0$ and the maximizing player should set $\tilde{w} = 0$ which would yield

an equilibrium. Two things remain to be done. First we have to work out how the minimizing player can control q_3. It will turn out that the minimizing player can control q_3 arbitrarily but not exactly, which is the reason why we find only an almost equilibrium and not an equilibrium (if D is injective we do not have q_3 and an exact equilibrium). Finally, it has to be shown that the above choices guarantee that the corresponding u and w are in the desired feedback classes. For this we have to use the second rank condition that P satisfies.

We shall use lemma 6.5 together with lemma 2.18, which turns out to be extremely useful for singular differential games, to prove theorem 6.3:

Proof of theorem 6.3 :   The matrix P satisfies the conditions of theorem 6.3. Let $\varepsilon > 0$ be given. First note that when we choose $\tilde{w} = 0$, i.e. $w_0 := E^\mathrm{T} Px$ then by lemma (6.5) we have:

$$\hat{\mathcal{J}}(v, w_0) \geq \xi^\mathrm{T} P\xi \geq \xi^\mathrm{T} P\xi - \varepsilon,$$

for all v such that (v, w_0) is an admissible pair. This is the second inequality in (6.3).

We have to construct u_0 such that for all $w \in \mathcal{L}_2$ the first inequality in (6.3) is satisfied. We have to do some preparatory work. We start by choosing $\tilde{v}_1 = 0$, i.e.

$$v_1 := -\left(\hat{D}^\mathrm{T} \hat{D}\right)^{-1} B_{11}^\mathrm{T} P_1 x_1.$$

We shall use the system dynamics given by equations (4.11)–(4.13). Since P satisfies the conditions of theorem 6.3 we know by corollary A.8 that the matrix $\tilde{A}_{11}$ is asymptotically stable. Assume that we have an initial value in some bounded set $\mathcal{V}$. Then the mapping from q_3 and w to x_1 is bounded, i.e. there are M_1, M_2, M such that for all q_3 and w in $\mathcal{L}_2$ and $\xi \in \mathcal{V}$ we have

$$\|x_1\|_2^2 \leq M_1 \|q_3\|_2^2 + M_2 \|w\|_2^2 + M. \tag{6.14}$$

Moreover, by theorem A.6 we know that $\bar{A}$ is stable and hence by equation (6.9) we find that M_3, M_4 and M_5 exist such that

$$\|x_1\|_2^2 \leq M_3 \|q_3\|_2^2 + M_4 \|\tilde{w}\|_2^2 + M_5. \tag{6.15}$$

Consider the system given by the differential equation (4.12) with input v_2, state x_2, q_3 and output q_3. It was shown in the proof of theorem 4.3

that this system is strongly controllable. We assumed that $\xi \in \mathcal{V}$ for some bounded set $\mathcal{V}$. Therefore by lemma 2.18 we know we can find a feedback

$$v_2 = F_1 \begin{pmatrix} x_2 \\ q_3 \end{pmatrix},$$

such that by applying that feedback in (6.10) (with once again $\tilde{v}_1 = 0$) we have (simultanuously !!!):

$$\|q_3\|_2^2 \leq (\|C_{23}\| + 1)^{-1} \varepsilon (\varepsilon M_1 + M + 1)^{-1} \left(\|x_1\|_2^2 + \|w\|_2^2 + 1 \right), \quad (6.16)$$

$$\|q_3\|_2^2 \leq (\|C_{23}\| + 1)^{-1} (M_1 + M_2 + 1)^{-1} \left(\|x_1\|_2^2 + \|w\|_2^2 + 1 \right), \quad (6.17)$$

$$\|q_3\|_2^2 \leq (M_4 + 1)^{-1} (\|E^{\mathrm{T}} P\| + 1)^{-1} \left(\|x_1\|_2^2 + \|w\|_2^2 + 1 \right), \quad (6.18)$$

for all $\xi \in \mathcal{V}$ and for all $w \in \mathcal{L}_2$ and $x_1 \in \mathcal{L}_2$. Moreover, we have $x_2 \in \mathcal{L}_2$. Combining (6.16) and (6.17) with (6.14) we find

$$\|C_{23} q_3\|_2^2 \leq \|w\|_2^2 + \varepsilon.$$

By choosing

$$v_2 = F_1 \begin{pmatrix} x_2 \\ q_3 \end{pmatrix} \ , \ v_0 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

we therefore find

$$\hat{\mathcal{J}}(v_0, w) \leq \xi^{\mathrm{T}} P \xi + \varepsilon.$$

It is also easy to check that for arbitrary $w \in \mathcal{L}_2^n$ we have $x \in \mathcal{L}_2$. This gives the first inequality in (6.3). It remains to be shown that (v_0, w_0) is an admissible pair. Choose v_0 and w_0 as defined above. Hence we have $\tilde{v}_1 = 0$ and $\tilde{w} = 0$ in the set of equations (6.9)–(6.11). Combining (6.15) with (6.18) we find that $x_1 \in \mathcal{L}_2$ and $q_3 \in \mathcal{L}_2$. Moreover F_1 was such that the remaining state $x_2 \in \mathcal{L}_2$. Therefore the pair (v_0, w_0) is admissible. By noting that for each bounded set of initial values the u_0 and w_0 are given by a static state feedback the proof is completed.   ∎

Remarks : The above proof is very simple when D is injective. Using remark (ii) after lemma 6.5 and since $R(P) = 0$ we find that the strategies

$$
\begin{aligned}
v_0 &:= -\left(D^{\mathrm{T}} D\right)^{-1} B^{\mathrm{T}} P x, \\
w_0 &:= -E^{\mathrm{T}} P x,
\end{aligned}
$$

define an (exact) equilibrium (note that the matrix $\bar{A}$ is stable since P is a stabilizing solution of the algebraic Riccati equation).

Corollary 6.6 : *Let $P \in \mathcal{R}^{n \times n}$ satisfy the conditions of theorem 6.3. We have the following equality,*

$$\xi^{\mathrm{T}} P \xi = \inf_{u \in \mathcal{U}_{st}^{m}(\xi)} \sup_{w \in \mathcal{L}_{2}^{l}} \mathcal{J}(u, w). \tag{6.19}$$

$\square$

Proof : For any $\varepsilon > 0$ we can choose $v = v_0$ as defined in the proof of theorem 6.3. We have $\tilde{v}_1 = 0$ and $\|C_{23} q_3\| \leq \|\tilde{w}\| + \varepsilon$ and using the equality in (6.5) we find

$$\inf_{v \in \mathcal{U}_{st}^{m}(\xi)} \sup_{w \in \mathcal{L}_{2}^{l}} \hat{\mathcal{J}}(v, w) \leq \xi^{\mathrm{T}} P \xi + \varepsilon. \tag{6.20}$$

As we mentioned earlier the transformation from u to v is not essential and can be easily reversed. Since (6.20) is true for all ε we find an inequality in (6.19). By choosing $\tilde{w} = 0$ for arbitrary u we find the opposite inequality and hence equality. $\blacksquare$

6.4   Necessary conditions for the existence of almost equilibria

In this section we shall derive necessary conditions for the existence of an almost equilibrium. This will be done by showing that the existence of an almost equilibrium implies the existence of a feedback which makes the H_∞ norm of the closed-loop system less than *or equal* to 1. Our main tool will be theorem 4.1 which gives necessary and sufficient conditions under which we can make the H_∞ norm of the closed-loop system strictly less than γ. However, we have to do some work to find necessary conditions for "less than or equal to γ".

We define the following class of input functions:

$$\mathcal{U}(\Sigma, w, \xi) = \left\{ u \in \mathcal{L}_{2}^{m} \mid \text{By applying } u, w \text{ in } \Sigma \text{ we have } x_{u,w,\xi} \in \mathcal{L}_{2}^{n} \right\}.$$

We shall use the following lemmas.

Lemma 6.7 : *Assume that for the system (6.2) with cost criterion (6.1) there exists an almost equilibrium. Moreover, let the initial condition $\xi = 0$.*

Under the above assumptions we have:

$$\inf_{u \in \mathcal{U}(\Sigma, w, 0)} \mathcal{J}(u, w) \leq 0, \qquad \forall w \in \mathcal{L}_2^l. \qquad (6.21)$$

$\square$

Proof : We know that for any $\varepsilon > 0$ there exists $u_0 \in \mathcal{U}_{fb}$ such that (6.3) is satisfied. Choose an arbitrary $w \in \mathcal{L}_2^l$. We have

$$\inf_{u \in \mathcal{U}(\Sigma, w, 0)} \mathcal{J}(u, w) \leq \mathcal{J}(u_0, w) \leq \mathcal{J}^*(0) + \varepsilon.$$

This implies that for arbitrary $\lambda > 0$ we have

$$\inf_{u \in \mathcal{U}(\Sigma, w, 0)} \mathcal{J}(\lambda u, \lambda w) = \inf_{u \in \mathcal{U}(\Sigma, \lambda w, 0)} \mathcal{J}(u, \lambda w) \leq \mathcal{J}(u_0, \lambda w) \leq \mathcal{J}^*(0) + \varepsilon.$$

But since $\xi = 0$ we have $\mathcal{J}(\lambda u, \lambda w) = \lambda^2 \mathcal{J}(u, w)$. Hence

$$\lambda^2 \inf_{u \in \mathcal{U}(\Sigma, w, 0)} \mathcal{J}(u, w) \leq \mathcal{J}^*(0) + \varepsilon.$$

Since this is true for all $\lambda > 0$ we find (6.21). $\blacksquare$

The following lemma is a straightforward consequence of the definition of an almost equilibrium and the definition of the feedback classes.

Lemma 6.8 : *Assume that $\mathcal{J}^*$ defines an almost equilibrium for the system (6.2) with cost criterion (6.1). For all $\xi \in \mathcal{R}^n$ we have:*

$$\inf_{u \in \mathcal{U}_{st}^m(\xi)} \sup_{w \in \mathcal{L}_2^l} \mathcal{J}(u, w) \leq \mathcal{J}^*(\xi). \qquad (6.22)$$

$\square$

Proof of the necessity part of theorem 6.4 : We define:

$$\mathcal{J}_\gamma(u, w) := \int_0^\infty z^{\mathrm{T}}(t) z(t) - \gamma^2 w^{\mathrm{T}}(t) w(t) \, dt = \|z\|_2^2 - \gamma^2 \|w\|_2^2.$$

It is easy to see that $\mathcal{J}_\gamma(u, w) = \mathcal{J}(u, w)$ for $\gamma = 1$. Let $\gamma > 1$ be given. We have:

$$\inf_{u \in \mathcal{U}(\Sigma, w, 0)} \mathcal{J}_\gamma(u, w) = \inf_{u \in \mathcal{U}(\Sigma, w, 0)} \mathcal{J}(u, w) - (\gamma^2 - 1)\|w\|_2^2,$$

and hence by lemma 6.7 we have

$$\inf_{u \in \mathcal{U}(\Sigma, w, 0)} \|z_{u,w,0}\|_2^2 - \gamma^2 \|w\|_2^2 \leq -(\gamma^2 - 1)\|w\|_2^2, \tag{6.23}$$

for all $w \in \mathcal{L}_2^l$. Therefore, by applying theorem 4.1 to the system Σ, we find that a positive semi-definite matrix P_γ exists such that $F_\gamma(P_\gamma) \geq 0$ where F_γ is defined by (4.2) (with D_1 replaced by D) and such that P_γ satisfies the following two rank conditions,

$$\text{rank} \qquad F_\gamma(P_\gamma) \qquad = \quad \text{rank}_{\mathcal{R}(s)} G_{ci},$$

$$\text{rank} \quad \begin{pmatrix} L_\gamma(P_\gamma, s) \\ F_\gamma(P_\gamma) \end{pmatrix} \quad = \quad n + \text{rank}_{\mathcal{R}(s)} G_{ci}, \qquad \forall s \in \mathcal{C}^+ \cup \mathcal{C}^0,$$

where L_γ is defined by (4.3). Since by theorem 6.3 P_γ is an almost equilibrium of a differential game with cost criterion $\mathcal{J}_\gamma$ which satisfies corollary 6.6 it is easy to see that if $\gamma \downarrow 1$ then P_γ increases i.e. $P_{\gamma_1} - P_{\gamma_2} \geq 0$ if $1 < \gamma_1 \leq \gamma_2$. On the other hand by lemma 6.8 we have:

$$\begin{aligned} \xi^{\text{T}} P_\gamma \xi \quad &= \quad \inf_{u \in \mathcal{U}_{st}^m(\xi)} \sup_{w \in \mathcal{L}_2^l} \quad \mathcal{J}_\gamma(u, w) \\ &\leq \quad \inf_{u \in \mathcal{U}_{st}^m(\xi)} \sup_{w \in \mathcal{L}_2^l} \quad \mathcal{J}_1(u, w) \\ &\leq \quad \mathcal{J}^*(\xi). \end{aligned}$$

Hence $\lim_{\gamma \downarrow 1} P_\gamma = P$ exists. Since rank $F_\gamma(X) \geq \text{rank}_{\mathcal{R}(s)} G_{ci}$ for all symmetric matrices X (see the proof of theorem A.6), by a continuity argument it can be shown that our limit P satisfies the rank condition (i) of theorem 6.4. In lemma A.6, part (iii), it has been shown that the rank condition (ii) of theorem 6.3 implies that a certain matrix is asymptotically stable. Therefore, again by a continuity argument, we know that in the limit this matrix has all its eigenvalues in the closed right half plane. This is equivalent with the rank condition (ii) of theorem 6.4 by lemma A.6, part (iii). ∎

6.5   The regular differential game

We shall now show the last part of theorem 6.4. The problem we have with the necessary conditions is that if we apply the u_0 and w_0 as we derived them in section 6.3, then (u_0, w_0) is no longer admissible. The trick we use is that we approximate w_0 by a function $\tilde{w}_0$ with compact support. Then $(u_0, \tilde{w}_0)$ is admissible and after some estimations it can

be shown that we indeed have an almost equilibrium. This has been recapitulated in the following theorem,

Theorem 6.9 : *Assume that we have the system (6.2) with a cost criterion given by (6.1). Furthermore assume that D is injective and that there are no invariant zeros on the imaginary axis for the system (A, B, C, D). If a matrix P exists such that $F(P) \geq 0$ and the rank conditions (i) and (ii) in theorem 6.4 are satisfied, then an almost equilibrium exists.* $\square$

Remarks :   This is an extension of the results in [Ma]. However there is an essential difference because we require stability. This prevents some nasty effects like the one mentioned in the example of [Ma]. However, in this chapter the set of admissible inputs is no longer a simple product space and this adds extra technical difficulties.

The proof will make use of two lemmas. The following lemma has been proven in chapter 4 after the statement of theorem 4.1.

Lemma 6.10 : *Assume that D is injective. Suppose a symmetric matrix P is given. Then the following two conditions are equivalent*

(i) $F(P) \geq 0$ *and* $\operatorname{rank} F(P) = \operatorname{rank}_{\mathcal{R}(s)} G_{ci}$,

(ii) $R(P) := PA + A^{\mathrm{T}}P + PEE^{\mathrm{T}}P + C^{\mathrm{T}}C$
$$- \left(PB + C^{\mathrm{T}}D\right)\left(D^{\mathrm{T}}D\right)^{-1}\left(B^{\mathrm{T}}P + D^{\mathrm{T}}C\right) = 0.$$

Moreover if P satisfies (i) (or equivalently (ii)), then the following two conditions are equivalent for all $s \in \mathcal{C}$:

(iii) $\operatorname{rank}\begin{pmatrix} L(P,s) \\ F(P) \end{pmatrix} = n + \operatorname{rank}_{\mathcal{R}(s)} G_{ci}$,

(iv) *the matrix $A + EE^{\mathrm{T}}P - B\left(D^{\mathrm{T}}D\right)^{-1}\left(B^{\mathrm{T}}P + D^{\mathrm{T}}C\right)$ has no eigenvalue in s.* $\square$

Note that in the case that D is injective we have $\operatorname{rank}_{\mathcal{R}(s)} G_{ci} = \operatorname{rank} D$. At this point we shall recall the following, known, result for the LQ-problem with stability (see [Wi7]).

Lemma 6.11 : *Consider the system (6.2) with cost criterion (6.1). Let $w = 0$. Assume that (A, B) is stabilizable, that (A, B, C, D) has no invariant zeros in $\mathcal{C}^0$ and that D injective. Then we have the following:*

$$\inf_{u \in \mathcal{U}(\Sigma, 0, \xi)} \mathcal{J}(u, 0) = \xi^{\mathrm{T}} L \xi.$$

Here $L \geq 0$ is the (unique) solution of the algebraic Riccati equation

$$A^{\mathrm{T}} L + LA + C^{\mathrm{T}} C - (PB + C^{\mathrm{T}} D)(D^{\mathrm{T}} D)^{-1}(B^{\mathrm{T}} P + D^{\mathrm{T}} C) = 0,$$

with the property that the matrix $A - B(D^{\mathrm{T}} D)^{-1}(B^{\mathrm{T}} P + D^{\mathrm{T}} C)$ is asymptotically stable.   □

Proof of theorem 6.9 : We know that we have a solution of $R(P) = 0$ such that the matrix $A + EE^{\mathrm{T}} P - B(D^{\mathrm{T}} D)^{-1}(B^{\mathrm{T}} P + D^{\mathrm{T}} C)$ has all its eigenvalues in the *closed* left half plane. P is related to the same linear quadratic control problem with stability as L only the C matrix should this time be replaced by $\tilde{C}$ satisfying $\tilde{C}^{\mathrm{T}} \tilde{C} = C^{\mathrm{T}} C + PEE^{\mathrm{T}} P$. This immediately implies that $P \geq L$.

Next, consider the following Riccati differential equation (RDE),

$$\dot{K} + KA + A^{\mathrm{T}} K + C^{\mathrm{T}} C = (KB + C^{\mathrm{T}} D)(D^{\mathrm{T}} D)^{-1}(B^{\mathrm{T}} K + D^{\mathrm{T}} C)$$

$$-KEE^{\mathrm{T}} K, \qquad K(0) = L.$$

Let $T > 0$ be such that the solution of the RDE exists on $[0, T]$. We know such a T exists. For the system (6.2) we shall consider the finite horizon differential game with endpoint-penalty. The cost criterion is given by

$$\mathcal{J}_T(u, w) = \int_0^T z^{\mathrm{T}}(t) z(t) - w^{\mathrm{T}}(t) w(t)\, dt \ + x^{\mathrm{T}}(T) L x(T).$$

It is well known (see [Ma]) that the optimal strategies for w and u are given by

$$
\begin{aligned}
u_0(t) &:= -(D^{\mathrm{T}} D)^{-1}(B^{\mathrm{T}} K(T - t) + C^{\mathrm{T}} D) x(t), \\
w_0(t) &:= E^{\mathrm{T}} K(T - t) x(t),
\end{aligned}
$$

and the corresponding equilibrium is $\mathcal{J}_T^*(\xi) = \xi^{\mathrm{T}} K(T) \xi$. It can be seen (using the interpretation of L as the cost defined in lemma 6.11) that this problem is equivalent with the original problem with cost criterion (6.1) when we add the additional constraint $\forall t > T, \ w(t) = 0$. This constraint is weakened for increasing T and hence it is clear that for increasing T the cost will increase since w is a maximizing player. That

is, for $T \geq t_1 \geq t_2$ we have $K(t_1) \geq K(t_2)$. Moreover, let $S(t)$ be the solution of the RDE with endpoint $S(T) = P$. Then S defines an equilibrium of a differential game over the same finite horizon but with endpoint-penalty $x(T)^{\mathrm{T}} P x(T)$. Since P is a stationary solution of the RDE we know $S(t) = P$ for all $t \in [0, T]$. Because the endpoint-penalties P and L of these two differential games satisfy $P \geq L$ we know that $P = S(t) \geq K(t) \quad \forall t < T$. Since $K(.)$ is an increasing solution of the RDE which is bounded from above we know that $K(.)$ exists for all t and converges to a matrix K_∞ which is a stationary solution of the RDE, i.e. $R(K_\infty) = 0$. Note that $K_\infty \geq L \geq 0$.

Next we claim that the matrix $A_1 := A - B \left(D^{\mathrm{T}} D\right)^{-1} \left(B^{\mathrm{T}} K_\infty + D^{\mathrm{T}} C\right)$ is asymptotically stable. To show this we rewrite the ARE in the following form.

$$K_\infty A_1 + A_1^{\mathrm{T}} K_\infty + K_\infty E E^{\mathrm{T}} K_\infty + C^{\mathrm{T}} C +$$

$$\left(K_\infty B + C^{\mathrm{T}} D\right) \left(D^{\mathrm{T}} D\right)^{-1} \left(B^{\mathrm{T}} K_\infty + D^{\mathrm{T}} C\right) = 0.$$

By applying an eigenvector x corresponding to an unstable eigenvalue λ to both sides of this equation we find $\mathrm{Re}\,\lambda \; x^{\mathrm{T}} K_\infty x = 0$, $E^{\mathrm{T}} K_\infty x = 0$, $Cx = 0$ and $\left(B^{\mathrm{T}} K_\infty + D^{\mathrm{T}} C\right) x = 0$. For $\mathrm{Re}\,\lambda > 0$ we find that $Ax = \lambda x$ and $K_\infty x = 0$. Since $K_\infty \geq L \geq 0$ this implies $Lx = 0$. This again implies that λ is also an unstable eigenvalue of $A - B \left(D^{\mathrm{T}} D\right)^{-1} \left(B^{\mathrm{T}} L + D^{\mathrm{T}} C\right)$. However, since by lemma 6.11 this matrix is stable we have a contradiction. If $\mathrm{Re}\,\lambda = 0$, then we have:

$$Ax = \lambda x, \qquad Cx = 0. \tag{6.24}$$

Because D is injective it can be shown that every unobservable eigenvalue of A is an invariant zero of (A, B, C, D). Hence (6.24) contradicts the fact that we have no invariant zeros on the imaginary axis. Therefore we know that A_1 is stable.

We are now in the position to show that $\mathcal{J}^*(\xi) = \xi^{\mathrm{T}} K_\infty \xi$ is an almost equilibrium of the system (6.2) with cost criterion (6.1). Let $\varepsilon > 0$ be given. Choose $T > 0$ such that $\xi^{\mathrm{T}} K_\infty \xi - \xi^{\mathrm{T}} K(T) \xi < \varepsilon$. The following u_0, w_0 turn out to satisfy (6.3):

$$\begin{aligned}
u_0(t) &:= -\left(D^{\mathrm{T}} D\right)^{-1} \left(B^{\mathrm{T}} K_\infty + D^{\mathrm{T}} C\right) x(t), \\
w_0(t) &:= \begin{cases} E^{\mathrm{T}} K(T - t) x(t) & \text{for } t < T, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Indeed for admissible pairs (u, w) we can now rewrite the cost criterion in the following way

$$\mathcal{J}(u, w) = \xi^{\mathrm{T}} K_\infty \xi - \int_0^\infty \| \left(w(t) - E^{\mathrm{T}} K_\infty x(t) \right) \|^2 dt$$

$$+ \int_0^\infty \| D \left(u(t) + (D^{\mathrm{T}} D)^{-1} \left(B^{\mathrm{T}} K_\infty + D^{\mathrm{T}} C \right) x(t) \right) \|^2 dt.$$

Since u_0 is a stabilizing feedback it can be seen from this equation that u_0 satisfies its requirements. Another way of rewriting the cost criterion when $w(t) = 0 \ \forall t > T$ is given by

$$\mathcal{J}(u, w) = \xi^{\mathrm{T}} K(T) \xi - x^{\mathrm{T}}(T) L x(T) + \int_T^\infty z^{\mathrm{T}}(t) z(t) dt$$

$$+ \int_0^T \| D \left(u(t) + (D^{\mathrm{T}} D)^{-1} \left(B^{\mathrm{T}} K(T - t) + D^{\mathrm{T}} C \right) x(t) \right) \|^2 dt$$

$$- \int_0^T \| \left(w(t) - E^{\mathrm{T}} K(T - t) x(t) \right) \|^2 dt.$$

Since, by lemma 6.11, the sum of the second and third term is non-negative and the first term differs from $\mathcal{J}^*(\xi)$ less than ε, it is easy to see that w_0 satisfies the second equation in (6.3). This proves that indeed an almost equilibrium exists. ∎

6.6 Conclusion

In this chapter the linear quadratic differential game was solved. We could derive necessary conditions as well as sufficient conditions for the existence of equilibria. However, for the derivation of the necessary conditions we made the extra assumption that there are no invariant zeros on the imaginary axis. Under this assumption we obtained necessary and sufficient conditions for the existence of equilibria for the case that D is injective (but not for the case that D is not injective).

Interesting points for future research would be to find necessary and sufficient conditions in the case that either D is not injective or if there are invariant zeros on the imaginary axis. Another point is the uniqueness of equilibria. In our opinion the almost equilibrium is unique but we have not been able to prove this claim. The almost equilibrium we find in theorem 6.3 can be shown to be the smallest possible.

An interesting feature is that under the assumptions of theorem 6.3 the existence of an almost equilibrium guarantees the existence of an

internally stabilizing controller which makes the H_∞ norm of the closed-loop system less than or equal to 1. Conversely, if there is an internally stabilizing controller which makes the H_∞ norm strictly less than 1, then an almost equilibrium exists. This shows the strong relationship between H_∞ control and differential games. It is the view of the author that in the initial surge of interest in H_∞ control theory (especially with respect to the state feedback case) the known results from the differential game literature have not been taken into account seriously enough. However, results as in this thesis are not trivial applications of these older results. Stability requirements had to be added. Moreover, in many cases the theory of differential games only supplies sufficient conditions for the existence of equilibria.

Chapter 7

The singular minimum entropy H_∞ control problem

7.1 Introduction

The H_∞ control problem has been discussed extensively in the previous chapters. However, H_∞ controllers are designed to minimize the peak value of the magnitude Bode diagram. It is well-known that using the frequency-domain approach to H_∞ control (see [Fr2, Kw, Kw2]) we find closed-loop transfer matrices with a completely flat magnitude Bode diagram. In the latter two papers this was even explicitly used to find suitable controllers. On the other hand the classical H_2 or Linear Quadratic Gaussian (LQG) control problem (see e.g. [AM]) the avarage value of the magnitude over all frequencies is minimized. The latter is not very much concerned with peaks as long as they have small width.

The above reasoning implies that H_∞ control is well-suited for robustness analysis via the small-gain theorem. On the other hand, the H_2 or LQG cost criteria might have very large values. This leads to a new problem: minimize the H_2 norm under an H_∞ norm bound. It is hoped that the H_∞ norm bound yields the desired level of robustness while performance is optimized simultaneously via the minimization of the H_2 norm.

The minimum entropy H_∞ control problem is defined as the problem of minimizing an entropy function under the constraint of internal stability of the closed-loop system and under the constraint of an upper bound 1 on the H_∞ norm of the closed-loop transfer matrix. This means that we want to minimize the entropy function evaluated for the same closed-loop transfer matrix as the one on which we imposed the H_∞ norm constraint. It should be noted that the interest in minimizing this entropy function is related to the fact that the entropy function is an upper bound to the

cost function used in the LQG control problem. Therefore, it is hoped that by minimizing this entropy function, one is minimizing the LQG cost criterion at the same time. This has, as far as we know, never been proven but this is the reason why the problem we discuss in this chapter is sometimes referred to as the mixed LQG/H_∞ or H_2/H_∞ control problem.

For the regular case, this problem was solved in [Gl5, Mu, Mu2]. It was shown that a minimizing controller, which is often called the "central controller", always exists. This terminology is due to the fact that the parametrization (given in section 5.5) of all internally stabilizing controllers which yield a closed-loop system with H_∞ norm less than 1, is centered around this specific controller. We shall not discuss extensions towards the more general case mentioned in subsection 1.6.3, i.e. the interconnection (1.6) where we search for a compensator from y to u such that the transfer matrix from w_2 to z_2 satisfies a certain H_∞ norm bound while the H_2 norm of the closed-loop transfer matrix from w_1 to z_1 is minimized over all internally stabilizing controllers. This extension is discussed in [BH, BH2, Do4, HB, HB2, Kh6, ZK3]. The paper [Kh6] has been extended to the singular case in [St14].

In this chapter we extend the results of [Mu] to the singular case. We still exclude invariant zeros on the imaginary axis. This chapter is in essence a combination of the results of chapters 4 and 5 with the results from [Mu]. The results of this chapter have already appeared in [St10]. Note that the singular LQG control problem has been investigated in [St12].

7.2   Problem formulation and results

Consider the linear time-invariant system:

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew, \\ y = C_1 x + D_1 w, \\ z = C_2 x + D_1 u, \end{cases} \tag{7.1}$$

Here A, B, E, C_1, C_2, D_1 and D_2 are real matrices of suitable dimension. Let G be a strictly proper real rational matrix which has no poles on the imaginary axis and which is such that

$$\|G\|_\infty < 1.$$

For such a transfer matrix G, we define the following entropy function:

$$\mathcal{J}(G) := -\frac{1}{2\pi} \int_{-\infty}^{\infty} \ln \det \left(I - G^\sim(i\omega) G(i\omega) \right) d\omega \tag{7.2}$$

where $G^\sim(s) := G^{\mathrm{T}}(-s)$. The following equality is derived easily using the Lebesgue dominated convergence theorem:

$$\mathcal{J}(G) := \lim_{s\to\infty} -\frac{1}{2\pi}\int_{-\infty}^{\infty} \ln\det\left(I - G^\sim(i\omega)G(i\omega)\right)\left(\frac{s^2}{s^2+\omega^2}\right)d\omega$$

The latter expression was used in [Mu]. The minimum entropy H_∞ control problem is then defined as:

> infimize $\mathcal{J}(G_{cl})$ over all controllers which yield a strictly proper, internally stable closed-loop transfer matrix G_{cl} with H_∞ norm strictly less than 1.

We shall investigate proper controllers of the form (2.4). It should be noted that this class of controllers is restrictive since it will be shown that in general for the singular problem the infimum is attained only by a non-proper controller.

A central role in our study of the above problem will be played by the *quadratic matrix inequality*. We shall use the notation and the main result of chapter 5.

We can now formulate the main result of this chapter:

Theorem 7.1 :   *Consider the system (7.1). Assume that the systems (A, B, C_2, D_2) and (A, E, C_1, D_1) have no invariant zeros on the imaginary axis and assume that a controller exists which is such that the closed-loop system is internally stable and has H_∞ norm strictly less than 1. The infimum of (7.2), over all internally stabilizing controllers of the form (2.4) which are such that the closed system has H_∞ norm strictly less than 1, is equal to:*

$$\text{Trace } E^{\mathrm{T}}PE + \text{Trace } (A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 + PEE^{\mathrm{T}}P)(I - QP)^{-1}Q$$

where P and Q are such that part (ii) of theorem 5.1 is satisfied. The infimum is attained if, and only if,

(i) $\text{Im}\, E_Q \subseteq (I - QP)\mathcal{V}_g(\Sigma_{ci}) + B\, \text{Ker}\, D_2$

(ii) $\text{Ker}\, C_{2,P} \supseteq (I - QP)^{-1}\mathcal{T}_g(\Sigma_{di}) \cap C_1^{-1}\, \text{Im}\, D_1$

(iii) $(I - QP)\mathcal{V}_g(\Sigma_{ci}) \supseteq \mathcal{T}_g(\Sigma_{di})$

where $\Sigma_{ci} = (A+EE^{\mathrm{T}}P, B, C_{2,P}, D_P)$ and $\Sigma_{di} = (A+QC_1^{\mathrm{T}}C_1, E_Q, C_1, D_Q)$. The matrices $C_{2,P}, D_P, E_Q, D_Q$ are arbitrary matrices satisfying

$$F(P) \;=\; \begin{pmatrix} C_{2,P}^{\mathrm{T}} \\ D_P^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} C_{2,P} & D_P \end{pmatrix}, \tag{7.3}$$

$$G(Q) \;=\; \begin{pmatrix} E_Q \\ D_Q \end{pmatrix} \begin{pmatrix} E_Q^{\mathrm{T}} & D_Q^{\mathrm{T}} \end{pmatrix}. \tag{7.4}$$

$\square$

Remarks:

(i) In the system (7.1) we have two direct-feedthrough matrices which are identical to zero. We can extend the above result to the more general case with all direct-feedthrough matrices possibly unequal to zero. Consider the system (2.1). Due to the fact that the closed-loop system must be strictly proper for the entropy to be finite, there must be a preliminary static output feedback which makes D_{22} equal to 0. We can remove D_{11} by the same argument as given in subsection 5.6.2.

(ii) It is straightforward to prove that the conditions (i)–(iii) are independent of the particular factorizations chosen in (7.3) and (7.4). It can also be shown that conditions (i)–(iii) are automatically satisfied if D_2 and D_1 are injective and surjective respectively.

(iii) In [Mu] an H_∞ norm bound $\gamma > 0$ is used and a corresponding entropy function depending on this bound γ. This result can be obtained easily by dividing the matrices E and D_1 by γ and by multiplying the entropy function by γ^2.

7.3 Properties of the entropy function

In this section we recall some basic properties of the entropy function as defined in (7.2). These properties were derived in [Mu] but we give separate proofs because we only investigate what is called in [Mu] "entropy at infinity" . This enables us to derive more straightforward proofs.

We can derive the following properties of our entropy function (7.2):

Lemma 7.2 : *Let G be a strictly proper, stable rational matrix such that $\|G\|_\infty \leq 1$. Then we have*

- $\mathcal{J}(G) \geq 0$ *and* $\mathcal{J}(G) = 0$ *implies* $G = 0$.
- $\mathcal{J}(G) = \mathcal{J}(G^\sim) = \mathcal{J}(G^{\mathrm{T}})$. $\qquad\qquad\square$

Proof : Straightforward. $\qquad\qquad\blacksquare$

Next, we relate our entropy function to the LQG cost criterion. First we define the LQG cost criterion:

Definition 7.3 : *Let Σ be given by*

$$\Sigma : \left\{ \begin{array}{l} dx = Ax\,dt + C\,dw, \\ z\ \ = Cx. \end{array} \right. \tag{7.5}$$

Assume that A is stable. Let w be a standard Wiener process and define the solution to the first equation (which is a stochastic differential equation) via Wiener integrals. Then the associated LQG cost is defined as:

$$\mathcal{C}(G) := \lim_{s \to \infty} \mathcal{E}\ \left\{ \frac{1}{s} \int_0^s z^{\mathrm{T}}(t)z(t)dt \right\},$$

where $\mathcal{E}$ denotes the expectation with respect to the noise. $\qquad\square$

Using the above definition of the LQG cost we find:

Lemma 7.4 : *Let Σ be defined by (7.5). Assume that A is stable and let G be the transfer matrix from dw to z. We have:*

$$\mathcal{J}(G) \geq \mathcal{C}(G). \qquad\qquad\square$$

Proof : It is well-known that the LQG cost is equal to Trace $B^{\mathrm{T}}\tilde{X}B$ where $\tilde{X}$ is the unique solution of the following Lyapunov equation:

$$\tilde{X}A + A^{\mathrm{T}}\tilde{X} + C^{\mathrm{T}}C = 0$$

A proof of lemma 7.4 can then be based upon corollary 7.7, by showing that $X \geq \tilde{X}$. ∎

Remarks : We can define, via scaling, entropy functions with an H_∞ norm bound of γ instead of 1. Then it can be shown that as $\gamma \to \infty$ the entropy cost function converges to the LQG cost.

Next, we give two key lemmas. Of the first lemma, the first part is equal to lemma 2.12 while the second part originates from [Mu]. We give a separate proof of the second part.

Lemma 7.5 : *Suppose that two systems Σ and Σ_2, both described by some state space representation, are interconnected in the following way:*

$$(7.6)$$

Assume that the system Σ is inner. Moreover, assume that its transfer matrix G has the following decomposition:

$$G \begin{pmatrix} w \\ u \end{pmatrix} =: \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} z \\ y \end{pmatrix} \qquad (7.7)$$

such that $G_{21}^{-1} \in H_\infty$ and such that G_{11}, G_{22} are strictly proper. Under the above assumptions the following two statements are equivalent:

- *(i) The closed-loop system (7.6) is internally stable and its closed-loop transfer matrix G_{cl} has H_∞ norm less than 1.*

- *(ii) The system Σ_2 is internally stable and its transfer matrix G_2 has H_∞ norm less than 1.*

Moreover, G_{cl} is strictly proper if, and only if, G_2 is strictly proper.
Finally, if (i) holds and G_2 is strictly proper then the following relation between the entropy functions for the different transfer matrices is satisfied:

$$\mathcal{J}(G_{cl}, 1) = \mathcal{J}(G_{11}, 1) + \mathcal{J}(G_2, 1). \qquad (7.8)$$

□

Proof : The first claim that the statements (i) and (ii) are equivalent, has been shown in lemma 2.12. We know that G_{11} and G_{22} are strictly proper. Combined with the fact that G is inner, this implies that G_{12} and G_{21} are bicausal. Using this, it is easy to check that G_2 is strictly proper if, and only if, G_{cl} is strictly proper.

