ISOSPECTRAL VIBRATING SYSTEMS

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Abstract

Two vibrating systems are said to be *isospectral* if they have the same natural frequencies. This paper reviews some recent results on isospectral conservative (i.e., undamped) discrete vibrating systems. The paper centres around two ways of creating isospectral systems: by QR factorisation with a shift, and by using the concept of isospectral flow. Both these procedures are illustrated by using FEM models.

Keywords: vibration, isospectral, QR factorisation, isospectral flow

1. Introduction

An undamped vibrating system has certain frequencies, called *natural* frequencies, at which it can vibrate freely, without the application of forces. An actual physical system has theoretically an infinity of such frequencies. A model of such a system may be either *continuous* or *discrete*, having respectively an infinity or a finite number of natural frequencies; in this paper we consider only discrete systems. Two systems with the same set of natural frequencies, *spectrum*, are said to be *isospectral*. In general, the spectrum of a system *mirrors* the system, but does not *specify* it completely: there can be many systems, an *isospectral family*, with the same spectrum. There are two broad classifications of problems relating to a system and its spectrum. In *inverse* problems, one attempts to construct a system with a given *spectrum*; in *isospectral* problems, one attempts to find another system, maybe a family of systems, having the same spectrum as a *given* system. In some ways, isospectral problems are easier than inverse problems: at least one is sure that there exists at least one system, the given system, with the specified spectrum; this is not always the case with inverse problems. For an in-depth discussion of inverse and isospectral problems, see Gladwell (2004).

Modelling of a physical system is usually done by means of some *finite element method* (FEM): the system is treated as a set of *elements*, connected in some way. Each element is specified by a set of generalised displacements, an *element stiffness matrix* $\mathbf{K}_e$, and an *element mass (inertia) matrix* $\mathbf{M}_e$. In the process of assembling the elements, one constructs overall or *global* stiffness and mass matrices, $\mathbf{K}$ and $\mathbf{M}$, and assembles the element displacements into a displacement vector $\mathbf{u}$. The natural frequencies of the system appear as (the square roots, $\omega = \lambda^{\frac{1}{2}}$, of) the eigenvalues $(\lambda_i)_1^n$ of the generalised eigenvalue problem

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{u} = \mathbf{0}. \quad (1)$$

A system is thus defined by a pair of matrices $(\mathbf{K}, \mathbf{M})$. We say that two systems $(\mathbf{K}, \mathbf{M})$ and $(\mathbf{K}', \mathbf{M}')$ are isospectral if $(\mathbf{K}' - \lambda\mathbf{M}')\mathbf{u}' = \mathbf{0}$ has the same spectrum of eigenvalues as (1).

In practice, the matrices $\mathbf{K}, \mathbf{M}$ have specific forms. Let M_n denote the set of square matrices of order n, and S_n denote the subset of symmetric matrices. If the system is conservative, then

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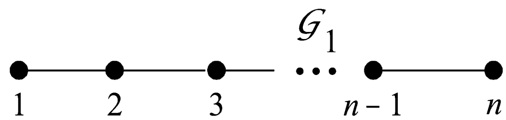

Fig. 1. The matrix J lies on the path $\mathcal{G}_1$.

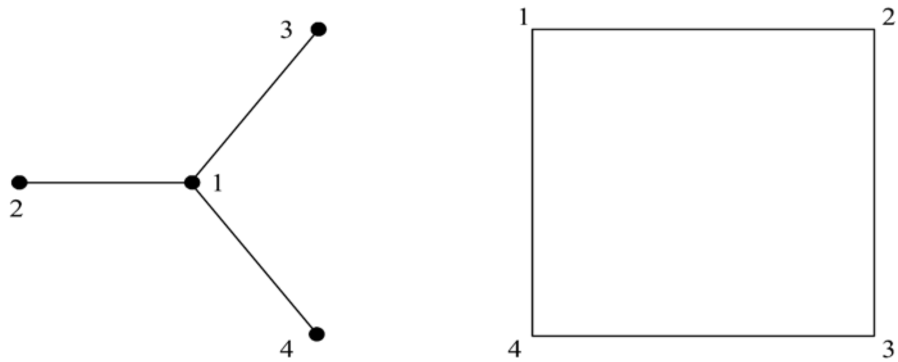

Fig. 2. A star, $\mathcal{G}_2$, on 4 vertices, and a ring, $\mathcal{G}_3$, on 4 vertices.

