CRACK INDUCED STRESS FIELD IN AN ELASTIC-PLASTIC PLATE

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Abstract

This paper presents a solution for stress and deformation fields induced by a central crack in an elasticplastic plate subject to tensile load. The solution is controlled by a crack opening parameter related to material modulus and far-field stress.

1. Introduction

Cracks in a structure cause stress at the tip to increase to a level that could lead to structural failure. Hence, knowledge of the stress distribution around crack tips is important to engineers and designers. Design requirement and need for a failure analysis was the reason for a rapid development in crack analyses especially since the World War II. Early work used small deformation theory, linear elastic behavior and relied on un-deformed geometry to satisfy traction boundary conditions. The result led to singular stress and strain fields at the crack tip. The contributions on this topic are part of fracture mechanics now known as linear elastic fracture mechanics (LEFM). Because of the inherent contradiction between small deformation and singular stress and strain fields at the tip, there is an implicit understanding that the result obtained under such assumptions does not apply at or close to the tip. LEFM solution does not apply at large distances from the crack tip either. The applicability of linear elastic fracture mechanics was thus limited to a finite domain surrounding but excluding the crack tip. However, crack tip analyses did produce the concept of stress intensity factor. This factor controls the stress field around the tip and its value was found to depend on crack length, far field stress and also on structural geometry. Naturally, determination of stress intensity factor became the focus of research and its critical value became a basis for structural design.

When stresses in metals exceed yield limit, they undergo plastic deformation. Since LEFM predicts high value for stresses in an area around crack tip, part of that area is subject to yielding and plastic deformation. Because the yield criterion limits stresses to remain within a finite value, the stress field within plastic zone cannot be singular. A new approach is therefore required to accommodate plastic behavior near crack tip.

It is possible to estimate plastic zone size on the basis of elastic analyses. Irwin (1957) proposed that the actual plastic zone is greater than this estimate. To obtain the actual size, he evaluated the load between the tip and yield point from the elastic analysis and redistributed it over the plastic zone. The proposal of Dugdale and Barenblatt (Barenblatt, 1962) to remove stress singularity at the tip is based on canceling two singularities, one from the elastic analysis and other associated with the wedge force due to yield stress. Both of these proposals considered perfectly plastic solid that allows no strain hardening. They also assume blunting of the crack tip. There is thus an implicit recognition that crack blunting in plastic deformation and non-singular stress field in plastic zone go hand in hand. In other words, singular stress field is incompatible with a blunt crack tip. The conclusion is obvious: stress at the crack tip is reduced to a finite value because of blunting of crack caused due to deformation. If that is the case, an analysis formulated in terms of deformed geometry that allows for blunting is expected to result in finite crack tip stress.

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Two analytical crack tip analyses for plastic solids involving work hardening nonlinear material are due to Hutchinson (1968) and Rice and Rosenberg (1968). They obtained singular stress field near crack tip using finite deformation theory of plasticity. They assume a stress field, known as HRR solution, consistent with singular strain energy density. Just like the LEFM solution, HRR solution is not valid at the tip because singular stress it predicts at the crack tip is incompatible with the limitation on stresses imposed by plasticity. HRR solution is not valid at large distance from the tip either. A finite element analysis by Mcmeeking et al. identifies the area over which HRR solution applies.

Singh et al. (1994) obtained stress, strain and displacement field around a crack in an infinite, isotropic and linear elastic plate. They used a non-classical small deformation theory. The classical theory assumes the displacement to be small such that replacing the deformed position of a particle by its initial un-deformed position is likely to induce negligible error in the solution. Singh et al. used the reverse argument that in an analysis of a problem involving small displacement, it is equally justified to use the deformed position of a particle, rather than its un-deformed position. Further, the consistency of analysis requires the use of deformed geometry if the boundary value problem is formulated in terms of true stress and true traction. On this basis, Singh et al. (1994) obtained a solution for the entire plate. They obtained the geometry of the deformed crack surface as part of their solution. In this presentation, we use their methodology to obtain stress and deformation field in an elastic-plastic plate. Solution uses deformed geometry and linearized stress-strain behavior.