We still have to prove equation (7.8). The following equality is easily derived using the property that Σ is inner:

$$I - G_{cl}^{\sim} G_{cl} = G_{21}^{\sim} \left(I - G_2^{\sim} G_{22}^{\sim}\right)^{-1} \left(I - G_2^{\sim} G_2\right) \left(I - G_{22} G_2\right)^{-1} G_{21}$$

Therefore, we find that

$$\ln \det \left(I - G_{cl}^{\sim} G_{cl}\right) = \ln \det \left(I - G_{11}^{\sim} G_{11}\right)$$

$$+ \ln \det \left(I - G_2^{\sim} G_2\right) - 2 \ln \det \left(I - G_{22} G_2\right) \qquad (7.9)$$

Moreover, if statement (i) is satisfied and if G_2 is strictly proper then we have

$$\mathcal{J}(G_2, 1) \quad = \quad -\frac{1}{2\pi} \int_{-\infty}^{\infty} \ln \det \left(I - G_2^{\sim}(i\omega) G_2(i\omega)\right) d\omega, \qquad (7.10)$$

$$\mathcal{J}(G_{11}, 1) \quad = \quad -\frac{1}{2\pi} \int_{-\infty}^{\infty} \ln \det \left(I - G_{11}^{\sim}(i\omega) G_{11}(i\omega)\right) d\omega. \qquad (7.11)$$

Using the fact that G_2 is strictly proper, stable and has H_{∞} norm strictly less than 1 and the fact that G_{22} is also stable, strictly proper and has H_{∞} norm less than or equal to 1, we know that $\ln \det \left(I - G_{22}(s) G_2(s)\right)$ is an analytic function in the open right half plane and that a constant M exists such that

$$| \ln \det \left(I - G_{22}(s) G_2(s)\right)| < \frac{M}{|s|^2} \qquad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$$

This implies, using Cauchy's theorem, that

$$\int_{-\infty}^{\infty} \ln \det \left(I - G_{22}(i\omega) G_2(i\omega)\right) d\omega = 0 \qquad (7.12)$$

Combining (7.9),(7.10),(7.11) and (7.12) we find (7.8).   ∎

The following lemma is an essential tool for actually calculating the entropy function for some specific system:

Lemma 7.6 : *Assume that a rational matrix G which has a detectable and stabilizable realization (A, B, C, D) with $\det D = 1$ is given. Also, assume that $G, G^{-1} \in H_\infty$ and G has H_∞ norm equal to 1. Then we have:*

$$\int_{-\infty}^{\infty} \ln |\det G(i\omega)| \, d\omega = -\pi \; \text{Trace } BD^{-1}C \tag{7.13}$$

$\square$

Proof : Denote the integral in (7.13) by $\mathcal{K}$ and define

$$a := - \text{ Trace } BD^{-1}C.$$

We have (remember that $\ln |z| = \text{Re} \ln z$):

$$\mathcal{K} = \text{Re} \left(\int_{-\infty}^{\infty} \ln \det G(i\omega) - \frac{a}{1 + i\omega} \, d\omega \right) + a \int_{-\infty}^{\infty} \frac{d\omega}{1 + \omega^2} \tag{7.14}$$

Next, it is easily checked that $p(s) := \ln \det G(s) - \frac{a}{1+s}$ is a bounded analytic function in $\mathcal{C}^+$ such that $p(s) = O(1/s^2)$ ($|s| \to \infty$, Re $s \geq 0$). Hence, using Cauchy's theorem we find

$$\int_{-\infty}^{\infty} p(i\omega) \, d\omega = 0. \tag{7.15}$$

Combining (7.14) and (7.15) yields (7.13). $\blacksquare$

Corollary 7.7 : *Let G be a strictly proper, stable transfer matrix with H_∞ norm strictly less than 1 and with stabilizable and detectable realization $(A, B, C, 0)$. Then we have:*

$$\mathcal{J}(G) = \text{ Trace } B^{\mathrm{T}}XB \tag{7.16}$$

where X is the unique solution of the algebraic Riccati equation:

$$XA + A^{\mathrm{T}}X + XBB^{\mathrm{T}}X + C^{\mathrm{T}}C = 0$$

such that $A + BB^{\mathrm{T}}X$ is asymptotically stable. $\square$

Proof : The existence and uniqueness of X is a well-known result (see e.g. [Ku]). It is easy to check that the transfer matrix M with realization $(A, B, -B^{\mathrm{T}}X, I)$ satisfies:

$$I - G^{\sim}G = M^{\sim}M$$

Moreover, $M, M^{-1} \in H_{\infty}$, i.e. M is a spectral factor of $I - G^{\sim}G$. We have

$$\mathcal{J}(G) = \frac{-1}{\pi} \int_{-\infty}^{\infty} \ln |\det M(i\omega)| \, d\omega$$

and therefore (7.16) is a direct consequence of applying lemma 7.6 to the above equation. ∎

7.4   A system transformation

Throughout this section we assume that there are matrices P and Q satisfying the conditions in theorem 5.1. Note that this is no restriction when proving theorem 7.1. The existence of such P and Q is implied by our assumption that an internally stabilizing controller exists which makes the H_{∞} norm strictly less than 1. We use the technique from chapter 5 of transforming the system twice such that the problem of minimizing the entropy function for the original system is equivalent to minimizing the entropy function for the new system we thus obtain. In the next section we shall show that this new system satisfies some desirable properties which enables us to solve the minimum entropy H_{∞} control problem for this new system and hence also for the original system.

We factorize $F(P)$ as in (7.3). This can be done since $F(P) \geq 0$. We define the following system:

$$\Sigma_P : \begin{cases} \dot{x}_P = A_P x_P + B u_P + E w_P, \\ y_P = C_{1,P} x_P + D_1 w_P, \\ z_P = C_{2,P} x_P + D_P u_P, \end{cases} \qquad (7.17)$$

where $A_P := (A + EE^{\mathrm{T}}P)$ and $C_{1,P} := (C_1 + D_1 E^{\mathrm{T}}P)$.

Lemma 7.8 : *Let Σ and Σ_P be defined by (7.1) and (7.17), respectively.*
For any system Σ_U of suitable dimensions consider the following inter-
connection:

$$(7.18)$$
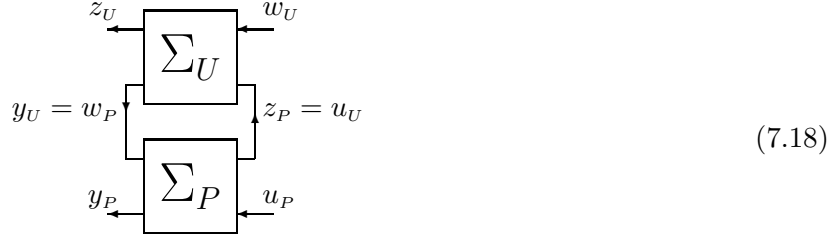
and decompose the transfer matrix U of Σ_U as follows:

$$U \begin{pmatrix} w_U \\ u_U \end{pmatrix} =: \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} w_U \\ u_U \end{pmatrix} = \begin{pmatrix} z_U \\ y_U \end{pmatrix},$$

compatible with the sizes of u_U, w_U, y_U, and z_U. Then the following holds:
a system Σ_U of suitable dimensions exists such that:

(i) The system Σ_U is inner

(ii) The transfer matrix U_{21}^{-1} is well-defined and stable

(iii) The transfer matrices U_{11} and U_{22} are strictly proper

(iv) $\mathcal{J}(U_{11}, 1) = $ Trace $E^{\mathrm{T}} P E$

(v) The system Σ and the interconnection in (7.18) have the same trans-
fer matrix.

(vi) The interconnection in (7.18) is detectable from y_P and stabilizable
by u_P. $\square$

Proof : In the basis of appendix A and if we have for the factorization of
$F(P)$ in (7.3) the matrices defined in (B.1) and (B.2) then such a system
Σ_U is given by (B.5). Note that this factorization is unique up to unitary
transformation and therefore it is immediate that the result remains true
for an arbitrary factorization of $F(P)$ in (7.3).

Note that U_{21} is a spectral factor for $I - U_{11}^{\sim} U_{11}$ which yields (iv) by
using the state space realization for U_{21} given in (B.5) and by applying
lemma 7.6.   ∎

Remarks :   A state space realization for a system Σ_U satisfying the
conditions of lemma 7.8 in case D_2 and D_1 are injective and surjective

respectively is given, without the need of the bases in appendix A, by (B.6).

Combining lemmas 7.5 and 7.8, we find the following theorem:

Theorem 7.9 : *Let the systems (7.1) and (7.17) be given. Moreover, let a compensator Σ_F of the form (2.4) be given. The following two conditions are equivalent:*

- *Σ_F is internally stabilizing for Σ such that the closed-loop transfer matrix G_{cl} is strictly proper and has H_∞ norm strictly less than 1.*

- *Σ_F is internally stabilizing for Σ_P such that the closed-loop transfer matrix $G_{cl,P}$ is strictly proper and has H_∞ norm strictly less than 1.*

Moreover, if Σ_F satisfies the above conditions then we have

$$\mathcal{J}(G_{cl}, 1) = \mathcal{J}(G_{cl,P}, 1) + \ \text{Trace } E^{\mathrm{T}} P E. \qquad \Box$$

Next, we make another transformation from Σ_P to $\Sigma_{P,Q}$. This transformation is exactly dual to the transformation from Σ to Σ_P. We know there exists a controller which is internally stabilizing for Σ_P which makes the H_∞ norm of the closed-loop system strictly less than 1. Therefore if we apply theorem 5.1 to Σ_P we find that a unique matrix Y exists such that $\bar{G}(Y) \geq 0$ and

(i) $\text{rank } \bar{G}(Y) = \text{rank}_{\mathcal{R}(s)} \bar{G}_{di}$,

(ii) $\text{rank} \left(\begin{array}{cc} \bar{M}(Y,s) & \bar{G}(Y) \end{array} \right) = n + \text{rank}_{\mathcal{R}(s)} \bar{G}_{di}, \qquad \forall \, s \in \mathcal{C}^0 \cup \mathcal{C}^+,$

where

$$\bar{G}(Y) \quad := \quad \begin{pmatrix} A_P Y + Y A_P^{\mathrm{T}} + EE^{\mathrm{T}} + Y C_{2,P}^{\mathrm{T}} C_{2,P} Y & Y C_{1,P}^{\mathrm{T}} + ED_1^{\mathrm{T}} \\ C_{1,P} Y + D_1 E^{\mathrm{T}} & D_1 D_1^{\mathrm{T}} \end{pmatrix},$$

$$\bar{M}(Y,s) \quad := \quad \begin{pmatrix} sI - A_P - Y C_{2,P}^{\mathrm{T}} C_{2,P} \\ -C_{1,P} \end{pmatrix},$$

$$\bar{G}_{di}(s) \quad := \quad C_{1,P} \left(sI - A_P \right)^{-1} E + D_1.$$

In lemma 5.5 it has been shown that $Y := (I - QP)^{-1} Q$ satisfies the above conditions. We factorize $\bar{G}(Y)$:

$$\bar{G}(Y) = \begin{pmatrix} E_{P,Q} \\ D_{P,Q} \end{pmatrix} \begin{pmatrix} E_{P,Q}^{\mathrm{T}} & D_{P,Q}^{\mathrm{T}} \end{pmatrix}. \tag{7.19}$$

where $E_{P,Q}$ and $D_{P,Q}$ are matrices of suitable dimensions. We define the following system:

$$\Sigma_{P,Q}: \begin{cases} \dot{x}_{P,Q} = A_{P,Q}x_{P,Q} + B_{P,Q}u_{P,Q} + E_{P,Q}w, \\ y_{P,Q} = C_{1,P}x_{P,Q} + D_{P,Q}w, \\ z_{P,Q} = C_{2,P}x_{P,Q} + D_P u_{P,Q}, \end{cases} \qquad (7.20)$$

where $A_{P,Q} := A_P + YC_{2,P}^{\mathrm{T}}C_{2,P}$ and $B_{P,Q} := B + YC_{2,P}^{\mathrm{T}}D_P$. Using theorem 7.9 and a dualized version for the transformation from Σ_P to $\Sigma_{P,Q}$ we can derive the following corollary:

Corollary 7.10 : *Let the systems (7.1) and (7.20) be given. Moreover, let a compensator Σ_F of the form (2.4) be given. The following two conditions are equivalent:*

- *Σ_F is internally stabilizing for Σ such that the closed-loop transfer matrix G_{cl} is strictly proper and has H_∞ norm strictly less than 1.*

- *Σ_F is internally stabilizing for $\Sigma_{P,Q}$ such that the closed-loop transfer matrix $G_{cl,P,Q}$ is strictly proper and has H_∞ norm strictly less than 1.*

Moreover, if Σ_F satisfies the above conditions then we have

$$\mathcal{J}(G_{cl}, 1) = \mathcal{J}(G_{cl,P,Q}, 1) + \text{ Trace } E^{\mathrm{T}}PE + \text{ Trace } C_{1,P}YC_{1,P}. \qquad \square$$

From this corollary it is immediate that it is sufficient to investigate $\Sigma_{P,Q}$ to prove the results in our main theorem 7.1. This is done in the next section.

7.5   (Almost) disturbance decoupling and minimum entropy

The first thing we would like to know is what we gain by our transformation from Σ to $\Sigma_{P,Q}$. It turns out that we obtain in this way a system of the form (7.1) such that part (ii) of theorem 5.1 is satisfied for $P = 0$ and $Q = 0$. This implies that:

$$\text{rank } \begin{pmatrix} sI - A_{P,Q} & -B_{P,Q} \\ C_{2,P} & D_P \end{pmatrix} = n + \text{rank } \begin{pmatrix} C_{2,P} & D_P \end{pmatrix}, \qquad (7.21)$$

$$\text{rank } \begin{pmatrix} sI - A_{P,Q} & -E_{P,Q} \\ C_{1,P} & D_{P,Q} \end{pmatrix} = n + \text{rank } \begin{pmatrix} E_{P,Q} \\ D_{P,Q} \end{pmatrix}, \qquad (7.22)$$

for all $s \in \mathcal{C}^0 \cup \mathcal{C}^+$. Using these conditions we can apply theorem 2.20. We obtain the following:

Theorem 7.11 : *Consider the system (7.1). Assume that the systems (A, B, C_2, D_2) and (A, E, C_1, D_1) have no invariant zeros on the imaginary axis and assume that a controller exists which is such that the closed-loop system is internally stable and has H_∞ norm strictly less than 1. The infimum of (7.2), over all internally stabilizing controllers of the form (2.4) which are such that the closed system has H_∞ norm strictly less than 1, is equal to:*

$$Trace \ E^{\mathrm{T}} P E + Trace \ (A^{\mathrm{T}} P + P A + C_2^{\mathrm{T}} C_2 + P E E^{\mathrm{T}} P)(I - QP)^{-1} Q$$

$$(7.23)$$

where P and Q are such that part (ii) of theorem 5.1 is satisfied.   □

Proof : By corollary 7.10 and the non-negativity of the entropy function, the infimum is always larger than or equal to (7.23). Next we choose a sequence of controllers $\Sigma_{F,n}$ satisfying the conditions of theorem 2.20. This is possible because of (7.21) and (7.22). These controllers applied to $\Sigma_{P,Q}$ yield internally stable closed-loop systems and it is straightforward to check that the closed-loop transfer matrices $\tilde{G}_{cl,n,P,Q}$ satisfy:

$$\mathcal{J}(\tilde{G}_{cl,n,P,Q}, 1) \to 0$$

as $n \to \infty$. By applying corollary 7.10 we find that if we apply the controllers $\Sigma_{F,n}$ to Σ then we find closed-loop systems which are internally stable and the closed-loop transfer matrices $G_{cl,n}$ are strictly proper, have H_∞ norm less than 1 and satisfy

$$\mathcal{J}(G_{cl,n}, 1) \to Trace \ E^{\mathrm{T}} P E +$$

$$Trace \ (A^{\mathrm{T}} P + P A + C_2^{\mathrm{T}} C_2 + P E E^{\mathrm{T}} P)(I - QP)^{-1} Q$$

as $n \to \infty$, which completes the proof of this corollary.   ■

Theorem 7.11 is in fact the main part of theorem 7.1. Exactly when the infimum of the entropy function is attained is still to be investigated. From the theory of almost disturbance decoupling (see [Tr]), it is well-known that there is often only a non-proper controller which attains the

infimum (note that because the interconnection should be well-posed it is possible that not even a non-proper controller can attain the infimum). However, we would like to know when it is possible to attain the infimum by a *proper* controller. Note that the infimum is attained if, and only if, we can find a controller for $\Sigma_{P,Q}$ which makes the closed-loop system internally stable and the closed-loop transfer matrix $\tilde{G}_{cl}$ has H_∞ norm less than 1 and its entropy is equal to 0. However, the entropy is 0 if, and only if, $\tilde{G}_{cl} = 0$ by lemma 7.2. This reduces our original problem to what is often called the disturbance decoupling problem with measurement feedback and internal stability (DDPMS), i.e. the problem of finding a stabilizing controller which makes the closed-loop transfer matrix equal to 0 as discussed in section 2.6.

After some extensive calculations the following two equalities can be derived:

$$\begin{aligned}
\mathcal{V}_g(A + EE^{\mathrm{T}}P, B, C_{2,P}, D_P) &= \mathcal{V}_g(A_{P,Q}, B_{P,Q}, C_{2,P}, D_P) \\
\mathcal{T}_g(A + QC_1^{\mathrm{T}}C_1, E_Q, C_1, D_Q) &= (I - QP)\mathcal{T}_g(A_{P,Q}, E_{P,Q}, C_{1,P}, D_{P,Q})
\end{aligned}$$

Applying theorem 2.19 then yields the proof of the final result in theorem 7.1.

7.6   Conclusion

In this chapter we have given a complete treatment of the singular minimum entropy H_∞ control problem. We have an explicit formula for the infimum. Moreover, we can characterize when the infimum is attained. The construction of a controller can be done using some tools from the geometric approach to control theory. However, this method is too extensive (though straightforward) to discuss in this chapter. Finally we would like to note that this chapter gives a nice structured approach to "entropy at infinity" with fewer technicalities than [Mu] where entropy at infinity is simply a special yet important case.

A main open problem remains the problem of invariant zeros on the imaginary axis. Moreover, we would like to have results on LQG itself instead of the upper bound given by the entropy. A first start has been made in [RK] but much more work has to be done. Finally, another interesting extension is the general setup of subsection 1.6.3 where we have two kinds of disturbances and two kinds of output to be controlled. From one disturbance input to one of the outputs to be controlled we want to satisfy an H_∞ norm bound. For the closed-loop transfer matrix from the other disturbance to the other output to be controlled we want to minimize an H_2 norm. In this way we can relate performance criteria to

robustness criteria with much more arbitrary structure for the parameter uncertainty. Some references are given in the introduction of this chapter.

Chapter 8

The finite horizon H_∞ control problem

8.1 Introduction

Most of the papers on the H_∞ control problem mentioned in the previous chapters discuss the "standard" H_∞ problem: minimize the $\mathcal{L}_2[0,\infty)$-induced operator norm of the closed-loop operator over all internally stabilizing feedback controllers.

A number of generalizations have appeared recently. One of these is the minimization of the $\mathcal{L}_2$-induced operator norm over a *finite* horizon (see [Li5, Ta]). As in the infinite horizon H_∞ problem, difficulties arise if the direct-feedthrough matrices do not satisfy certain assumptions (the so-called singular case). This chapter will use the techniques from chapters 4 and 5 to handle this problem for the finite-horizon case.

The following problem will be considered: given a finite-dimensional system on a bounded time-interval $[0,T]$, together with a positive real number γ, find necessary and sufficient conditions for the existence of a dynamic compensator such that the $\mathcal{L}_2[0,T]$-induced norm of the resulting closed-loop operator is smaller than γ. In [Ta] and [Li5] such conditions were formulated in terms of the existence of solutions to certain Riccati differential equations. Of course, in order to guarantee the existence of these Riccati differential equations, certain coefficient matrices of the system under consideration should have *full rank* (the *regular* case). The present paper addresses the problem formulated above *without* these full rank assumptions. We find necessary and sufficient conditions in terms of a pair of quadratic matrix differential inequalities.

The results of this chapter have already appeared in [St7, Tr2]. The detailed proof of the results is highly technical and can be found in [St7]. This paper discusses time-varying systems which satisfy a number of con-

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ditions. These conditions are more or less requirements that all the geo-
metric subspaces we use in appendix A and their dual versions, which we
can define for each t, are independent of time. For two important cases
these conditions are always satisfied:

- If the system is time-invariant (i.e. all coefficient matrices are con-
  stant, independent of time).

- If the problem is regular in the sense as explained above.

In this chapter, contrary to [Li5, Ta], we shall assume that the system
is time-invariant. We shall try to give the main ideas of the proof which
follows the same lines as the proof of the results on the singular, infinite
horizon, H_∞ control problem discussed in the previous chapters. However,
we shall not give all the technical details. For this, we refer to [St7]

The outline of this chapter is as follows: in section 8.2 we shall for-
mulate our problem and present our main result. In section 8.3 it will be
noted that if a controller exists which makes the $\mathcal{L}_2[0,T]$-induced opera-
tor norm of the closed-loop operator less than 1 then matrices functions
P and Q exist satisfying a pair of quadratic matrix differential inequali-
ties, two corresponding rank conditions and two boundary conditions. In
the same section we shall introduce a system transformation with the,
by now standard, property: a controller "works" for this new system if,
and only if, the same controller "works" for the original system. Using
this transformation we shall show that another necessary condition for
the existence of the desired controller is that P and Q satisfy a cou-
pling condition: $I - PQ$ is invertible for all t. In section 8.4 we shall
apply a second transformation, dual to the first, which will show that
the necessary conditions derived are also sufficient. This will be done by
showing that the system we obtained by our two transformations satisfies
the following condition: for all $\varepsilon > 0$ a controller exists which makes the
$\mathcal{L}_2[0,T]$-induced norm of the closed-loop operator less than ε. We shall
end the chapter with a couple of concluding remarks.

8.2   Problem formulation and main results

Consider the linear time-invariant system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew, \\ y = C_1 x + D_1 w, \\ z = C_2 x + D_2 u. \end{cases} \tag{8.1}$$

where $x \in \mathcal{R}^n$ is the state, $u \in \mathcal{R}^m$ the control input, $w \in \mathcal{R}^l$ an unknown disturbance input, $y \in \mathcal{R}^p$ the measured output and $z \in \mathcal{R}^q$ the output to be controlled. A, B, E, C_1, D_1, C_2 and D_2 are constant real matrices of appropriate dimensions. In addition, we assume that some fixed time interval $[0, T]$ is given. We shall be concerned with the existence and construction of dynamic compensators of the form

$$
\Sigma_F : \begin{cases} \dot{p}(t) = K(t)p(t) + L(t)y(t), \\ u(t) = M(t)p(t) + N(t)y(t). \end{cases} \tag{8.2}
$$

where K, L, M and N are real, matrix valued, continuous functions on $[0, T]$. The feedback interconnection of Σ and Σ_F is the linear time-varying system Σ_{cl} which yields a Volterra integral operator $\mathcal{G}_{cl}$ from w to z as defined on page 19.

Thus, the influence of disturbances $w \in \mathcal{L}_2^l[0, T]$ on the output z can be measured by the operator norm of $\mathcal{G}_{cl}$, given in the usual way by

$$
\|\mathcal{G}_{cl}\|_\infty := \sup \left\{ \frac{\|\mathcal{G}_F w\|_2}{\|w\|_2} \mid 0 \neq w \in \mathcal{L}_2^l[0, T] \right\}.
$$

Here, $\|x\|_2$ denotes the $\mathcal{L}_2[0, T]$ norm of the function x. The problem that we shall discuss in this chapter is the following:

> given $\gamma > 0$, find necessary and sufficient conditions for the existence of a dynamic compensator Σ_F such that $\|G_{cl}\|_\infty < \gamma$.

The problem as posed here will be referred to as the *finite horizon H_∞ control problem by measurement feedback*. This problem was studied before in [Li5, Ta]. However, these references assume that the following conditions hold: D_1 is surjective, D_2 is injective. In the present chapter we shall extend the results obtained in [Li5, Ta] to the case that D_1 and D_2 are arbitrary.

A central role in our study of the problem posed is played by what we shall call the quadratic differential inequality. Let $\gamma > 0$ be given. For any differentiable matrix function $P : [0, T] \to \mathcal{R}^{n \times n}$, define $F_\gamma(P) : [0, T] \to \mathcal{R}^{(n+m) \times (n+m)}$ by

$$
F_\gamma(P) := \begin{pmatrix} \dot{P} + A^{\mathrm{T}}P + PA + C_2^{\mathrm{T}}C_2 + \frac{1}{\gamma^2}PEE^{\mathrm{T}}P & PB + C_2^{\mathrm{T}}D_2 \\ B^{\mathrm{T}}P + D_2^{\mathrm{T}}C_2 & D_2^{\mathrm{T}}D_2 \end{pmatrix}.
$$

$$\tag{8.3}$$

If $F_\gamma(P)(t) \geq 0$ for all $t \in [0, T]$, then we shall say that P satisfies the quadratic differential inequality (at γ). A dual version of (8.3) will also

be important to us: for any differentiable $Q : [0,T] \to \mathcal{R}^{n \times n}$ define $G_\gamma(Q) : [0,T] \to \mathcal{R}^{(n+p) \times (n+p)}$ by:

$$G_\gamma(Q) := \begin{pmatrix} -\dot{Q} + AQ + QA^{\mathrm{T}} + EE^{\mathrm{T}} + \frac{1}{\gamma^2}QC_2^{\mathrm{T}}C_2Q & QC_1^{\mathrm{T}} + ED_1^{\mathrm{T}} \\ C_1Q + D_1E^{\mathrm{T}} & D_1D_1^{\mathrm{T}} \end{pmatrix}.$$

If $G_\gamma(Q)(t) \geq 0$ for all $t \in [0,T]$ then we shall say that Q satisfies the dual quadratic differential inequality (at γ). In the sequel let

$$\begin{aligned} G_{ci}(s) &:= C_2(Is - A)^{-1}B + D_2, \\ G_{di}(s) &:= C_1(Is - A)^{-1}E + D_1 \end{aligned}$$

denote the open-loop transfer matrices from u to z and w to y, respectively. We are now ready to state our main result:

Theorem 8.1 : *Let $\gamma > 0$. The following two statements are equivalent:*

(i) *A compensator Σ_F of the form (8.2) exists such that $\|\mathcal{G}_{cl}\|_\infty < \gamma$,*

(ii) *Differentiable matrix functions P and $Q : [0,T] \to \mathcal{R}^{n \times n}$ exist such that*

   (a) $F_\gamma(P)(t) \geq 0 \ \forall t \in [0,T]$ *and* $P(T) = 0$,

   (b) *rank* $F_\gamma(P)(t) = rank_{\mathcal{R}(s)}G_{ci} \ \forall t \in [0,T]$,

   (c) $G_\gamma(Q)(t) \geq 0 \ \forall t \in [0,T]$ *and* $Q(0) = 0$,

   (d) *rank* $G_\gamma(Q)(t) = rank_{\mathcal{R}(s)}G_{di} \ \forall t \in [0,T]$,

   (e) $\gamma^2 I - Q(t)P(t)$ *is invertible for all* $t \in [0,T]$. $\square$

The aim of this chapter is to outline the main steps and ideas involved in a proof of the latter theorem. For a more detailed discussion we would like to refer to [St7].

   It can be shown that, in general, if $F_\gamma(P) \geq 0$ on $[0,T]$ then

$$\text{rank } F_\gamma(P)(t) \geq \text{rank}_{\mathcal{R}(s)}G_{ci} \qquad \text{on} [0,T]$$

and, likewise, $G_\gamma(Q) \geq 0$ on $[0,T]$ then

$$\text{rank } G_\gamma(Q)(t) \geq \text{rank}_{\mathcal{R}(s)}G_{di} \qquad \text{on} [0,T]$$

This means that conditions (a) and (b) can be reformulated as: P is a rank-minimizing solution of the quadratic differential inequality at γ, satisfying the end-condition $P(T) = 0$. A similar statement is valid for

the conditions (c),(d). It can also be shown that if P satisfies (a), (b) then it is *unique*. Also, this unique solution turns out to be *symmetric* for all $t \in [0, T]$. The same holds for Q satisfying (c) and (d).

It can be shown that for the special case when D_1 and D_2 are assumed to be surjective and injective, respectively, our theorem 8.1 specializes to the results obtained before in [Li5, Ta].

It can also be shown that if the conditions in the statement of 8.1 part (ii) indeed hold, then it is always possible to find a suitable compensator with dynamic order equal to n, the dynamic order of the system to be controlled.

8.3   Completion of the squares

In this section we shall outline the proof of the implication (i) $\Rightarrow$ (ii) of theorem 8.1. Consider the system Σ. Our starting point is the following lemma:

Lemma 8.2 : *Let $\gamma > 0$. Assume that there exists $\delta < \gamma$ such that for all $w \in \mathcal{L}_2^l[0, T]$ we have*

$$\inf \left\{ \|z_{u,w}\|_2 - \delta \|w\|_2 \mid u \in \mathcal{L}_2^m[0, T] \right\} \le 0. \tag{8.4}$$

Then a differentiable matrix function $P : [0, T] \to \mathcal{R}^{n \times n}$ exists such that $P(T) = 0$, $F_\gamma(P)(t) \ge 0$ and rank $F_\gamma(P)(t) = rank_{\mathcal{R}(s)} G_{ci}$ for all $t \in [0, T]$. $\qquad\square$

Proof : A proof of this can be given by using the techniques from chapter 3 (solve the regular case) and 4 (extend to the singular case). $\qquad\blacksquare$

Now, assume that the condition (i) in the statement of theorem 8.1 holds, i.e. assume that a dynamic compensator Σ_F exists such that $\|G_{cl}\|_\infty =: \delta < \gamma$. Therefore condition (8.4) is satisfied: let $w \in \mathcal{L}_2^l[0, T]$ and let z be the closed-loop output with $x(0) = 0$ and $p(0) = 0$. Then $z = z_{\bar{u},w}$, where $\bar{u}$ is the output of Σ_F. Clearly

$$\frac{\|z\|_2}{\|w\|_2} \le \|G_{cl}\|_\infty$$

and hence $\|z_{\bar{u},w}\|_2 - \delta \|w\|_2 \le 0$. Then also the infimum in (8.4) is less than or equal to 0. Therefore we may conclude that, indeed, a differentiable

matrix function P exists that satisfies conditions (a) and (b) of theorem 8.1.

The fact that (c) and (d) of theorem 8.1 also hold for a certain matrix function Q can be proven by the following dualization argument. Consider the dual system

$$\Sigma' : \begin{cases} \dot\xi = A^{\mathrm{T}}\xi + \ C_1^{\mathrm{T}}\nu + C_2^{\mathrm{T}}d, \\ \eta = B^{\mathrm{T}}\xi + \qquad\qquad D_2^{\mathrm{T}}d, \\ \zeta = E^{\mathrm{T}}\xi + D_1^{\mathrm{T}}\nu. \end{cases}$$

and apply to Σ' the time-varying compensator

$$\Sigma'_F : \begin{cases} \dot q(t) = K^{\mathrm{T}}(T-t)q(t) + M^{\mathrm{T}}(T-t)\eta(t), \\ \nu(t) = \ L^{\mathrm{T}}(T-t)q(t) + \ N^{\mathrm{T}}(T-t)\eta(t). \end{cases}$$

It can be shown that if we denote by $\tilde{\mathcal{G}}_{cl}$ the closed-loop operator of $\Sigma' \times \Sigma'_F$ (with $\zeta(0) = 0, q(0) = 0$), and if $\mathcal{G}_{cl}^{\star}$ denotes the adjoint operator of $\mathcal{G}_{cl}$ then the following equality holds:

$$\tilde{\mathcal{G}}_{cl} = R \circ \mathcal{G}_{cl}^{\star} \circ R, \tag{8.5}$$

where R denotes the time-reversal operator $(Rx)(t) := x(T-t)$. Now, if $\|\mathcal{G}_{cl}\|_{\infty} < \gamma$ then also $\|\mathcal{G}_{cl}^{\star}\|_{\infty} < \gamma$ and therefore, by (8.5), $\|\tilde{\mathcal{G}}_{cl}\|_{\infty} < \gamma$. We can therefore conclude that the quadratic differential inequality associated with Σ' has an appropriate solution, say $\tilde{P}(t)$, on $[0, T]$. By defining $Q(t) := \tilde{P}(T-t)$ we obtain a function Q that satisfies (c) and (d) of theorem 8.1.

Finally, we have to show that condition (e) of theorem 8.1 holds. We shall need the following lemma:

Lemma 8.3 : *Assume that $P : [0,T] \to \mathcal{R}^{n \times n}$ exists such that $F_\gamma(P)(t) \geq 0$, $\forall t \in [0,T]$, and rank $F_\gamma(P)(t) = rank_{\mathcal{R}(s)}G$, $\forall t \in [0,T]$. Then continuous matrix functions $C_{2,P}$ and D_P exist such that for all $t \in [0,T]$*

$$F_\gamma(P)(t) = \begin{pmatrix} C_{2,P}^{\mathrm{T}}(t) \\ D_P^{\mathrm{T}}(t) \end{pmatrix} \begin{pmatrix} C_{2,P}(t) & D_P(t) \end{pmatrix}. \tag{8.6}$$

$\square$

Note that the non-trivial part of the above lemma is that this factorization can be done with continuous factors. Assume that $F_\gamma(P)$ is factorized as in (8.6). Introduce a new system, say Σ_P, by

$$
\Sigma_P : \begin{cases}
\dot{x}_P = & \left(A + \frac{1}{\gamma^2}EE^\mathrm{T}P\right)x_P + & Bu_P + & Ew_P, \\
y_P = \left(C_1 + \frac{1}{\gamma^2}D_1E^\mathrm{T}P\right)x_P + & & D_1^\mathrm{T}w_P, \\
z_P = & C_{2,P}x_P + D_Pu_P.
\end{cases} \tag{8.7}
$$

We stress that Σ_P is a time-varying system with continuous coefficient matrices. For any dynamic compensator Σ_F of the form (8.2), let $\mathcal{G}_{cl,P}$ denote the operator from w_P to z_P obtained by interconnecting Σ_P and Σ_F.

The crucial observation now is that $\|\mathcal{G}_{cl}\|_\infty < \gamma$ if, and only if, $\|\mathcal{G}_{cl,P}\|_\infty < \gamma$, that is, *a compensator Σ_F "works" for Σ if, and only if, it "works" for Σ_P!* A proof of this can be based on the following "completion of the squares" argument:

Lemma 8.4 : *Assume that P satisfies (a) and (b) of theorem 8.1. Assume that $x_P(0) = x(0) = 0$, $u_P(t) = u(t)$ for all $t \in [0,T]$ and suppose that w_P and w are related by $w_P(t) = w(t) - \gamma^{-2}E^\mathrm{T}P(t)x(t)$ for all $t \in [0,T]$. Then for all $t \in [0,T]$ we have*

$$
\|z(t)\|^2 - \gamma^2\|w(t)\|^2 = \frac{d}{dt}\left(x^\mathrm{T}(t)P(t)x(t)\right) + \|z_P(t)\|^2 - \gamma^2\|w_P(t)\|^2.
$$

Consequently:

$$
\|z\|_2^2 - \gamma^2\|w\|_2^2 = \|z_P\|_2^2 - \gamma^2\|w_P\|_2^2. \tag{8.8}
$$

$\square$

Proof : This can be proven by straightforward calculation, using the factorization (8.6). ∎

We could not use this argument in chapter 5 since the stability requirement makes a formal proof along these lines relatively hard. In the finite-horizon case we do not have stability requirements and hence this argument can be applied.