$\mathbf{K}, \mathbf{M} \in S_n$, $\mathbf{K}$ is *positive semi-definite* (PSD), i.e., $\mathbf{u}^T\mathbf{K}\mathbf{u} \geq 0$; $\mathbf{M}$ is positive definite (PD), i.e., $\mathbf{u}^T\mathbf{M}\mathbf{u} > 0$, for all $\mathbf{u} \neq \mathbf{0}$.

The matrices $\mathbf{K}$, $\mathbf{M}$, and in particular their *structure*, i.e., the pattern of zero and non-zero entries, will depend on the choices of finite elements, and on how these elements are connected. It is convenient to use concepts from graph theory: a (simple, undirected) graph $\mathcal{G}$ is a set of vertices P_i in a *vertex set* $\mathcal{V}$, connected by edges $(i, j)(= (j, i))$ in an *edge set* $\mathcal{E}$.

A matrix $\mathbf{A} \in S_n$ is said to lie *on* $\mathcal{G}$ if $a_{ij} = 0$ whenever $(i, j) \notin \mathcal{E}$. For example, a symmetric tridiagonal matrix, sometimes called a Jacobi matrix $\mathbf{J}$, and written

$$\mathbf{J} = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix} \tag{2}$$

lies on a graph $\mathcal{G}_1$ that is a *path* with $\mathcal{V} = \{1, 2, \ldots, n\}$ and $\mathcal{E} = \{(1, 2), (2, 3), \ldots, (n-1, n)\}$, as shown in Figure 1.

Figure 2 shows two simple graphs: a star, $\mathcal{G}_2$, and a ring, $\mathcal{G}_3$. The matrices $\mathbf{A}_2$ and $\mathbf{A}_3$ lie on $\mathcal{G}_2$, $\mathcal{G}_3$ respectively.

$$\mathbf{A}_2 = \begin{bmatrix} a_1 & b_2 & b_3 & b_4 \\ b_2 & a_2 & 0 & 0 \\ b_3 & 0 & a_3 & 0 \\ b_4 & 0 & 0 & a_4 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} a_1 & b_1 & 0 & b_4 \\ b_1 & a_2 & b_2 & 0 \\ 0 & b_2 & a_3 & b_3 \\ b_4 & 0 & b_3 & a_4 \end{bmatrix}. \tag{3}$$

The matrix $\mathbf{A}_2$ is an example of a *bordered* matrix, $\mathbf{A}_3$ is called a *periodic Jacobi matrix*. A Jacobi matrix is a particular case of a *band* matrix – it is a symmetric matrix with bandwidth 1. An important

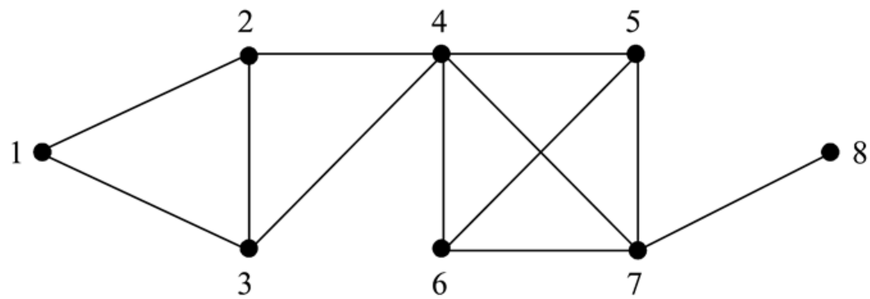

Fig. 3. The matrix $\mathbf{A}_4$ lies on $\mathcal{G}_4$.