2. Background

Consider a plate with a centrally located crack subject to far field tensile stress S. for the purpose of analysis; choose a Cartesian coordinate system with origin at the crack center, x-axis along the crack and y-axis perpendicular to it. Suppose the crack extends from $x = -a_0$ to $x = a_0$ on $y = 0$. Let

$$z = a_0 \cosh \xi, \quad \xi = \alpha + i\beta, \quad i = \sqrt{-1}$$

with $a = 0$ describing the initial geometry of a crack in elliptical coordinates. It is a special case of general transformation

$$z = x + iy = z(\xi)$$

that relates Cartesian coordinates (x, y) to elliptical coordinates (α, β). In the new system, crack surface $\alpha = 0$ extends from $\beta = 0$ to $\beta = 2\pi$. Because of the symmetry, it is sufficient to consider the upper right quadrant $(x \geq 0, y \geq 0)$ or $(\alpha \geq 0, 0 \leq \beta \leq \pi/2)$ of the plate.

The plate under external tensile stress normal to the crack is expected to develop a non-uniform stress field that can be obtained from solving the equation of equilibrium

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0. \tag{1}$$

These equations are transformed to complex plane then combined and expressed in the form

$$\frac{\partial}{\partial z}(\sigma_{xx} + \sigma_{yy}) + \frac{\partial}{\partial z}(\sigma_{xx} - \sigma_{yy} - 2i\sigma_{xy}) = 0.$$

It has a solution in terms of a stress function Φ such that

$$\sigma_{xx} + \sigma_{yy} = \frac{\partial^2 \Phi}{\partial z \partial \bar{z}},$$

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = -\frac{\partial^2 \Phi}{\partial \bar{z} \partial \bar{z}}. \tag{2}$$

The main task now is to find a suitable stress function. The task becomes easier if the material behavior is linear and isotropic. For such a solid,

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = \frac{\partial u}{\partial z} + \frac{\partial \bar{u}}{\partial \bar{z}} = \frac{1 - \upsilon - 2k\upsilon^2}{E}\left(\frac{\partial^2 \Phi}{\partial z \partial \bar{z}}\right),$$

$$\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} + i\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) = 2\frac{\partial u}{\partial \bar{z}} = -\frac{1+\upsilon}{E}\left(\frac{\partial^2 \Phi}{\partial \bar{z} \partial \bar{z}}\right),$$

where u_x and u_y are the x- and y-components of displacement u $(= u_x + iu_y)$; $k = 0$ or 1 depending on whether the plate is in plane stress or plane strain. The material is linear if the mechanical parameters E and υ are constant. Otherwise, the material is considered non-linear. The above equations show that for a linear solid, the function Φ must be real and bi-harmonic, and hence $\partial^2\Phi/\partial z \partial \bar{z}$ must be real and harmonic. For a linear solid therefore, choose

$$\sigma_{xx} + \sigma_{uu} = \frac{\partial^2 \Phi}{\partial z \partial \bar{z}} = \frac{\partial f}{\partial z} + \overline{\left(\frac{\partial f}{\partial z}\right)}, \tag{3}$$

where f is an analytic function. The above equation can be integrated to yield

$$\frac{\partial \Phi}{\partial \bar{z}} = f + z\overline{\left(\frac{\partial f}{\partial z}\right)} + \bar{g}, \tag{4}$$

where g is another analytic function.

Suppose a particular choice of analytic function f and g solves a given boundary value problem for an isotropic linear solid. The same analytic functions can also be used to solve the problem of another isotropic linear solid if it is subject to the same boundary conditions. This feature can be exploited even for non-linear material provided its non-linear stress strain behavior can be replaced by piece-wise linear approximation. In the simplest case, non-linear material response can be replaced by bi-linear stress strain behavior involving mechanical parameters (υ, E) and (υ_n, E_n). Suppose the first segment I represents elastic behavior and the second segment n involves plastic deformation characterized by the plastic modulus E_p and the ratio $\upsilon_p = 1/2$ such that

$$\frac{1}{E_n} = \frac{1}{E} + \frac{1}{E_p}, \quad \frac{\upsilon_n}{E_n} = \frac{\upsilon}{E} + \frac{1}{2E_p}.$$