Theorem 8.5 : *Let P satisfy (a) and (b) of theorem 8.1. Let Σ_F be a dynamic compensator of the form (8.2). Then*

$$\|\mathcal{G}_{cl}\|_\infty < \gamma \ \Leftrightarrow \ \|\mathcal{G}_{cl,P}\|_\infty < \gamma. \qquad\qquad \Box$$

Proof : Assume that $\|G_{cl,P}\|_\infty < \gamma$ and consider the interconnection of Σ and Σ_F.
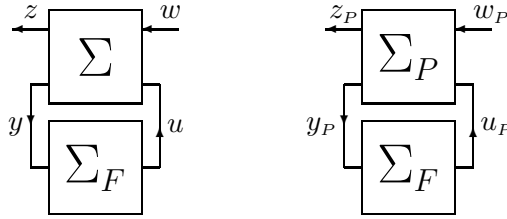
Let $0 \neq w \in \mathcal{L}_2^l[0,T]$, let x be the corresponding state trajectory of Σ and define $w_P := w - \gamma^{-2}E^{\mathrm{T}}Px$. Then clearly $y_P = y$, $x_P = x$ and therefore $u_P = u$. This implies that the equality (8.7) holds. We also have

$$\|z_P\|_2^2 - \gamma^2\|w_P\|_2^2 \leq (\|\mathcal{G}_{cl,P}\|_\infty^2 - \gamma^2)\|w_P\|_2^2. \qquad\qquad (8.9)$$

Next, note that the mapping $w_P \to w_P + \gamma^{-2}E^{\mathrm{T}}Px_P$ defines a bounded operator from $\mathcal{L}_2^l[0,T]$ to $\mathcal{L}_2^l[0,T]$. Hence a constant $\mu > 0$ exists such that $\|w_P\|_2^2 < \mu\|w\|_2^2$. Define $\delta > 0$ by $\delta^2 := \gamma^2 - \|\mathcal{G}_{cl,P}\|_\infty^2$. Combining (8.7) and (8.8) then yields

$$\|z\|_2^2 - \gamma^2\|w\|_2^2 \leq -\delta^2\mu\|w\|_2^2.$$

Obviously, this implies that $\|\mathcal{G}_{cl}\|_\infty^2 \leq \gamma^2 - \delta^2\mu < \gamma^2.$ ■

We shall now prove that condition (e) of theorem 8.1 holds. Again assume that Σ_F yields $\|\mathcal{G}_{cl}\|_\infty < \gamma$. By applying a version of lemma 8.2 for time-varying systems it can then be proven that the *dual quadratic differential inequality* associated with Σ_P:

$$\bar{G}_\gamma(Y) := \begin{pmatrix} -\dot{Y} + A_P Y + Y A_P^{\mathrm{T}} + EE^{\mathrm{T}} + \frac{1}{\gamma^2}Y C_{2,P}^{\mathrm{T}} C_{2,P} Y & Y C_{1,P}^{\mathrm{T}} + ED_1^{\mathrm{T}} \\ C_{1,P} Y + D_1 E^{\mathrm{T}} & D_1 D_1^{\mathrm{T}} \end{pmatrix} \geq 0$$

has a solution $Y(t)$ on $[0,T]$, satisfying $Y(0) = 0$ and

$$\mathrm{rank}\,\bar{G}_\gamma(Y)(t) = \mathrm{rank}_{\mathcal{R}(s)} \begin{pmatrix} Is - A_P(t) & -E \\ C_{1,P}(t) & D_1 \end{pmatrix} - n \qquad\qquad (8.10)$$

for all $t \in [0,T]$. Here, we have denoted $A_P = A + \gamma^{-2}EE^{\mathrm{T}}P$ and $C_{1,P} = C_1 + \gamma^{-2}D_1E^{\mathrm{T}}P$. Furthermore, it can be shown that Y is *unique* on each interval $[0,t_1]$ $(t_1 \le T)$. On the other hand, it can be proven that on each interval $[0,t_1]$ on which $I - QP$ is invertible, the function $\tilde{Y} := (I - QP)^{-1}Q$ satisfies $\bar{G}_\gamma(Y)(t) \ge 0$, $\tilde{Y}(0) = 0$ and the rank condition (8.10). Thus on any such interval $[0,t_1]$ we must have $Y(t) = \tilde{Y}(t)$. Clearly, since $Q(0) = 0$, $t_1 : 0 < t \le T$ exists such that $I - QP$ is invertible on $[0,t)$. Assume now that $t_1 > 0$ is the smallest real number such that $I - Q(t_1)P(t_1)$ is not invertible. Then on $[0,t_1)$ we have

$$Q(t) = (I - Q(t)P(t))\,\tilde{Y}(t)$$

and hence, by continuity

$$Q(t_1) = (I - Q(t_1)P(t_1))\,\tilde{Y}(t_1). \tag{8.11}$$

$x \ne 0$ exists such that $x^{\mathrm{T}}(I - Q(t_1)P(t_1)) = 0$. By (8.11) this yields $x^{\mathrm{T}}Q(t_1) = 0$ whence $x^{\mathrm{T}} = 0$, which is a contradiction. We must conclude that $I - Q(t)P(t)$ is invertible for all $t \in [0,T]$.

This completes our proof of the implication (i) $\Rightarrow$ (ii) of theorem 8.1

8.4   Existence of compensators

In the present section we shall sketch the main ideas of our proof of the implication (ii) $\Leftarrow$ (i) of theorem 8.1. The main idea is as follows: starting from the original system Σ we shall define a new system, $\Sigma_{P,Q}$, which has the following important properties:

(1) Let Σ_F be any compensator. The closed-loop operator $\mathcal{G}_{cl}$ of the interconnection $\Sigma \times \Sigma_F$ satisfies $\|\mathcal{G}_{cl}\|_\infty < \gamma$ if, and only if, the closed-loop operator of $\Sigma_{P,Q} \times \Sigma_F$, say $\mathcal{G}_{cl,P,Q}$, satisfies $\|\mathcal{G}_{cl,P,Q}\|_\infty < \gamma$.

(2) The system $\Sigma_{P,Q}$ is almost disturbance decouplable by dynamic measurement feedback, i.e. for *all* $\varepsilon > 0$ there exists Σ_F such that $\|\mathcal{G}_{cl,P,Q}\|_\infty < \varepsilon$.

Property (1) states that a compensator Σ_F "works" for Σ if, and only if, it "works" for $\Sigma_{P,Q}$. On the other hand, property (2) states that, *indeed*, there is a compensator Σ_F that "works" for $\Sigma_{P,Q}$: take any $\varepsilon \le \gamma$ and take a compensator Σ_F such that $\|\mathcal{G}_{cl,P,Q}\|_\infty < \varepsilon$. Then by, property (1), $\|\mathcal{G}_{cl}\|_\infty < \gamma$ so Σ_F works for Σ. This would clearly establish a proof of the implication (ii) $\Leftarrow$ (i) in theorem 8.1.

We shall now describe how the new system $\Sigma_{P,Q}$ is defined. Assume that P and Q exist satisfying (a) to (e) of theorem 8.1. Apply lemma 8.3 to obtain a continuous factorization (8.6) of $F_\gamma(P)$ and let the system Σ_P be defined by (8.7). Next, consider the dual quadratic differential inequality $\bar{G}_\gamma(Y) \geq 0$ associated with the system Σ_P, together with the conditions $Y(0) = 0$ and the rank condition (8.10). As was already noted in the previous section, the conditions (c), (d) and (e) assure that there is a unique solution Y on $[0, T]$ (In fact, $Y(t) = (\gamma^2 I - Q(t)P(t))^{-1}Q(t)$). Now, it can be shown that there exists a factorization

$$\bar{G}_\gamma(Y)(t) = \begin{pmatrix} E_{P,Q}(t) \\ D_{P,Q}(t) \end{pmatrix} \begin{pmatrix} E_{P,Q}^{\mathrm{T}}(t) & D_{P,Q}^{\mathrm{T}}(t) \end{pmatrix},$$

with $E_{P,Q}$ and $D_{P,Q}$ continuous on $[0, T]$. Denote

$$\begin{aligned} A_{P,Q}(t) &:= A_P(t) + Y(t)C_{2,P}^{\mathrm{T}}(t)C_{2,P}(t), \\ B_{P,Q}(t) &:= B + Y(t)C_{2,P}^{\mathrm{T}}(t)D_P(t). \end{aligned}$$

Then, introduce the new system $\Sigma_{P,Q}$ by:

$$\Sigma_{P,Q} : \begin{cases} \dot{x}_{P,Q} = A_{P,Q}x_{P,Q} + B_{P,Q}u_{P,Q} + E_{P,Q}w, \\ y_{P,Q} = C_{1,P}x_{P,Q} \qquad\qquad\quad + D_{P,Q}w, \\ z_{P,Q} = C_{2,P}x_{P,Q} + D_P u_{P,Q}, \end{cases}$$

Again, $\Sigma_{P,Q}$ is a time-varying system with continuous coefficient matrices. We note that $\Sigma_{P,Q}$ is in fact obtained by first transforming Σ into Σ_P and by subsequently applying the dual of this transformation to Σ_P. We shall now first show that property (1) above holds. If Σ_F is a dynamic compensator, then let $\mathcal{G}_{cl,P,Q}$ be the closed-loop operator from $w_{P,Q}$ to $z_{P,Q}$ in the interconnection of $\Sigma_{P,Q}$ with Σ_F:
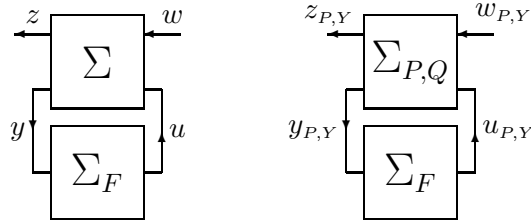
Recall that $\mathcal{G}_{cl}$ denotes the closed-loop operator from w to z in the interconnection of Σ and Σ_F. We have the following:

Theorem 8.6 :

$$\|\mathcal{G}_{cl}\|_\infty < \gamma \Leftrightarrow \|\mathcal{G}_{cl,P,Q}\|_\infty < \gamma. \qquad \qquad \qquad \square$$

Proof : Assume that Σ_F yields $\|\mathcal{G}_{cl}\|_\infty < \gamma$. By theorem 8.5 then also $\|\mathcal{G}_{cl,P}\|_\infty < \gamma$, i.e. Σ_F interconnected with Σ_P (given by (8.7)) also yields a closed-loop operator with norm less than γ. It is easy to see that the dual compensator Σ'_F (see section 8.3), interconnected with the dual of Σ_P:

$$\Sigma'_{P,Q} : \begin{cases} \dot\xi = A_P^{\mathrm{T}}(T-t)\xi + C_{1,P}^{\mathrm{T}}(T-t)\nu + C_{2,P}^{\mathrm{T}}(T-t)d \\ \eta = \qquad\quad B^{\mathrm{T}}\xi + \qquad\qquad\qquad D_P^{\mathrm{T}}(T-t)d \\ \zeta = \qquad\quad E^{\mathrm{T}}\xi + \qquad\quad D_1^{\mathrm{T}}\nu \end{cases}$$

yields a closed-loop operator $\tilde{\mathcal{G}}_{cl,P}$ (from d to ζ) with $\|\tilde{\mathcal{G}}_{cl,P}\|_\infty < \gamma$. Now, the quadratic differential inequality associated with Σ'_P is the transposed, time-reversed version of the inequality $\bar{G}_\gamma(Y) \geq 0$ and therefore has a unique solution $\tilde{Y}(t) = Y(T-t)$ such that $\tilde{Y}(T) = 0$ and the corresponding rank condition (8.10) holds. By applying theorem 8.5 to the system Σ'_P we may then conclude that the interconnection of Σ'_F with the dual $\Sigma'_{P,Q}$ of $\Sigma_{P,Q}$ yields a closed-loop operator with norm less than γ. Again by dualization we then conclude that $\|\mathcal{G}_{cl,P,Q}\|_\infty < \gamma$. The converse implication is proven analogously. ∎

Property (2) is stated formally in the following theorem:

Theorem 8.7 : *For all $\varepsilon > 0$ a time-varying dynamic compensator Σ_F exists such that $\|\mathcal{G}_{cl,P,Q}\|_\infty < \varepsilon$.* $\qquad\qquad\square$

Due to space limitations, for a proof of the latter theorem we refer to [St7]. The main difficulty is that $\Sigma_{P,Q}$ is time-varying and therefore it is necessary to derive a time-varying version of theorem 2.20 (we do not need (2.48) but we do need (2.47)). The basic steps are all the same but the proof is based on [Tr, Theorem 3.25] of which we had to derive a time-varying version.

By combining theorems 8.6 and 8.7 we immediately obtain a proof of the implication (ii) $\Leftarrow$ (i) in theorem 8.1.

8.5 Conclusion

In this chapter we have studied the finite horizon H_∞ control problem by dynamic measurement feedback. We have noted that the results obtained can be specialized to re-obtain results that were obtained before (see [Li5, Ta]). The development of our theory runs analogously to the theory developed in chapters 4 and 5 around the standard H_∞ control problem (the *infinite* horizon version of the problem studied in the present chapter). In the latter chapters the main tools are the so-called *quadratic matrix inequalities*, the algebraic versions of the differential inequalities used in the present paper. It should be noted that the geometric approach from appendix A is not well-suited to treat time-varying systems. This is why we cannot treat the time-varying case as in [Li5, Ta]. The formal proof for the results in this chapter is also highly technical, mainly due to the fact that the systems Σ_P and $\Sigma_{P,Q}$ are time-varying. The reason why the approach via appendix A still works is that the singular structure of Σ_P and $\Sigma_{P,Q}$ is time-invariant. A major open problem is how to treat the time-varying singular H_∞ or LQG control problem. A different approach will be needed to derive the general result.

Chapter 9

The discrete time full-information H_∞ control problem

9.1 Introduction

As already mentioned in the previous chapters, in recent years a considerable number of papers have appeared on the H_∞ optimal control problem. However, most of these papers discuss the continuous time case. In the next two chapters we shall discuss the discrete time case.

In the papers on H_∞ control with continuous time several methods were used to solve the H_∞ control problem as discussed in section 5.1. Recently, a paper has appeared solving the discrete time H_∞ control problem using frequency-domain techniques (see [Gu]). The polynomial approach has also been applied to discrete time systems (see [Gri]). In addition, a couple of papers have appeared using a time-domain approach (see [Bas, Li4, Ya]).

With respect to the papers which use a time-domain approach it should be noted that the references [Bas, Li4, Ya] do not contain a proof of the results obtained for the infinite horizon case and the papers [Bas, Ya] make a number of extra assumptions on the system under consideration. In [Bas, Li4, Ya] the authors first investigate the finite horizon problem and then derive a solution of the infinite horizon problem by considering it as a kind of limiting case as the endpoint tends to infinity.

In contrast, in the next two chapters we investigate the infinite horizon case using a direct approach. We shall use time-domain techniques which are reminiscent of those used in chapter 3 and the paper [Ta] which deal with the continuous time case. The method used in the next two chapters was derived independently of [Bas, Li4, Ya] and has already appeared in [St5, St6]. After the appearance of the above papers, several new papers and a book appeared on this subject using a state space ap-

proach (see e.g. [BB, Ig, Wa]). Very recently the J-spectral factorization approach has also been extended to discrete time systems (see [LMK]).

In this chapter we assume that we deal with the special cases that either both disturbance and state are available for feedback or only the state is available for feedback. This distinction has to be made since there is a more essential difference between these two cases than in the continuous time case. The more general case of measurement feedback is discussed in the next chapter.

The assumptions we shall make are weaker than the assumptions made in [Gu, Ya] and the same as the ones made in [Li4]. For the full-information case we have to make two assumptions which are exactly the discrete time analogues of the assumptions made in the regular continuous time H_∞ control problem as discussed in chapter 3. Firstly, the subsystem from control input to output should be left-invertible, and secondly, this subsystem should have no invariant zeros on the unit circle.

As in the regular continuous time H_∞ problem the necessary and sufficient conditions for the existence of an internally stabilizing controller such that the closed-loop system has H_∞ norm less than 1 involve a positive semi-definite stabilizing solution of a given algebraic Riccati equation. However, other than for the continuous time case, P has to satisfy an additional assumption: a matrix depending on P should be positive definite (in the continuous time case there also was a matrix which should be positive definite but in that case this matrix was independent of P).

Another difference with the continuous time case is that in the discrete time case, even if $D_2 = 0$, we cannot always achieve our goal with a static state feedback. In general, we also need a static feedback depending on the disturbance. Moreover, in the continuous time case we could not do better by allowing for non-causal compensators. This is *not* true in the discrete time case.

The outline of this chapter is as follows: in section 9.2 we shall formulate the problem and give the main results. In section 9.3 we shall derive necessary conditions under which an internally stabilizing feedback exists which makes the H_∞ norm less than 1. In section 9.4 we shall show that these conditions are also sufficient. After that, in section 9.5 we shall give but not prove some results on the discrete algebraic Riccati equation. We shall investigate properties like uniqueness of solutions and methods to calculate the solutions. We shall end with some concluding remarks in section 9.6.

9.2 Problem formulation and main results

We consider the following shift-invariant system:

$$\Sigma : \begin{cases} \sigma x = Ax + Bu + Ew, \\ z \quad = Cx + D_1 u + D_2 w, \end{cases} \tag{9.1}$$

where for each k, $x(k) \in \mathcal{R}^n$ is the state, $u(k) \in \mathcal{R}^m$ is the control input, $w(k) \in \mathcal{R}^l$ is the unknown disturbance and $z(k) \in \mathcal{R}^p$ is the output to be controlled. Moreover, A, B, E, C, D_1 and D_2 are matrices of appropriate dimensions. Our objective is to find a compensator Σ_F described by a static feedback law $u(k) = F_1 x(k) + F_2 w(k)$ such that the closed-loop system is internally stable and for the closed-loop system the ℓ_2-induced operator norm from disturbance w to the output z is less than 1, i.e. the H_∞ norm of the closed-loop system is less than 1.

In this chapter we shall derive necessary and sufficient conditions for the existence of a suitable compensator Σ_F. Moreover, in the case that such a compensator exists we give an explicit formula for one static feedback law which yields a closed-loop system which is internally stable and which has H_∞ norm less than 1. We shall also derive similar conditions for the existence of a suitable static *state* feedback. A characterization of all suitable *dynamic* compensators will be given in section 10.5.

We first give a definition:

Definition 9.1 : *An function $f : \ell_2 \to \ell_2$, $w \to f(w)$ is called causal if for any $w_1, w_2 \in \ell_2$ and $k \in \mathcal{N}$:*

$$w_1|_{[0,k]} = w_2|_{[0,k]} \quad \Rightarrow \quad f(w_1)|_{[0,k]} = f(w_2)|_{[0,k]}.$$

Such a function f is called strictly causal if for any $w_1, w_2 \in \ell_2$ and $k \in \mathcal{N}$ we have

$$w_1|_{[0,k-1]} = w_2|_{[0,k-1]} \quad \Rightarrow \quad f(w_1)|_{[0,k]} = f(w_2)|_{[0,k]}. \qquad \square$$

A controller of the form (2.9) always defines a causal operator. In the case that $N = 0$ this operator is strictly causal.

We now formulate our main result:

Theorem 9.2 : *Consider the system (9.1) and assume that the system (A, B, C, D_1) has no invariant zeros on the unit circle and is left-invertible. The following three statements are equivalent:*

(i) *A compensator Σ_F described by a static feedback law of the form $u = F_1 x + F_2 w$ exists such that the closed-loop system is internally stable and has H_∞ norm less than 1, i.e. the closed-loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$.*

(ii) *(A, B) is stabilizable and for the system (9.1) a causal operator $f : \ell_2^l \to \ell_2^m$ and $\delta < 1$ exist such that for all $w \in \ell_2^l$ with $u = f(w)$ we have $x_{u,w} \in \ell_2^n$ and $\|z_{u,w}\|_2 \leq \delta\|w\|_2$.*

(iii) *A symmetric matrix $P \geq 0$ exists such that*

(a) $V > 0, \qquad R > 0,$ (9.2)

where

$$V := D_1^\mathrm{T} D_1 + B^\mathrm{T} P B,$$
$$R := I - D_2^\mathrm{T} D_2 - E^\mathrm{T} P E$$
$$+ \left(E^\mathrm{T} P B + D_2^\mathrm{T} D_1\right) V^{-1} \left(B^\mathrm{T} P E + D_1^\mathrm{T} D_2\right).$$

This implies that the matrix $G(P)$ is invertible, where:

$$G(P) := \begin{pmatrix} D_1^\mathrm{T} D_1 & D_1^\mathrm{T} D_2 \\ D_2^\mathrm{T} D_1 & D_2^\mathrm{T} D_2 - I \end{pmatrix} + \begin{pmatrix} B^\mathrm{T} \\ E^\mathrm{T} \end{pmatrix} P \begin{pmatrix} B & E \end{pmatrix}.$$ (9.3)

(b) *P satisfies the following discrete algebraic Riccati equation:*

$$P = A^\mathrm{T} P A + C^\mathrm{T} C$$
$$- \begin{pmatrix} B^\mathrm{T} P A + D_1^\mathrm{T} C \\ E^\mathrm{T} P A + D_2^\mathrm{T} C \end{pmatrix}^\mathrm{T} G(P)^{-1} \begin{pmatrix} B^\mathrm{T} P A + D_1^\mathrm{T} C \\ E^\mathrm{T} P A + D_2^\mathrm{T} C \end{pmatrix}.$$ (9.4)

(c) *The matrix A_{cl} is asymptotically stable, where:*

$$A_{cl} := A - \begin{pmatrix} B & E \end{pmatrix} G(P)^{-1} \begin{pmatrix} B^\mathrm{T} P A + D_1^\mathrm{T} C \\ E^\mathrm{T} P A + D_2^\mathrm{T} C \end{pmatrix}.$$ (9.5)

Moreover, in the case that P satisfies part (iii), then the compensator Σ_F described by the static feedback law $u(k) = F_1 x(k) + F_2 w(k)$ where

$$F_1 := -\left(D_1^\mathrm{T} D_1 + B^\mathrm{T} P B\right)^{-1} \left(B^\mathrm{T} P A + D_1^\mathrm{T} C\right),$$ (9.6)
$$F_2 := -\left(D_1^\mathrm{T} D_1 + B^\mathrm{T} P B\right)^{-1} \left(B^\mathrm{T} P E + D_1^\mathrm{T} D_2\right),$$ (9.7)

satisfies the requirements in part (i).   □

Remarks :

(i) Necessary and sufficient conditions under which we can find an internally stabilizing feedback which makes the H_∞ norm of the closed-loop system less than some, a priori given, upper bound γ can easily be derived from theorem 9.2 by scaling. We only prove the result for $D_2 = 0$. The general result can be proven along the same lines and this proof can be found in [St6].

(ii) Note that part (ii) of theorem 9.2 is equivalent to the requirement that a causal operator f exists such that the feedback law $u = f(x, w)$ satisfies part (ii). This follows from the fact that, after applying the feedback, there exists a causal operator g mapping w to x and therefore we could have started with the causal operator $u = f(g(w), w)$ in the first place. Conversely if we have the feedback $u = f(w)$, then we define $f_1(x, w) := f(w)$ which then satisfies the requirements of the reformulated part (ii).

(iii) In the continuous time case (except for the singular case with $D_2 \neq 0$) we could not really do better when minimizing the H_∞ norm of the closed-loop system under the constraint of internal stability by allowing for non-causal feedbacks. This is not true in the discrete time case. Consider, e.g. the following system:

$$\begin{cases} \sigma x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 10 \end{pmatrix} u + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w, \\ z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u. \end{cases}$$

The feedback $u = -\sigma w / 10$ makes the H_∞ norm of the closed-loop system equal to 0.1. On the other hand, by a causal feedback we cannot make the H_∞ norm of the closed-loop system less than 1.

(iv) If we compare these conditions with the conditions for the continuous time case as given in chapter 3 we note that condition (9.2) (which is comparable to the condition (3.2) in theorem 3.1) now depends on P and is still present if $D_2 = 0$. A simple example showing that this assumption is not superfluous is given by the system:

$$\begin{cases} \sigma x = \qquad\qquad u + 2w, \\ z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u. \end{cases}$$

No feedback Σ_F satisfying part (i) of theorem 9.2 exists but there is a positive semi-definite matrix P satisfying (9.4) and such that $A_{cl} = 0$ and hence asymptotically stable, namely $P = 1$. However for this P we have $R = -1$.

The general outline of the proof will be reminiscent of the proof given in [Ta] and chapter 3 for the continuous time case. The extra condition (9.2), the invertibility of (9.3) and the requirement of left-invertibility instead of assuming that D_1 is injective will give rise to a substantial increase in the amount of intricacies in the proof.

From the above result we can easily derive necessary and sufficient conditions when the system (A, B, C, D) has H_∞ norm strictly less than 1. This is stated in the following corollary:

Corollary 9.3 : *Let the system (A, B, C, D) be given with A stable. The corresponding transfer matrix G has H_∞ norm strictly less than 1 if, and only if, there exists a matrix $P \geq 0$ such that:*

- $N := I - D^\mathrm{T}D - B^\mathrm{T}PB > 0$,
- $P = A^\mathrm{T}PA + C^\mathrm{T}C + (A^\mathrm{T}PB + C^\mathrm{T}D)N^{-1}(B^\mathrm{T}PA + D^\mathrm{T}C)$, *and*
- $A + BN^{-1}(B^\mathrm{T}PA + D^\mathrm{T}C)$ *is asymptotically stable.*   □

As in the continuous time case it is interesting to know whether we really need the disturbance feedback component or not. The following theorem gives necessary and sufficient conditions under which we can obtain internal stability and an H_∞ norm of the closed-loop system less than 1 by using a static state feedback.

Theorem 9.4 : *Consider the system (9.1) and assume that (A, B, C, D_1) has no invariant zeros on the unit circle and is left-invertible. The following statements are equivalent:*

(i) *A compensator Σ_F described by a static state feedback law exists such that the closed-loop system is internally stable and has H_∞ norm less than 1, i.e. the closed-loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$.*

(ii) *A symmetric matrix $P \geq 0$ exists such that*

(a) *We have*

$$D_1^T D_1 + B^T P B > 0,$$

$$R_1 := I - D_2^T D_2 - E^T P E > 0. \tag{9.8}$$

   This implies that the matrix $G(P)$ defined by (9.3) is invertible.

(b) *P satisfies the discrete algebraic Riccati equation (9.4).*

(c) *The matrix A_{cl} defined by (9.5) is asymptotically stable.*

Moreover, in the case that P satisfies part (ii), then the compensator Σ_F described by the static feedback law $u(k) = Fx(k)$ where

$$F := H^{-1} \left(B^T P A + D_1^T C + \left(B^T P E + D_1^T D_2 \right) R_1^{-1} \left(E^T P A + D_2^T C \right) \right) \tag{9.9}$$

satisfies the requirements in part (i) where

$$H := B^T P B + D_1^T D_1 + \left(B^T P E + D_1^T D_2 \right) R_1^{-1} \left(E^T P B + D_2^T D_1 \right). \qquad \square$$

Remarks : Note that the only difference with the conditions of theorem 9.2 is that the condition $R > 0$ is replaced by the condition (9.8). Note that the latter condition indeed implies $R > 0$.

   In the continuous time case we only needed the extra condition $D_2^T D_2 < I$. This is not sufficient in discrete time as can be seen from the following system:

$$\begin{cases} \sigma x = \quad\; u + 1/2\, w, \\ \;\; z = 8x + u. \end{cases}$$

The matrix $P = 84$ satisfies the conditions in part (iii) of theorem 9.2 but condition (9.8) is not satisfied. By state feedback the H_∞ norm of the closed-loop system cannot be made less than 3 while maintaining internal stability. On the other hand with disturbance feedback we can make the H_∞ norm equal to 1/2.

   The above theorem will only be proven for the special case that $D_2 = 0$. This is done to make the formulae more tractable. It yields no extra difficulties to extend the proof to the general case.

9.3   Existence of a stabilizing solution of the Riccati equation

In this section we shall show that the existence of a causal operator f and $\delta < 1$ satisfying part (ii) of theorem 9.2 implies that a positive semi-definite solution of the discrete algebraic Riccati equation (9.4) exists

such that (9.5) is asymptotically stable and (9.2) is satisfied. We also prove the implication (i) $\Rightarrow$ (ii) of theorem 9.4. As mentioned before we shall assume throughout the next two sections that $D_2 = 0$. The method used has a great similarity with the proof given in section 3.4. We shall assume that

$$D_1^{\mathrm{T}} C = 0, \tag{9.10}$$

for the time being and we shall derive the more general statement later. Note that part (iii) of the remarks after theorem 9.2 guarantees that, contrary to the continuous time case, we have to be concerned with causal dependence of u on w. In order to prove the existence of the desired P we shall investigate the following sup-inf problem:

$$\mathcal{C}^*(\xi) := \sup_{w \in \ell_2^l} \inf_u \ \left\{ \ \|z_{u,w,\xi}\|_2^2 - \|w\|_2^2 \ \mid \ u \in \ell_2^m \text{ such that } x_{u,w,\xi} \in \ell_2^n \ \right\},$$

$$\tag{9.11}$$

for arbitrary initial state ξ. It turns out that the conditions of part (ii) of theorem 9.2 imply that $\mathcal{C}^*(\xi)$ is finite for every ξ. Moreover, it will be shown that, as in section 3.4, a $P \geq 0$ exists such that $\mathcal{C}^*(\xi) = \xi^{\mathrm{T}} P \xi$. At the end of this section we then prove that this P exactly satisfies conditions (a)–(c) of theorem 9.2. We shall first infimize, for given $w \in \ell_2$ and $\xi \in \mathcal{R}^n$, the function $\|z_{u,w,\xi}\|_2^2 - \|w\|_2^2$ over all $u \in \ell_2$ for which $x_{u,w,\xi} \in \ell_2$. After that we shall maximize over $w \in \ell_2$.

As in chapter 3 our proof is based on Pontryagin's maximum principle. We shall use the ideas from [LM], together with our stability requirement $x_{u,w,\xi} \in \ell_2$ to adapt the proof to the infinite horizon discrete time case.

We start by constructing a solution of the adjoint Hamilton–Jacobi equation which is a natural starting point if one wants to use Pontryagin's maximum principle.

Let L be such that $D_1^{\mathrm{T}} D_1 + B^{\mathrm{T}} L B$ is invertible and such that L is the positive semi-definite solution of the following discrete algebraic Riccati equation:

$$L = A^{\mathrm{T}} L A + C^{\mathrm{T}} C - A^{\mathrm{T}} L B \left(D_1^{\mathrm{T}} D_1 + B^{\mathrm{T}} L B \right)^{-1} B^{\mathrm{T}} L A, \tag{9.12}$$

for which

$$A_L := A - B \left(D_1^{\mathrm{T}} D_1 + B^{\mathrm{T}} L B \right)^{-1} B^{\mathrm{T}} L A,$$

is asymptotically stable. The existence of such L is guaranteed under the assumption that (A, B, C, D_1) has no invariant zeros on the unit circle

and is left-invertible and, moreover, (A, B) is stabilizable (see [Si]). We define

$$r(k) := -\sum_{i=k}^{\infty} [X_1 A^{\mathrm{T}}]^{i-k} X_1 LEw(i), \tag{9.13}$$

where

$$X_1 := I - LB \left(D_1^{\mathrm{T}} D_1 + B^{\mathrm{T}} LB\right)^{-1} B^{\mathrm{T}}.$$

Note that r is well-defined since the matrix $A_L = X_1^{\mathrm{T}} A$ is asymptotically stable which implies that $X_1 A^{\mathrm{T}}$ is asymptotically stable. Next we define the functions $y, \tilde{x}$ and η by:

$$
\begin{align}
y &:= M^{-1} B^{\mathrm{T}} \left[A^{\mathrm{T}} \sigma r - LEw\right], \tag{9.14} \\
\sigma \tilde{x} &= A_L \tilde{x} + By + Ew, \qquad \tilde{x}(0) = \xi, \tag{9.15} \\
\eta &:= -X_1 LA\tilde{x} + r, \tag{9.16}
\end{align}
$$

where $M := D_1^{\mathrm{T}} D_1 + B^{\mathrm{T}} LB$. Since A_L and $X_1 A^{\mathrm{T}}$ are asymptotically stable, it can be checked straightforwardly that, given $\xi \in \mathcal{R}^n$ and $w \in \ell_2^l$, we have $r, \tilde{x}, \eta \in \ell_2$.

After some standard calculations, we find the following lemma:

Lemma 9.5 : *Let $\xi \in \mathcal{R}^n$ and $w \in \ell_2^l$ be given. The function $\eta \in \ell_2^n$ is a solution of the following backward difference equation:*

$$\sigma^{-1}\eta = A^{\mathrm{T}}\eta - C^{\mathrm{T}}C\tilde{x}, \qquad \lim_{k\to\infty} \eta(k) = 0. \tag{9.17}$$

Here η is extended to a function from $\mathcal{N} \cup \{-1\}$ to $\mathcal{R}^n$ by choosing $\eta(-1)$ such that (9.17) is satisfied. □

In the statement of Pontryagin's maximum principle this equation is the so-called "adjoint Hamilton–Jacobi equation" and η is called the "adjoint state variable". We have constructed a solution to this equation and we shall show that this η indeed yields a minimizing u. Note the difference with the continuous time case where we could derive a differential equation forward in time, while in discrete time we can only derive a difference equation forward in time when A is invertible. To prevent this kind of difficulty it is assumed in [Gu] that A is invertible. The proof that η yields a minimizing u is adapted from the proof of lemma 3.4:

Lemma 9.6 : *Let the system (9.1) be given. Moreover let w and ξ be fixed. Then*

$$
\begin{aligned}
\tilde{u} \; := \; & -\left(D_1^{\mathrm{T}} D_1 + B^{\mathrm{T}} L B\right)^{-1} B^{\mathrm{T}} L A \tilde{x} + y \\
= \; & \arg \inf_{u} \; \{\; \|z_{u,w,\xi}\|_2 \mid u \in \ell_2^m \text{ such that } x_{u,w,\xi} \in \ell_2^n \;\}. \qquad \square
\end{aligned}
$$

Proof : It is easy to check that $\tilde{x} = x_{\tilde{u},w,\xi}$. Define

$$
\mathcal{J}_T(u) := \sum_{i=0}^{T} \|C x_{u,w,\xi}(i) + D_1 u(i)\|^2.
$$

Let $u \in \ell_2^m$ be an arbitrary control input such that $x_{u,w,\xi} \in \ell_2^n$. We find

$$
\mathcal{J}_T(u) - \mathcal{J}_{T-1}(u) - 2\eta^{\mathrm{T}}(T)x(T+1) + 2\eta^{\mathrm{T}}(T-1)x(T) =
$$

$$
\|Cx(T)\|^2 + \left[D_1^{\mathrm{T}} D_1 u(T) - 2B^{\mathrm{T}}\eta(T)\right]^{\mathrm{T}} u(T)
$$

$$
-2\eta^{\mathrm{T}}(T)Ew(T) - 2x^{\mathrm{T}}(T)C^{\mathrm{T}}C\tilde{x}(T).
$$

We also find

$$
\mathcal{J}_T(\tilde{u}) - \mathcal{J}_{T-1}(\tilde{u}) - 2\eta^{\mathrm{T}}(T)\tilde{x}(T+1) + 2\eta^{\mathrm{T}}(T-1)\tilde{x}(T) =
$$

$$
-\|C\tilde{x}(T)\|^2 + \left[D_1^{\mathrm{T}} D_1 \tilde{u}(T) - 2B^{\mathrm{T}}\eta(T)\right]^{\mathrm{T}} \tilde{u}(T) - 2\eta^{\mathrm{T}}(T)Ew(T).
$$

Hence, if we sum the last two equations from zero to infinity and subtract from each other, we find:

$$
\|z_{\tilde{u},w,\xi}\|_2^2 - \|z_{u,w,\xi}\|_2^2 = \sum_{i=0}^{\infty} -\|C\left(x(i) - \tilde{x}(i)\right)\|^2 +
$$

$$
+ \sum_{i=0}^{\infty} \left[D_1^{\mathrm{T}} D_1 \tilde{u}(i) - 2B^{\mathrm{T}}\eta(i)\right]^{\mathrm{T}} \tilde{u}(i) - \left[D_1^{\mathrm{T}} D_1 u(i) - 2B^{\mathrm{T}}\eta(i)\right]^{\mathrm{T}} u(i).
$$

It is easy to check that $B^{\mathrm{T}}\eta(i) = D_1^{\mathrm{T}} D_1 \tilde{u}(i)$ for all i. Therefore, for every i we have

$$
\left[D_1^{\mathrm{T}} D_1 \tilde{u}(i) - 2B^{\mathrm{T}}\eta(i)\right]^{\mathrm{T}} \tilde{u}(i) = \inf_{u} \left[D_1^{\mathrm{T}} D_1 u - 2B^{\mathrm{T}}\eta(i)\right]^{\mathrm{T}} u.
$$

Together the last two equations imply that:

$$
\|z_{\tilde{u},w,\xi}\|_2^2 \leq \|z_{u,w,\xi}\|_2^2,
$$

which is exactly what we had to prove. Since (A, B, C, D_1) is left-invertible it can easily be shown that the minimizing u is unique. ∎

We are now going to maximize over $w \in \ell_2^l$. This will then yield $\mathcal{C}^*(\xi)$. Define $\mathcal{F}(\xi, w) := (\tilde{x}, \tilde{u}, \eta)$ and $\mathcal{G}(\xi, w) := z_{\tilde{u}, w, \xi} = C\tilde{x} + D_1\tilde{u} + D_2 w$. It is clear from the previous lemma that $\mathcal{F}$ and $\mathcal{G}$ are bounded linear operators. Define

$$
\begin{aligned}
\mathcal{C}(\xi, w) &:= \|\mathcal{G}(\xi, w)\|_2^2 - \|w\|_2^2, \\
\|w\|_C &:= (-\mathcal{C}(0, w))^{1/2}.
\end{aligned}
$$

It is easy to show that $\|.\|_C$ defines a norm on ℓ_2^l. Using the conditions in part (ii) of theorem 9.2 it can be shown straightforwardly that

$$
\|w\|_2 \geq \|w\|_C \geq \rho\|w\|_2, \tag{9.18}
$$

where $\rho > 0$ is such that $\rho^2 = 1 - \delta^2$ and δ is such that the conditions of part (ii) of theorem 9.2 are satisfied. Hence $\|.\|_C$ and $\|.\|_2$ are equivalent norms. We have

$$
\mathcal{C}^*(\xi) = \sup_{w \in \ell_2^l} \mathcal{C}(\xi, w).
$$

We can derive the following properties of $\mathcal{C}^*$. The proof is identical to the proof of lemma 3.5 and is therefore omitted.

Lemma 9.7 :

(i) *For all $\xi \in \mathcal{R}^n$ we have*

$$
0 \leq \xi^{\mathrm{T}} L \xi \leq \mathcal{C}^*(\xi) \leq \frac{\xi^{\mathrm{T}} L \xi}{1 - \delta^2},
$$

where δ is such that part (ii) of theorem 9.2 is satisfied.

(ii) *For all $\xi \in \mathcal{R}^n$ a unique $w_* \in \ell_2^l$ exists such that $\mathcal{C}^*(\xi) = \mathcal{C}(\xi, w_*)$.* □

Define $\mathcal{H} : \mathcal{R}^n \to \ell_2^l$, $\xi \to w_*$. Unlike the explicit expression for $\tilde{u}$ we can only derive an implicit formula for w_*. However, we can show that w_* is the unique solution of a linear equation. This time we also omit the proof. It is similar to the proof of lemma 3.6. The adaptations necessary are the same as the difference between the proofs of lemma 3.4 and lemma 9.6.