subset of band matrices is the set of *staircase* matrices, an example of which is shown in (4); $\mathbf{A}_4$ lies of the graph $\mathcal{G}_4$.

$$\mathbf{A}_4 = \begin{bmatrix} x & x & x & & & & & \\ x & x & x & x & & & & \\ x & x & x & x & & & & \\ & x & x & x & x & x & x & \\ & & & x & x & x & x & \\ & & & x & x & x & x & \\ & & & x & x & x & x & x \\ & & & & & & x & x \end{bmatrix}. \tag{4}$$

The general isospectral problem is this: Given a system $(\mathbf{K}, \mathbf{M})$ with $\mathbf{K}, \mathbf{M} \in S_n$, and $\mathbf{K}, \mathbf{M}$, lying on a graph $\mathcal{G}$, find another (or all) isospectral system(s) $(\mathbf{K}', \mathbf{M}')$ with $\mathbf{K}', \mathbf{M}'$ lying on the same graph $\mathcal{G}$.

This problem is very difficult, and is still open. To simplify it somewhat, we suppose that the systems have lumped mass, so that $\mathbf{M}, \mathbf{M}'$ are *diagonal*. In that case, if $\mathbf{K}$ is PD (PSD) then the λ_i will be positive (non-negative) and $\mathbf{K}'$ will be PD (PSD).

In practice, the problem is even more difficult because, instead of being just PD (PSD), $\mathbf{K}$ will have to satisfy other, usually *positivity* constraints, that state that the system is physically realisable. We need some more concepts from matrix theory.

Suppose $\mathbf{A} \in M_n$. Let $\alpha = \{i_1, i_2, \ldots, i_k\}$ be a sequence of k numbers taken from $\{1, 2, \ldots, n\}$. The *submatrix* of $\mathbf{A}$ with rows taken from $\alpha = \{i_1, i_2, \ldots, i_k\}$, and columns taken from $\beta = \{j_1, j_2, \ldots, j_k\}$ is denoted by $A(\alpha|\beta)$. The determinnant

$$\det(A(\alpha|\beta)) = A(\alpha; \beta)$$

is called a *minor* of A. A minor $A(\alpha; \alpha)$ is called a *principal* minor of $\mathbf{A}$.

Suppose $\mathbf{A} \in M_n$:

- $\mathbf{A}$ is *totally positive*, TP, if all its minors are *positive*,
- $\mathbf{A}$ is *totally non-negative*, TN, if all its minors are non-negative; $\mathbf{A}$ is NTN if it is *non-singular* and TN,
- $\mathbf{A}$ is *oscillatory*, O, if A is TN and a power of $\mathbf{A}$, $\mathbf{A}^P$, is TP.

It may be shown that $\mathbf{A}$ is O iff it is NTN, *and* its immediately off-diagonal entries $a_{i,i+1}$ and $a_{i+1,i}$, $i = 1, 2, \ldots, n-1$ are *positive*, see Gladwell (1998). Note that TP is much stronger than PD: $\mathbf{A} \in S_n$ is PD iff its *principal* minors are positive.

Note that the definition of TP, TN and O matrices applies to *any* matrix in M_n, not just to symmetric matrices, those in S_n. Such matrices have many important properties.

Define $\mathbf{Z} = diag(+1, -1, \ldots, (-)^{n-1})$; the operation $\mathbf{A} \rightarrow \mathbf{ZAZ} = \tilde{\mathbf{A}}$ changes the signs of the entries of $\mathbf{A}$ in a chequered pattern.

We list three properties:

– if $\mathbf{A}, \mathbf{B}$ are O, so is $\mathbf{AB}$,
– $\mathbf{A}^{-1}$ is O iff $\tilde{\mathbf{A}}$ is O; we say $\mathbf{A}$ is *sign-oscillatory*, SO,
– if $\mathbf{A}$ is O then it *has n* eigenvalues, and they are *positive* and *distinct*.