Once the analytic functions f and g are known or have been found, stress field can be obtained from (1). The displacement field in the elastic domain I is obtained from

$$u = z - Z = C + \frac{8f}{E} - 2\frac{1+\upsilon}{E}\frac{\partial \Phi}{\partial \bar{z}},$$

where C is a constant and z is the deformed position of a particle that initially, in the un-deformed configuration, occupied a position Z. The relation

$$u = z - Z = C_n + \frac{8f}{E_n} - 2\frac{1+\upsilon_n}{E_n}\frac{\partial \Phi}{\partial \bar{z}}$$

yields displacement in the plastic domain. The constants C and C_n, and the functions f and g must be chosen to ensure continuity of displacement and stress across the elastic plastic interface.

In the case of perfectly plastic material, the yield stress is constant; the stress strain curve is linear and horizontal (parallel to the strain axis) with slope $E_n = 0$. Since the stress strain curve is linear in the plastic domain, the sum $\sigma_{xx} + \sigma_{xx}$ can still be considered analytic. However, it is necessary only to choose an analytic function g for determination of the stress field in plastic domain. The yield criterion provides a second condition that can be used for this purpose. It is necessary only to choose an analytic function that determines $\sigma_{xx} + \sigma_{xx}$ such that the stress field it generates in conjunction with the yield criterion leaves the crack surface free traction. It is also necessary to maintain continuity between the stress fields of the elastic and plastic solutions across the common boundary.

3. Stress and Strain Field around Crack

To show how the proposal works, assume that the stress field in the elastic domain has been obtained from LEFM analysis. Irwin uses the stress field on the crack line $y = 0$ in the form

$$\sigma_{11} = \sigma_{22} = \frac{K_I}{\sqrt{2\pi r}}, \quad \sigma_{12} = 0,$$

where K_I $(= S\sqrt{\pi a})$ is the stress intensity factor and r is the distance from the crack tip and S is the far-field stress. Consider plane stress, Mises yield criterion and use the effective stress

$$\sigma_e = \sqrt{\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{12}}.$$

Suppose the material yields at $r = r_Y$ where the effective stress equals the yield stress σ_Y. $K_I = S\sqrt{2\pi r_Y}$. The load transmitted across the surface $y = 0$ in the plastic zone is $P = 2\sigma_Y r_Y$. If the condition $\sigma_{11} = \sigma_{22}$ holds along the crack line even in the plastic zone, the equilibrium condition predicts the plastic zone size of

$$l_p = \frac{P}{\sigma_Y} = 2r_Y = \left(\frac{S}{\sigma_Y}\right)^2 a. \tag{5}$$

To accommodate the increased plastic zone, one of the three conditions must hold: (1) the crack tip is displaced towards the center of the crack, (2) the elastic-plastic boundary moves away from the crack center, or (3) or a combination of the above two occurs. In Irwin's scheme, condition (2) is assumed to hold. That is, the elastic-plastic boundary is assumed to move such that the stress field of LEFM begins at a distance of $2r_Y$ from the crack tip. In order for stress field (2) to hold, the crack is assumed to be located at $a_{\text{eff}} = a + r_Y$ and the crack opens to a blunt configuration at the tip. The crack tip opening displacement (CTOD) is calculated from the formula

$$\frac{x^2}{a_{\text{eff}}^2} + \frac{y^2}{b^2} = 1$$

of the elastic solution. Thus CTOD at $x = a$ is

$$2\delta = 2\frac{b}{a + r_Y}\sqrt{2ar_Y + r_Y^2}.$$

If $r_Y = 1/2(S/\sigma_Y)^2 a \ll a$, the Singh solution for b along with the assumption that $S \ll E$ yields

$$2\delta = 4\frac{S}{E}\frac{S}{\sigma_Y}a.$$

Note that the method of evaluating crack opening displacement involves the use of elastic solution even though a part of the material undergoes plastic deformation. Moreover, the crack opening displacement of elastic solution indicates that the material particles initially on the crack surface ($\alpha = 0$ or $-a_0 < x < a_0$, $y = 0$) has opened into ($\alpha > 0$), otherwise there will be no crack opening. Obviously, if the boundary conditions are expressed in terms of true stress, the traction boundary conditions must be satisfied not on the un-deformed ($\alpha = 0$) surface but on the deformed ($\alpha > 0$) surface it opens into. This shows blunting and requires σ_{11}, vanishing at the crack tip. Further, in the case of plane stress, $\sigma_{22} = \sigma_Y$. This information can be used to select an analytic function f for the plastic zone as follows.