Lemma 9.8 : *Let $\xi \in \mathcal{R}^n$ be given. Then $w_* = \mathcal{H}\xi$ is the unique ℓ_2-function w satisfying:*

$$w = -E^{\mathrm{T}}\eta, \tag{9.19}$$

where $(x, u, \eta) = \mathcal{F}(\xi, w)$. □

Next, we shall show that $\mathcal{C}^*(\xi) = \xi^{\mathrm{T}}P\xi$ for some matrix P. In order to do that we first show that u_*, η_* and w_* are linear functions of x_*:

Lemma 9.9 : *There exist constant matrices K_1, K_2 and K_3 such that*

$$
\begin{aligned}
u_* &= K_1 x_*, & (9.20) \\
\eta_* &= K_2 x_*, & (9.21) \\
w_* &= K_3 x_*. & (9.22)
\end{aligned}
$$

□

Proof : We shall first consider time 0. By lemma 9.8 it is easy to see that $\mathcal{H} : \xi \to w_*$ is linear. Hence the mapping from ξ to $w_*(0)$ is also linear. This implies the existence of a matrix K_3 such that $w_*(0) = K_3\xi$. From (9.17) and lemma 9.6 it can be seen that u_* and η_* are linear functions of ξ and w_*. This implies, since w_* is a linear function of ξ, that $u_*(0)$ and $\eta_*(0)$ are linear functions of ξ and hence K_1 and K_2 exist such that $u_*(0) = K_1\xi$ and $\eta_*(0) = K_2\xi$.

We shall now look at time t. The sup-inf problem starting at time t with initial value $x(t)$ can now be solved. Due to the time-invariance property we see that w_* restricted to $[t, \infty)$ satisfies (9.19) and hence for this problem the optimal x and η are x_* and η_*. But since t is the initial time for this optimization problem, which is exactly equal to the original optimization problem, we find equations (9.20)–(9.22) at time t with the same matrices K_1, K_2 and K_3 as at time 0. Since t was arbitrary this completes the proof. ■

Lemma 9.10 : *A matrix P exists such that $\sigma^{-1}\eta_* = -Px_*$. Moreover for this P we find*

$$\mathcal{C}^*(\xi) = \xi^{\mathrm{T}}P\xi. \tag{9.23}$$

□

Proof : We have

$$
\begin{aligned}
\sigma^{-1}\eta_* &= A^{\mathrm{T}}\eta_* - C^{\mathrm{T}}Cx_* \\
&= \left(A^{\mathrm{T}}K_2 - C^{\mathrm{T}}C\right)x_*
\end{aligned}
$$

We define $P := -\left(A^{\mathrm{T}}K_2 - C^{\mathrm{T}}C\right)$ using the matrices defined in lemma 9.9. We shall prove that this P satisfies (9.23). We can derive the following equation

$$
\|z_{u_*,w_*,\xi}(T)\|^2 - \|w_*(T)\|^2 - 2\eta_*^{\mathrm{T}}(T)x_*(T+1) + 2\eta_*(T-1)^{\mathrm{T}}x_*(T) =
$$

$$
\|w_*(T)\|^2 - \|z_{u_*,w_*,\xi}(T)\|^2.
$$

We sum this equation from zero to infinity. Since $\lim_{T\to\infty}\eta_*(T) = 0$ and $\lim_{T\to\infty}x_*(T) = 0$ we find

$$
\mathcal{C}(\xi, w_*) + 2\eta_*^{\mathrm{T}}(-1)x_*(0) = -\mathcal{C}(\xi, w_*).
$$

Since $\mathcal{C}(\xi, w_*) = \mathcal{C}^*(\xi)$ and $\eta_*(-1) = -P\xi$ we find (9.23). ■

We shall now show that this matrix P satisfies conditions (a)–(c) of theorem 9.2. We first show part (a). Since we do not know yet if P is symmetric we have to be a little bit careful. This essential step in our derivation is new compared to the method used in chapter 3. This will be the step where we have to take causality into account.

Lemma 9.11 : *Let P be given by lemma 9.10. The matrices V and R as defined in the part (iii) of theorem 9.2, condition (a) satisfy:*

$$
\begin{aligned}
V + V^{\mathrm{T}} &> 0, \\
R + R^{\mathrm{T}} &> 0.
\end{aligned}
$$
□

Proof : By lemma 9.7 and lemma 9.10, we know $(P + P^{\mathrm{T}})/2 \geq L$ and therefore we find $(V + V^{\mathrm{T}})/2 \geq D_1^{\mathrm{T}}D_1 + B^{\mathrm{T}}LB$. The latter matrix is positive definite and hence $(V + V^{\mathrm{T}})/2$ is positive definite, i.e. $V + V^{\mathrm{T}} > 0$. We shall now look at the following "sup–inf–sup–inf" problem for initial condition 0:

$$
\mathcal{J}(0) := \sup_{w(0)} \inf_{u(0)} \sup_{w^+} \inf_{u^+} \|z_{u,w}\|^2 - \|w\|^2, \tag{9.24}
$$

where $w^+ := w|_{[1,\infty)}$ and $u^+ := u|_{[1,\infty)}$. We shall always implicitly add the constraint that u^+ is such that the resulting state x is in ℓ_2.

We know a causal operator f exists satisfying part (ii) of theorem 9.2 and hence this function makes the ℓ_2-induced operator norm strictly less than 1 under the constraint $x \in \ell_2^n$. In (9.24) we set $u = f(w)$. This is possible since by causality we know that $u(0)$ only depends on $w(0)$ and u^+ depends on the whole function w. Thus we get:

$$
\begin{aligned}
\mathcal{J}(0) &= \sup_{w(0)} \inf_{u(0)} \sup_{w^+} \inf_{u^+} \|z_{u,w}\|_2^2 - \|w\|_2^2 \\
&\leq \sup_{w} \|z_{f(w),w}\|_2^2 - \|w\|_2^2 \tag{9.25} \\
&\leq 0. \tag{9.26}
\end{aligned}
$$

By lemma 9.10 we have:

$$
\sup_{w^+} \inf_{u^+} \|z_{u^+,w^+,x(1)}\|_2^2 - \|w^+\|_2^2 = x(1)^{\mathrm{T}} P x(1). \tag{9.27}
$$

Therefore, we can reduce (9.24) to the following "sup–inf" problem:

$$
\sup_{w(0)} \inf_{u(0)} \begin{pmatrix} u(0) \\ w(0) \end{pmatrix}^T \begin{pmatrix} V & B^{\mathrm{T}} P E \\ E^{\mathrm{T}} P B & E^{\mathrm{T}} P E - I \end{pmatrix} \begin{pmatrix} u(0) \\ w(0) \end{pmatrix}.
$$

When we define

$$
\tilde{u}(0) = u(0) - E^{\mathrm{T}} P B V^{-1} w(0),
$$

then we get

$$
\mathcal{J}(0) = \sup_{w(0)} \inf_{\tilde{u}(0)} \begin{pmatrix} \tilde{u}(0) \\ w(0) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} V & 0 \\ 0 & -R \end{pmatrix} \begin{pmatrix} \tilde{u}(0) \\ w(0) \end{pmatrix}. \tag{9.28}
$$

Since, by (9.26), $\mathcal{J}(0)$ is finite we immediately find that a necessary condition is that $w(0)^{\mathrm{T}} R w(0) \geq 0$ for all $w(0)$, i.e. $R + R^{\mathrm{T}} \geq 0$. We also note that $\mathcal{J}(0) = 0$.

Assume that $R + R^{\mathrm{T}}$ is not invertible. Then a $v \neq 0$ exists such that $v^{\mathrm{T}} R v = 0$. Let $w^+(u(0))$ be the ℓ_2-function which attains the optimum in the optimization (9.27) with initial state $x(1) = B u(0) + E v$. We define the function w by

$$
[w(u(0))](t) := \begin{cases} v & \text{if } t = 0, \\ w^+(u(0))(t) & \text{otherwise.} \end{cases} \tag{9.29}
$$

Assume that δ and f are such that part (ii) of theorem 9.2 is satisfied. Define u by

$$
u = f[w(u(0))]. \tag{9.30}
$$

Since the map from u to w defined by (9.29) is strictly causal and since f is causal, u is uniquely defined by (9.30). In order to prove this note that $u(0)$ only depends on $w(u(0))(0) = v$ and hence w^+ as a function of $u(0)$ is uniquely defined which, in turn, yields u. Denote u and w obtained in this way by u_1 and w_1. By (9.25), (9.28) and since $Rv = 0$ we find that, for this particular choice of w_1 and u_1, we have:

$$
\begin{aligned}
\|z_{u_1,w_1}\|_2^2 &- \|w_1\|_2^2 \\
&\geq \inf_u \; \|z_{u,w(u(0))}\|_2^2 - \|w(u(0))\|_2^2 \\
&= \inf_{\tilde{u}(0)} \; \tilde{u}^{\mathrm{T}}(0) V \tilde{u}(0) - v^{\mathrm{T}} R v \\
&= 0.
\end{aligned}
\tag{9.31}
$$

On the other hand, using part (ii) of theorem 9.2 we find:

$$
\|z_{u_1,w_1}\|_2^2 - \|w_1\|_2^2 < \left(\delta^2 - 1\right) \|w_1\|_2^2 < 0,
$$

since $w_1(0) = v \neq 0$. Therefore we have a contradiction and hence our assumption that $R + R^{\mathrm{T}}$ is not invertible was incorrect. Together with $R + R^{\mathrm{T}} \geq 0$ this yields $R + R^{\mathrm{T}} > 0$. ∎

Lemma 9.12 : *Assume that (A, B, C, D_1) has no invariant zeros on the unit circle and is left-invertible. Moreover, assume that $D_1^{\mathrm{T}} C = 0$. If the statement in part (ii) of theorem 9.2 is satisfied, then a symmetric matrix $P \geq 0$ exists satisfying (a)–(c) of part (iii) of theorem 9.2.* □

Proof : We define the matrices

$$
\begin{aligned}
M &:= D_1^{\mathrm{T}} D_1 + B^{\mathrm{T}} L B > 0, \\
Z &:= I - E^{\mathrm{T}} X_1 L E.
\end{aligned}
$$

We know that $-(R + R^{\mathrm{T}})/2$ is the Schur complement of $(V + V^{\mathrm{T}})/2$ in $G((P+P^{\mathrm{T}})/2)$. By lemma 9.11 we know that $R + R^{\mathrm{T}} > 0$ and $V + V^{\mathrm{T}} > 0$. Therefore $G((P+P^{\mathrm{T}})/2)$ has m eigenvalues on the positive real axis and l eigenvalues on the negative real axis. We know $G((P+P^{\mathrm{T}})/2) - G(L) \geq 0$ since $(P + P^{\mathrm{T}})/2 \geq L$. An easy consequence of the theorem of Courant-Fischer (see [Bel]) then tells us that $G(L)$ has at least l eigenvalues on the negative real axis. Since $-Z$ is the Schur complement of $M > 0$ in $G(L)$ this implies that $Z > 0$.

By lemma 9.10 we have $\eta_* = -\sigma P x_*$. By combining lemma 9.6 and lemma 9.8 and rewriting the equations we find that u_* and w_* satisfy the following equations:

$$
\begin{aligned}
w_* &= Z^{-1} \left\{ E^{\mathrm{T}} X_1 \left(P - L \right) \sigma x_* + E^{\mathrm{T}} X_1 L A x_* \right\}, \\
u_* &= -M^{-1} B^{\mathrm{T}} \left\{ \left(P - L \right) \sigma x_* + L A x_* + L E w_* \right\}.
\end{aligned}
$$

Thus we get

$$
\left\{ I + \left[B M^{-1} B^{\mathrm{T}} - X_1^{\mathrm{T}} E Z^{-1} E^{\mathrm{T}} X_1 \right] \left(P - L \right) \right\} x_*(k+1) =
$$

$$
X_1^{\mathrm{T}} \left\{ A + E Z^{-1} E^{\mathrm{T}} X_1 L A \right\} x_*(k). \tag{9.32}
$$

Since, by lemma 9.11, R as defined in theorem 9.2 is invertible, it can be shown that the matrix on the left is invertible and hence (9.32) uniquely defines $x_*(k+1)$ as a function of $x_*(k)$. It turns out that (9.32) can be rewritten in the form $\sigma x_* = A_{cl} x_*$ with A_{cl} as defined by (9.5). Since $x_* \in \ell_2^n$ for every initial state ξ we know that A_{cl} is asymptotically stable. Next we show that P satisfies the discrete algebraic Riccati equation (9.4). From the backwards difference equation in (9.17) combined with lemma 9.10 and the formula given above for w_* we find:

$$
P = A^{\mathrm{T}} P A_{cl} + C^{\mathrm{T}} C
$$

By some extensive calculations this equation turns out to be equivalent to the discrete algebraic Riccati equation (9.4). Next we show that P is symmetric. Note that both P and P^{T} satisfy the discrete algebraic Riccati equation. Using this we find that:

$$
\left(P - P^{\mathrm{T}} \right) = A_{cl}^{\mathrm{T}} \left(P - P^{\mathrm{T}} \right) A_{cl}.
$$

Since A_{cl} is asymptotically stable this implies that $P = P^{\mathrm{T}}$. P can be shown to be positive semi-definite by combining lemma 9.7 and (9.23). It remains to be shown that P satisfies (9.2). Since P is symmetric we know that V and R are symmetric. (9.2) is then an immediate consequence of lemma 9.11. ∎

As we did in chapter 3, we can extend this result to systems which do not satisfy (9.10). A proof can be derived similarly to the proof of corollary 3.10.

Corollary 9.13 : *Assume that (A, B, C, D_1) has no invariant zeros on the unit circle and is left-invertible. If part (ii) of theorem 9.2 is satisfied, then a symmetric matrix $P \geq 0$ exists satisfying (a)–(c) of part (iii) of theorem 9.2.* □

Next, we show the implication (i) $\Rightarrow$ (ii) in theorem 9.4.

Proof of the implication (i) $\Rightarrow$ (ii) in theorem 9.4 : First note that by corollary 9.13 we know a matrix P exists satisfying (a)–(c) of part (iii) of theorem 9.2. It only remains to be shown that (9.8) is satisfied. We use the same kind of argument as used in lemma 9.11.

Note that a consequence of the fact that there is an internally stabilizing feedback law $u = Fx$ which makes the H_∞ norm less than 1 is that a function f exists satisfying part (ii) of theorem 9.2 where f has the extra property that, for initial condition 0, the fact that $x(t) = 0$ implies that $u(t) = [f(w)](t) = 0$. One suitable choice for f is given by

$$u(t) = [f(w)](t) := Fx(t) = \sum_{i=1}^{t} F(A + BF)^{t-i} Ew(i).$$

Next, we investigate the criterion (9.24) once again with initial condition $\xi = 0$ but this time we restrain u by requiring that $u(0) = 0$. The inequalities (9.25) and (9.26) then still hold because of our extra requirement on f. Using (9.27) we can then reduce our "sup–inf–sup–inf" problem to the following problem

$$\mathcal{J}(0) = \sup_{w(0)} w^{\mathrm{T}}(0) \left(E^{\mathrm{T}} PE - I \right) w(0).$$

Since we know by (9.26) that $\mathcal{J}(0)$ is finite we find

$$S := E^{\mathrm{T}} PE - I \leq 0.$$

It remains to be shown that this matrix is invertible. Assume not, then a vector $v \neq 0$ exists such that $Sv = 0$. We define by w^+ the function which attains the supremum in the optimization problem (9.27) with initial condition $x(1) = Ev$. This function is well-defined by our previous results. Next, we define the function w_1 by

$$w_1(t) := \begin{cases} v & \text{if } t = 0, \\ w^+(t) & \text{otherwise.} \end{cases} \tag{9.33}$$

Moreover, we define the function u_1 by $u_1 := f(w_1)$. Because of our extra condition on f and since $\xi = 0$, we know that $u_1(0) = 0$. We know that $\mathcal{J}(0) = 0$ and hence by the inequality (9.25) we know that

$$\inf_u \ \|z_{u_1,w_1}\|_2^2 - \|w\|_2^2 \geq 0. \tag{9.34}$$

On the other hand, using part (ii) of theorem 9.2 we find:

$$\|z_{u_1,w_1}\|_2^2 - \|w_1\|_2^2 < \left(\delta^2 - 1\right) \|w_1\|_2^2.$$

Combined with (9.34) this implies that $w_1 = 0$. However $w_1(0) = v \neq 0$. This yields a contradiction and therefore S is invertible. ∎

9.4   Sufficient conditions for the existence of suboptimal controllers

In this section we shall show that if there is a P satisfying the conditions of theorem 9.2, then the feedback as suggested by theorem 9.2 satisfies condition (i). In order to do this we first need a number of preliminary results.

We define the following system:

$$\Sigma_U : \begin{cases} \sigma x_U = A_U x_U + B_U u_U + E_U w, \\ y_U = C_{1,U} x_U + \qquad\qquad + D_{12,U} w, \\ z_U = C_{2,U} x_U + D_{21,U} u_U + D_{22,U} w, \end{cases} \tag{9.35}$$

where

$$\begin{aligned} A_U &:= A - BV^{-1}\left(B^{\mathrm{T}}PA + D_1^{\mathrm{T}}C\right), \\ B_U &:= BV^{-1/2}, \\ E_U &:= E - BV^{-1}\left(B^{\mathrm{T}}PE\right), \\ C_{1,U} &:= -R^{-1/2}\left(E^{\mathrm{T}}PA - E^{\mathrm{T}}PBV^{-1}[B^{\mathrm{T}}PA + D_1^{\mathrm{T}}C]\right), \\ C_{2,U} &:= C - D_1 V^{-1}\left(B^{\mathrm{T}}PA + D_1^{\mathrm{T}}C\right), \\ D_{12,U} &:= R^{1/2}, \\ D_{21,U} &:= D_1 V^{-1/2}, \\ D_{22,U} &:= D_2 - D_1 V^{-1}\left(B^{\mathrm{T}}PE + D_1^{\mathrm{T}}D_2\right), \end{aligned}$$

and V and R as defined in part (iii) of theorem 9.2 with $D_2 = 0$.

Lemma 9.14 : *The system Σ_U as defined by (9.35) is inner. Denote the transfer matrix of Σ_U by U. We decompose U:*

$$U \begin{pmatrix} w \\ u_U \end{pmatrix} =: \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} w \\ u_U \end{pmatrix} = \begin{pmatrix} z_U \\ y_U \end{pmatrix},$$

compatible with the sizes of w, u_U, z_U and y_U. Then U_{21} is invertible and its inverse is in H_∞. Moreover U_{22} is strictly proper. □

Proof : It is easy to check that P as defined by conditions (a)–(c) of theorem 9.2 satisfies the conditions (i)–(iii) of lemma 2.13 for the system (9.35) with input (w, u_U) and output (z_U, y_U). Part (i) of lemma 2.13 turns out to be equivalent to the discrete algebraic Riccati equation (9.4). Parts (ii) and (iii) follow by simply writing out the equations in the original system parameters of system (9.1).

Next we note that $P \geq 0$ and

$$P = A_U^{\mathrm{T}} P A_U + \begin{pmatrix} C_{1,U}^{\mathrm{T}} & C_{2,U}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} C_{1,U} \\ C_{2,U} \end{pmatrix}. \tag{9.36}$$

Since $A_{cl} = A_U - E_U D_{12,U}^{-1} C_{1,U}$ is asymptotically stable we know that the pair $(C_{1,U}, A_U)$ is detectable. Using standard Lyapunov theory it can then be shown that A_U is asymptotically stable.

To show that U_{21}^{-1} is an H_∞ function we write down a realization for U_{21}^{-1} and remember once again that A_{cl} is asymptotically stable. The proof is then trivial. ∎

Lemma 9.15 : *Assume that a matrix P exists satisfying the conditions in part (iii) of theorem 9.2. In this case the compensator Σ_F described by the static feedback law $u = F_1 x + F_2 w$ where F_1 and F_2 are given by (9.6) and (9.7) satisfies condition (i) of theorem 9.2.* □

Proof : First note that G_F as given by (2.7) for this particular F is equal to U_{11} and moreover $A + BF_1$ is equal to A_U. This implies that Σ_F is internally stabilizing and G_F as a submatrix of an inner matrix satisfies $\|G_F\| \leq 1$. Using the fact that U_{21} is invertible in H_∞, it can be shown that the inequality is strict. ∎

Finally it can be quite easily seen that theorem 9.2 is simply a combination of corollary 9.13 and lemma 9.15. Therefore the main result has been proven. Next, we complete the proof of theorem 9.4.

Lemma 9.16 : *Assume that there is a P satisfying the conditions in part (ii) of theorem 9.4. In this case the compensator Σ_F described by the static state feedback law $u = Fx$ where F as given by (9.9) satisfies condition (i) of theorem 9.2.*   □

Proof : We apply the feedback $u = \tilde{F}x$ to the system Σ_U where

$$\tilde{F} := V^{-1/2}B^{\mathrm{T}}PER^{-1/2}.$$

We have, using (9.8):

$$\tilde{F}^{\mathrm{T}}\tilde{F} = I - R^{-1/2}\left(I - E^{\mathrm{T}}PE\right)R^{-1/2} < I.$$

By lemma 2.14 we then know that the resulting closed-loop system is internally stable and has H_∞ norm less than 1. This closed-loop system is the same as the one we obtain by applying the feedback law $u = Fx$ to our original system where F as defined in theorem 9.4. We conclude that F indeed has the desired properties, i.e. F satisfies part (i) of theorem 9.4. ∎

The choice of $\tilde{F}$ looks like a wild guess. Clearly it is not. The derivation of the proofs of the last two lemmas was originally based on Σ_P as defined in the next chapter. It is shown in lemma 10.3 that the problem of finding feedbacks for Σ is equal to the same problem for Σ_P. We have to be careful since this transformation also changes the measurement equation. The compensator suggested in lemma 9.15 makes the closed-loop transfer matrix of Σ_P equal to 0. The feedback $u = \tilde{F}x$ makes the closed-loop transfer matrix of Σ_P equal to the direct-feedthrough matrix from w_P to z_P. Since we only allow for state feedback we cannot do any better. This was the way we originally derived the specific choices for the different feedbacks. However, the above formal proofs are much more straightforward.

The previous lemma yields the implication (ii) $\Rightarrow$ (i) in theorem 9.4 as a corollary. This also completes the proof of theorem 9.4.

9.5   Discrete algebraic Riccati equations

In this section we shall investigate uniqueness of stabilizing solutions of
the discrete algebraic Riccati equation and methods to check whether
such solutions exist and, if they exist, how we can find them. We shall
not prove the results of this section because these results are not needed
for our main results and to prevent a too extensive book. For details we
refer to [St15].

   We investigate the existence of a matrix P such that:

- The matrix $G(P)$ defined by (9.3) is invertible.

- P satisfies the discrete algebraic Riccati equation (9.4).

- The matrix A_{cl} defined by (9.5) is asymptotically stable.

In [Ig, Wa] solvability of the discrete time algebraic Riccati equation in-
troduced in this chapter is reduced to a generalized eigenvalue problem
for a symplectic pair. We show that this can be done if D_1 is injective.
In general, no method is available for such a reduction. We first list a
number of properties for matrices satisfying these conditions.

Lemma 9.17 : *A matrix P such that $G(P)$ is invertible, the discrete
algebraic Riccati equation (9.4) is satisfied and A_{cl} is asymptotically stable
is uniquely defined by these three conditions.*   □

The above result is more or less the discrete time analogue of corollary
A.7 where we showed that a solution of the quadratic matrix inequality
satisfying two corresponding rank conditions is unique. The continuous
time result was proved by using the already known fact that in continuous
time a stabilizing solution of a algebraic Riccati equation is unique. This
is proven by reducing solvability to properties of modal subspaces of the
associated Hamiltonian matrix. (A modal subspace of H is simply the
largest H-invariant subspace $\mathcal{S}$ such that $\sigma(A|\mathcal{S})$ is contained in some
prespecified area of the complex plane.) This cannot be used as a proof
of lemma 9.17. In discrete time we have only been able to prove the
following result:

Lemma 9.18 : *We assume that the matrices S and $\tilde{A}$ are invertible where S and $\tilde{A}$ are defined by:*

$$S \quad := \quad \begin{pmatrix} D_1^{\mathrm{T}} D_1 & D_1^{\mathrm{T}} D_2 \\ D_2^{\mathrm{T}} D_1 & D_2^{\mathrm{T}} D_2 - I \end{pmatrix}, \tag{9.37}$$

$$\tilde{A} \quad := \quad A - \begin{pmatrix} B & E \end{pmatrix} S^{-1} \begin{pmatrix} D_1^{\mathrm{T}} \\ D_2^{\mathrm{T}} \end{pmatrix} C. \tag{9.38}$$

A matrix P satisfies the conditions of lemma 9.17 if, and only if, the subspace

$$\mathcal{X} := Im \begin{pmatrix} I \\ P \end{pmatrix}, \tag{9.39}$$

is the modal subspace of H associated to the open unit ball where H is defined by

$$H := \begin{pmatrix} \tilde{A} + X\tilde{A}^{-\mathrm{T}} Q & -X\tilde{A}^{-\mathrm{T}} \\ -\tilde{A}^{-\mathrm{T}} Q & \tilde{A}^{-\mathrm{T}} \end{pmatrix}.$$

where

$$X \quad := \quad \begin{pmatrix} B & E \end{pmatrix} S^{-1} \begin{pmatrix} B^{\mathrm{T}} \\ E^{\mathrm{T}} \end{pmatrix},$$

$$Q \quad := \quad C^{\mathrm{T}} C - C^{\mathrm{T}} \begin{pmatrix} D_1 & D_2 \end{pmatrix} S^{-1} \begin{pmatrix} D_1^{\mathrm{T}} \\ D_2^{\mathrm{T}} \end{pmatrix} C.$$

The matrix H is a symplectic matrix, i.e.

$$H \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} H^{\mathrm{T}} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

In particular, H has the property that λ is an eigenvalue of H if, and only if, λ^{-1} is an eigenvalue of H. □

Because of the symplectic property of H we know that there is at most one subspace which is H-invariant, for which the restriction of H to this subspace is asymptotically stable and which is n-dimensional. We can then conclude that a result of the above lemma is that a solution P of our three conditions is unique. But only under the assumption that the matrices S and $\tilde{A}$ are invertible. For the proof of lemma 9.17 another kind of proof was needed.

In the case that the matrix $\tilde{A}$ is not necessarily invertible we can derive an extension of the previous lemma. We first need a definition:

Definition 9.19 : *A vector v is called an eigenvector of the matrix pair (H_1, H_2) with eigenvalue λ if $v \neq 0$ and $H_1 v = \lambda H_2 v$. We call v an eigenvector of the pair (H_1, H_2) with eigenvalue ∞ if $v \neq 0$ and $H_2 v = 0$. A subspace $\mathcal{X}$ is called a modal subspace with respect to $\mathcal{D}^-$ of the pair (H_1, H_2) if $\mathcal{X}$ is the largest subspace $\mathcal{V}$ for which a matrix S and an asymptotically stable matrix $\bar{A}$ exist such that $\mathcal{X} = \mathrm{im}\, S$ and $H_1 S = H_2 S \bar{A}$.* □

Now we are able to give our extension of lemma 9.18:

Lemma 9.20 : *Assume that S, defined by (9.37), is invertible. A matrix P satisfies the three above-mentioned conditions if, and only if, the subspace defined by (9.39) is the modal subspace with respect to the open unit ball for the generalized symplectic pair (H_1, H_2) where*

$$H_1 := \begin{pmatrix} \tilde{A} & 0 \\ -Q & I \end{pmatrix},$$

$$H_2 := \begin{pmatrix} I & X \\ 0 & \tilde{A}^{\mathrm{T}} \end{pmatrix},$$

where $\tilde{A}$ as defined by (9.38) and X and Q as defined in lemma 9.18. Moreover, (H_1, H_2) is a symplectic pair, i.e.

$$H_1 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} H_1^{\mathrm{T}} = H_2 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} H_2^{\mathrm{T}}.$$

In particular, λ is an eigenvalue of (H_1, H_2) if, and only if, λ^{-1} is an eigenvalue of (H_1, H_2). The same is true for $\lambda = 0$ and $\lambda = \infty$ respectively. □

We can reduce the condition that S should be invertible in our specific case:

Lemma 9.21 : *Assume that condition (i) of theorem 9.2 is satisfied. In that case S defined by (9.37) is invertible if, and only if, D_1 is injective.* □

Proof : Assume D_1 is injective. Clearly if condition (i) of theorem 9.2 is satisfied then there exists a matrix F which solves the H_∞ problem "at infinity", i.e. $\|D_2 + D_1 F\| < 1$. This implies, using the orthogonal projector $I - D_1^{\mathrm{T}}(D_1^{\mathrm{T}} D_1)^{-1} D_1$, that

$$I - D_2^{\mathrm{T}} \left(I - D_1^{\mathrm{T}}(D_1^{\mathrm{T}} D_1)^{-1} D_1 \right) D_2 > 0.$$

Therefore, the Schur complement of $D_1^{\mathrm{T}} D_1$ in S in invertible. Thus we find that also S is invertible.

It is trivial that S invertible implies that D_1 is injective. ∎

Hence if D_1 is injective we have the possibility of calculating the solution P of our three conditions. However, for discrete time Riccati equations it is sometimes more desirable to have recursive algorithms as, e.g. is given in [Fa] for a different type of Riccati equation. This is derived in [WA, St15]. An extension of the use of symplectic pairs to the case that D_1 is not injective is useful. This is a subject of current research.

9.6 Conclusion

In this chapter the *discrete time* full-information H_∞ control problem has been investigated. As in the continuous time case the solvability is related to an algebraic Riccati equation. However, in contrast to the continuous time case, it turns out that, even in the case that $D_2 = 0$ the feedback law we find is in general not a state feedback but also requires a disturbance feedback part. Another interesting feature is the condition $R > 0$ where R now depends on the solution of the algebraic Riccati equation.

The assumptions made in this chapter are exactly the discrete time versions of the two main assumptions which are made in the chapter 3.

Naturally, this chapter is a preliminary step towards the measurement feedback case which will be elaborated in the next chapter.

Chapter 10

The discrete time H_∞ control problem with measurement feedback

10.1 Introduction

The H_∞ control problem with measurement feedback in continuous time
has been thoroughly investigated, for example in chapter 5 and the refer-
ences mentioned there. In this chapter we shall extend the result of the
previous chapter to the discrete time H_∞ control problem with measure-
ment feedback.

One approach to this problem is to transform the system into a con-
tinuous time system, to derive controllers for the latter system and then
transform back to discrete time. However, in our opinion, it is more natu-
ral to have the formulae directly available in terms of the original physical
parameters, so their effect on the solution is transparent. This possibil-
ity might otherwise be blurred by the transformation to the continuous
time. Also, the numerical problems arising with this transformation and
the fact that discrete time systems with a pole in 1 are transformed into
non-proper systems are arguments in favour of a more direct approach.
Finally, consider sampled-data design where we take the sample and hold
functions of the digital implementation of our controller into account in
the design of our controller. In [Ch, KH, SK]) it has been shown that if we
use sampled-data design for a H_∞ control problem of a continuous time
system then this can be reduced to a discrete time H_∞ control problem.

The present chapter is reminiscent of chapter 5 which deals with the
continuous time case. The results of this chapter already appeared in
[St6].

We make the assumptions which yield an H_∞ control problem which

is the exact discrete time analogue of the regular H_∞ control problem with measurement feedback. The subsystem from the control input to the output should be left-invertible and should not have invariant zeros on the unit circle. Moreover, the subsystem from the disturbance to the measurement should be right-invertible and again should not have invariant zeros on the unit circle. Note that the left- and right-invertibility assumptions are automatically satisfied if the corresponding direct-feedthrough matrices are injective and surjective, respectively.

As in the regular continuous time case, the necessary and sufficient conditions for the existence of suitable controllers involve positive semi-definite stabilizing solutions of two algebraic Riccati equations. Again, the quadratic term in these algebraic Riccati equations is indefinite. However, other than in the continuous time case, the solutions of these equations have to satisfy another assumption: matrices depending on these solutions should be positive definite. Another complication is that this time the Riccati equations are coupled. The second Riccati equation is the discrete time analogue of the equation for Y (see lemma 5.5). However, this time we did not succeed in expressing the existence and the solution in terms of the solutions of the full-information Riccati equation and its dual version. Recently it has been shown (see [Ig, Wa]) that if the direct-feedthrough matrices from u to z and from w to y are injective and surjective respectively then we can reduce our result to a result similar to theorem 5.1: two uncoupled Riccati equations and a coupling condition (a bound on the spectral radius of the product of the two solutions). As for the regular H_∞ control problem with measurement feedback (see section 5.5) we find an explicit expression for all stable closed-loop systems with H_∞ norm less than 1 and an explicit expression for all internally stabilizing controllers which result in such a closed-loop system.

The outline of this chapter is as follows. In section 10.2 we shall formulate the problem and give our main results. In section 10.3, we shall show the existence of stabilizing solutions of the two algebraic Riccati equations and complete the proof that our conditions are necessary. This is done by transforming the original system into a new system with the property that a controller "works" for the new system if, and only if, it "works" for the original system. In section 10.4 it is shown that our conditions are also sufficient. It turns out that the system transformation of section 10.3 repeated in a dual form gives exactly the desired results. These transformations will lead to a characterization of all suitable closed-loop systems and all suitable controllers in section 10.5. We shall end with some concluding remarks in section 10.6.

10.2   Problem formulation and main results

We consider the following shift-invariant system:

$$
\Sigma : \begin{cases}
\sigma x = Ax + Bu + Ew, \\
y = C_1 x + + D_{12}w, \\
z = C_2 x + D_{21}u + D_{22}w,
\end{cases}
\tag{10.1}
$$

where for each k we have $x(k) \in \mathcal{R}^n$ is the state, $u(k) \in \mathcal{R}^m$ is the control input, $y(k) \in \mathcal{R}^q$ is the measurement, $w(k) \in \mathcal{R}^l$ the unknown disturbance and $z(k) \in \mathcal{R}^p$ the output to be controlled. $A, B, E, C_1, C_2, D_{12}, D_{21}$ and D_{22} are matrices of appropriate dimension.

In this chapter we shall derive necessary and sufficient conditions for the existence of a dynamic compensator Σ_F of the form (2.9) which is internally stabilizing and which is such that the closed-loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$. By scaling the plant we can thus, in principle, find the infimum of the closed-loop H_∞ norm over all stabilizing controllers. This will involve a search procedure. Furthermore, if a stabilizing Σ_F exists which makes the H_∞ norm less than 1, then we derive an explicit formula for one particular Σ_F satisfying these requirements. Note that the loop-shifting arguments in section 5.6 allow us to extend our results to the more general system, defined by (2.8) (we could also assume for the moment that $D_{22} = 0$ and include it later via loop-shifting).