The last property is particularly important: recall that if $\mathbf{A} \in S_n$, all we can say is that it *has n* real eigenvalues; they may not be distinct. If we know only that $\mathbf{A} \in M_n$, then we do not know *a priori*, how many eigenvalues it has.

2. QR Factorisation

Recall that a matrix $\mathbf{Q} \in M_n$ is *orthogonal* if $\mathbf{QQ}^T = \mathbf{I} = \mathbf{Q}^T\mathbf{Q}$. If $\mathbf{A} \in M_n$ is non-singular, then it may be factorised in the form $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q}$ is orthogonal, and $\mathbf{R}$ is upper triangular with *positive* diagonal terms. This factorisation is equivalent to the Gram-Schmidt process in which the columns of $\mathbf{A}$ are expressed as linear combinations of orthonormal vectors $\mathbf{q}_i$, the columns of $\mathbf{Q}$.

QR factorisation gives us a procedure for getting another matrix, $\mathbf{A}^*$, isospectral to $\mathbf{A}$: if $\mathbf{A} = \mathbf{QR}$, then $\mathbf{A}^* = \mathbf{RQ}$ may be written $\mathbf{A}^* = \mathbf{Q}^T(\mathbf{QR})\mathbf{Q} = \mathbf{Q}^T\mathbf{AQ}$: $\mathbf{A}^*$ is orthogonally equivalent to $\mathbf{A}$. If $\mathbf{A}$ is symmetric, so that $\mathbf{A}^T = \mathbf{A}$, then $(\mathbf{A}^*)^T = (\mathbf{Q}^T\mathbf{AQ})^T = \mathbf{Q}^T\mathbf{A}^T\mathbf{Q} = \mathbf{Q}^T\mathbf{AQ} = \mathbf{A}^*$: $\mathbf{A}^*$ is symmetric; $\mathbf{A}$ and $\mathbf{A}^*$ are isospectral.

We can apply $QR \rightarrow RQ$ transformation with a *shift*. Suppose μ is *not* an eigenvalue of $\mathbf{A} \in S_n$, write

$$\mathbf{A} - \mu\mathbf{I} = \mathbf{QR} \tag{5}$$

and construct $\mathbf{A}^*$ from

$$\mathbf{A}^* - \mu\mathbf{I} = \mathbf{RQ}. \tag{6}$$

Again, $\mathbf{A}^* = \mu\mathbf{I} + \mathbf{RQ} = \mathbf{Q}^T(\mu\mathbf{I} + \mathbf{QR})\mathbf{Q} = \mathbf{Q}^T\mathbf{AQ}$, so that $\mathbf{A}$ and $\mathbf{A}^*$ are isospectral; we write $\mathbf{A}^* = \mathcal{G}_\mu(\mathbf{A})$.

The simplest application of this result is to isospectral in-line spring-mass systems. Now $\mathbf{K}$ is a Jacobi matrix with negative off-diagonal and $\mathbf{M}$ is diagonal: $\mathbf{M} = diag(m_1, m_2, \ldots, m_n)$. We write $\mathbf{M} = \mathbf{P}^2$, where $\mathbf{P} = diag(p_1, p_2, \ldots, p_n)$; Equation (1) may be written

$$\mathbf{P}^{-1}(\mathbf{K} - \lambda\mathbf{P}^2)\mathbf{P}^{-1}\mathbf{Pu} = \mathbf{0},$$

or

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}, \quad \mathbf{A} = \mathbf{P}^{-1}\mathbf{KP}^{-1}, \quad \mathbf{x} = \mathbf{Pu}.$$

The matrix A is a Jacobi matrix with negative off-diagonal. We form $\mathbf{A}^*$ from (5), (6) and then factorise $\mathbf{A}^*$ in the form $\mathbf{A}^* = \mathbf{P}^{*-1}\mathbf{K}^*\mathbf{P}^{*-1}$ to obtain a new in-line spring-mass system. The details of the analysis may be found in Gladwell (1995). The extension to a system in which $\mathbf{K}, \mathbf{M}$ are both Jacobi matrices, with negative, positive off-diagonals respectively, may be found in Gladwell (1999).