4. Stresses in Plastic Zone

In view of the fact that in plane stress, $\sigma_{11} = 0$, $\sigma_{22} = \sigma_Y$ at the blunted crack tip and $\sigma_{11} + \sigma_{22}$ must be harmonic, choose

$$\sigma_{11} + \sigma_{22} = \sigma_Y(\sin p(z - x_0) + \sin p(\bar{z} - x_0)).$$

where p is a constant and $z = x + iy$. Rewrite the above equation in the form

$$\sigma_{11} + \sigma_{22} = 2\sigma_Y \sin p(x - x_0) \cosh py. \tag{6}$$

On x-axis, $y = 0$ and $\sigma_{11} + \sigma_{22} = 2\sigma_Y \sin p(x - x_0)$. For a Mises solid therefore,

$$\begin{aligned}\sigma_{22} &= \sigma_Y \left\{ \sin p(x - x_0) + \frac{1}{\sqrt{3}} \cos p(x - x_0) \right\}, \\ \sigma_{11} &= \sigma_Y \left\{ \sin p(x - x_0) - \frac{1}{\sqrt{3}} \cos p(x - x_0) \right\}.\end{aligned}$$

To satisfy $\sigma_{11} = 0$ at the crack tip $x = x_t$, choose p such that $p(x_t - x_0) = \pi/6$. Note that on the extended crack-line in the plastic zone, $\sigma_{11} \neq \sigma_{22}$ except at the point where $p(x - x_0) = \pi/2$.

To reconcile the chosen plastic field with LEFM field in the elastic domain, choose $\sigma_{11} = \sigma_{22} = \sigma_Y$ at the yield point ($x = x_Y = r_Y$, $y = 0$). This is possible provided $p(x_Y - x_)) = \pi/2$, or $x_Y = 3x_t - 2x_0$. Therefore, the plastic zone is $l_p = x_Y - x_t = 2(x_t - x_0) = \pi/3p$. To find its value, evaluate $P = \int_{x_t}^{x_Y} \sigma_{22} ds$ and equate it with $P = 2\sigma_Y r_Y$ obtained from LEFM. The result is $p = 1/(x_Y\sqrt{3})$ and the plastic zone size is

$$l_p = \frac{\pi}{\sqrt{3}} r_Y = \frac{\pi}{2\sqrt{3}} \left(\frac{S}{\sigma_Y}\right)^2 a$$

with p, l_p and x_Y known, it is easy to evaluate $x_0 = x_Y - \pi/2p$ and $x_t = x_Y - l_p$.

To find the crack opening in the plastic observe that on the crack surface $x = x_c$, $y = y_c$, $\sigma_{11} = \sigma_{22} = \sigma_Y$ and hence the equation of the deformed crack is

$$\sin p(x_c - x_0) \cosh py_c = 0.5. \tag{7}$$

Since the value of p is already known, the above equation can be used to find the deformed crack surface in the plastic domain. The surface in the elastic domain is obtained from the solution of Singh et al. and for $S \ll E$, it can be expressed in the form

$$x_e = a \cos\beta, \quad y_e = \frac{2S}{E} \sin\beta.$$

5. Crack Geometry

Consider the symmetry requirements on the deformed crack surface. Symmetry requires that $x = 0$ at $\beta = \pi/2$, and $y = 0$ at $\beta = 0$. The constraint imposed by continuity is that the elastic and have the same value at the interface ($\alpha = \alpha_f, \beta = \beta_Y$) between two domains as a consequence, the points on the elastic segment of the deformed crack are found to have coordinates

$$x_1 = \left(a_0 - \frac{cS}{E}\cosh\alpha_f\right)\cos\beta,$$

$$y_1 = cS(\sinh\alpha_f + 2\cosh\alpha_f)\left(\frac{\sin\beta - \sin\beta_Y}{E} + \frac{\sin\beta_Y}{E_n}\right).$$