We can now formulate our main result:

Theorem 10.1 : *Consider the system (10.1). Assume that the system (A, B, C_2, D_{21}) has no invariant zeros on the unit circle and is left-invertible. Moreover, assume that the system (A, E, C_1, D_{12}) has no invariant zeros on the unit circle and is right-invertible. The following statements are equivalent:*

(i) *A dynamic compensator Σ_F of the form (2.9) exists such that the resulting closed-loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$ and the closed-loop system is internally stable.*

(ii) *Symmetric matrices $P \geq 0$ and $Y \geq 0$ exist such that*

(a) *We have*

$$
V > 0, \qquad R > 0,
\tag{10.2}
$$

where

$$
\begin{aligned}
V &:= B^{\mathrm{T}}PB + D_{21}^{\mathrm{T}}D_{21}, \\
R &:= I - D_{22}^{\mathrm{T}}D_{22} - E^{\mathrm{T}}PE \\
&\qquad + \left(E^{\mathrm{T}}PB + D_{22}^{\mathrm{T}}D_{21}\right)V^{-1}\left(B^{\mathrm{T}}PE + D_{21}^{\mathrm{T}}D_{22}\right).
\end{aligned}
$$

This implies that the matrix $G(P)$ is invertible where:

$$
G(P) := \begin{pmatrix} D_{21}^{\mathrm{T}}D_{21} & D_{21}^{\mathrm{T}}D_{22} \\ D_{22}^{\mathrm{T}}D_{21} & D_{22}^{\mathrm{T}}D_{22} - I \end{pmatrix} + \begin{pmatrix} B^{\mathrm{T}} \\ E^{\mathrm{T}} \end{pmatrix} P \begin{pmatrix} B & E \end{pmatrix}. \qquad (10.3)
$$

(b) **P satisfies the discrete algebraic Riccati equation:**

$$
\begin{aligned}
P &= A^{\mathrm{T}}PA + C_2^{\mathrm{T}}C_2 \\
&\quad - \begin{pmatrix} B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C_2 \\ E^{\mathrm{T}}PA + D_{22}^{\mathrm{T}}C_2 \end{pmatrix}^{\mathrm{T}} G(P)^{-1} \begin{pmatrix} B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C_2 \\ E^{\mathrm{T}}PA + D_{22}^{\mathrm{T}}C_2 \end{pmatrix} \qquad (10.4)
\end{aligned}
$$

(c) **The matrix $A_{cl,P}$ is asymptotically stable where:**

$$
A_{cl,P} := A - \begin{pmatrix} B & E \end{pmatrix} G(P)^{-1} \begin{pmatrix} B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C_2 \\ E^{\mathrm{T}}PA + D_{22}^{\mathrm{T}}C_2 \end{pmatrix}. \qquad (10.5)
$$

Moreover, if, given the matrix P satisfying (a)–(c), we define the following matrices:

$$
\begin{aligned}
Z &:= E^{\mathrm{T}}PA + D_{22}^{\mathrm{T}}C_2 - \left[E^{\mathrm{T}}PB + D_{22}^{\mathrm{T}}D_{21}\right]V^{-1}\left[B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C_2\right], \\
A_P &:= A + ER^{-1}Z, \\
E_P &:= ER^{-1/2}, \\
C_{1,P} &:= C_1 + D_{12}R^{-1}Z, \\
C_{2,P} &:= V^{-1/2}\left(B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C_2\right) + V^{-1/2}\left[B^{\mathrm{T}}PE + D_{21}^{\mathrm{T}}D_{22}\right]R^{-1}Z, \\
D_{12,P} &:= D_{12}R^{-1/2}, \\
D_{21,P} &:= V^{1/2}, \\
D_{22,P} &:= V^{-1/2}\left(B^{\mathrm{T}}PE + D_{21}^{\mathrm{T}}D_{22}\right)R^{-1/2},
\end{aligned}
$$

then the matrix Y should satisfy conditions (d)–(f):

(*d*) *We have*

$$W > 0, \qquad S > 0, \tag{10.6}$$

where

$$
\begin{aligned}
W \;\; &:= \;\; D_{12,P} D_{12,P}^{\mathrm{T}} + C_{1,P} Y C_{1,P}^{\mathrm{T}}, \\
S \;\; &:= \;\; I - D_{22,P} D_{22,P}^{\mathrm{T}} - C_{2,P} Y C_{2,P}^{\mathrm{T}} + \\
& \qquad \left(C_{2,P} Y C_{1,P}^{\mathrm{T}} + D_{22,P} D_{12,P}^{\mathrm{T}} \right) W^{-1} \left(C_{1,P} Y C_{2,P}^{\mathrm{T}} + D_{12,P} D_{22,P}^{\mathrm{T}} \right).
\end{aligned}
$$

This implies that the matrix $H_P(Y)$ is invertible where:

$$
H(Y) := \begin{pmatrix} D_{12,P} D_{12,P}^{\mathrm{T}} & D_{12,P} D_{22,P}^{\mathrm{T}} \\ D_{22,P} D_{12,P}^{\mathrm{T}} & D_{22,P} D_{22,P}^{\mathrm{T}} - I \end{pmatrix} + \begin{pmatrix} C_{1,P} \\ C_{2,P} \end{pmatrix} Y \begin{pmatrix} C_{1,P}^{\mathrm{T}} & C_{2,P}^{\mathrm{T}} \end{pmatrix}. \tag{10.7}
$$

(*e*) *Y satisfies the following discrete algebraic Riccati equation:*

$$
Y = A_P Y A_P^{\mathrm{T}} + E_P E_P^{\mathrm{T}}
$$

$$
- \begin{pmatrix} C_{1,P} Y A_P^{\mathrm{T}} + D_{12,P} E_P^{\mathrm{T}} \\ C_{2,P} Y A_P^{\mathrm{T}} + D_{22,P} E_P^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} H(Y)^{-1} \begin{pmatrix} C_{1,P} Y A_P^{\mathrm{T}} + D_{12,P} E_P^{\mathrm{T}} \\ C_{2,P} Y A_P^{\mathrm{T}} + D_{22,P} E_P^{\mathrm{T}} \end{pmatrix}. \tag{10.8}
$$

(*f*) *The matrix $A_{cl,P,Y}$ is asymptotically stable where:*

$$
A_{cl,P,Y} := A_P - \begin{pmatrix} C_{1,P} Y A_P^{\mathrm{T}} + D_{12,P} E_P^{\mathrm{T}} \\ C_{2,P} Y A_P^{\mathrm{T}} + D_{22,P} E_P^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} H(Y)^{-1} \begin{pmatrix} C_{1,P} \\ C_{2,P} \end{pmatrix}. \tag{10.9}
$$

Where $P \geq 0$ and $Y \geq 0$ exist satisfying part (ii), a controller of the form (2.9) satisfying the requirements in part (i) is given by:

$$
\begin{aligned}
N \;\; &:= \;\; -D_{21,P}^{-1} \left(C_{2,P} Y C_{1,P}^{\mathrm{T}} + D_{22,P} D_{12,P}^{\mathrm{T}} \right) W^{-1}, \\
M \;\; &:= \;\; -(D_{21,P}^{-1} C_{2,P} + N C_{1,P}), \\
L \;\; &:= \;\; BN + \left(A_P Y C_{1,P}^{\mathrm{T}} + E_P D_{12,P}^{\mathrm{T}} \right) W^{-1}, \\
K \;\; &:= \;\; A_{cl,P} - L C_{1,P}. \qquad\qquad\qquad\qquad\qquad\qquad \square
\end{aligned}
$$

Remarks :

(i) Necessary and sufficient conditions for the existence of an internally stabilizing feedback compensator which makes the H_∞ norm less than some, a priori given, upper bound $\gamma > 0$ can be easily derived from theorem 10.1 by scaling.

(ii) If we compare these conditions with the conditions for the continuous time case (see [Do5, St2]), then we note that our Riccati equations are coupled. In [Ig, Wa] it has been shown that if D_{21} is injective and D_{12} is surjective then our conditions (d)-(f) are equivalent to the existence of a matrix Q which is dual to P in the sense that Q satisfies the same conditions as P but for the system Σ^{T} such that $\rho(PQ) < 1$. Moreover, in this case $Y = Q(I - PQ)^{-1}$. In other words, these papers derive a discrete time version of lemma 5.5.

(iii) Note that conditions (10.2) and (10.6) are this time depending on P and Y (the continuous time conditions appear only if $D_{22} \neq 0$ and in that case they are independent of the solutions of the two algebraic Riccati equations). A simple example showing that the assumption $G(P)$ invertible is not sufficient is given in the previous chapter. Note that if R is not positive semi-definite, then matrices like E_P are ill-defined and we can not even look for a matrix Y satisfying (10.6)–(10.9).

The proofs in this chapter will be strongly reminiscent of the proofs given in chapter 5. They will depend on the results as derived in the previous chapter. The details of the proof are much easier since we do not need the bases from appendix A. This is a consequence of the fact that we work with the discrete time analogue of a regular problem.

10.3 A first system transformation

Using the results of the previous chapter we can derive the following result:

Lemma 10.2 : *Assume that there exists a controller satisfying the conditions in part (i) of theorem 10.1. Moreover, assume that (A, B, C_2, D_{21}) has no invariant zeros on the unit circle and is left-invertible. Then a positive semi-definite matrix P exists satisfying conditions (a)–(c) of part (ii) of theorem 10.1.* $\qquad\square$

Proof : If there is a dynamic controller which is internally stabilizing and which makes the H_∞ norm of the closed-loop system less than 1 for the problem with measurement feedback, then certainly condition (ii) of theorem 9.2 holds. This implies that part (iii) of theorem 9.2 also holds, which exactly yields the desired result.   ■

We assume throughout this section that a positive semi-definite matrix P exists which satisfies the conditions (a)–(c) of part (ii) of theorem 10.1. By the above lemma we know that such a P exists when part (i) of theorem 10.1 is satisfied. But naturally this matrix P with the desired properties also exists when part (ii) of theorem 10.1 is satisfied.

In order to proceed with the proof of theorem 10.1 in this section we shall transform our original system (10.1) into a new system. The problem of finding an internally stabilizing feedback which makes the H_∞ norm less than 1 for the original system is equivalent to the problem of finding an internally stabilizing feedback which makes the H_∞ norm less than 1 for the new transformed system. However, this new system has some very desirable properties which make it much easier to work with. In particular, the disturbance decoupling problem with measurement feedback is solvable. We shall perform the transformation in two steps. First we shall perform a transformation related to the full-information H_∞ problem and next a transformation related to the filtering problem. These two transformations are as the transformations used in chapter 5. However, because our problem is regular, we do not need the technical hardware from appendices A and B.

We define the following system:

$$\Sigma_P : \begin{cases} \sigma x_P = & A_P x_P + & B u_P + & E_P w_P, \\ y_P & = C_{1,P} x_P + & & + D_{12,P} w_P, \\ z_P & = C_{2,P} x_P + D_{21,P} u_P + D_{22,P} w_P, \end{cases} \tag{10.10}$$

where the matrices are as defined in the statement of theorem 10.1. Furthermore, we define the following system

$$\Sigma_U : \begin{cases} \sigma x_U = & A_U x_U + & B_U u_U + & E_U w, \\ y_U & = C_{1,U} x_U + & & + D_{12,U} w, \\ z_U & = C_{2,U} x_U + D_{21,U} u_U + D_{22,U} w, \end{cases} \tag{10.11}$$

where

$$\begin{aligned} A_U & := A - B V^{-1} \left(B^{\mathrm{T}} P A + D_{21}^{\mathrm{T}} C_2 \right), \\ B_U & := B V^{-1/2}, \end{aligned}$$

$$
\begin{aligned}
E_U &:= E - BV^{-1}\left(B^{\mathrm{T}}PE + D_{21}^{\mathrm{T}}D_{22}\right), \\
C_{1,U} &:= -R^{-1/2}Z, \\
C_{2,U} &:= C_2 - D_{21}V^{-1}\left(B^{\mathrm{T}}PA + D_{21}^{\mathrm{T}}C_2\right), \\
D_{12,U} &:= R^{1/2}, \\
D_{21,U} &:= D_{21}V^{-1/2}, \\
D_{22,U} &:= D_{22} - D_{21}V^{-1}\left(B^{\mathrm{T}}PE + D_{21}^{\mathrm{T}}D_{22}\right),
\end{aligned}
$$

and V, R and Z are as defined in theorem 10.1.

Note that this system Σ_U is the same as the system Σ_U as used in section 9.4 where C and D_1 are replaced by C_2 and D_{21} respectively while $D_{22} = 0$ (since we assumed in the proofs of the previous chapter that $D_2 = 0$). A minor extension of the results in the previous chapter (to the case that D_{22} is not necessarily 0) gives us that Σ_U is inner and satisfies the other properties of lemma 9.14.

We shall now formulate our key lemma.

Lemma 10.3 : *Let P satisfy theorem 10.1 part (ii) (a)–(c). Moreover, let Σ_F be an arbitrary dynamic compensator in the form (2.9). Consider the following two systems, where the system on the left is the interconnection of (10.1) and (2.9) and the system on the right is the interconnection of (10.10) and (2.9):*
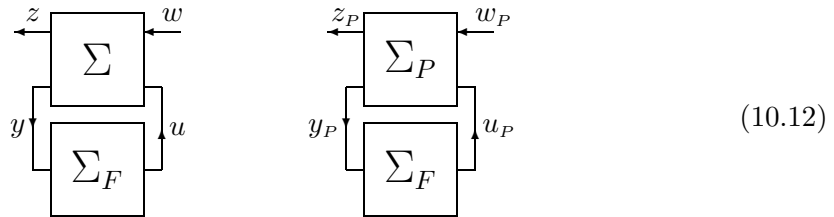
$$(10.12)$$

Then the following statements are equivalent :

(*i*) *The system on the left is internally stable and its transfer matrix from w to z has H_∞ norm less than 1.*

(*ii*) *The system on the right is internally stable and its transfer matrix from w_P to z_P has H_∞ norm less than 1.*   □

Remarks : Note the relation with lemma 5.4. Although the systems Σ_P and Σ_U in this chapter are discrete time systems and in chapter 5 we used continuous time systems the proof of lemma 5.4 can still be used to prove

the discrete time version with some minor alterations. We shall add the proof for the sake of completeness.

Proof : We investigate the following systems:

$$(10.13)$$
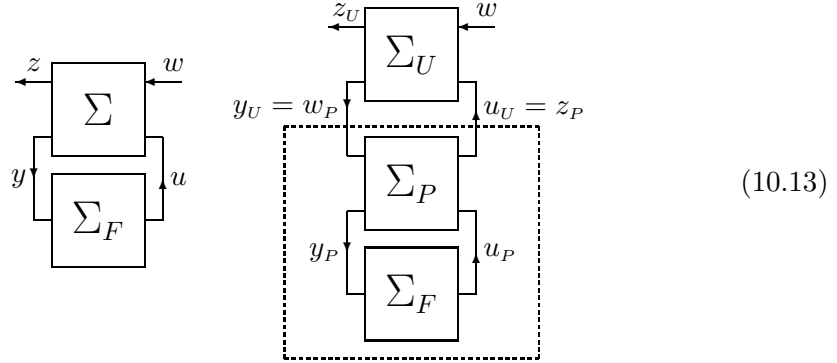
The system on the left is the same as the system on the left in (10.12) and the system on the right is described by the system (10.11) interconnected with the system on the right in (10.12). A realization for the system on the right is given by:

$$\sigma \begin{pmatrix} e \\ x_P \\ p \end{pmatrix} = \begin{pmatrix} A_{cl,P} & 0 & 0 \\ * & A+BNC_1 & BM \\ * & LC_1 & K \end{pmatrix} \begin{pmatrix} e \\ x_P \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ E+BND_{12} \\ LD_{12} \end{pmatrix} w$$

$$z_U = \begin{pmatrix} * & C_2+D_{21}NC_1 & D_{21}M \end{pmatrix} \begin{pmatrix} e \\ x_P \\ p \end{pmatrix} + (D_{22}+D_{21}ND_{12}) w$$

where $A_{cl,P}$ is defined by (10.5) and $e = x_U - x_P$. The $*$'s denote matrices which are unimportant for this argument. The system on the right is internally stable if, and only if, the system described by the above set of equations is internally stable. If we also derive the system equations for the system on the left in (10.13), then we see immediately that, since $A_{cl,P}$ is asymptotically stable, the system on the left is internally stable if, and only if, the system on the right is internally stable. Moreover, if we take zero initial conditions and both systems have the same input w, then we have $z = z_U$, i.e. the input–output behaviour of both systems is equivalent. Hence the system on the left has H_∞ norm less than 1 if, and only if, the system on the right has H_∞ norm less than 1.

We know Σ_U is inner and satisfies the other properties necessary in order to apply lemma 2.14 to the system on the right in (10.13) and hence

we find that the closed-loop system is internally stable and has H_∞ norm less than 1 if, and only if, the dashed system is internally stable and has H_∞ norm less than 1.

Since the dashed system is exactly the system on the right in (10.12) and the system on the left in (10.13) is exactly equal to the system on the left in (10.12), we have completed the proof. ∎

Using the previous lemma, we know that we only have to investigate the system Σ_P. This new system has some very nice properties which we shall exploit. First we shall look at the Riccati equation for the system Σ_P. It can be checked immediately that $X = 0$ satisfies conditions (a)–(c) of theorem 10.1 for the system Σ_P.

We now dualize Σ_P. We know that (A, E, C_1, D_{12}) is right-invertible and has no invariant zeros on the unit circle. It is easy to verify that this implies that $(A_P, E, C_{1,P}, D_{12})$ is right-invertible and has no invariant zeros on the unit circle. Hence for the dual of Σ_P we know that $(A_P^{\mathrm{T}}, C_{1,P}^{\mathrm{T}}, E^{\mathrm{T}}, D_{21}^{\mathrm{T}})$ is left-invertible and has no invariant zeros on the unit circle. If there is an internally stabilizing feedback for the system Σ which makes the H_∞ norm less than 1, then the same feedback is internally stabilizing and makes the H_∞ norm less than 1 for the system Σ_P. If we dualize this feedback and apply it to the dual of Σ_P, then it is again internally stabilizing and again it makes the H_∞ norm less than 1. We can now apply corollary 10.2 which exactly guarantees the existence of a matrix Y satisfying conditions (d)–(f) of theorem 10.1. Thus we derived the following lemma, which shows that the conditions of theorem 10.1 are necessary:

Lemma 10.4 : *Let the system (10.1) be given with zero initial state. Assume that (A, B, C_2, D_{21}) has no invariant zeros on the unit circle and is left-invertible. Moreover, assume that (A, E, C_1, D_{12}) has no invariant zeros on the unit circle and is right-invertible. If part (i) of theorem 10.1 is satisfied, then matrices P and Y exist satisfying (a)–(f) of part (ii) of theorem 10.1.* □

This completes the proof (i) $\Rightarrow$ (ii). In the next section we shall prove the reverse implication. Moreover, in the case that the desired Σ_F exists we shall derive an explicit formula for one choice for F which satisfies all requirements.

10.4 The transformation into a disturbance decoupling problem with measurement feedback

In this section we shall assume that matrices P and Y exist satisfying part (ii) of theorem 10.1 for the system (10.1). We shall transform our original system Σ into another system $\Sigma_{P,Y}$. We shall show that a compensator is internally stabilizing and makes the H_∞ norm less than 1 for the system Σ if, and only if, the same compensator is internally stabilizing and makes the H_∞ norm less than 1 for our transformed system $\Sigma_{P,Y}$. After that we shall show that $\Sigma_{P,Y}$ has a very special property: the Disturbance Decoupling Problem with Measurement feedback and internal Stability (DDPMS, as defined in section 2.6) is solvable.

We first define $\Sigma_{P,Y}$. We transform Σ into Σ_P. Then we apply the dual transformation on Σ_P to obtain $\Sigma_{P,Y}$:

$$\Sigma_{P,Y} : \begin{cases} \sigma x_{P,Y} = A_{P,Y} x_{P,Y} + B_{P,Y} u_{P,Y} + E_{P,Y} w_{P,Y}, \\ y_{P,Y} = C_{1,P} x_{P,Y} + \phantom{B_{P,Y} u_{P,Y}} + D_{12,P,Y} w_{P,Y}, \\ z_{P,Y} = C_{2,P,Y} x_{P,Y} + D_{21,P,Y} u_{P,Y} + D_{22,P,Y} w_{P,Y}, \end{cases} \quad (10.14)$$

where

$$
\begin{aligned}
\tilde{Z} \quad &:= \quad A_P Y C_{2,P}^{\mathrm{T}} + E_P D_{22,P}^{\mathrm{T}} \\
&\qquad - \left(A_P Y C_{1,P}^{\mathrm{T}} + E_P D_{12,P}^{\mathrm{T}} \right) W^{-1} \left(C_{1,P} Y C_{2,P}^{\mathrm{T}} + D_{12,P} D_{22,P}^{\mathrm{T}} \right), \\
A_{P,Y} \quad &:= \quad A_P + \tilde{Z} S^{-1} C_{2,P}, \\
B_{P,Y} \quad &:= \quad B + \tilde{Z} S^{-1} D_{21,P}, \\
E_{P,Y} \quad &:= \quad \left(A_P Y C_{1,P}^{\mathrm{T}} + E_P D_{12,P}^{\mathrm{T}} \right) W^{-1/2} \\
&\qquad + \tilde{Z} S^{-1} \left(C_{2,P} Y C_{1,P}^{\mathrm{T}} + D_{22,P} D_{12,P}^{\mathrm{T}} \right) W^{-1/2}, \\
C_{2,P,Y} \quad &:= \quad S^{-1/2} C_{2,P}, \\
D_{12,P,Y} \quad &:= \quad W^{1/2}, \\
D_{21,P,Y} \quad &:= \quad S^{-1/2} D_{21,P}, \\
D_{22,P,Y} \quad &:= \quad S^{-1/2} \left(C_{2,P} Y C_{1,P}^{\mathrm{T}} + D_{22,P} D_{12,P}^{\mathrm{T}} \right) W^{-1/2},
\end{aligned}
$$

and $W > 0$ and $S > 0$ as defined in part (ii) of theorem 10.1. When we first apply lemma 10.3 on the transformation from Σ to Σ_P and then the dual version of lemma 10.3 on the transformation from Σ_P to $\Sigma_{P,Y}$ we find:

Lemma 10.5 : *Let P satisfy theorem 10.1 part (ii) (a)–(c). Moreover let an arbitrary dynamic compensator Σ_F be given, described by (2.9). Consider the following two systems, where the system on the left is the interconnection of (10.1) and (2.9) and the system on the right is the interconnection of (10.14) and (2.9):*
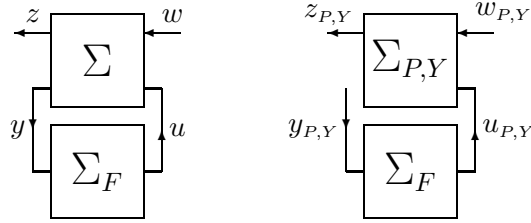
Then the following statements are equivalent :

(i) The system on the left is internally stable and its transfer matrix from w to z has H_∞ norm less than 1.

(ii) The system on the right is internally stable and its transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ has H_∞ norm less than 1. □

It remains to be shown that for $\Sigma_{P,Y}$ the disturbance decoupling problem with internal stability and measurement feedback is solvable:

Lemma 10.6 : *Let Σ_F be given by:*

$$\sigma p \;\; = A_{P,Y}p + B_{P,Y}u + E_{P,Y}D_{12,P,Y}^{-1}(y_{P,Y} - C_{1,P}p)$$
$$u_{P,Y} = -D_{21,P,Y}^{-1}C_{2,P,Y}p + D_{21,P,Y}^{-1}D_{22,P,Y}D_{12,P,Y}^{-1}(y_{P,Y} - C_{1,P}p),$$

The interconnection of Σ_F and $\Sigma_{P,Y}$ is internally stable and the closed-loop transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ is zero. □

Remarks :   Note that Σ_F has the structure of an observer interconnected with a full-information feedback when applied to $\Sigma_{P,Y}$. When applied to Σ it is still an observer interconnected with a full-information feedback but the observer is a worst-case observer with an extra term representing the "worst" disturbance.

Proof : We can write out the formulae for a state space representation of the interconnection of $\Sigma_{P,Y}$ and Σ_F. We then apply the following basis transformation:

$$\begin{pmatrix} x_{P,Y} - p \\ p \end{pmatrix} = \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} x_{P,Y} \\ p \end{pmatrix}.$$

After this transformation one immediately sees that the closed-loop transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ is zero. Moreover the closed-loop state matrix (2.6) after this transformation is given by:

$$\begin{pmatrix} A_{cl,P,Y} & 0 \\ L_{P,Y}C_{1,P} & A_{cl,P} \end{pmatrix}.$$

Since $A_{cl,P,Y}$ and $A_{cl,P}$ are asymptotically stable matrices, this implies that Σ_F is indeed internally stabilizing. ∎

This controller is the same as the controller described in the statement of theorem 10.1. We know Σ_F is internally stabilizing and the resulting closed-loop system has H_∞ norm less than 1 for the system $\Sigma_{P,Y}$. Hence, by applying lemma 10.5, we find that Σ_F satisfies part (i) of theorem 10.1. This completes the proof of (ii) $\Rightarrow$ (i) of theorem 10.1. We have already shown the reverse implication and hence the proof of theorem 10.1 is completed.

10.5   Characterization of all suitable controllers and closed-loop systems

In this section we shall give a characterization of all controllers which stabilize the system such that the closed-loop system has H_∞ norm strictly less than 1. Moreover, we characterize all suitable closed-loop systems obtained in this way. Note that we can do this because we have the discrete time anologue of a regular problem. The derivation of the results in this section is as the derivation in section 5.5 and will therefore be omitted.

We define two systems:

$$\Sigma_C : \begin{cases} \dot{x}_U = Ax_U + B\left[N\left(C_1 x_U + D_{12}w\right) + Me + V^{-1/2}S^{1/2}z_Q \right] + Ew, \\ \dot{e} = Ke + L(C_1 x_U + D_{12}w) + \left(BV^{-1/2}S^{1/2} + \tilde{Z}S^{-1/2} \right)z_Q, \\ w_Q = W^{-1/2}\left(C_1 x_U + D_{12}w - C_{1,P}e \right), \\ z = C_2 x_U + D_{21}\left[N\left(C_1 x_U + D_{12}w\right) + Me + V^{-1/2}S^{1/2}z_Q \right], \end{cases}$$

where we used the definitions of theorem 10.1 while $\tilde{Z}$ is defined on page 200. Similarly we define:

$$\Sigma_E : \left\{ \begin{array}{l} \dot{x}_1 = Kx_1 + Ly - \left(B + \tilde{Z}S^{-1}V^{1/2} \right) S^{-1/2}V^{1/2}z_Q, \\ w_Q = W^{-1/2} \left(y - C_{1,P}e \right), \\ u = Mx_1 + Ny - S^{-1/2}V^{1/2}z_Q, \end{array} \right.$$

where we used the same definitions. Next, we construct the following two interconnections:

$$(10.15)$$
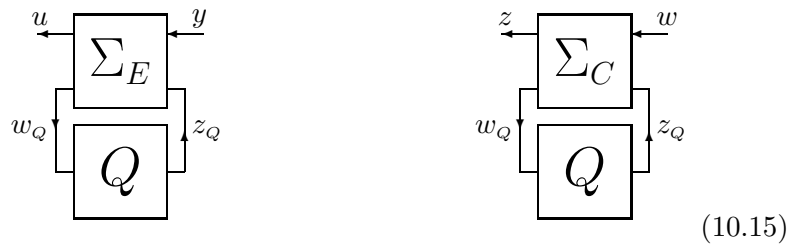
We can then derive the following theorem:

Theorem 10.7 : *Let Σ be given by (10.1). Assume that the conditions (ii) of theorem 10.1 are satisfied for Σ. Finally, let Σ_C and Σ_E denote the two systems defined above.*

Any controller of the form (2.9) which stabilizes Σ admits a realization in the form of the left interconnection in picture (10.15) where Q is a suitably chosen stable shift-invariant system with H_∞ norm strictly less than 1. The interconnection on the right is a realization for the corresponding closed-loop system. Conversely, for any stable shift-invariant system Q with H_∞ norm strictly less than 1 the interconnection on the left is a controller which stabilizes Σ such that the closed-loop system has H_∞ norm strictly less than 1. Moreover, this closed-loop system is given by the interconnection on the right in picture (10.15). □

The above theorem gives us a complete characterization of all suitable controllers and of all suitable closed-loop systems. This result is completely analogous to the continuous time result.

It should be noted that if we set $Q = 0$ then we obtain the central controller and the corresponding closed-loop system. Note that the central controller can be given an interpretation as minimizing the entropy (a discrete time version of the entropy function discussed in chapter 7) and as the optimal controller for the linear exponential Gaussian stochastic control problem (see [Wh]).

10.6   Conclusion

In this chapter we have solved the discrete time H_∞ control problem with measurement feedback. It is shown that the techniques for the continuous time case can be applied to the discrete time case. Unfortunately the formulae are much more complex, but since we are investigating the discrete time analogue of a regular system, it is possible to give a characterization of all closed-loop systems and all controllers satisfying the requirements. The extension to the finite horizon discrete time case (similar to chapter 8) is discussed in [Bas, Li4].

It would be interesting to find two dual Riccati equations and a coupling condition as in [Ig, Wa] for the case that either D_{21} is not injective or D_{12} is not surjective.

Nevertheless the results presented in this chapter show that it is possible to solve discrete time H_∞ problems directly, instead of transforming them to continuous time problems. The assumption of left-invertibility is not very restrictive. When the system is not left-invertible this implies that there are several inputs which have the same effect on the output and this non-uniqueness can be factored out (see, for a continuous time treatment, [S]). The assumption of right-invertibility can be removed by dualizing this reasoning. However, at this moment it is unclear how to remove the assumptions concerning zeros on the unit-circle.

Chapter 11

Applications

11.1 Introduction

In this book we have derived several results concerning the H_∞ control problem. In section 11.2, we apply the results of this book to a number of problems related to robustness. However, these are again theoretical results.

    After that, in section 11.3 we shall discuss practical applications of H_∞ control theory. We shall put emphasis on giving references to a number of practical applications in literature. To conclude we give one design example, an inverted pendulum, and shall study aspects of its robustness and especially the limitations which H_∞ control theory still posseses.

11.2 Application of our results to certain robustness problems

In section 1.3 we discussed stabilization of uncertain systems. In this section we are going to apply the results of this book to three different types of uncertainty:

- Additive perturbations
- Multiplicative perturbations
- Perturbations in the realization of a system.

Each time we find a problem which can be reduced to an H_∞ control problem. The first two problems can be found in [MF, Vi]. The last problem is discussed in [Hi]. The second and third problems will in general yield *singular* H_∞ control problems.

11.2.1   Additive perturbations

Essentially the results of this subsection have already appeared in [Gl2]. Assume that we have a continuous time system Σ being an imperfect model of a certain plant. We assume that the error is additive, i.e. we assume that the plant can be exactly described by the following interconnection:

$$(11.1)$$
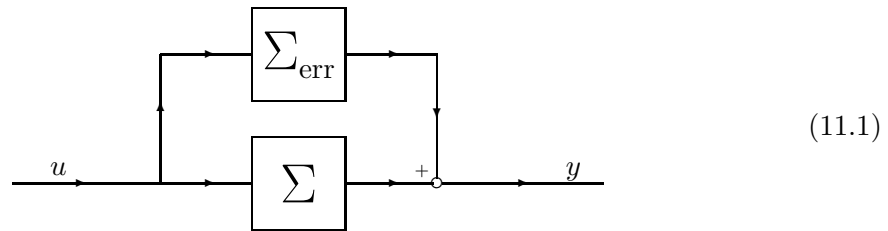
Here Σ_{err} is some arbitrary system such that Σ and $\Sigma + \Sigma_{err}$ have the same number of unstable poles. Thus we assume that the plant is described by the system Σ interconnected as in diagram (11.1) with another system Σ_{err}. The system Σ_{err} represents the uncertainty and is hence, by definition, unknown. In this subsection we derive conditions under which a controller Σ_F of the form (2.4) from y to u exists such that the interconnection (11.1) is stabilized by this controller for all systems Σ_{err} which do not change the number of unstable poles and which have $\mathcal{L}_\infty$ norm less than *or equal to* some, a priori given, number γ. It should be noted that an assumption like fixing the number of unstable poles is needed since otherwise there are always arbitrary small perturbations that destabilize the closed-loop system. In [Vi] the following result is given:

Lemma 11.1 :   *Let a controller Σ_F of the form (2.4) be given. The following conditions are equivalent:*

(i) *If we apply the controller Σ_F from y to u to the interconnection (11.1), then the closed-loop system is well-posed and internally stable for every system Σ_{err} such that*

    (a) *Σ_{err} has $\mathcal{L}_\infty$ norm less than or equal to γ.*

    (b) *Σ and $\Sigma + \Sigma_{err}$ have the same number of unstable poles.*

(ii) *Σ_F internally stabilizes Σ and if G_{ci} and G_F denote the transfer matrices of Σ and Σ_F, respectively, then $I - G_{ci}G_F$ is invertible as a proper rational matrix and $\|G_F(I - G_{ci}G_F)^{-1}\|_\infty < \gamma^{-1}$.* $\square$

Assume that a stabilizable and detectable realization of Σ is given. Hence Σ is described by some quadruple (A, B, C, D) with (A, B) stabilizable and (C, A) detectable. We define a new system:

$$\Sigma_{na} : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + \ Du + w, \\ z = \qquad\quad u. \end{cases} \tag{11.2}$$

It is easy to verify that a controller Σ_F of the form (2.4) applied to Σ yields a closed-loop system that is well-posed and internally stable if, and only if, the same controller Σ_F from y to u applied to Σ_{na} yields a closed-loop system which is well-posed and internally stable.

Moreover, assume that a controller Σ_F from y to u of the form (2.4) is given which, when applied to Σ_{na}, yields a well-posed and internally stable closed-loop system. Then the resulting closed-loop transfer matrix is equal to $G_F(I - G_{ci}G_F)^{-1}$ where G_{ci} and G_F denote the transfer matrices of Σ and Σ_F, respectively.

Using the above reasoning we find that our original problem formulation is equivalent to the problem of finding an internally stabilizing controller for Σ_{na} which makes the H_∞ norm of the closed-loop system less than γ^{-1}.

Next, we define P_a and Q_a as the unique positive semi-definite matrices satisfying the following two Riccati equations:

$$P_a A + A^{\mathrm{T}} P_a = P_a B B^{\mathrm{T}} P_a,$$
$$Q_a A^{\mathrm{T}} + A Q_a = Q_a C^{\mathrm{T}} C Q_a,$$

such that the matrices $A - BB^{\mathrm{T}}P_a$ and $A - Q_a C^{\mathrm{T}}C$ are asymptotically stable. Since (A, B) is stabilizable and (C, A) is detectable, existence and uniqueness of P_a and Q_a is guaranteed by standard linear quadratic control (see [Ku]).

After applying theorems 5.1, 5.11 and 11.1 we find the following theorem:

Theorem 11.2 : *Assume that a system Σ is given with a stabilizable and detectable realization (A, B, C, D) such that A has no eigenvalues on the imaginary axis. We define the related system Σ_{na} by (11.2) and let $\gamma > 0$ be given. The following three conditions are equivalent:*

(i) *A controller Σ_F from y to u of the form (2.4) exists which, when applied to the interconnection (11.1), yields a closed-loop system that is well-posed and internally stable for all systems Σ_{err} such that:*

(*a*) Σ_{err} *has* $\mathcal{L}_\infty$ *norm less than or equal to* γ.

(*b*) Σ *and* $\Sigma + \Sigma_{err}$ *have the same number of unstable poles.*

(*ii*) *A controller* Σ_F *from* y *to* u *of the form (2.4) exists which, when applied to the system* Σ_{na}, *yields a closed-loop system that is well-posed, internally stable and has* H_∞ *norm less than* γ^{-1}.

(*iii*) *We have* $\rho(P_a Q_a) < \gamma^{-2}$.

Moreover, if P_a *and* Q_a *satisfy part (iii), then a controller of the form (2.4) satisfying both part (i) as well as part (ii) is described by:*

$$
\begin{aligned}
N &:= 0, \\
M &:= -B^{\mathrm{T}} P_a, \\
L &:= \left(I - \gamma^2 Q_a P_a\right)^{-1} Q_a C^{\mathrm{T}}, \\
K &:= A - BB^{\mathrm{T}} P_a - L(C - DB^{\mathrm{T}} P_a). \qquad \square
\end{aligned}
$$

Remarks:

(*i*) Naturally the class of perturbations we have chosen is rather artificial. However, it is easy to show that, if we allow for perturbations which add an extra unstable pole, then there are arbitrarily small perturbations which destabilize the closed-loop system. On the other hand, our class of perturbations does include all stable systems Σ_{err} with H_∞ norm less than or equal to γ.

(*ii*) We want to find a controller satisfying part (i) for a γ which is as large as possible. Note that part (iii) shows that for every γ smaller than the bound $[\rho(P_a Q_a)]^{-1/2}$ we can find a suitable controller satisfying part (i). In fact, we can find a controller which makes the H_∞ norm equal to $[\rho(P_a Q_a)]^{-1/2}$ (see [Gl2]) which is clearly the best we can do.

(*iii*) It can be shown that the bound $[\rho(P_a Q_a)]^{-1/2}$ depends only on the antistable part of Σ. Hence we could assume a priori that A has only eigenvalues in the open right half complex plane (we still have to exclude eigenvalues on the imaginary axis). In that case it can be shown that P_a and Q_a are the inverses of X and Y, respectively, where X and Y are the unique positive definite solutions of the following two Lyapunov equations:

$$
\begin{aligned}
AX + XA^{\mathrm{T}} &= BB^{\mathrm{T}} \\
A^{\mathrm{T}} Y + YA &= C^{\mathrm{T}} C
\end{aligned}
$$

Then it is easy to derive that our bound is equal to the smallest Hankel singular value of Σ. This result was already known (see, e.g. [Gl2]).

11.2.2 Multiplicative perturbations

The results of this subsection have already appeared in [St8]. We assume that we once again have a continuous time system Σ being an imperfect model of a certain plant. This time, however, we assume that the error is multiplicative, i.e. we assume that the plant is exactly described by the following interconnection:

$$(11.3)$$
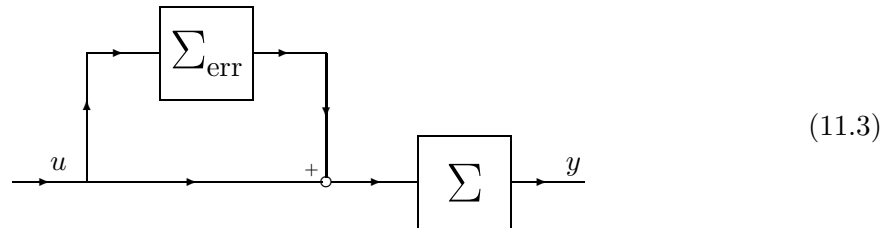
Here Σ_{err} is some arbitrary system such that the interconnection (11.3) has the same number of unstable poles as Σ. Thus we assume that the plant is described by the system Σ interconnected as in diagram (11.3) with another system Σ_{err}. The system Σ_{err} represents the uncertainty. As for additive perturbations, our goal is to find conditions under which a controller Σ_F of the form (2.4) from y to u exists such that the interconnection (11.3) is stabilized by this controller for all systems Σ_{err} which do not change the number of unstable poles of the interconnection (11.3) and which have $\mathcal{L}_\infty$ norm less than *or equal to* some, a priori given, number γ. In [Vi] there is also a result for multiplicative perturbations:

Lemma 11.3 : *Let a system $\Sigma = (A, B, C, D)$ and a controller Σ_F of the form (2.4) be given. The following conditions are equivalent:*

(i) *If we apply the controller Σ_F from y to u to the interconnection (11.3), then the closed-loop system is well-posed and internally stable for every system Σ_{err} such that*

  (a) *Σ_{err} has $\mathcal{L}_\infty$ norm less than or equal to γ.*

  (b) *The interconnection (11.3) and Σ have the same number of unstable poles.*

(ii) Σ_F *internally stabilizes* Σ *and if* G_{ci} *and* G_F *denote the transfer matrices of* Σ *and* Σ_F, *respectively, then* $I - G_{ci}G_F$ *is invertible as a proper rational matrix and* $\|G_{ci}G_F(I - G_{ci}G_F)^{-1}\|_\infty < \gamma^{-1}$. $\qquad\square$

Assume that a stabilizable and detectable realization of Σ is given. Hence Σ is described by some quadruple (A, B, C, D) with (A, B) stabilizable and (C, A) detectable. We define a new system:

$$\Sigma_{nm} : \begin{cases} \dot{x} = Ax + Bu + Bw \\ y = Cx + Du + Dw, \\ z = \qquad\quad u. \end{cases} \tag{11.4}$$

As we did for additive perturbations we can once again rephrase our problem formulation in terms of our new system Σ_{nm}. We first need some definitions:

$$G(Q) \quad := \quad \begin{pmatrix} AQ + QA^{\mathrm{T}} + BB^{\mathrm{T}} & QC^{\mathrm{T}} + BD^{\mathrm{T}} \\ CQ + DB^{\mathrm{T}} & DD^{\mathrm{T}} \end{pmatrix}$$

$$M(Q, s) \quad := \quad \begin{pmatrix} sI - A \\ -C \end{pmatrix}$$

$$G_{ci}(s) \quad := \quad C(sI - A)^{-1}B + D$$

Moreover, we define P_m and Q_m as the unique positive semi-definite matrices satisfying the following conditions:

(i)   • $A^{\mathrm{T}}P_m + P_mA = P_mBB^{\mathrm{T}}P_m$

  • $A - BB^{\mathrm{T}}P_m$ is asymptotically stable.