This application depends on the fact that the operation $\mathcal{G}_\mu$ defined by (5), (6) changes a Jacobi matrix $\mathbf{A}$ with negative (positive) off-diagonal into a Jacobi matrix $\mathbf{A}^*$ with negative (positive) off-diagonal. It may be verified that $\mathbf{A}^*$ is PD (PSD) iff $\mathbf{A}$ is PD (PSD). This is a special case of a general result, proved in Gladwell (1998): Suppose $\mathbf{A} \in S_n$, and μ is *not* an eigenvalue of $\mathbf{A}$ (both these conditions are necessary), then $\mathbf{A}$, $\mathbf{A}^*$ have the same staircase pattern, and $\mathbf{A}^*$ is TP, NTN, O or SO, iff $\mathbf{A}$ is TP, NTN, O or SO, respectively.

This general result may be used to find an isospectral family of FEM models of a thin straight rod in longitudinal vibration. Now $\mathbf{K}$, $\mathbf{M}$ are Jacobi matrices with negative, positive, off-diagonals respectively. We start from Equation (1), factorise $\mathbf{M} = \mathbf{B}\mathbf{B}^T$ where $\mathbf{B}$ is a bi-diagonal upper triangle matrix, and reduce (1) to standard form

$$(\mathbf{B}^{-1}\mathbf{K}\mathbf{B}^{-T} - \lambda\mathbf{I})\mathbf{B}^T\mathbf{u} = \mathbf{0}.$$

It may be shown that $\mathbf{A} = \mathbf{B}^{-1}\mathbf{K}\mathbf{B}^{-T}$ is SO. We find $\mathbf{A}^* = \mathcal{G}_\mu(A)$, and then factorise $\mathbf{A}^* = \mathbf{B}^{*-1}\mathbf{K}^*\mathbf{B}^{*-T}$ to form a new FEM model, $\mathbf{K}^*$, $\mathbf{M}^*$ with $\mathbf{M}^* = \mathbf{B}^*\mathbf{B}^{*T}$; $\mathbf{K}^*$, $\mathbf{M}^*$, like $\mathbf{K}$, $\mathbf{M}$ are Jacobi matrices with negative, positive, off-diagonals respectively, corresponding to a FEM model of a new rod, as described in Gladwell (1997).

The operation $\mathcal{G}_\mu$ is essentially tied to staircase matrices: $\mathbf{A}^*$ is a staircase iff $\mathbf{A}$ is a staircase. To obtain a wider class of isospectral matrices, we must consider the concept of *isospectral flow*.

3. Isospectral Flow

If $\mathbf{A}$ is symmetric, i.e., $\mathbf{A} \in S_n$, it has n eigenvalues $(\lambda_i)_1^n$ and n corresponding orthonormal eigevectors $\mathbf{q}_i$ that span R^n. The matrix $\mathbf{A}$ may be written

$$\mathbf{A} = \mathbf{Q}\wedge\mathbf{Q}^T,$$

where

$$\wedge = diag(\lambda_1, \lambda_2, \ldots, \lambda_n), \quad \mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n], \quad \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}.$$

The family of matrices with all possible orthogonal matrices $\mathbf{Q}$ forms an isospectral family, the complete family with the given spectrum $(\lambda_i)_1^n$.