For an elastic solid, $E = E_n$ and hence

$$y_1 = cS(\sinh\alpha_f + 2\cosh\alpha_f)\left(\frac{\sin\beta}{E}\right) = c\sinh\alpha_f\sin\beta.$$

The crack opening parameter is therefore related to Young's modulus E and far-field stress S. In fact

$$\tan\alpha_f = \frac{2S}{E - S}.$$

However, this equation is not available if the behavior is elastic plastic.

To satisfy the continuity of stress across the elastic-plastic interface, we assume the same value for crack opening parameter on both elastic and plastic segments. Hence, the deformed position of the points on the plastic segment is

$$x_n = \left(a_0 - \frac{cS}{E}\cosh\alpha_f\right)\cos\beta + \left(a_0 - \frac{cS}{E_n}\cosh\alpha_f\right)(\cos\beta - \cos\beta_Y),$$

$$y_n = c\sinh\alpha_f\sin\beta = \frac{cS}{E_n}(\sinh\alpha_f + 2\cosh\alpha_f)\sin\beta.$$

The second of the above equation yield

$$\tan\alpha_f = \frac{2S}{E_n - S}.$$

Therefore, the value

$$\alpha_f = 0.5\ln\left(\frac{E_n + S}{E_n - 3S}\right)$$

of the crack opening parameter depends on the modulus E_n of the material at the tip and the far-field stress S.

For further generalization, divide the non-linear stress-strain curve into N segments. At the same time, divide the crack surface also in N domains such that points in k obey the stress-strain rule of the segment k, and $\beta = \beta_k$ is the interface between domains $k - 1$ and k.

Therefore, the deformed position of this interface is

$$x_k = \sum_{i=1}^{k}\left(a_0 - \frac{cS}{E_i}\cosh\alpha_f\right)(\cos\beta_i - \cos\beta_{i-1}),$$

$$y_k = \sum_{i=1}^{k} cS(\sinh\alpha_f + 2\cosh\alpha_f)\left(\frac{\sin\beta_{N-i} - \sin\beta_{N-i+1}}{E_{N-i+1}}\right).$$

For the purpose of book-keeping, β_N and β_0 in this equation corresponds, respectively, to the tip ($\beta = 0$) and the crown ($\beta = \pi/2$) and E_i is the modulus of the material in the ith segment. It is of course possible to convert the summed terms into integral forms by letting the number of segments become infinitely large such that, in the limit, each domain shrinks to a point and

$$\sum_{i=1}^{k}\left(a_0 - \frac{cS}{E_i}\cosh\alpha_f\right)(\cos\beta_i - \cos\beta_{i-1}) = -\int_{\pi/2}^{\beta}\left(a_0 - \frac{cS}{E_t}\cosh\alpha_f\right)\sin\beta d\beta,$$

$$\sum_{i=1}^{k} cS(\sinh\alpha_f + 2\cosh\alpha_f)\left(\frac{\sin\beta_{N-i} - \sin\beta_{N-i+1}}{E_{N-i+1}}\right)$$
$$= \int_0^{\beta}(\sinh\alpha_f + 2\cosh\alpha_f)\frac{cS\cos\beta}{E_t}d\beta,$$

where E_t is the tangent modulus. The right-hand side can be integrated provided a relationship between the tangent modulus and β can be established. But such a relation does exist. Recall the first of the two expressions for stress field in (5). Since the crack surface is free of traction, stress normal to it must vanish. Thus, in plane stress, $\sigma_{yy} + \sigma_{xx} = \sigma_t$ is the only non-zero stress, and Equation $(5)_3$ can be rearranged in the form

$$\cos 2\beta = \cos 2\alpha_f - \frac{\sinh 2\alpha_f(e^{2\alpha f} + 1)}{e^{2\alpha_f} + \sigma_t/S}.$$