(ii) • $G(Q_m) \geq 0$

  • $\operatorname{rank} G(Q_m) = \operatorname{rank}_{\mathcal{R}(s)} G_{ci}$

  • $\operatorname{rank} \begin{pmatrix} M(Q_m, s) & G(Q_m) \end{pmatrix} = n + \operatorname{rank}_{\mathcal{R}(s)} G_{ci} \quad \forall\, s \in \mathcal{C}^+ \cup \mathcal{C}^0$.

The existence of such a P_m was already discussed in the previous subsection and follows from standard regular Linear Quadratic control.

The first two conditions on Q_m require that Q_m is a rank-minimizing solution of a linear matrix inequality. The same linear matrix inequality is also appearing in the *singular* filtering problem (see [SH4], this problem is dual to the singular linear quadratic control problem). Via the reduced order Riccati equation associated with this linear matrix inequality (see appendix A) it can be shown that the largest solution of the linear matrix

inequality, whose existence is guaranteed since (C,A) is detectable (dualize the results in [Gee, Wi7]), satisfies all the requirements on Q_m. This shows existence of Q_m. Uniqueness was already guaranteed by corollary A.7.

The above enables us to formulate our main result:

Theorem 11.4 : *Let a system Σ be given with stabilizable and detectable realization (A, B, C, D) and state space $\mathcal{R}^n$. Moreover, let $\gamma > 0$ be given. Assume that A has no eigenvalues on the imaginary axis and assume that (A, B, C, D) has no invariant zeros on the imaginary axis. We define the auxiliary system Σ_{nm} by (11.2). Under the above assumptions the following three conditions are equivalent:*

(i) *A controller Σ_F from y to u of the form (2.4) exists which, when applied to the interconnection (11.3), yields a closed-loop system that is well-posed and internally stable for all systems Σ_{err} such that:*

   (a) *Σ_{err} has $\mathcal{L}_\infty$ norm less than or equal to γ.*

   (b) *The interconnection (11.3) and Σ have the same number of unstable poles.*

(ii) *A controller Σ_F from y to u of the form (2.4) exists which, when applied to the system Σ_{nm}, yields a closed-loop system that is well-posed, internally stable and has H_∞ norm less than γ^{-1}.*

(iii) *Either A is stable or $1 + \rho(P_m Q_m) < \gamma^{-2}$.*   □

Proof : The equivalence between (i) and (ii) is a direct result from lemma 11.3. To show the equivalence between (ii) and (iii) we have to investigate four cases:

- *A is stable*: the matrices $P := 0$ and $Q := Q_m$ satisfy condition (iii) of theorem 5.1.

- *A is not stable and $\gamma < 1$*: we define

$$P := \frac{P_m}{1 - \gamma^2}, \qquad Q := Q_m \tag{11.5}$$

   Then it is straightforward to check that P and Q are the unique matrices satisfying all requirements of condition (iii) of theorem 5.1

for Σ_{nm} except for possibly the condition on the spectral radius of PQ for Σ_{nm}. Moreover, P and Q satisfy condition (iii) of theorem 11.4 if, and only if, P and Q satisfy the requirement $\rho(PQ) < \gamma^{-2}$ of condition (iii) of theorem 5.1 (note that we have to replace γ by γ^{-1}).

- *A is not stable and $\gamma = 1$*: the stability requirement for P reduces to the requirement that A is stable which, by assumption, is not true.

- *A is not stable and $\gamma > 1$*: the unique matrices P and Q that satisfy most of the conditions of part (iii) of theorem 5.1 are given by (11.5). However, note that $P \leq 0$ ($P \neq 0$).

The equivalence between (ii) and (iii) is then for each of the above cases a direct result from theorem 5.1.   ■

Remarks:

(i) Our class of perturbations is rather artificial but it includes all stable systems Σ_{err} with H_∞ norm less than or equal to γ if $\gamma < 1$. The restriction $\gamma < 1$ does not matter since we know that part (iii) from our theorem is only satisfied if $\gamma < 1$ (or A is stable but in that case Σ_{err} should be stable anyway).

(ii) As for additive perturbations, we have an explicit bound for the allowable size of perturbations: part (iii) shows that for every γ smaller than the bound $[1 + \rho(P_m Q_m)]^{-1/2}$ we can find a suitable controller satisfying part (i).

(iii) For additive perturbations it could be shown that the upper bound $[\rho(P_a Q_a)]^{-1/2}$ depends only on the antistable part of Σ. It should be noted that this is *not* true for the bound $[1 + \rho(P_m Q_m)]^{-1/2}$ which we obtained for multiplicative perturbations.

(iv) Note that because this is, in general, a singular problem we have not been able to find an explicit formula for a controller satisfying part (i). Note that we know that a controller satisfies part (i) if, and only if, this controller satisfies part (ii).

11.2.3   Perturbations in the realization of a system

The previous two classes of perturbations were directly concerned with the perturbations of input–output operators or, as an alternative formulation, with perturbations of transfer matrices. In contrast with the above, the

theory of complex stability radii is concerned with perturbations of state space realizations (an interesting overview article is [Hi]).

Assume some autonomous system is given:

$$\dot{x} = (A + D\Delta E)\, x.$$

$A \in \mathcal{R}^{n \times n}, D \in \mathcal{R}^{n \times l}$ and $E \in \mathcal{R}^{p \times n}$ are given matrices and Δ expresses the uncertainty which is structured by the matrices D and E. The complex stability radius of the triple (A, D, E) is then defined as

$$r_\mathcal{C}(A, D, E) := \inf \left\{ \|\Delta\| \mid \Delta \in \mathcal{C}^{l \times p} \text{ such that } A + D\Delta E \text{ is not stable} \right\}.$$

Naturally by allowing for complex perturbations the class of perturbations is not very natural. But there are two good reasons for investigating this complex stability radius. First of all, we can derive very elegant results for the complex stability radius which we cannot obtain for the real stability radius (defined as the complex stability radius but with the restriction that Δ should be a real matrix). Moreover, if we define Σ_Δ as the feedback interconnection of some stable system $\Delta = (F, G, H, J)$ with $\Sigma_{ci} = (A, D, E, 0)$, i.e.

$$\Sigma_\Delta : \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A + DJE & DH \\ GE & F \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}, \tag{11.6}$$

and the real dynamic stability radius $r_{\mathcal{R},d}$ by

$$r_{\mathcal{R},d}(A, D, E) := \inf \{ \|\Delta\|_\infty \mid \Delta = (F, G, H, J) \in \mathcal{S} \text{ is such that}$$

$$\Delta \times \Sigma_{ci} \text{ described by (11.6) is not stable.} \},$$

where $\mathcal{S}$ denotes the class of quadruples of real matrices which define an asymptotically stable system, then it is shown in [Hi] that the complex stability radius is equal to the real dynamic stability radius. This makes an investigation of the complex stability radius more important because we investigate the *real* dynamic stability radius at the same time.

In [Hi] the following relation between H_∞ control and the complex stability radius is given:

Lemma 11.5 : *We have*

$$r_\mathcal{C}(A, D, E) = \|G\|_\infty^{-1},$$

where G denotes the transfer matrix of $(A, D, E, 0)$.   □

Next we investigate the problem of maximization of the complex stability radius. Let the system Σ_{ci} be given by:

$$\Sigma_{ci}: \ \dot{x} = (A + D\Delta E)\,x + Bu.$$

We search for a static state feedback $u = Fx$ such that the closed-loop stability radius $r_{\mathcal{C}}(A + BF, D, E)$ is larger than γ. For any matrix $P \in \mathcal{R}^{n \times n}$ we define :

$$
\begin{aligned}
F_\gamma(P) \quad &:= \quad \begin{pmatrix} PA + A^{\mathrm{T}}P + E^{\mathrm{T}}E + \gamma^2 PDD^{\mathrm{T}}P & PB \\ B^{\mathrm{T}}P & 0 \end{pmatrix}, \\
L_\gamma(P, s) \quad &:= \quad \begin{pmatrix} sI - A - \gamma^2 DD^{\mathrm{T}}P & -B \end{pmatrix}, \\
G_{ci}(s) \quad &:= \quad E(sI - A)^{-1}B.
\end{aligned}
$$

Using lemma 11.5 and theorem 4.1 we find the following result:

Theorem 11.6 : _Assume that a system $\Sigma_{ci} = (A, D, E, 0)$ is given which does not have invariant zeros on the imaginary axis. Let $\gamma > 0$. Then the following two conditions are equivalent:_

(i) _There exists a matrix $F \in \mathcal{R}^{m \times n}$ such that_

$$r_{\mathcal{C}}(A + BF, D, E) > \gamma.$$

(ii) _There exists a positive semi-definite matrix P such that_

(a) $F_\gamma(P) \geq 0$,

(b) $rank\, F_\gamma(P) = rank_{\mathcal{R}(s)} G_{ci}$,

(c) $rank \begin{pmatrix} L_\gamma(P, s) \\ F_\gamma(P) \end{pmatrix} = n + rank_{\mathcal{R}(s)} G_{ci}, \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+.$ □

11.3 Practical applications

Many design problems can be reformulated in terms of constraints on the largest and smallest singular values of the closed-loop transfer matrix for each frequency (see, e.g. [Do, Hor].) This clearly gives an immediate starting point for rephrasing these problems as H_∞ control problems. To reformulate the performance requirements into a standard H_∞ control

problem, as we are studying in this book, requires the choice of appropriate weights. Certainly for multiple-input/multiple-output (MIMO) systems this is not straightforward.

Basically the above translation step is easy for single-input/single-output (SISO) systems where we can use the graphical tools like Bode- and Nyquist diagrams; this is not possible for MIMO systems. Moreover, directional information might be important for the performance criteria and is clearly lost when using singular values as performance criteria. This requires MIMO weights which translate directional information in terms of singular values. The above should clarify why practical problems are not solved as soon as you have read this book. A good understanding of design trade-offs (as discussed in, e.g. [FL]) and your system are required to construct weights which yield an H_∞ control problem which suitably reflects all your criteria.

Several examples have been treated in literature. These are physical examples and reflect what you can and cannot do with H_∞ control.

- *An aircraft autopilot design:* see [SLH, SC]

- *Vertical plane dynamics of an aircraft:* see [MF, M]

- *Large space structure:* see [SC2]

- *Attitude control of a flexible space platform:* see [MF]

- *Gas turbine control:* see [BW]

- *Control design for a 90 MW coal fired fluidized bed boiler:* see [Cr].

Basically the above examples discuss the choice of weights, after which they simply apply the results as discussed in this book. The only difference is the book [MF]. These authors discuss a design method combining loop-shaping with robustness requirements. The method is still relatively *ad-hoc* and needs a more thorough foundation before guaranteeing the kind of problems for which the method can be used.

One major class of problems in design is related to robustness, which can be treated by H_∞ control via the small gain theorem. However, for real parameter variations and structured uncertainty the small gain theorem is very conservative. On the other hand, for high frequency uncertainty, like bending modes and, in general, discarded dynamics, the theory yields very good results. These problems are discussed more specifically via the example of the inverted pendulum in the next section.

Up to to this moment we can, in general, only guarantee the existence of controllers with McMillan degree equal to the sum of the McMillan degrees of the plant and all the weights. This implies that complex weights

yield high-order controllers. However, in practice we would like to know whether a controller of lower McMillan degree exists which still gives an acceptable closed-loop performance. A main area of research, therefore, lies in reduced order H_∞ controllers and in model reduction with an H_∞ criterion (see [BH, HB, Mu2]).

A possible theoretical extension of the theory given in this book can be used to incorporate some specific performance constraints:

- **Steady state disturbance rejection:** the requirement that the closed-loop transfer matrix is zero in zero.

- **High gain roll-off:** the requirement that the closed-loop transfer matrix is strictly proper (or, in general that the k'th ($k \geq 0$) derivative of G is strictly proper to guarantee a sufficiently fast roll-off).

These kind of conditions are all based on the requirement that the closed-loop transfer matrix satisfies certain constraints on the imaginary axis. Without proof we shall give the following result. This result stems basically from [HSK] and [Vi, lemma 6.5.9].

Theorem 11.7 : *Let a system of the form (2.1) be given. Assume that a compensator of the form (2.4) which is internally stabilizing exists such that the closed-loop system has H_∞ norm less than one. Moreover, assume that for $s_i \in \mathcal{C}^0$ with $i = 1, 2, \ldots, j$ a static controller K_i exists such that the resulting closed-loop transfer matrix G_i satisfies $G_i(s_i) = 0$. In this case there is a time-invariant compensator Σ_F of the form (2.4) and with MacMillan degree less than $n + j$ such that the closed-loop system is internally stable, the closed-loop transfer matrix G_{cl} has H_∞ norm less than one and $G_{cl}(s_i) = 0$ for $i = 1, 2, \ldots, j$.*   □

In the previous theorem $s_i = \infty$ is allowed but in this case we should replace $G_{cl}(s_i)$ by the corresponding direct-feedthrough matrix of G_{cl}. Also, if we want the first k derivatives of G to be 0 then this can be incorporated but it will yield an increase of k in the McMillan degree of the controller.

Basically these requirements can also be incorporated via weights. However, results like this facilitate the construction of weights. Specific performance criteria like the ones above can be left out while constructing weights and can be incorporated at a later stage.

Finally it should be noted that there are now several toolboxes available (see, e.g. [CS, Va]) using the software package Matlab which contain

built-in tools to design controllers based on LQG and H_∞ control techniques. Singular systems cannot be treated with these toolboxes but software is being developed also using Matlab (see [SE]).

11.4   A design example: an inverted pendulum on a cart

We shall consider the following physical example of an inverted pendulum on a cart:
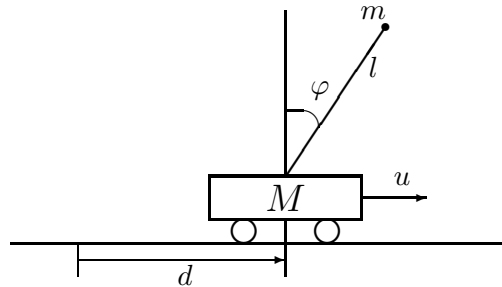
We assume the mass of the pendulum to be concentrated in the top with mass m. l is the length of the pendulum and M is the mass of the cart. To describe the position, d and φ express the distance of the cart from some reference point and the angle of the pendulum with respect to the vertical axis. The input u is the horizontal force applied to the cart. All motions are assumed to be in the plane. We assume that the system is completely stiff and the friction between the cart and the ground has friction coefficient F. Finally let g denote the accelaration of gravity. We then have the following non-linear model for this system:

$$
\begin{aligned}
(M + m)\ddot{d} + ml\ddot{\varphi}\cos\varphi - ml(\dot{\varphi})^2\sin\varphi + F\dot{d} &= u \\
l\ddot{\varphi} - g\sin\varphi + \ddot{d}cos\varphi &= 0
\end{aligned}
$$

If $l \neq 0$ and $M \neq 0$, then linearization around $\varphi = 0$ yields the following linear model:

$$
\begin{pmatrix} \dot{d} \\ \ddot{d} \\ \dot{\varphi} \\ \ddot{\varphi} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{F}{lM} & \frac{g(m+M)}{lM} & 0 \end{pmatrix} \begin{pmatrix} d \\ \dot{d} \\ \varphi \\ \dot{\varphi} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{lM} \end{pmatrix} u
$$

Denote these matrices by A and B respectively. Since $g \neq 0$ our linearized system is always controllable. Assuming our measurements are d and $\dot{d}$

then the system is observable if $m \neq 0$. If $m = 0$ the system is not even detectable (which has a clear physical interpretation). However, it turned out that although the system is observable, for many choices of the parameters an unstable pole and zero almost cancel out. This makes the system impossible to control. Therefore we have to add the angle φ as a measurement, i.e. our measurement vector equals $y = (d, \dot{d}, \varphi)$.

We would like to track a reference signal for the position. Moreover, we require our controller to yield a robustly stable system with respect to the several uncertainties that affect our system:

(i) Discarded non-linear dynamics

(ii) Uncertainties in the parameters F, m, M, l

(iii) Flexibility in the pendulum.

Finally we have to take into account the limit on the bandwidth and gain of our controller. This is essential due to limitations on the sampling rate for the digital implementation as well as limitation in the speed of the actuators.

We look at the following setup:

$$(11.7)$$
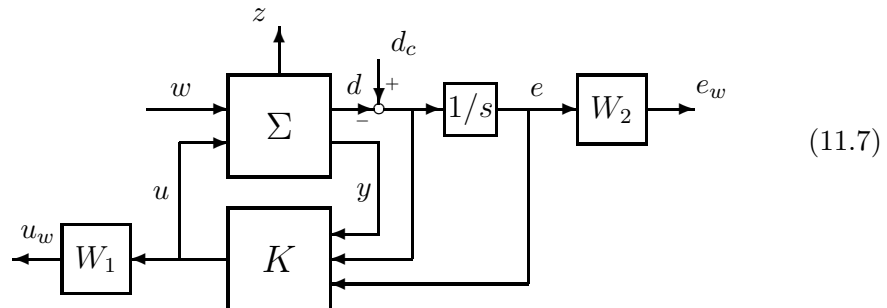
- d_c is the command signal for the position d. We minimize the weighted integrated tracking error where W_2 is a first order weight of the form

$$W_2(s) := \varepsilon \frac{1 + \alpha s}{1 + \beta s}$$

  By choosing $\alpha << \beta$ we obtain a low pass filter. This expresses that we are only interested in tracking low-frequency signals. The integrator also expresses our interest in low frequencies: it is one way to guarantee zero steady state tracking error. Finally ε in the weight is used to express the relative importance of tracking over the other goals we have for the system.

- We also minimize u_w which is the weighted control input. W_1 has the same structure as W_2. However, this time we choose $\alpha >> \beta$ to obtain a high pass filter. We would like to constrain the open-loop bandwidth and gain of the controller. This facilitates digital implementation and it also prevents us pushing our actuators beyond their capabilities. Finally, it prevents the controller stimulating high-frequency uncertainty in the system dynamics, such as bending modes of the pendulum. Since we cannot incorporate open-loop limitations directly in an H_∞ design, we push the bandwidth and gain down indirectly via this weight on u. However, it turned out to be very hard to make the bandwidth small.

- z and w are new inputs and outputs we add to the system Σ to express robustness requirements, i.e. the system Σ is of the form:

$$
\Sigma: \begin{cases} \dot{x} = & Ax + Bu + Ew, \\ y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x, \\ z = & C_2 x. \end{cases} \tag{11.8}
$$

The matrices A and B are as defined before. On the other hand E and C_2 still have to be chosen. We shall use the technique of complex stability radii described in subsection 11.2.3. For instance if we want to guard against fluctuations in the parameters F and m, the friction and the mass of the pendulum, then we choose E and C_2 as:

$$
E := \begin{pmatrix} 0 \\ -1/M \\ 0 \\ 1/lM \end{pmatrix}, \qquad C_2 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \end{pmatrix}. \tag{11.9}
$$

A similar definition can be made to guard against fluctuations in all parameters of the 2 differential equations due to discarded non-linearities or flexibility of the beam. The latter are dynamic uncertainties. It should be noted that the complex stability radius might be conservative with respect to (real) parameter fluctuations. On the other hand, for discarded dynamics or non-linearities the complex stability radius is not conservative. We decided to guard

against all fluctuations in the two differential equations. That is, instead of (11.9), we choose E and C_2 as:

$$E := \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C_2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11.10}$$

On the basis of the above we start designing a controller K for the interconnection (11.7) to minimize the H_∞ norm from (w, d_c) to (z, u_w, e_w) where we manipulated, by hand, the parameters of the weights W_1 and W_2 on the basis of the properties of the controllers. It turned out that the system Σ is in general much more sensitive to perturbations in all parameters of the differential equation then to perturbations in the parameters F and m. Therefore we incorporate in our design robustness against all parameters of the differential equation. In other words Σ is given by (11.8) where E and C_2 are given by (11.10).

We would like to stress that optimal controllers have a tendency of ruining every nice property of the system which is not explicitly taken into account. Our design incorporated a γ iteration. However, we implemented the minimum-entropy controller discussed in chapter 7 for a γ approximately 10% larger than the infimum over all stabilizing controllers of the closed-loop H_∞ norm. It is our experience that this reduces the bandwidth of the resulting controllers.

We shall give some facts illustrated by frequency and time-responses of our controller and the resulting closed-loop system. We choose $M = 1, m = .1, l = 1$ and $F = 0.1$.

(i) The open-loop transfer matrix from u to x and from u to ϕ are given by

$$\begin{aligned} G_{xu}(s) &:= \frac{(s + 3.13)(s - 3.13)}{(s + 3.29)(s - 3.28)(s + 0.09)} \\ G_{\phi u}(s) &:= \frac{-s^2}{(s + 3.29)(s - 3.28)(s + 0.09)} \end{aligned}$$

Hence, it is immediate that because of the near pole-zero cancellation in the right half plane that we really need a measurement of the angle ϕ.

(ii) Although not for every choice of the parameters, our final controller, whose Bode magnitude diagram is depicted in figure 11.1, is stable.

At this moment this desirable property cannot, however, be incorporated in our design criteria. We obtained a stable controller by playing around with the parameters. The eigenvalues of the controller are $-45.7, -22.8, -4.04, -2.37, -1.3 \pm 0.51i$ and -0.667. The fact that the controller has both very fast and relatively slow modes suggests that via singular perturbation theory (see, e.g. [OM]) we can reduce the order of the controller. There is, however, no theory available for such an approach.

(iii) Our final controller can, according to the theory developed in subsection 11.2.3, stand fluctuations in the parameters F and m of 400%. Moreover, fluctuations in the parameters of the differential equations of size less than 0.025 are allowed. The latter is not very much but since most parameters are 0 they are in general quite sensitive to fluctuations. Without taking robustness into account the final controller could only stand fluctuations of size less than 0.000025. Some simulations suggested good robustness properties.

(iv) The time-responses as given in figures 11.5, 11.6 and 11.7 show little overshoot and therefore the controller is expected to work well on the non-linear model.

(v) The final controller still has quite a large gain and bandwidth as can be seen from figure 11.1. The gain is for a large part due to the fact that the controller is translating angles in radians into forces in Newtons. In general the angles are much smaller than the forces, which is to be expected. The large bandwidth is needed to be able to stand sudden fluctuations in the angles. The bandwidth and gain of the transfer matrix from d_c to u is much smaller as shown in figure 11.3. This transfer matrix is weighted more strongly in our cost criterion than the effect of fluctuations in ϕ (since the angle is not steered directly via a command signal).

(vi) Figure 11.2 shows that we have good tracking properties of low-frequency command signals as required. Tracking can be improved to 1 rad/sec. However, this results in a controller of much larger bandwidth and gain.

(vii) The loop gain if we break the loop at the control input is given in figure 11.4. The surprising part here is the cross-over angle which is very small. The angle of the Bode diagram at the cross-over frequency is related to the phase margin. The fact that this angle is very small suggests a good phase margin.
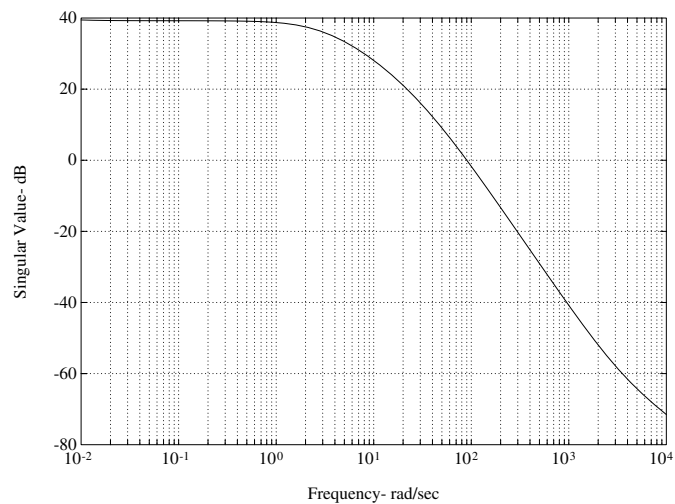
Figure 11.1: open loop magnitude Bode diagram of the controller
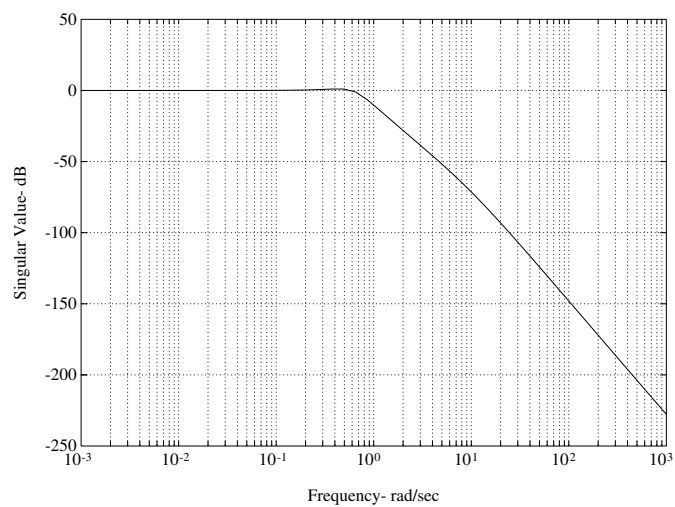
Figure 11.2: magnitude Bode diagram from d_c to d
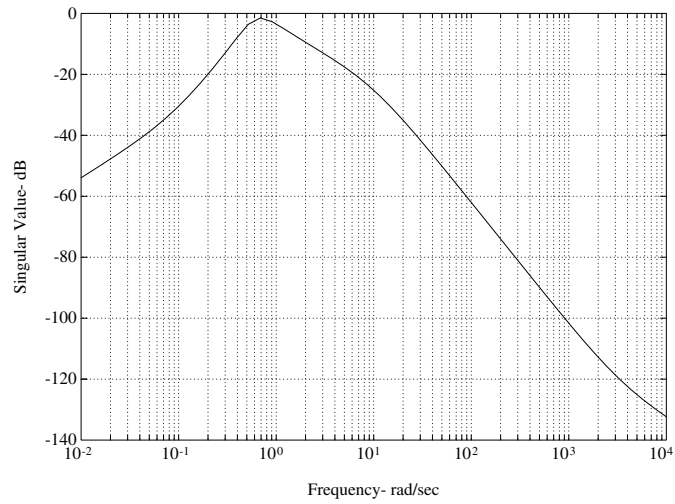
Figure 11.3: magnitude Bode diagram from d_c to u
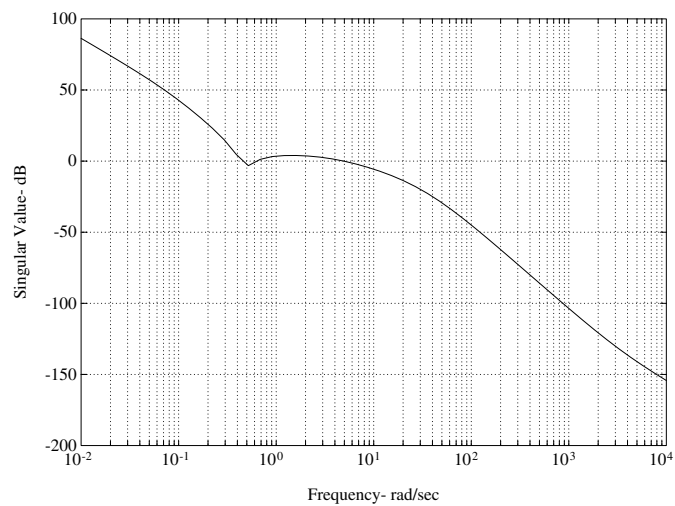
Figure 11.4: magnitude Bode diagram of the loop gain from u_c to u
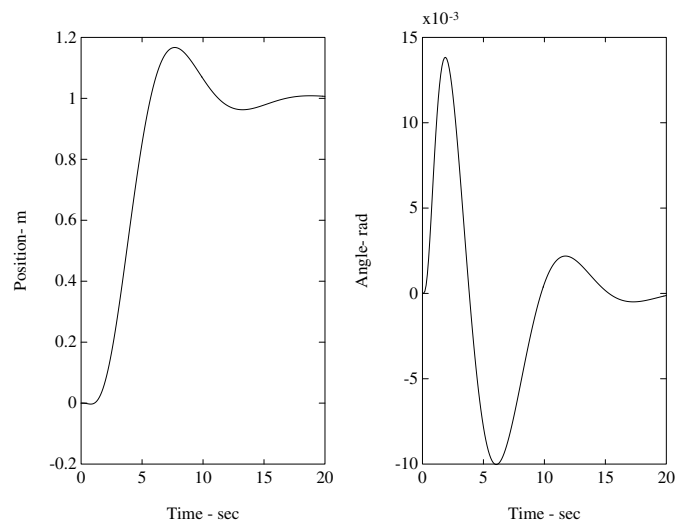
Figure 11.5: step response from d_c to d and ϕ
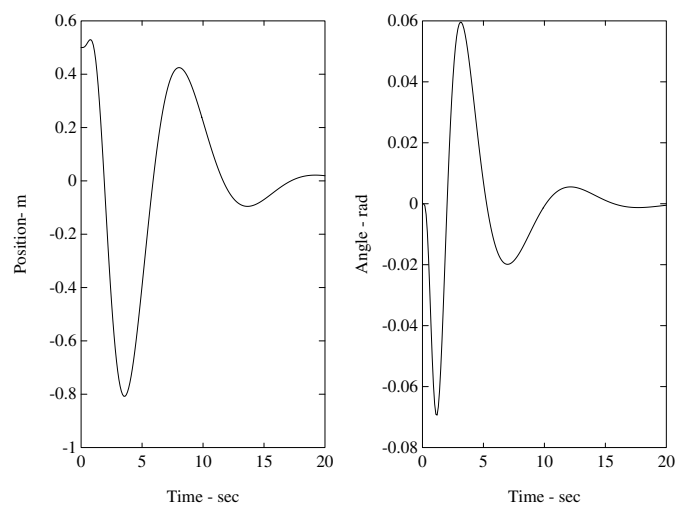
Figure 11.6: impulse response from d_c to d and ϕ
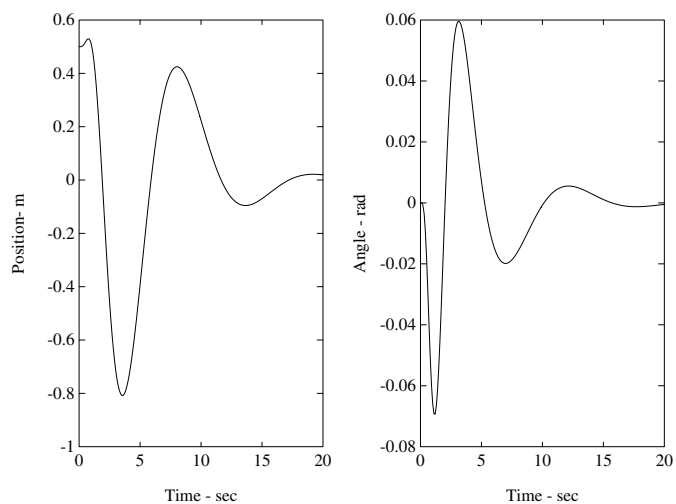
Figure 11.7: impulse response from ϕ to d and ϕ

To conclude, we would like to note that we can design controllers via H_∞ control similar as one used to do it via LQG control. However, it is our belief that the H_∞ norm makes it easier and more transparent to incorporate several performance requirements in our cost criterion, especially those performance requirements which are directly related to robustness and magnitude Bode diagrams. However a lot of work still needs to be done. In particular we have to gain experience similar to that we have gained for LQG to translate our criteria into a well-formulated H_∞ problem.

Chapter 12

Conclusion

In this book we have derived several results concerning the H_∞ control problem. Naturally the research in this area is not completed by this book and several interesting open problems remain. In this chapter we first devote a section to a discussion of the main contributions of this book to the active research area of H_∞ control. In section 12.2, we discuss some open problems which could be of interest for future research. Then, in the last section, we give our final concluding remarks on this book.

12.1 Summary of results obtained

The results derived in this book are related to a number of research areas within the framework of H_∞ control:

- Singular systems

- Differential games

- Finite horizon H_∞ control problem

- Minimum entropy H_∞ control problem

- H_∞ control for discrete time systems.

We shall briefly discuss our contributions to these five subjects.

12.1.1 Singular systems

For systems which are not necessarily regular, we derived necessary and sufficient conditions for the existence of an internally stabilizing controller which makes the H_∞ norm strictly less than some, a priori given, number γ. For the measurement-feedback case these conditions are in terms of

solutions of two quadratic matrix inequalities and four associated rank conditions. This can be shown to be equivalent to the existence of two stabilizing solutions of reduced order Riccati equations.

In our opinion, our proof yields a nice understanding of the structure of the H_∞ control problem. However, our algorithm for finding a suitable controller, if one exists, is still not completely satisfactory.

At this point we shall outline the main steps of the proof of theorem 5.1, which is our main result for singular systems. This is done because the main steps give nice insight in the structure of the H_∞ control problem as we see it.

We start with a system of the form (5.1). We first investigate how well we can regulate the system if we have all information of the state and the disturbance available. We can derive necessary and sufficient conditions under which we can find a controller which makes the H_∞ norm less than some bound γ (a so-called suitable controller), by investigating a sup-inf problem:

$$\sup_w \inf_u \left\{ \|z_{u,w,\xi}\|_2^2 - \gamma^2 \|w\|_2^2 \ \mid u \in \mathcal{L}_2^m, w \in \mathcal{L}_2^l \text{ such that } x_{u,w,\xi} \in \mathcal{L}_2^n \right\}$$

(12.1)

for arbitrary initial state $x(0) = \xi$. We solved this problem using ideas supplied by Pontryagin's Maximum Principle. We did this explicitly for regular systems and we extended it to general systems by some decompositions of the state space, the input space and the output space (that this extension to the general case is related to the same sup-inf problem can be seen more easily by investigating the results of chapter 6). We find necessary and sufficient conditions under which a suitable controller exists: there should be a positive semi-definite solution to a quadratic matrix inequality which satisfies two rank conditions. This can be shown to be equivalent to the existence of a stabilizing solution to an algebraic Riccati equation (a reduced order Riccati equation if the system is singular). Although our proof might not be the most simple one available it has one clear advantage. It shows that we cannot do better with respect to minimizing the H_∞ norm by allowing more general (e.g. non-linear or non-causal) controllers.

We now return to the general case where we might have only partial information on the state and the disturbance. If a suitable controller with measurement feedback exists, then there is certainly a suitable controller for the full-information feedback case. Hence, there is a positive semi-definite solution to our quadratic matrix inequality and the corresponding rank conditions. Then we can transform our original system Σ into a new

system Σ_P. A controller is suitable for Σ if, and only if, a controller is suitable for Σ_P (this is true for controllers with measurement feedback and for controllers with state feedback). However this new system Σ_P has the interesting property that we can find state feedbacks which are internally stabilizing and which make the H_∞ norm of the closed-loop system arbitrarily small (ADDPS, as discussed in section 2.6, is solvable).

The problem of whether we can measure the state well enough to find a suitable controller with measurement feedback still remains. It turns out that this problem is exactly dual to the problem of full-information H_∞ control: minimizing the induced norm from w to z for some system Σ_t by full-information feedback is essentially the same problem as building an observer for the state of Σ_t^T which minimizes the induced norm of the operator from w to $x - \hat{x}$ where $\hat{x}$ denotes the estimated state. Hence, we immediately find necessary and sufficient conditions under which we can observe the state of Σ_P well enough: there should be a positive semi-definite solution to a second quadratic matrix inequality which has to satisfy two rank conditions (remember that we already had a solution to one quadratic matrix inequality before our transformation from Σ to Σ_P). Thus we obtain a necessary condition for the existence of a suitable controller: we should have solutions of two quadratic matrix inequalities and four corresponding rank conditions. These conditions are also sufficient. To show this we apply a second transformation, dual to the first, from Σ_P to $\Sigma_{P,Q}$. Again, a controller is suitable for Σ_P if, and only if, this controller is suitable for $\Sigma_{P,Q}$. For $\Sigma_{P,Q}$ we can observe the state arbitrarily well. Surprisingly enough $\Sigma_{P,Q}$ still has the same nice property Σ_P has: there is an internally stabilizing state feedback which makes the H_∞ norm of the closed-loop system arbitrarily small (ADDPS is solvable). This implies that we can find a controller with measurement feedback which is internally stabilizing and which makes the H_∞ norm of the closed-loop system associated with $\Sigma_{P,Q}$ arbitrarily small (ADDPMS is solvable). We can make the H_∞ norm of the closed-loop system equal to 0 if the system Σ is regular (DDPMS is solvable), but we cannot do this in general.