Instead of seeking the complete family, we look for a family in which $\mathbf{Q}$ depends on a single parameter t.

$$\mathbf{A}(t) = \mathbf{Q}(t)\wedge\mathbf{Q}^T(t).$$

Differentiating w.r.t. t we find

$$\dot{\mathbf{A}} = \mathbf{Q}\wedge\dot{\mathbf{Q}}^T + \dot{\mathbf{Q}}\wedge\mathbf{Q}^T.$$

Since $\mathbf{Q}$ is orthogonal, we may write

$$\dot{\mathbf{A}} = (\mathbf{Q}\wedge\mathbf{Q}^T)(\mathbf{Q}\dot{\mathbf{Q}}^T) + (\dot{\mathbf{Q}}\mathbf{Q}^T)(\mathbf{Q}\wedge\mathbf{Q}^T).$$

Put $\mathbf{Q}\dot{\mathbf{Q}}^T = \mathbf{S}$ then, since $\mathbf{Q}\wedge\mathbf{Q}^T = \mathbf{A}$, we have

$$\dot{\mathbf{A}} = \mathbf{A}\mathbf{S} + \mathbf{S}^T\mathbf{A}.$$

Now, $\mathbf{Q}$ is orthogonal, so that $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$, and hence

$$\mathbf{Q}\dot{\mathbf{Q}}^T + \dot{\mathbf{Q}}\mathbf{Q}^T = \mathbf{0} : \mathbf{S} + \mathbf{S}^T = \mathbf{0}.$$

This means that

$$\dot{\mathbf{A}} = \mathbf{AS} - \mathbf{SA} \tag{7}$$

and that $\mathbf{S}$ is a skew-symmetric matrix. We note that (7) is a so-called *autonomous* differential equation: the parameter t does not appear explicitly; it appears implicitly because $\mathbf{S}$ depends on $\mathbf{A}$, i.e., $\mathbf{S} = \mathbf{S}(\mathbf{A})$, and $\mathbf{A}$ depends on t.

Most importantly, we may argue conversely: if $\mathbf{S}$ is skew symmetric and $\mathbf{A}$ varies according to (7) then $\mathbf{A}(t)$ keeps the same eigenvalues, those for $\mathbf{A}(0)$. We may choose $\mathbf{S}$ in many ways; different choices will lead to different isospectral families.

The autonomous differential equation (7), called the *Toda flow* equation, was investigated first for tridiagonal matrices, with the choice

$$\mathbf{A}(t) = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix}, \quad \mathbf{S}(t) = \begin{bmatrix} 0 & b_1 & & & \\ -b_1 & 0 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & -b_{n-1} & 0 \end{bmatrix}, \tag{8}$$

as in Symes (1982). Watkins (1984) gives a survey of the general theory. See also Chapter 7 of Gladwell (2004). In this case, it may easily be shown that $\mathbf{AS} - \mathbf{SA}$ is tridiagonal, so that if $\mathbf{A}(0)$ is tridiagonal, then $\mathbf{A}(t)$ will be tridiagonal. This is a special case of he result that if $\mathbf{A}(0)$ is a staircase matrix and if

$$\mathbf{S}(t) = \mathbf{A}^{+}(t) - \mathbf{A}^{+T}(t), \tag{9}$$

where $\mathbf{A}^{+}(t)$ denotes the upper triangle of $\mathbf{A}(t)$, then Equation (7) constrains $\mathbf{A}(t)$ to remain a staircase matrix, with the same staircase dimensions as $\mathbf{A}(0)$. In general, even if $\mathbf{A}(0)$ is a staircase with holes, these holes will eventually be filled in. Gladwell (2002) showed that the Toda flow (7), with S given by (9) maintains the properties TP, NTN, O and SO.

There are two important engineering structures for which the stiffness matrix is a staircase matrix: the rod in logitudinal vibration, already mentioned; an Euler–Bernoulli beam in flexure, for which the stiffness matrix is pentadiagonal, see Gladwell (2002b).

To obtain an isospectral flow for more general cases, we must pose the question: How may we construct an isospectral flow that constrains $\mathbf{A}$ to lie on a given graph $\mathcal{G}$?