This equation relates β on the crack surface to the tangential stress σ_t which in turn is related to the tangential modulus via the stress strain law. In that case, the stress-strain law need not be linearized. However, it must be emphasized that the purpose of linearization was to make the stress function biharmonic and allow the use of analytic function f and g in the analysis. It is of course possible to assume a form for stress and strain fields on some other ground in such a manner that the stress field is an approximate solution of the equations of equilibrium. This is the course adopted by Hutchinson (1968) and Rice and Rosenberg (1968). In our case, we have opted in favor of approximating the stress-strain law but, at the same time, choosing a stress field that satisfies the equations of equilibrium.

6. Results and Discussion

Let us first examine if the use of deformed configuration in the analysis has any influence on stresses. Consider the basic assumptions. Both linear elastic fracture mechanics and the classical infinitesimal strain theory assume small deformation. Therefore, displacement z–Z is considered small and hence $z \cong Z$ in lieu of deformed position z on the basis that this choice is expected to introduce negligible error. In view of this argument, the two theories use $\alpha_f = 0$ for the crack line, and assume the crack tip at $\alpha = \beta = 0$. Or, in terms of cylindrical polar coordinates with origin at the crack tip, the two theories use $\theta = \pi$ for the crack surface, and assume the crack-tip at $r = 0$. The two stress fields $(5)_3$ and $(5)_4$ both display singular behavior in the limit as $\alpha, \beta \to 0$ or as the crack-tip is approached. Both linear elastic fracture mechanics and HRR solutions exhibit such singularity (Barenblatt, 1962; Hutchinson, 1968; Irwin, 1957).

We adopt another view point. Let us assume small deformation and consider the displacement z–Z to be small such that $z \cong Z$. In that case, we propose to use the deformed position z of the particle in place of its un-deformed position Z in solving a boundary value problem. This particular choice can be justified on the basis of the argument employed in the classical theory that replacing one by the other is expected to introduce negligible error. There is yet another reason in favor of the present choice. If the boundary value problem is formulated in terms of true stress and true traction, and if the analysis is to be consistent with the formulation, solution must depend on the deformed geometry. Note that true stress and true traction are defined in terms of load over unit current or deformed area. The use of undeformed geometry in boundary value problems formulated in terms of true stress is only a convenient approximation. It should not be considered a requirement that must be imposed.

Suppose the crack opens under and the crack opening parameter α_f has non-zero value, implying crack tip blunting. The tip is initially at ($\alpha = 0$, $\beta = 0$) and is displaced under deformation to ($\alpha = \alpha_f$, $\beta = 0$). The stress field in (5) is no longer singular even though the stresses near the tip are still high. Stress distribution depends on α_f, which can be used as a parameter that controls stresses. Under sufficiently high stresses, the crack tip and the area around it yield and undergo plastic deformation. The extent of plasticity along the crack surface can be determined from $(5)_3$.

The problem involving elastic behavior was solved by Singh et al. (1994). They obtained the value

$$\alpha_e = 0.5 \ln \left(\frac{E + S}{E - 3S} \right)$$

for the crack opening parameter. With $\alpha_f = \alpha_e$ in Equations $(5)_3$ and $(5)_4$, stresses can be obtained from (6).

Equation $(5)_1$ with $C = 0$ and $u = c \cosh(\alpha_e + i\beta) - a_0 \cos\beta$ yields the location

$$x = c \cosh \alpha_e \cos \beta = \frac{E}{E + S} a_0 \cos \beta,$$

$$y = c \sinh \alpha_e \sin \beta = \frac{2S}{E - S} a_0 \sin \beta$$

of points on the deformed crack surface.