The conditions we thus obtained can be reformulated to yield the conditions of theorem 5.1. The problem is that our second quadratic matrix inequality as defined above is in terms of the system Σ_P. Rewriting this second matrix inequality with two corresponding rank conditions in terms of Σ yields one quadratic matrix inequality, two rank conditions and a coupling condition.

12.1.2   Differential games

Differential games are not worked out in much detail in this book. However, some of the results obtained yield nice intuition.

Firstly we note that the quadratic form associated with the solution of our quadratic matrix inequality yields an *almost* Nash equilibrium while for the regular problem we obtain an (exact) Nash equilibrium.

Moreover, as we noted in the previous subsection, the results for the regular state feedback H_∞ control problem were derived by investigating a sup-inf problem (12.1). The value of (12.1) is, as a function of ξ, equal to the quadratic form given by the solution of the algebraic Riccati equation. The general full-information H_∞ control problem with state feedback was solved by reducing the general problem to the regular H_∞ control problem. In chapter 6, while discussing differential games, we find as a side result that the quadratic form associated with the solution of the quadratic matrix inequality again yields the value of (12.1) as a function of ξ. This is a nice result to complete the picture we have of the H_∞ control problem.

Another interesting fact is that a necessary condition for the existence of an almost equilibrium is the possibility of making the H_∞ norm less than or equal to 1 and a sufficient condition is the possibility of making the H_∞ norm strictly less than 1. All of this holds under the constraint of internal stability and with state feedback. This provides a natural starting point for research for testing whether we can make the H_∞ norm less than or equal to 1, as done in [Gl6] for the regular case.

12.1.3   The minimum entropy H_∞ control problem

In chapter 7 we discuss the minimum entropy H_∞ control problem. We only discussed a very limited case. In the general setup of subsection 1.6.3 we only discuss the case that we have *one* closed-loop transfer matrix on which we want to satisfy an H_∞ norm bound while minimizing the H_2 norm of this same transfer matrix. Not even that, we used only an upper bound for the LQG cost criterion. This problem therefore appears in one of the coming sections as one of the major open problems.

What did we do? By restraining attention to "entropy at infinity" we could make the proofs of [Mu] more straightforward and less technical. Moreover, we showed how you could extend this result to singular systems. This chapter is meant to be a starting point for future research in this area. In our opinion this chapter sketches ideas which could also be applied to the more general case of subsection 1.6.3.

12.1.4 The finite horizon H_∞ control problem

In chapter 8 we discussed the $\mathcal{L}_2[0, T]$-induced operator norm instead of the usual H_∞ norm which is equal to the $\mathcal{L}_2[0, \infty)$-induced operator norm. We derived conditions under which we could find a time-varying controller which makes the $\mathcal{L}_2[0, T]$-induced operator norm less than some a priori given number $\gamma > 0$. Instead of a Riccati differential equation we obtain, for singular systems, a quadratic differential inequality together with a number of rank conditions. Solvability of this quadratic differential inequality can be tested by using the basis of appendix A. It turns out to be equivalent to a reduced-order Riccati differential equation.

For regular systems the techniques used can equally well be applied to time-varying systems. This is not true for singular systems. Our proofs are strongly based on the bases of appendix A and we have to make a number of assumptions to make sure these bases are not time-varying. This is not worked out in detail in this book but can be found in [St7].

12.1.5 Discrete time systems

For discrete time systems which are discrete time analogues of *regular* continuous time systems, we were able to derive nice conditions: a suitable controller exists if, and only if, there are positive semi-definite stabilizing solutions to two algebraic Riccati equations. This certainly shows that transformations to the continuous time case are not necessary and that H_∞ control can be applied directly to discrete time systems. However, we should spend some time on deriving numerically reliable methods to check whether these Riccati equations indeed have solutions or not and, if they exist, to calculate the solutions.

The techniques used to derive this result were completely similar to the techniques we used to derive the results for the regular continuous time H_∞ control problem. However, for discrete time systems there are a number of extra technicalities and the formulae are rather complex and cumbersome. On the other hand, there is one essential difference: for continuous time systems we could do no better by allowing for non-causal feedbacks. This is not true for discrete time systems: it is possible to attain H_∞ norms of the closed-loop system via non-causal controllers which cannot be attained nor approximated by causal controllers.

12.2 Open problems

We shall mention some problems of interest in the next two subsections. Clearly this list is not exhaustive and several other open problems remain.

12.2.1 Invariant zeros

Throughout this book, we have excluded invariant zeros on the imaginary axis for continuous time systems. For discrete time systems, invariant zeros on the unit disc have been excluded. In this subsection we want to discuss the difficulty of these invariant zeros *intuitively*. The reader should not expect formal proofs in this section.

A treatment of invariant zeros on the imaginary axis for continuous time systems is given in several papers, [HSK, S3, S4]. For the regular state feedback case (no problems at infinity) the conditions in [S3] can, in principle, be reformulated as: a solution of a Riccati equation exists for which the matrix A_{cl} as given in theorem 3.1 has all eigenvalues in the open left half plane or in points on the imaginary axis which are invariant zeros. Besides that, separate extra conditions have to be satisfied for each invariant zero on the imaginary axis.

However, it is worthwhile to look for alternative formulations of the results in [HSK, S3, S4] in order to obtain a better insight of what the extra conditions which have to be satisfied when we have an invariant zero on the imaginary axis look like.

Moreover, a numerical reliable way of finding a suitable controller, if one exists, is needed. As in this book, in [S3, S4] a geometric approach is chosen. This yields good understanding of the nature of the problems but gives results which are in general numerically not very reliable.

Next, we shall try to show the difficulty of invariant zeros on the imaginary axis for continuous time systems. Our approach is based on the paper [HSK], which only treats the regular one block problem (D_{12} and D_{21} in (2.1) are both square invertible matrices).

The H_∞ control problem with measurement feedback can be reduced to the so-called model-matching problem (see [Fr2]). This is the following problem:

$$\inf_{Q \in H_\infty} \ \|T_1 - T_2 Q T_3\|_\infty, \tag{12.2}$$

where T_1, T_2 and T_3 are given matrices in H_∞. For the sole reason of an easy exposition of the problems we assume that T_2 and T_3 are, as rational matrices, right- and left-invertible, respectively. In this case for $Q_1 := T_2^+ T_1 T_3^+$ (+ denotes a right cq. left inverse) we have

$$T_1 - T_2 Q_1 T_3 = 0.$$

However, in general, Q_1 will not be in H_∞. The invariant zeros of the two subsystems (A, B, C_2, D_2) and (A, E, C_1, D_1) (using the definitions of theorem 5.1) are the only points in the complex plane where T_2 and T_3,

respectively, might have a zero, i.e. lose rank. The right cq. left inverse will then have a pole in these points. Hence if we have only invariant zeros in the open left half plane, then Q_1 will be in H_∞.

Invariant zeros in the open left half plane do not give rise to constraints on the attainable closed-loop system $T_1 - T_2 Q T_3$. On the other hand, invariant zeros in the open right half plane do yield constraints on the attainable closed-loop system. A zero on the imaginary axis can, however, be cancelled approximately by choosing Q such that it has a pole in the left half plane arbitrarily close to the zero on the imaginary axis. It turns out that for these points the only constraint is a condition in terms of the transfer matrices evaluated in this point itself. If we have only invariant zeros on the imaginary axis we find the constraint that the infimum in (12.2) is less than 1 if, and only if, for all invariant zeros $s \in \mathcal{C}^0$ a constant matrix K_s exists such that

$$\|T_1(s) - T_2(s) K_s T_3(s)\| < 1.$$

For invariant zeros in the open right half plane this condition is necessary but not sufficient for the infimum to be less than 1.

This shows that zeros on the imaginary axis and zeros in the right half plane need a different approach and that is exactly the difficulty we have in treating the most general case with both invariant zeros on the imaginary axis as well as zeros in the open right half plane.

We expect that for the general case without assumptions on the system and with continuous time, we can find the following necessary and sufficient conditions for the existence of a suitable controller (we shall use the notation as was already used in theorem 5.1): we think that the existence of matrices P and Q which satisfy the quadratic matrix inequalities $F_\gamma(P) \geq 0$ and $G_\gamma(Q) \geq 0$ and which satisfy the rank conditions (a) and (b) of theorem 5.1 are conditions which are always needed.

For any s in the closed right half plane which is not an invariant zero of either (A, B, C_2, D_2) or (A, E, C_1, D_1), rank conditions (c) or (d) of theorem 5.1 should still be satisfied.

The condition for invariant zeros of (A, B, C_2, D_2) or (A, E, C_1, D_1) in the open right half plane is that rank conditions (c) and (d) of theorem 5.1 should still be satisfied. However, for invariant zeros of (A, B, C_2, D_2) or (A, E, C_1, D_1) on the imaginary axis the rank conditions (c) and (d) should be replaced by some *weaker* condition.

12.2.2   Mixed $H_\infty / \dots$ problems

As we already noted in section 1.1 robustness plays a key role in control theory. In this book we investigated H_∞ control problems which play an

essential role in robustness of internal stability as outlined in section 11.2. However, it often happens that we are not only interested in robustness of internal stability but in robustness analysis of other preformance criteria as well.

Mixed Linear Quadratic Gaussian (LQG) and H_∞ control problems as introduced in subsection 1.6.3 are quite well suited for the above objective. We made a start with the discussion of this problem in chapter 7. For regular systems a quite extensive literature is already available (see section 7.1). However all these papers discuss an entropy function instead of the LQG criterion. The real mixed LQG/H_∞ control problem is still completely unsolved.

There are also other kinds of mixed problems which are of interest but which have not received much attention. For instance, in the above problems the sensitivity of the LQG criterion to system perturbations is *not* incorporated in the controller design. Only the sensitivity of internal stability to system perturbations is taken into account. The main problem of course remains the trade-off between robustness and performance. This also yields problems with respect to the required model accuracy and bounds on the McMillan degree of the controller.

12.3   Conclusion of this book

In this book we have generalized existing theorems on the standard H_∞ control problem. Interesting open problems at this moment are more specific applications of H_∞ control which are non-standard, e.g. the above-mentioned mixed problems. The main open problem at this moment is related to the practical application of H_∞ control. Although we feel that the theoretical H_∞ control problem and its structure are well understood at the moment, the practical issue of the design of weights for multi-input, multi-output systems lacks a systematic approach. Besides that, the H_∞ control problems with either invariant zeros on the imaginary axis (unit disc for discrete time systems) or with problems at infinity are now reasonably well understood but they need numerically more reliable algorithms to check existence of suitable controllers and to calculate them, if they exist.

We feel that this book and the large number of papers mentioned in the references give a very thorough basic understanding of the H_∞ control problem. Nevertheless the subject will remain an important research area in the near future. This is due to the fact that we still need to make the step from basic understanding to design (how to apply all this nice theory in practice). Moreover, because of the extreme importance of

robustness and its strong correlation with H_∞, its relation with almost all aspects of system and control theory is interesting. We could think of identification with H_∞ error bounds (see, e.g. [HJN, GK, GK2]) and other combinations which at the moment might be far-fetched. However, control engineers working in practice will be the first to acknowledge that the main reason why they are sceptical about applying modern control schemes is their concern about robustness. They sometimes even claim that all modern schemes including controllers built with H_∞ performance criteria are not robust and only classical P(I)(D) controllers can be trusted to control their expensive machinery. The control engineer in practice should realize that, although we present this new tool of H_∞ control in a very mathematical setting, in fact H_∞ control is nothing other then applying the available theoretical machinery to the problem of obtaining the desired shape of the closed-loop Bode diagram.

It is our belief that in the future H_∞ will lead to robust and reliable controllers but a lot of work has to be done in order to be ready to tackle real-life control problems (which will often be MIMO systems) in a structured and reliable way.

Appendix A

Preliminary basis transformations

A.1 A suitable choice of bases

In this section we shall choose bases to identify for systems (A, B, C, D) with D not injective, a regular subsystem which contains *all* essential features of the H_∞ control problem. We start by applying a suitable state feedback transformation $u = F_0 x + v$ to the system $\Sigma_{ci} = (A, B, C, D)$. It is transformed into a system $\Sigma_{ci,F_0} := (A + BF_0, B, C + DF_0, D)$ with a very particular structure. We shall display this structure by writing down the matrices of the mappings $A + BF_0$, B, $C + DF_0$, D with respect to suitable bases in the input space $\mathcal{R}^m$, the state space $\mathcal{R}^n$ and the output space $\mathcal{R}^p$. In this section we shall use the notation $A_{F_0} := A + BF_0$ and $C_{F_0} := C + DF_0$.

Our basic tool is the strongly controllable subspace. This subspace has been defined and some of its properties have been given in section 2.4. We now define the bases which will be used in the sequel.

First choose a basis of the input space $\mathcal{R}^m$ as follows. Let $u_1, u_2, \ldots, u_m$ be a basis such that $u_1, u_2, \ldots, u_i$ is a basis of $\ker D$ ($0 \leq i \leq m$). In other words, decompose $\mathcal{R}^m = \mathcal{U}_1 \oplus \mathcal{U}_2$ such that $\mathcal{U}_2 = \ker D$ and $\mathcal{U}_1$ arbitrary. Next, choose a basis of the output space $\mathcal{R}^p$ as follows. Let $z_1, z_2, \ldots, z_p$ be an *orthonormal* basis such that $z_1, \ldots, z_j$ is an orthonormal basis of $\operatorname{im} D$ and $z_{j+1}, \ldots, z_p$ is an orthonormal basis of $(\operatorname{im} D)^\perp$. ($0 \leq j \leq p$). In other words, write $\mathcal{R}^p = \mathcal{Z}_1 \oplus \mathcal{Z}_2$ with $\mathcal{Z}_1 = \operatorname{im} D$ and $\mathcal{Z}_2 = (\operatorname{im} D)^\perp$. Because this is an orthonormal basis this basis transformation does not change the norm $\|z\|$.

With respect to these decompositions the mapping D has the form

$$D = \begin{pmatrix} \hat{D} & 0 \\ 0 & 0 \end{pmatrix},$$

with $\hat{D}$ invertible. Moreover B and C can be partitioned as

$$B = \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \quad C = \begin{pmatrix} \hat{C}_1 \\ \hat{C}_2 \end{pmatrix}.$$

It is easy to see that $B_2 = B \ker D$ and $\ker \hat{C}_2 = C^{-1} \operatorname{im} D$. Next define a linear mapping by

$$F_0 := \begin{pmatrix} -\hat{D}^{-1} \hat{C}_1 \\ 0 \end{pmatrix}. \tag{A.1}$$

Then we have

$$C + DF_0 = \begin{pmatrix} 0 \\ \hat{C}_2 \end{pmatrix}.$$

We now choose a basis of the state space $\mathcal{R}^n$. Let $x_1, x_2, \ldots, x_n$ be a basis such that $x_{s+1}, \ldots, x_r$ is a basis of $\mathcal{T}(\Sigma_{ci}) \cap C^{-1} \operatorname{im} D$ and $x_{s+1}, \ldots, x_n$ is a basis of $\mathcal{T}(\Sigma_{ci})$. $(0 \le s \le r \le n)$ In other words, write $\mathcal{R}^n = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ with $\mathcal{X}_2 = \mathcal{T}(\Sigma_{ci}) \cap C^{-1} \operatorname{im} D$, $\mathcal{X}_2 \oplus \mathcal{X}_3 = \mathcal{T}(\Sigma_{ci})$ and $\mathcal{X}_1$ arbitrary.

It turns out that with respect to the bases introduced above A_{F_0}, B and C_{F_0} have a particular form. This is a consequence of the following lemma:

Lemma A.1 : *Let F_0 be given by (A.1). Then we have*

(i) $(A + BF_0)(\mathcal{T}(\Sigma_{ci}) \cap C^{-1} \operatorname{im} D) \subseteq \mathcal{T}(\Sigma_{ci})$

(ii) $\operatorname{im} B_2 \subseteq \mathcal{T}(\Sigma_{ci})$

(iii) $\mathcal{T}(\Sigma_{ci}) \cap C^{-1} \operatorname{im} D \subseteq \ker \hat{C}_2$. $\qquad\qquad \square$

Proof :

(i) $\mathcal{T}(\Sigma_{ci})$ is (C_{F_0}, A_{F_0})-invariant by lemma 2.6. This implies that

$$A_{F_0}(\mathcal{T}(\Sigma_{ci}) \cap \ker C_{F_0}) \subseteq \mathcal{T}(\Sigma_{ci}).$$

Since $\ker C_{F_0} = \ker \hat{C}_2 = C^{-1} \operatorname{im} D$, the result follows.

(ii) Let $\mathcal{T}_i(\Sigma_{ci})$ be the sequence defined by (2.15). Then we know that $\mathcal{T}_1(\Sigma_{ci}) = B \ker D = \operatorname{im} B_2$. Since $\mathcal{T}_i(\Sigma_{ci})$ is non-decreasing this proves our claim.

(iii) This follows immediately from the fact that $C^{-1}\operatorname{im}D = \ker \hat{C}_2$.   ∎

By applying this lemma we find that the matrices $A + BF_0$, B, $C + DF_0$ and D with respect to these bases have the following form.

$$\begin{cases} A + BF_0 = \begin{pmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{pmatrix}, \\[4mm] C + DF_0 = \begin{pmatrix} 0 & 0 & 0 \\ C_{21} & 0 & C_{23} \end{pmatrix}, \quad D = \begin{pmatrix} \hat{D} & 0 \\ 0 & 0 \end{pmatrix}. \end{cases} \tag{A.2}$$

We apply the preliminary feedback $u = F_0 x + v$ to the system Σ, given by

$$\Sigma \; : \; \begin{cases} \dot{x} = Ax + Bu + Ew, \\ z = Cx + Du. \end{cases} \tag{A.3}$$

Denote the resulting system by Σ_{F_0}. We decompose x,v and z corresponding to the bases in state, input and output space, i.e.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

We also decompose E and $\hat{C}_1$ corresponding to the bases defined:

$$\hat{C}_1 = \begin{pmatrix} C_{11} & C_{12} & C_{13} \end{pmatrix}, \qquad E = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}. \tag{A.4}$$

Hence we have

$$C = \begin{pmatrix} \hat{C}_1 \\ \hat{C}_2 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & 0 & C_{23} \end{pmatrix}.$$

In our new bases the system Σ_{F_0} then has the following form:

$$\dot{x}_1 = A_{11}x_1 + \begin{pmatrix} B_{11} & A_{13} \end{pmatrix} \begin{pmatrix} v_1 \\ x_3 \end{pmatrix} + E_1 w, \tag{A.5}$$

$$\begin{pmatrix} \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix} v_2 + \begin{pmatrix} B_{21} & A_{21} \\ B_{31} & A_{31} \end{pmatrix} \begin{pmatrix} v_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} E_2 \\ E_3 \end{pmatrix} w, \text{(A.6)}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ C_{21} \end{pmatrix} x_1 + \begin{pmatrix} \hat{D} & 0 \\ 0 & C_{23} \end{pmatrix} \begin{pmatrix} v_1 \\ x_3 \end{pmatrix}. \tag{A.7}$$

As already suggested by the way in which we arranged these equations, the system Σ_{F_0} can be considered as the interconnection of two subsystems. This is depicted in the following diagram:
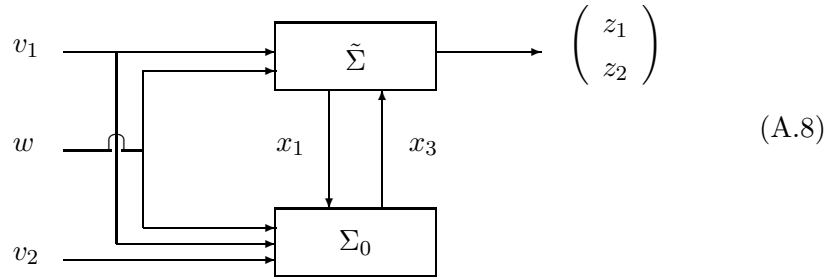
In the picture (A.8), $\tilde{\Sigma}$ is the system given by the equations (A.5) and (A.7). It has input space $\mathcal{U}_1 \times \mathcal{X}_3 \times \mathcal{R}^l$, state space $\mathcal{X}_1$ and output space $\mathcal{R}^p$. The system Σ_0 is given by equation (A.6). It has input space $\mathcal{R}^m \times \mathcal{X}_1 \times \mathcal{R}^l$, state space $\mathcal{X}_2 \oplus \mathcal{X}_3$ and output space $\mathcal{X}_3$. The interconnection is made via x_1 and x_3 as in the diagram. Note that $\tilde{\Sigma}$ and Σ_{F_0} have the same output equation. However, in Σ_{F_0} the variable x_3 is generated by Σ_0 while in $\tilde{\Sigma}$ it is considered as an input and is free.

The systems $\tilde{\Sigma}$ and Σ_0 turn out to have some nice structural properties:

Lemma A.2 :

(i) C_{23} is injective,

(ii) The system,

$$\Sigma_1 := \left[\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}, \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix}, \begin{pmatrix} 0 & I \end{pmatrix}, 0 \right] \tag{A.9}$$

with input space $\mathcal{U}_2$, state space $\mathcal{X}_2 \oplus \mathcal{X}_3$ $(= \mathcal{T}(\Sigma_{ci}))$ and output space $\mathcal{X}_3$ is strongly controllable.   □

Proof :

(*i*) Let $(x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, x_3^{\mathrm{T}})^{\mathrm{T}}$ be the coordinate vector of a given $x \in \mathcal{R}^n$. Assume that $C_{23} x_3 = 0$. Let $\tilde{x} \in \mathcal{R}^n$ be the vector with coordinates $(0^{\mathrm{T}}, 0^{\mathrm{T}}, x_3^{\mathrm{T}})$. Then $\tilde{x} \in \mathcal{X}_3$. In addition, $\tilde{x} \in \mathcal{T}(\Sigma_{ci}) \cap \ker \hat{C}_2 = \mathcal{X}_2$. Thus $\tilde{x} = 0$ so $x_3 = 0$.

(*ii*) Let $\mathcal{T}(\Sigma_1)$ be the strongly controllable subspace of the system Σ_1 given by (A.9). We shall prove $\mathcal{T}(\Sigma_1) = \mathcal{X}_2 \oplus \mathcal{X}_3$. First note that there exists $G := \begin{pmatrix} G_2^{\mathrm{T}} & G_3^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$ such that

$$\left[\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} + \begin{pmatrix} G_2 \\ G_3 \end{pmatrix} \begin{pmatrix} 0 & I \end{pmatrix} \right] \mathcal{T}(\Sigma_1) \subseteq \mathcal{T}(\Sigma_1).$$

Also note that

$$\mathrm{im} \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix} \subseteq \mathcal{T}(\Sigma_1).$$

Now assume that $\mathcal{T}(\Sigma_1) \subset \mathcal{X}_2 \oplus \mathcal{X}_3$ with *strict* inclusion. Define $\mathcal{V} \subseteq \mathcal{R}^n$ by

$$\mathcal{V} := \left\{ \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \in \mathcal{T}(\Sigma_1) \right\}.$$

Clearly, $\mathcal{V} \subset \mathcal{T}(\Sigma_{ci})$ with *strict* inclusion. We claim that a linear map $G_0 : \mathcal{R}^p \to \mathcal{R}^n$ exists such that

$$(A + G_0 C) \mathcal{V} \subseteq \mathcal{V}, \tag{A.10}$$

and

$$\mathrm{im} \ (B + G_0 D) \subseteq \mathcal{V}. \tag{A.11}$$

Indeed, let C_{23}^+ be any left inverse of C_{23} and define

$$G_0 := \begin{pmatrix} B_{11} & -A_{13} \\ B_{21} & G_2 \\ B_{31} & G_3 \end{pmatrix} \begin{pmatrix} -\hat{D}^{-1} & 0 \\ 0 & C_{23}^+ \end{pmatrix}.$$

It is then straightforward to check (A.10) and (A.11). This, however, contradicts the fact that $\mathcal{T}(\Sigma_{ci})$ is the smallest subspace $\mathcal{V}$ for which (A.10) and (A.11) hold (see definition 2.3). We conclude that $\mathcal{X}_2 \oplus \mathcal{X}_3 = \mathcal{T}(\Sigma_1)$. ∎

Our next result states that the zero structure of $\Sigma_{ci} = (A, B, C, D)$ is completely determined by the zero structure of $\tilde{\Sigma}_{ci}$ defined by

$$
\tilde{\Sigma}_{ci} := \left[A_{11}, \left(\, B_{11} \quad A_{13} \,\right), \left(\begin{array}{c} 0 \\ C_{21} \end{array}\right), \left(\begin{array}{cc} \hat{D} & 0 \\ 0 & C_{23} \end{array}\right) \right]. \qquad (A.12)
$$

Note that $\tilde{\Sigma}_{ci}$ is a subsystem of $\tilde{\Sigma}$ (where $\tilde{\Sigma}$ is described by (A.5) and (A.7)) in the same way as Σ_{ci} is a subsystem of Σ (where Σ is described by (A.3)).

Lemma A.3 : *The non-trivial transmission polynomials of Σ_{ci} and $\tilde{\Sigma}_{ci}$, respectively, coincide.* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Box$

Proof : According to section 2.3 the transmission polynomials of $\Sigma_{ci} = (A, B, C, D)$ and $\Sigma_{ci,F_0} = (A_{F_0}, B, C_{F_0}, D)$ coincide. Thus, in order to prove the lemma, it suffices to show that the system matrix P_{ci,F_0} of Σ_{ci,F_0} is unimodularly equivalent to a polynomial matrix of the form

$$
\left(\begin{array}{ccc} \tilde{P}_{ci} & 0 & 0 \\ 0 & I & 0 \end{array}\right),
$$

where $\tilde{P}_{ci}$ is the system matrix of $\tilde{\Sigma}_{ci}$. Since Σ_1, as defined by (A.9), is strongly controllable and $(\, 0 \quad I \,)$ is surjective, by lemma 2.8 the Smith form of the system matrix P_1 of Σ_1 is equal to $(\, I \quad 0 \,)$. In addition we clearly have

$$
\begin{aligned}
P_1 \quad &\sim \quad \left(\begin{array}{ccc} sI - A_{22} & 0 & -B_{22} \\ -A_{22} & 0 & -B_{32} \\ 0 & I & 0 \end{array}\right) \\
&\sim \quad \left(\begin{array}{ccc} sI - A_{22} & -B_{22} & 0 \\ -A_{22} & -B_{32} & 0 \\ 0 & 0 & I \end{array}\right),
\end{aligned}
$$

so we conclude that

$$
\left(\begin{array}{cc} sI - A_{22} & -B_{22} \\ -A_{22} & -B_{32} \end{array}\right),
$$

is unimodularly equivalent to $(\ \ I \quad 0\ \ )$. The proof is then completed by noting that

$$
P_{ci,F_0} \ \sim \
\begin{pmatrix}
sI - A_{11} & -B_{11} & -A_{13} & 0 & 0 \\
0 & \hat{D} & 0 & 0 & 0 \\
C_{21} & 0 & C_{23} & 0 & 0 \\
-A_{21} & -B_{21} & -A_{23} & sI - A_{22} & -B_{22} \\
-A_{31} & -B_{31} & sI - A_{33} & -A_{32} & -B_{32}
\end{pmatrix}
$$

$$
\sim \
\begin{pmatrix}
\tilde{P}_{ci} & 0 & 0 \\
0 & I & 0
\end{pmatrix}. \qquad \blacksquare
$$

A consequence of the above lemma is that the invariant zeros of Σ_{ci} and $\tilde{\Sigma}_{ci}$, respectively, coincide.

Our next lemma states that the normal rank of the transfer matrix

$$
G_{ci}(s) := C(sI - A)^{-1}B + D, \tag{A.13}
$$

of the system Σ_{ci} is equal to the number (rank D + dim $\mathcal{X}_3$) or, equivalently,

Lemma A.4 : *We have*

$$
rank_{\mathcal{R}(s)}G_{ci} = rank
\begin{pmatrix}
C_{23} & 0 \\
0 & \hat{D}
\end{pmatrix}. \qquad \square
$$

Proof : Define $L(s) := sI - A$. Then we have

$$
\text{rank}_{\mathcal{R}(s)}
\begin{pmatrix}
L & 0 \\
0 & G_{ci}
\end{pmatrix} = n + \text{rank}_{\mathcal{R}(s)}G_{ci}. \tag{A.14}
$$

We also have

$$
\begin{pmatrix}
L(s) & 0 \\
0 & G_{ci}(s)
\end{pmatrix} \sim
\begin{pmatrix}
sI - A_{F_0} & -B \\
C_{F_0} & D
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
sI - A_{11} & 0 & -A_{13} & -B_{11} & 0 \\
-A_{21} & sI - A_{22} & -A_{23} & -B_{21} & -B_{22} \\
-A_{31} & -A_{32} & sI - A_{33} & -B_{31} & -B_{32} \\
0 & 0 & 0 & \hat{D} & 0 \\
C_{21} & 0 & C_{23} & 0 & 0
\end{pmatrix}.
$$

Since C_{23} and $\hat{D}$ are injective we can make the (1,3), (1,4), (2,4) and (3,4) blocks zero by unimodular transformations. Furthermore we make a basis transformation on the output such that C_{23} has the form $(\ I_r\quad 0\ )^{\mathrm{T}}$ where $r = \ \mathrm{rank}\ C_{23}$. Thus, after suitable permutation of blocks, the normal rank of the latter matrix turns out to be equal to the normal rank of

$$
\begin{pmatrix}
\boxed{sI - \tilde{A}_{11}} & 0 & 0 & 0 & 0 \\
-A_{21} & sI - A_{22} & -A_{23} & -B_{22} & 0 \\
-A_{31} & -A_{32} & sI - A_{33} & -B_{32} & 0 \\
C_{211} & 0 & I_r & 0 & 0 \\
C_{212} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boxed{\hat{D}}
\end{pmatrix} .
$$

Here $\tilde{A}_{11}$ is a given matrix. Since, by lemma 4.2, the matrix in the centre box has full row rank for all $s \in \mathcal{C}$ and $\mathrm{rank}_{\mathcal{R}(s)}(sI - \tilde{A}_{11}) = \ \dim \mathcal{X}_1$ we find

$$
\mathrm{rank}_{\mathcal{R}(s)}
\begin{pmatrix}
L & 0 \\
0 & G_{ci}
\end{pmatrix}
= n + \ \mathrm{rank}
\begin{pmatrix}
C_{23} & 0 \\
0 & \hat{D}
\end{pmatrix} .
$$

Combining this with (A.14) gives the desired result.   ∎

To conclude we want to note that if D is injective, then the subspace $\mathcal{U}_2$ in the decomposition of $\mathcal{R}^m$ vanishes. Consequently, the partitioning of B reduces to a single block and the partitioning of D reduces to $(\ \hat{D}^{\mathrm{T}}\quad 0\ )^{\mathrm{T}}$ with $\hat{D}$ invertible. It is left as an exercise to the reader to show that $\mathcal{T}(\Sigma_{ci}) = \{0\}$ if, and only if, $\ker D \subset \ker B$. Thus, if D is injective, then also $\mathcal{T}(\Sigma_{ci}) = \{0\}$. In that case the subspaces $\mathcal{X}_2$ and $\mathcal{X}_3$ appearing in the decomposition of the state space $\mathcal{R}^n$ both vanish and the partitioning of A_{F_0} reduces to a single block.

A.2   The quadratic matrix inequality

We shall now derive some properties for matrices satisfying the *quadratic matrix inequality*, i.e. matrices P such that

$$
F(P) :=
\begin{pmatrix}
A^{\mathrm{T}}P + PA + PEE^{\mathrm{T}}P + C^{\mathrm{T}}C & C^{\mathrm{T}}D + PB \\
D^{\mathrm{T}}C + B^{\mathrm{T}}P & D^{\mathrm{T}}D
\end{pmatrix}
\geq 0.
$$

We first derive a result which is one of the key lemmas of this book.

Lemma A.5 : *Assume that a symmetric P is a solution of $F(P) \geq 0$. We have $P \, \mathcal{T}(\Sigma_{ci}) = 0$, i.e in our decomposition P can be written as*

$$P = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.15}$$

$\square$

Proof : Let F_0 be given by (A.1). Since $D^{\mathrm{T}} C_{F_0} = 0$ (this can be checked easily) we may apply lemma 2.7. We define

$$M(P) := \begin{pmatrix} I & F_0^{\mathrm{T}} \\ 0 & I \end{pmatrix} F(P) \begin{pmatrix} I & 0 \\ F_0 & I \end{pmatrix}. \tag{A.16}$$

If $F(P) \geq 0$, then also

$$M(P) = \begin{pmatrix} P A_{F_0} + A_{F_0}^{\mathrm{T}} P + P E E^{\mathrm{T}} P + C_{F_0}^{\mathrm{T}} C_{F_0} & P B \\ B^{\mathrm{T}} P & D^{\mathrm{T}} D \end{pmatrix} \geq 0$$

We claim $B \ker D \subseteq \ker P$. Let $u \in \mathcal{R}^m$ be such that $Du = 0$. Then we find

$$\begin{pmatrix} 0^{\mathrm{T}} & u^{\mathrm{T}} \end{pmatrix} M(P) \begin{pmatrix} 0 \\ u \end{pmatrix} = 0$$

and hence, since $M(P) \geq 0$, we find

$$M(P) \begin{pmatrix} 0 \\ u \end{pmatrix} = 0.$$

This implies $PB = 0$. Next we have to show that $\ker P$ is (C_{F_0}, A_{F_0}) invariant. Assume that $x \in \ker P \cap \ker C_{F_0}$. Then

$$x^{\mathrm{T}} \left(P A_{F_0} + A_{F_0}^{\mathrm{T}} P + P E E^{\mathrm{T}} P + C_{F_0}^{\mathrm{T}} C_{F_0} \right) x = 0.$$

Hence, by applying x to one side only, we find $P A_{F_0} = 0$ and therefore $A_{F_0} x \in \ker P$. Since $\mathcal{T}(\Sigma_{ci})$ is the smallest space with these two properties we have $\mathcal{T}(\Sigma_{ci}) \subseteq \ker P$. $\blacksquare$

Using the above we can derive the following result. Note that G_{ci} is the transfer matrix defined by (A.13).

Theorem A.6 : *Let $P \in \mathcal{R}^{n \times n}$ be symmetric. The following two statements are equivalent:*

(i) *P is a symmetric solution to the quadratic matrix inequality, i.e. we have $F(P) \geq 0$. Moreover, rank $F(P) = rank_{\mathcal{R}(s)} G_{ci}$.*

(ii) *If P is in the form (A.15) and if we define*

$$R(P_1) := P_1 A_{11} + A_{11}^{\mathrm{T}} P_1 + P_1 \left(E_1 E_1^{\mathrm{T}} - B_{11} \left(\hat{D}^{\mathrm{T}} \hat{D} \right)^{-1} B_{11}^{\mathrm{T}} \right) P_1 +$$

$$C_{21}^{\mathrm{T}} C_{21} - \left(A_{13}^{\mathrm{T}} P_1 + C_{23}^{\mathrm{T}} C_{21} \right)^{\mathrm{T}} \left(C_{23}^{\mathrm{T}} C_{23} \right)^{-1} \left(A_{13}^{\mathrm{T}} P_1 + C_{23}^{\mathrm{T}} C_{21} \right),$$

$$\text{(A.17)}$$

then we have $R(P_1) = 0$.

Let $s_0 \in \mathcal{C}$ be given. If (i) holds (or equivalently (ii)), then the following statements are equivalent:

(iii) *P satisfies*

$$rank \left(\begin{array}{c} L(P, s_0) \\ F(P) \end{array} \right) = n + rank_{\mathcal{R}(s)} G_{ci}.$$

(iv) *The matrix $Z(P_1)$ defined by*

$$Z(P_1) := A_{11} + E_1 E_1^{\mathrm{T}} P_1 - B_{11} \left(\hat{D}^{\mathrm{T}} \hat{D} \right)^{-1} B_{11}^{\mathrm{T}} P_1$$

$$- A_{13} \left(C_{23}^{\mathrm{T}} C_{23} \right)^{-1} \left(A_{13}^{\mathrm{T}} P_1 + C_{23}^{\mathrm{T}} C_{21} \right) \quad \text{(A.18)}$$

has <u>no</u> eigenvalue in s_0. $\square$

Proof : By (A.16) we have $M(P) \geq 0$ if, and only if, $F(P) \geq 0$ and we also know that these matrices have the same rank. Assume that a symmetric P satisfies $M(P) \geq 0$ and rank $M(P) = \text{rank}_{\mathcal{R}(s)} G_{ci}$. By lemma A.5 we know that in our new basis we can write P in the form (A.15). If we also use the decompositions (A.2) and (A.4) for the other matrices we find that $M(P)$ is equal to

$$\left(\begin{array}{ccccc} P_1 A_{11} + A_{11}^{\mathrm{T}} P_1 + C_{21}^{\mathrm{T}} C_{21} + P_1 E_1 E_1^{\mathrm{T}} P_1 & 0 & P_1 A_{13} + C_{21}^{\mathrm{T}} C_{23} & P_1 B_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A_{13}^{\mathrm{T}} P_1 + C_{23}^{\mathrm{T}} C_{21} & 0 & \boxed{\begin{array}{cc} C_{23}^{\mathrm{T}} C_{23} & 0 \\ 0 & \hat{D}^{\mathrm{T}} \hat{D} \end{array}} & 0 \\ B_{11}^{\mathrm{T}} P_1 & 0 & & & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

According to lemma A.4 the rank of this matrix equals the rank of the en-
circled matrix. Thus the Schur complement of this matrix must be equal
to 0. Since this condition exactly yields the algebraic Riccati equation
$R(P_1) = 0$ where R is defined by (A.17), we find that P_1 is a solution of
$R(P_1) = 0$.