Consider a very simple graph, the star on n vertices, as shown in Figure 4; a matrix on $\mathcal{G}$ has the form $\mathbf{A}_5$ in Equation (10); the only non-zero entries are those on the *borders*, i.e., the first row and column, and the diagonal. The matrix S must be skew-symmetric, so that we need consider only its upper triangle. We choose $s_{ij} = a_{ij}$ for entries in the first row, and then find the remaining entries below the first row and above the diagonal by making $\dot{a}_{ij} = 0$ for those entries; there are $m = (n-1)(n-2)/2$ algebraic equations for the m unknown s_{ij}.

$$\mathbf{A}_5 = \begin{bmatrix} a_1 & b_2 & b_3 & \dots & b_n \\ b_1 & a_2 & & & \\ b_3 & & a_3 & & \\ \vdots & & & \ddots & \\ b_n & & & & a_n \end{bmatrix}, \quad \mathbf{S}_5 = \begin{bmatrix} 0 & b_2 & b_3 & \dots & b_n \\ & 0 & s_{23} & \dots & s_{2n} \\ & & 0 & & \\ skew & & & \ddots & s_{n-1,n} \\ & & & & 0 \end{bmatrix} \tag{10}$$

The equations for the s_{ij} are separable, that for s_{ij} is

$$(a_i - a_j)s_{ij} + 2b_ib_j = 0 \quad i = 2, \dots, n, \quad j = i+1, \dots, n. \tag{11}$$

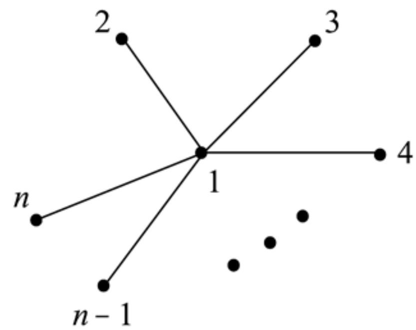

Fig. 4. A star on n vertices.

These m algebraic equations are combined with the equations

$$\dot{a}_1 = -2\sum_{i=2}^{n} b_i^2, \quad \dot{a}_j = 2b_j^2, \quad j = 2, \ldots, n \tag{12}$$

$$\dot{b}_j = (a_1 - a_j)b_j + \sum_{i=2}^{j-1} s_{ij}b_i - \sum_{i=j+1}^{n} s_{ji}b_i, \quad j = 2, \ldots, n. \tag{13}$$

On substituting for s_{ij} from (11) into (13), we find

$$\dot{b}_j = b_j \left\{ a_1 - a_j + 2\sum_{k=2}^{n}{}' \frac{b_k^2}{a_j - a_k} \right\}, \quad j = 2, \ldots, n, \tag{14}$$

where $'$ denotes $k \neq j$. It may be shown that if the $a_i(0)$, $i = 2, \ldots, n$ are *distinct*, and the $b_i(0)$, $i = 2, \ldots, n$ are *non-zero*, then the $a_i(t)$, $i = 2, \ldots, n$ will be distinct, and the $b_i(t)$ will be non-zero, so that the denominators in (14) will remain non-zero.

This procedure may be generalised: in the upper triangle of S, take $s_{ij} = a_{ij}$ when $(i, j) \in \mathcal{E}$; find s_{ij} for the remaining entries $(i, j) \notin \mathcal{E}$ by demanding that $\dot{a}_{ij} = 0$ when $(i, j) \notin \mathcal{E}$; this gives a set of p algebraic equations for the p entries, s_{ij} which are combined with the remaining equations for $\dot{a}_{ij}$, $(i, j) \in \mathcal{E}$, and $\dot{a}_{ii}$, $i = 1, 2, \ldots, n$. The algebraic equations for the s_{ij} will have coefficients that are linear combinations of the a_{ij}, as in Equation (11) for the star. In general, unlike for the star, we cannot assume that the p equations for the p entries s_{ij} will always admit a solution; we will have to fall back on continuity arguments – if they admit a solution when $t = 0$, they will admit a solution for some small interval of t around $t = 0$.

The application of this procedure to a typical FEM model, a triangular model of a membrane, is the topic of a forthcoming paper.

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