Conversely, if P_1 is a solution of $R(P_1) = 0$, then the Schur comple-
ment of the encircled submatrix of the above matrix is 0. Therefore it
satisfies the matrix inequality $M(P) \geq 0$ and the rank of the matrix is
equal to the normal rank of G. Hence P given by (A.15) satisfies the
required properties.

Now assume that (i) or (ii) holds. We shall prove the equivalence
of (iii) and (iv). Define $Z(P_1)$ by (A.18). We shall apply the following
unimodular transformation to the matrix in (iii):

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & F_0^{\mathrm{T}} \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} L(P,s) \\ F(P) \end{pmatrix} \begin{pmatrix} I & 0 \\ F_0^{\mathrm{T}} & I \end{pmatrix}.$$

Using the decomposition (A.2) the latter matrix turns out to be equal to

$$\begin{pmatrix}
sI - A_{11} - E_1 E_1^{\mathrm{T}} P_1 & 0 & -A_{13} & -B_{11} & 0 \\
-A_{21} - E_2 E_1^{\mathrm{T}} P_1 & sI - A_{22} & -A_{23} & -B_{21} & -B_{22} \\
-A_{31} - E_3 E_1^{\mathrm{T}} P_1 & -A_{32} & sI - A_{33} & -B_{31} & -B_{32} \\
P_1 A_{11} + A_{11}^{\mathrm{T}} P_1 + C_{21}^{\mathrm{T}} C_{21} + P_1 E_1 E_1^{\mathrm{T}} P_1 & 0 & P_1 A_{13} + C_{21}^{\mathrm{T}} C_{23} & P_1 B_{11} & 0 \\
0 & 0 & 0 & 0 & 0 \\
A_{13}^{\mathrm{T}} P_1 + C_{23}^{\mathrm{T}} C_{21} & 0 & C_{23}^{\mathrm{T}} C_{23} & 0 & 0 \\
B_{11}^{\mathrm{T}} P_1 & 0 & 0 & \hat{D}^{\mathrm{T}} \hat{D} & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

By using Schur complements we can get the Riccati equation $R(P_1) = 0$
in the 4,1 position and the matrix $Z(P_1)$ in the 1,1 position of the above
matrix. Furthermore, since $\hat{D}^{\mathrm{T}} \hat{D}$ is invertible we can make the 2,4 and
3,4 blocks equal to zero by a unimodular transformation. Since P_1 is a
solution of $R(P_1) = 0$, the 4,1 block becomes 0. Thus we find that the
above matrix is unimodularly equivalent to

$$\begin{pmatrix}
sI - Z(P_1) & 0 & 0 & 0 & 0 \\
* & sI - A_{22} & -A_{23} & 0 & -B_{22} \\
* & -A_{32} & sI - A_{33} & 0 & -B_{32} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & C_{23}^{\mathrm{T}} C_{23} & 0 & 0 \\
0 & 0 & 0 & \hat{D} & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

where $*$ denotes matrices which are unimportant for this argument.

Now, by lemma A.2, the encircled matrices together form the system matrix of a strongly controllable system. Hence this system matrix is unimodularly equivalent to a constant matrix (I $\quad$ 0), where I denotes the identity matrix of appropriate size. Therefore we can make the 2,1 and 3,1 blocks zero by a unimodular transformation.

Thus after reordering we find,

$$
\begin{pmatrix} L(P,s) \\ F(P) \end{pmatrix} \sim
\begin{pmatrix}
\boxed{sI - Z(P_1)} & 0 & 0 & 0 & 0 \\
0 & \boxed{\begin{matrix} sI - A_{22} & -A_{23} & -B_{22} \\ -A_{32} & sI - A_{33} & -B_{32} \\ 0 & C_{23}^{\mathrm{T}} C_{23} & 0 \end{matrix}} & & & \begin{matrix} -B_{22} \\ -B32 \\ 0 \end{matrix} \\
0 & 0 & 0 & 0 & \boxed{\hat{D}^{\mathrm{T}}\hat{D}} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

It follows that the matrix on the left has rank $n + \mathrm{rank}_{\mathcal{R}(s)} G$ for some $s_0 \in \mathcal{C}$ if, and only if, s_0 is not an eigenvalue of $Z(P_1)$. This proves that (iii) and (iv) are equivalent. ∎

Corollary A.7 : *If there exists a symmetric matrix P such that $F(P) \geq 0$ and moreover:*

(i) rank $F(P) = rank_{\mathcal{R}(s)} G_{ci}$,

(ii) rank $\begin{pmatrix} L(P,s) \\ F(P) \end{pmatrix} = n + rank_{\mathcal{R}(s)} G_{ci}, \qquad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+,$

then this matrix is uniquely defined by the above inequality and the corresponding two rank conditions. □

Proof : By lemma A.6 a solution P must be of the form (A.15) where P_1 is a solution of the algebraic Riccati equation $R(P_1) = 0$ such that $Z(P_1)$ is asymptotically stable. Denote the Hamiltonian matrix corresponding to this algebraic Riccati equation by H. Then we have:

$$
H \begin{pmatrix} I \\ P_1 \end{pmatrix} = \begin{pmatrix} I \\ P_1 \end{pmatrix} Z(P_1).
$$

Since a Hamiltonian matrix has the property that λ is an eigenvalue if, and only if, $-\lambda$ is an eigenvalue of H we know that an n-dimensional H-invariant subspace $\mathcal{W}$ such that $H \mid \mathcal{W}$ is asymptotically stable, must be unique. This implies that P_1 is unique and hence P is also unique. ∎

We also have the following corollary of theorem A.6. The proof of this result is strongly related to the proof that A_U is stable in lemma 3.11. Only we have a different and notationally more difficult Riccati equation.

Corollary A.8 : *If a matrix $P \geq 0$ exists such that $F(P) \geq 0$ and moreover:*

(i) $rank\, F(P) = rank_{\mathcal{R}(s)}\, G_{ci}$,

(ii) $rank \begin{pmatrix} L(P,s) \\ F(P) \end{pmatrix} = n + rank_{\mathcal{R}(s)}\, G_{ci}, \qquad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+,$

then P is in the form (A.15) and P_1 is such that the matrix

$$A_{11} - B_{11} \left(\hat{D}^{\mathrm{T}} \hat{D} \right)^{-1} B_{11}^{\mathrm{T}} P_1 - A_{13} \left(C_{23}^{\mathrm{T}} C_{23} \right)^{-1} \left(A_{13}^{\mathrm{T}} P_1 + C_{23}^{\mathrm{T}} C_{21} \right) \quad \text{(A.19)}$$

is asymptotically stable. □

Proof : By theorem A.6 we know P_1 is such that $R(P_1) = 0$ and $Z(P_1)$ is asymptotically stable where R is defined by (A.17) and Z is defined by (A.18). Denote the matrix in (A.19) by $\tilde{A}$. It is easy to check that:

$$P_1 \tilde{A} + \tilde{A}^{\mathrm{T}} P_1 + P_1 N_1 P_1 + N_2 = 0,$$

where

$$N_1 := \quad E_1 E_1^{\mathrm{T}} + B_{11} \left(\hat{D}^{\mathrm{T}} \hat{D} \right)^{-1} B_{11}^{\mathrm{T}} + A_{13} \left(C_{23}^{\mathrm{T}} C_{23} \right)^{-1} A_{13}^{\mathrm{T}} \quad \geq 0,$$

$$N_2 := \quad C_{21}^{\mathrm{T}} \left(I - C_{23} \left(C_{23}^{\mathrm{T}} C_{23} \right)^{-1} C_{23}^{\mathrm{T}} \right) C_{21} \quad \geq 0.$$

Assume that λ is an eigenvalue of $\tilde{A}$ with corresponding eigenvector $x \neq 0$. Then

$$2\mathrm{Re}\, \lambda x^{\mathrm{T}} P_1 x = -x^{\mathrm{T}} \left(P_1 N_1 P_1 + N_2 \right) x.$$

Since $P_1 \geq 0$, $N_1 \geq 0$ and $N_2 \geq 0$ this implies that if $\mathrm{Re}\, \lambda \geq 0$, then $N_1 P_1 x = 0$ which implies that $E_1 E_1^{\mathrm{T}} P_1 x = 0$. Thus, if $\mathrm{Re}\, \lambda \geq 0$ we find

$$\lambda x = \tilde{A}x = \left(\tilde{A} + E_1 E_1^{\mathrm{T}} P_1 \right) x = Z(P_1)x.$$

However, since $Z(P_1)$ is asymptotically stable, this yields a contradiction. Thus we have established that $\mathrm{Re}\, \lambda < 0$ which in turn yields that $\tilde{A}$ is asymptotically stable. ∎

Appendix B

Proofs concerning the system transformations

In this appendix we shall prove two key lemmas of chapter 5 for which the proofs are rather technical and are therefore deferred to this appendix.

Throughout this appendix we shall assume that we have chosen the bases described in appendix A with D replaced by D_2 and C replaced by C_2. Thus we know the matrices have the special form as given in (A.2) and (A.4). In case D_2 is injective we show what the results look like without the use of the bases of appendix A.

B.1 Proof of lemma 5.4

We first have to do some preparatory work.

Let the matrix P satisfy the conditions of lemma 5.2 part (i). Hence we know that in our new bases P has the form (A.15). It is easily shown that it is sufficient to prove lemma 5.4 for one specific choice of $C_{2,P}$ and D_P. We define the following matrices:

$$
C_{2,P} \;\; := \;\; \begin{pmatrix} \hat{D}\left(\hat{D}^{\mathrm{T}}\hat{D}\right)^{-1} B_{11}^{\mathrm{T}} P_1 + C_{11} & C_{12} & C_{13} \\[2mm] C_{23}\left(C_{23}^{\mathrm{T}} C_{23}\right)^{-1}\left(A_{13}^{\mathrm{T}} P_1 + C_{23}^{\mathrm{T}} C_{21}\right) & 0 & C_{23} \end{pmatrix}, \;\; \text{(B.1)}
$$

$$
D_P \;\; := \;\; \begin{pmatrix} \hat{D} & 0 \\ 0 & 0 \end{pmatrix} (\,= D_2)\,. \qquad\qquad \text{(B.2)}
$$

By writing down $F(P)$ in terms of the chosen bases and by using the fact that P_1 satisfies the algebraic Riccati equation $R(P_1) = 0$, where $R(P_1)$ is defined by (A.17), it can be verified, after some effort, that these matrices

indeed satisfy (5.3). In case D_2 is injective we define the matrices $C_{2,P}$ and D_P without making use of the bases of appendix A by

$$
\begin{aligned}
C_{2,P} &:= D_2 \left(D_2^{\mathrm{T}} D_2\right)^{-1} \left(B^{\mathrm{T}} P + D_2^{\mathrm{T}} C_2\right), &\text{(B.3)} \\
D_P &:= D_2. &\text{(B.4)}
\end{aligned}
$$

Next, we define the following matrices:

$$
\begin{aligned}
\tilde{A} &:= A_{11} - A_{13} \left(C_{23}^{\mathrm{T}} C_{23}\right)^{-1} \left(A_{13}^{\mathrm{T}} P_1 + C_{23}^{\mathrm{T}} C_{21}\right) \\
&\qquad\qquad\qquad\qquad\qquad - B_{11} \left(\hat{D}^{\mathrm{T}} \hat{D}\right)^{-1} B_{11}^{\mathrm{T}} P_1, \\
\tilde{C}_1 &:= -\left(\hat{D}^{\mathrm{T}}\right)^{-1} B_{11}^{\mathrm{T}} P_1, \\
\tilde{C}_2 &:= C_{21} - C_{23} \left(C_{23}^{\mathrm{T}} C_{23}\right)^{-1} \left(A_{13}^{\mathrm{T}} P_1 + C_{23}^{\mathrm{T}} C_{21}\right), \\
\tilde{B}_{11} &:= B_{11} \hat{D}^{-1}, \\
\tilde{B}_{12} &:= A_{13} \left(C_{23}^{\mathrm{T}} C_{23}\right)^{-1} C_{23}^{\mathrm{T}} - P_1^{\dagger} C_{21}^{\mathrm{T}} \left(I - C_{23} \left(C_{23}^{\mathrm{T}} C_{23}\right)^{-1} C_{23}^{\mathrm{T}}\right),
\end{aligned}
$$

where † denotes the Moore–Penrose inverse. We now define the following system:

$$
\Sigma_U : \left\{
\begin{aligned}
\dot{x}_U &= \tilde{A} x_U + \left(\begin{array}{cc} \tilde{B}_{11} & \tilde{B}_{12} \end{array}\right) u_U + E_1 w_U, \\
y_U &= -E_1^{\mathrm{T}} P_1 x_U + \qquad\qquad\qquad\qquad w_U, \\
z_U &= \left(\begin{array}{c} \tilde{C}_1 \\ \tilde{C}_2 \end{array}\right) x_U + \left(\begin{array}{cc} I & 0 \\ 0 & I \end{array}\right) u_U.
\end{aligned}
\right.
\qquad\text{(B.5)}
$$

In case D_2 and D_1 are injective and surjective respectively we can define the above system Σ_U without the bases of appendix A by

$$
\Sigma_U : \left\{
\begin{aligned}
\dot{x}_U &= \tilde{A} x_U + B(D_2^{\mathrm{T}} D_2)^{-1} D_2^{\mathrm{T}} u_U + E w_U, \\
y_U &= -E^{\mathrm{T}} P x_U + \qquad\qquad\qquad w_U, \\
z_U &= \tilde{C}_2 x_U + \qquad\qquad u_U,
\end{aligned}
\right.
\qquad\text{(B.6)}
$$

where

$$
\begin{aligned}
\tilde{A} &:= A - B(D_2^{\mathrm{T}} D_2)^{-1}(B^{\mathrm{T}} P + D_2^{\mathrm{T}} C_2), \\
\tilde{C}_2 &:= C_2 - D_2(D_2^{\mathrm{T}} D_2)^{-1}(B^{\mathrm{T}} P + D_2^{\mathrm{T}} C_2).
\end{aligned}
$$

We have the following properties of the system Σ_U:

Lemma B.1 : *The system Σ_U is inner. Let U denote the transfer matrix of Σ_U. If we decompose U:*

$$U \begin{pmatrix} w_U \\ u_U \end{pmatrix} =: \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} w_U \\ u_U \end{pmatrix} = \begin{pmatrix} z_U \\ y_U \end{pmatrix},$$

compatible with the sizes of u_U, w_U, y_U, and z_U then we have $U_{21}^{-1} \in H_\infty$ and U_{22} is strictly proper. □

Proof : By corollary A.8 we know that $\tilde{A}$ is asymptotically stable and hence Σ_U is internally stable. Moreover by theorem A.6 the matrix $Z(P_1) = \tilde{A} + E_1 E_1^{\mathrm{T}} P_1$ is asymptotically stable and therefore we have $U_{21}^{-1} \in H_\infty$. The fact that U_{22} is strictly proper is trivial. It is easy to check, using lemma A.6 part (ii), that P_1 is the observability gramian of Σ_U. Moreover we have

$$\left(\begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \\ \begin{pmatrix} 0 & 0 \end{pmatrix} \end{pmatrix} \right) \begin{pmatrix} -E_1^{\mathrm{T}} P_1 \\ \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} \tilde{B}_{11}^{\mathrm{T}} \\ \tilde{B}_{12}^{\mathrm{T}} \\ E_1^{\mathrm{T}} \end{pmatrix} \end{pmatrix} P_1 = 0.$$

This can be checked by simply writing out and using the fact that

$$\ker P_1 \subset \ker \left(I - C_{23} \left(C_{23}^{\mathrm{T}} C_{23} \right)^{-1} C_{23}^{\mathrm{T}} \right) C_{21}.$$

The result that Σ_U is inner then follows by applying lemma 2.10. ∎

Proof of lemma 5.4 :   We have our special choice of $C_{2,P}$ and D_P given by (B.1) and (B.2). As already noted, taking this special choice for C_P and D_P is not essential. We shall first compare the following two systems:

(B.7)
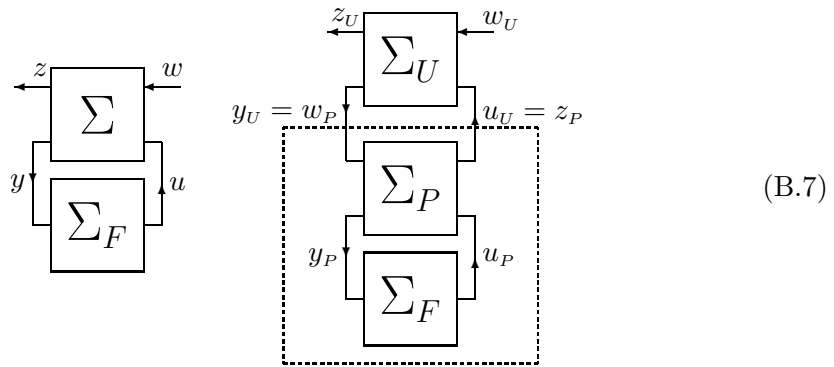
The system on the left is the same as the system on the left in (5.5) and the system on the right is described by the system (B.5) interconnected with the system on the right in (5.5). We decompose the state of Σ, x into x_1, x_2 and x_3 according to the choice of bases described in appendix A and decompose the state of Σ_P into $x_{1,P}, x_{2,P}, x_{3,P}$ with respect to the same basis (note that Σ and Σ_P have the same state space $\mathcal{R}^n$). Writing out all the differential equations using the decompositions of the matrices given in (A.2),(A.4) we find the following realization for the system on the right in (B.7):

$$\begin{cases} \begin{pmatrix} \dot{x}_U - \dot{x}_{1,P} \\ \dot{x}_P \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \tilde{A} + E_1 E_1^{\mathrm{T}} P_1 & 0 & 0 \\ * & A + BNC_1 & BM \\ * & LC_1 & K \end{pmatrix} \begin{pmatrix} x_U - x_{1,P} \\ x_P \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ E + BND_1 \\ LD_1 \end{pmatrix} w_U \\[4mm] z = \begin{pmatrix} * & C_2 + D_2 NC_1 & D_2 M \end{pmatrix} \begin{pmatrix} x_U - x_{1,P} \\ x_P \\ p \end{pmatrix} + D_2 ND_1 w_U \end{cases}$$

The * denotes matrices which are unimportant for this argument. If we also derive the system equations for the system on the left in (B.7) we see immediately that, since $\tilde{A} + E_1 E_1^{\mathrm{T}} P_1$ is asymptotically stable, the system on the left is internally stable if, and only if, the system on the right is internally stable. Moreover, if we take zero initial conditions and both systems have the same input w, then we have $z = z_U$, i.e. the input–output behaviour of both systems is identical. Hence the system on the left has H_∞ norm less than 1 if, and only if, the system on the right has H_∞ norm less than 1.

By lemma B.1 we may apply lemma 2.12 to the system on the right in (B.7) and hence we find that the closed-loop system is internally stable and has H_∞ norm less than 1 if, and only if, the dashed system is internally stable and has H_∞ norm less than 1.

Since the dashed system is exactly the system on the right in (5.5) and the system on the left in (B.7) is exactly equal to the system on the left in (5.5) we have completed the proof.   ∎

B.2   Proof of lemma 5.5

We are now going to prove lemma 5.5. In fact, we shall prove the dual version of this lemma since this is much more convenient for us. Otherwise we would have to introduce a decomposition dual to the one used in

appendix A and based on the weakly unobservable subspace. We first factorize $G(Q)$:

$$G(Q) := \begin{pmatrix} E_Q \\ D_Q \end{pmatrix} \begin{pmatrix} E_Q^{\mathrm{T}} & D_Q^{\mathrm{T}} \end{pmatrix}.$$

Define $A_Q := A + QC_2^{\mathrm{T}}C_2$ and $B_Q := B + QC_2^{\mathrm{T}}D_2$ and the system:

$$\Sigma_Q : \begin{cases} \dot{x}_Q = A_Q x_Q + B_Q u_Q + E_Q w_Q, \\ y_Q = C_1 x_Q \qquad\qquad + D_Q w_Q, \\ z_Q = C_2 x_Q + D_2 u_Q. \end{cases} \tag{B.8}$$

First of all, we know that Σ_F stabilizes Σ if, and only if, Σ_F^{T} stabilizes Σ^{T}. Moreover, we know that $\|G\|_\infty = \|G^{\mathrm{T}}\|_\infty$. Hence we can derive the following dualized version of lemma 5.4 for this dual system Σ_Q:

Lemma B.2 : *Let Q satisfy lemma 5.2 part (ii). Moreover let an arbitrary dynamic compensator Σ_F be given, described by (2.4). Let the following two systems be given where the system on the left is the interconnection of (5.1) and (2.4) and the system on the right is the interconnection of (B.8) and (2.4).*
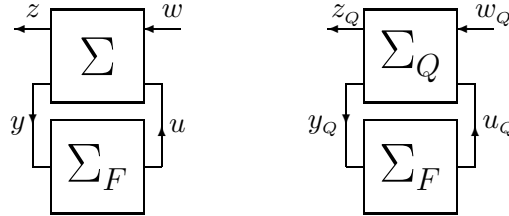
Then the following statements are equivalent :

(*i*) *The system on the left is internally stable and its transfer matrix has H_∞ norm less than 1.*

(*ii*) *The system on the right is internally stable and its transfer matrix has H_∞ norm less than 1.* □

We shall now investigate what the matrices appearing in the matrix inequality and the rank conditions look like for this new system Σ_Q:

$$\tilde{F}(X) := \begin{pmatrix} A_Q^{\mathrm{T}}X + XA_Q + C_2^{\mathrm{T}}C_2 + XE_Q E_Q^{\mathrm{T}}X & XB_Q + C_2^{\mathrm{T}}D_2 \\ B_Q^{\mathrm{T}}X + D_2^{\mathrm{T}}C_2 & D_2^{\mathrm{T}}D_2 \end{pmatrix},$$

$$\tilde{G}(Y) := \left(\begin{array}{cc} A_Q Y + Y A_Q^{\mathrm{T}} + E_Q E_Q^{\mathrm{T}} + Y C_2^{\mathrm{T}} C_2 Y & Y C_1^{\mathrm{T}} + E_Q D_Q^{\mathrm{T}} \\ C_1 Y + D_Q E_Q^{\mathrm{T}} & D_Q D_Q^{\mathrm{T}} \end{array} \right),$$

$$\tilde{L}(X, s) := \left(\begin{array}{cc} sI - A_Q - E_Q E_Q^{\mathrm{T}} X & -B_Q \end{array} \right),$$

$$\tilde{M}(Y, s) := \left(\begin{array}{c} sI - A_Q - Y C_2^{\mathrm{T}} C_2 \\ -C_1 \end{array} \right).$$

Moreover, we define two new transfer matrices:

$$\begin{aligned} \tilde{G}_{ci}(s) &:= C_2 \left(sI - A_Q \right)^{-1} B_Q + D_2, \\ \tilde{G}_{di}(s) &:= C_1 \left(sI - A_Q \right)^{-1} E_Q + D_Q. \end{aligned}$$

Using these definitions we have the following result:

Lemma B.3 :   *Let Q satisfy lemma 5.2 part (ii). Then $Y = 0$ is the unique solution of the quadratic matrix inequality $\tilde{G}(Y) \geq 0$ satisfying the following rank conditions*

(i) *rank* $\tilde{G}(Y) = rank_{\mathcal{R}(s)} \tilde{G}_{di}$,

(ii) *rank* $\left(\begin{array}{cc} \tilde{M}(Y, s) & \tilde{G}(Y) \end{array} \right) = n + rank_{\mathcal{R}(s)} \tilde{G}_{di}$, $\forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$.   □

Proof : It is trivial to check that $\tilde{G}(0) \geq 0$. Moreover, since $\tilde{G}(0) = G(Q)$ and $\tilde{M}(0, s) = M(Q, s)$ it remains to be shown that $\tilde{G}_{di}$ and G_{di} have the same normal rank. We have

$$\begin{aligned} \mathrm{rank}_{\mathcal{R}(s)} \tilde{G}_{di} &= \mathrm{rank}_{\mathcal{R}(s)} \left(\begin{array}{cc} sI - A_Q & E_Q \\ -C_1 & D_Q \end{array} \right) - n \\[2mm] &= \mathrm{rank}_{\mathcal{R}(s)} \left(\begin{array}{ccc} sI - A_Q & E_Q E_Q^{\mathrm{T}} & D_Q E_Q^{\mathrm{T}} \\ -C_1 & D_Q E_Q^{\mathrm{T}} & D_Q D_Q^{\mathrm{T}} \end{array} \right) - n \\[2mm] &= \mathrm{rank}_{\mathcal{R}(s)} \left(\begin{array}{cc} M(Q, s) & G(Q) \end{array} \right) - n \\[2mm] &= \mathrm{rank}_{\mathcal{R}(s)} G_{di}. \end{aligned}$$

The matrix Y is unique by the dualized version of corollary A.7. This is exactly what we had to prove.   ■

Lemma B.4 : *There is a solution X of the matrix inequality $\tilde{F}(X) \geq 0$ satisfying the following two rank conditions:*

(i) $\operatorname{rank} \tilde{F}(X) = \operatorname{rank}_{\mathcal{R}(s)} \tilde{G}_{ci}$,

(ii) $\operatorname{rank} \begin{pmatrix} \tilde{L}(X,s) \\ \tilde{F}(X) \end{pmatrix} = n + \operatorname{rank}_{\mathcal{R}(s)} \tilde{G}_{ci}, \qquad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+,$

if, and only if, $I - PQ$ is invertible. Moreover in that case the solution is unique and is given by $X = (I - PQ)^{-1} P$. We have $X \geq 0$ if, and only if:

$$\rho\left(PQ\right) < 1. \qquad\qquad\qquad\qquad\qquad \square$$

Proof : We first make a transformation on $\tilde{F}(X)$:

$$
F_{tr}(X) \;\; := \;\; \begin{pmatrix} I & (I+XQ)\,F_0^{\mathrm{T}} \\ 0 & I \end{pmatrix} \tilde{F}(X) \begin{pmatrix} I & 0 \\ F_0\,(I+QX) & I \end{pmatrix}
$$

$$
= \;\; \begin{pmatrix} \bar{A}^{\mathrm{T}}X + X\bar{A} + \bar{C}_2^{\mathrm{T}}\bar{C}_2 + XMX & XB \\ B^{\mathrm{T}}X & D_2^{\mathrm{T}}D_2 \end{pmatrix}
$$

where,

$$
\begin{aligned}
\bar{A} &:= A + BF_0 + Q\,(C_2 + D_2 F_0)^{\mathrm{T}}\,(C_2 + D_2 F_0)\,, \\
\bar{C}_2 &:= C_2 + D_2 F_0, \\
M &:= (A + BF_0)\,Q + Q\,(A^{\mathrm{T}} + F_0^{\mathrm{T}}B^{\mathrm{T}}) + EE^{\mathrm{T}} + Q\bar{C}_2^{\mathrm{T}}\bar{C}_2 Q,
\end{aligned}
$$

and F_0 as defined in (A.1). We also transform the second matrix appearing in the rank conditions:

$$
W(X,s) \;\; := \;\; \begin{pmatrix} I & 0 & -QF_0^{\mathrm{T}} \\ 0 & I & (I+XQ)\,F_0^{\mathrm{T}} \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \tilde{L}(X,s) \\ \tilde{F}(X) \end{pmatrix} \begin{pmatrix} I & 0 \\ F_0\,(I+QX) & I \end{pmatrix}
$$

$$
= \;\; \begin{pmatrix} sI - \bar{A} - MX & -B \\ \bar{A}^{\mathrm{T}}X + X\bar{A} + \bar{C}_2^{\mathrm{T}}\bar{C}_2 + XMX & XB \\ B^{\mathrm{T}}X & D_2^{\mathrm{T}}D_2 \end{pmatrix}.
$$

We have the following equality:

$$
\begin{aligned}
\text{rank}_{\mathcal{R}(s)} \tilde{G}_{ci} &= \text{rank}_{\mathcal{R}(s)} \begin{pmatrix} sI - A_Q & -B_Q \\ C_2 & D_2 \end{pmatrix} - n \\
&= \text{rank}_{\mathcal{R}(s)} \begin{pmatrix} I & QC_2^{\mathrm{T}} \\ 0 & I \end{pmatrix} \begin{pmatrix} sI - A_Q & -B_Q \\ C_2 & D_2 \end{pmatrix} - n \\
&= \text{rank}_{\mathcal{R}(s)} \begin{pmatrix} sI - A & -B \\ C_2 & D_2 \end{pmatrix} - n = \text{rank}_{\mathcal{R}(s)} G_{ci}.
\end{aligned}
$$

Therefore the conditions that $X \geq 0$ has to satisfy can be reformulated as:

(*i*) $F_{tr}(X) \geq 0$

(*ii*) rank $F_{tr}(X) = \text{rank}_{\mathcal{R}(s)} G_{ci}$

(*iii*) rank $W(X, s) = \text{rank}_{\mathcal{R}(s)} G_{ci} + n, \qquad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+.$

Moreover, we note that $\mathcal{T}(A, B, C_2, D_2) = \mathcal{T}(\bar{A}, B, \bar{C}_2, D_2)$. This can be shown by using the fact that the new system is obtained by a state feedback and an output injection (note that $B = B + Q(C_2 + D_2 F_0)^{\mathrm{T}} D_2$) and the well-known fact that the strongly controllable subspace is invariant under state feedback and output injection. This can be easily shown using the algorithm (2.15). We now choose the bases from appendix A where again C is replaced by C_2 and D is replaced by D_2. By lemma A.5 we know that if X exists, then it will have the form:

$$
X = \begin{pmatrix} X_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{B.9}
$$

for some positive semi-definite matrix X_1. Note that there is small difference since M is not necessarily positive semi-definite but it is easy to see from the proof of lemma A.5 that this difference is not important. We use this decomposition for X and the corresponding decompositions for P and Q:

$$
P = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix}.
$$

Together with the decompositions for the other matrices as given in (A.2) and (A.4) we can decompose $F_{tr}(X)$ correspondingly:

$$\begin{pmatrix} X_1\tilde{A}_{11} + \tilde{A}_{11}^{\mathrm{T}}X_1 + C_{21}^{\mathrm{T}}C_{21} + X_1 M_{11} X_1 & 0 & X_1\tilde{A}_{13} + C_{21}^{\mathrm{T}}C_{23} & X_1 B_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \tilde{A}_{13}^{\mathrm{T}}X_1 + C_{23}^{\mathrm{T}}C_{21} & 0 & C_{23}^{\mathrm{T}}C_{23} & 0 & 0 \\ B_{11}^{\mathrm{T}}X_1 & 0 & 0 & \hat{D}^{\mathrm{T}}\hat{D} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\begin{aligned} \tilde{A}_{11} &:= A_{11} + Q_{11} C_{21}^{\mathrm{T}} C_{21} + Q_{13} C_{23}^{\mathrm{T}} C_{21}, \\ \tilde{A}_{13} &:= A_{13} + Q_{11} C_{21}^{\mathrm{T}} C_{23} + Q_{13} C_{23}^{\mathrm{T}} C_{23}, \\ M_{11} &:= A_{11} Q_{11} + A_{13} Q_{13}^{\mathrm{T}} + Q_{11} A_{11}^{\mathrm{T}} + Q_{13} A_{13}^{\mathrm{T}} + E_1 E_1^{\mathrm{T}} \\ &\quad + Q_{11} C_{21}^{\mathrm{T}} \left(C_{21} Q_{11} + C_{23} Q_{13} \right) + Q_{13} C_{23}^{\mathrm{T}} \left(C_{21} Q_{11} + C_{23} Q_{13} \right). \end{aligned}$$

The rank condition: rank $F_{tr}(X) = \mathrm{rank}_{\mathcal{R}(s)} G_{ci}$ is, according to lemma A.4, equivalent with the condition that the rank of the above matrix is equal to the rank of the submatrix:

$$\begin{pmatrix} C_{23}^{\mathrm{T}} C_{23} & 0 \\ 0 & \hat{D}^{\mathrm{T}}\hat{D} \end{pmatrix}.$$

Therefore the Schur complement with respect to this submatrix should be zero. This implies that if we define:

$$\begin{aligned} \tilde{R}(X_1) &:= X_1\tilde{A}_{11} + \tilde{A}_{11}^{\mathrm{T}}X_1 + C_{21}^{\mathrm{T}}C_{21} \\ &\quad + X_1 \left(M_{11} - B_{11} \left(\hat{D}^{\mathrm{T}}\hat{D} \right)^{-1} B_{11}^{\mathrm{T}} \right) X_1 \\ &\quad - \left(X_1\tilde{A}_{13} + C_{21}^{\mathrm{T}}C_{23} \right) (C_{23}^{\mathrm{T}}C_{23})^{-1} \left(\tilde{A}_{13}^{\mathrm{T}}X_1 + C_{23}^{\mathrm{T}}C_{21} \right) \end{aligned}$$

then X_1 should satisfy $\tilde{R}(X_1) = 0$. Moreover, if we decompose $W(X, s)$ correspondingly, then we can show by using elementary row and column operations that for any matrix X in the form (B.9), where X_1 satisfies $\tilde{R}(X_1) = 0$, and for all $s \in \mathcal{C}$, the matrix $W(X, s)$ has the same rank as

the following matrix:

$$\begin{pmatrix} sI - \tilde{Z}(X_1) & 0 & 0 & 0 & 0 \\ * & sI - A_{22} & -A_{23} & 0 & -B_{22} \\ * & -A_{32} & sI - A_{33} & 0 & -B_{32} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{B.10}$$

where

$$\begin{aligned} \tilde{Z}(X_1) \quad &:= \quad \tilde{A}_{11} + M_{11}X_1 - B_{11}\left(\hat{D}^{\mathrm{T}}\hat{D}\right)^{-1}B_{11}^{\mathrm{T}}X_1 \\ &\qquad\qquad -\tilde{A}_{13}\left(C_{23}^{\mathrm{T}}C_{23}\right)^{-1}\left(\tilde{A}_{13}^{\mathrm{T}}X_1 + C_{23}^{\mathrm{T}}C_{21}\right). \end{aligned}$$

The matrix:

$$\begin{pmatrix} sI - A_{22} & -A_{23} & -B_{22} \\ -A_{32} & sI - A_{33} & -B_{32} \\ 0 & I & 0 \end{pmatrix},$$

has full row rank for all $s \in \mathcal{C}$ by lemma A.2 part (ii) and lemma 2.8. Hence the rank of the matrix (B.10) is $n + \mathrm{rank}_{\mathcal{R}(s)}G_{ci}$ for all $s \in \mathcal{C}^+ \cup \mathcal{C}^0$ if, and only if, the matrix $\tilde{Z}(X_1)$ is asymptotically stable. Using this we can now reformulate the conditions that $X_1 \geq 0$ has to satisfy:

(*i*) $\tilde{R}(X_1) = 0$

(*ii*) $\tilde{Z}(X_1)$ is asymptotically stable.

In other words X_1 should be the positive semi-definite stabilizing solution of the algebraic Riccati equation $\tilde{R}(X_1) = 0$. Denote the Hamiltonian corresponding to this ARE by H_{new}. We know that P_1 is the stabilizing solution of the algebraic Riccati equation $R(P_1) = 0$ as given by (A.17). Denote the Hamiltonian corresponding to this algebraic Riccati equation by H_{old}. Then it can be checked that:

$$H_{old} = \begin{pmatrix} I & Q_{11} \\ 0 & I \end{pmatrix} H_{new} \begin{pmatrix} I & -Q_{11} \\ 0 & I \end{pmatrix}. \tag{B.11}$$

Since P_1 is the stabilizing solution of the Riccati equation corresponding to the Hamiltonian H_{old} we know that the modal subspace of H_{old} corresponding to the open left half plane is given by:

$$\mathcal{X}_g(H_{old}) = \mathrm{Im} \begin{pmatrix} I \\ P_1 \end{pmatrix}. \tag{B.12}$$

Combining (B.11) and (B.12) we find:

$$\mathcal{X}_g(H_{new}) = \mathrm{Im} \begin{pmatrix} I & -Q_{11} \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ P_1 \end{pmatrix} = \mathrm{Im} \begin{pmatrix} I - Q_{11}P_1 \\ P_1 \end{pmatrix}.$$

Therefore we know that a stabilizing solution to the algebraic Riccati equation $\tilde{R}(X_1) = 0$ exists if, and only if, $I - Q_{11}P_1$ is invertible and in this case the solution is given by $X_1 = P_1 (I - Q_{11}P_1)^{-1}$. This implies that $X = P (I - QP)^{-1} = (I - PQ)^{-1} P$. The requirement $X \geq 0$ is satisfied if, and only if, $\rho(PQ) < 1$, which can be checked straightforwardly. This completes the proof. ∎

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