For a bilinear solid, the crack tip opening and consequent blunting is governed by the modulus at the tip. The parameter

$$\alpha_n = 0.5 \ln \left(\frac{E_n + S}{E_n - 3S} \right)$$

controls stresses in the plastic domain and determines the deformed shape of the crack opening at the tip. Since the tangential stress along the crack surface must remain continuous across elastic-plastic boundary, the parameter α must have a common value α_b at that point. The parameter α_e that determines the shape of the elastic part of the crack is no longer linked to Young's modulus E. It is therefore possible to assume $\alpha_f = \alpha_e = \alpha_n = \alpha_b$. In that case, the parameter α_n controls the stress field. The choice also ensures the continuity of stresses across the elastic-plastic boundary.

Since $\sigma_{yy} = E_n$ at the crack tip, a solid with lower modulus entails larger crack opening parameter, more blunting and lower stress at the tip. However, this solution is not applicable if the yield stress dominates stress field of the plastic domain.

In the case of ideal plastic behavior, there is no strain hardening and therefore $E_n = E$. It is inconceivable that Young's modulus will control stress field in the plastic domain. It is more likely to be controlled by the yield stress σ_Y which is expected to play an important role in stress analysis in the plastic zone.

Let us assume that the plate is in a state of plane stress and obeys Mises yield criterion. Consider Mises effective stress $\sigma_e = \sigma_Y$ and rewrite it in the form

$$3\frac{(\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy})}{2}\,\frac{(\sigma_{yy} - \sigma_{xx} - 2i\sigma_{xy})}{2} = \left(\sigma_Y^2 - \left(\frac{\sigma_{yy} + \sigma_{xx}}{2}\right)^2\right).$$

To satisfy the above equation, choose

$$\sigma_{yy} + \sigma_{xx} = 2\sigma_Y \sin t,$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = \frac{2\sigma_Y}{\sqrt{3}} \cos t e^{i\varphi}.$$

Note that we use linearized stress-strain curve in the case of work hardening material. The curve is linear even for non-hardening material. Since the sum $\sigma_{yy} + \sigma_{xx}$ known to be harmonic for linear solids, choose an analytic function $f_p(z)$ in the plastic domain such that

$$\sigma_{yy} + \sigma_{xx} = (f_p' + \bar{f}_p').$$

Therefore,

$$\mathrm{Re}\, f_p' = \sigma_Y \sin t,$$

where Re f_p' is the real part of $\partial f_p/\partial z$.

Since the crack surface is traction free, the only non-zero stress on it is tangential. It can be identified with the effective stress. In other words, $\sigma_{yy} + \sigma_{xx} = \sigma_Y$ or $t = \pi/6$, in the plastic domain of the crack surface.

On the surface $y = 0$, the shear stress must vanish because of symmetry. Therefore, x- and y-surfaces on the extended crack line are principal planes on which the principal stresses are

$$\sigma_2 = \sigma_Y(\sin t + \cos t/\sqrt{3}),$$

$$\sigma_1 = \sigma_Y(\sin t - \cos t/\sqrt{3}).$$

Some general conclusions regarding the stress distribution for $x \geq \alpha$ on the line $y = 0$ can be drawn even if the exact form of the function f_p' is unknown. For example, at the crack tip, $\sigma_1 = 0$, $\sigma_2 = \sigma_Y$, hence Re $f_p = \sigma_Y/2$. Thereafter, both principal stresses increase with Re f_p' until at Re $f_p' = \sqrt{3/4}\,\sigma_Y$, $\sigma_1 = \sigma_Y/\sqrt{3}$ and σ_2 attains a maximum value of $2\sigma_Y/\sqrt{3}$. Subsequently, σ_2 decreases while σ_1 continues to increase until the elastic plastic boundary is reached.

In view of the piece-wise linear stress-strain behavior assumed in the analysis, the analytic function f, or rather its derivative $\partial f/\partial z$ of the elastic domain can be used in the plastic domain as well. However, the analytic function g has no role in the plastic domain in which the yield criterion must be used for evaluating deviatoric stresses. It means that an additional term must be added to $\partial f/\partial z$ of the elastic domain to obtain the corresponding function of the plastic domain. Accordingly, choose

$$f_p' = A_1 + A_2 \coth \xi + A_3/\sinh \xi.$$

This choice immediately leads to

$$\sigma_{yy} + \sigma_{xx} = 8A_1 + 4\frac{A_2 \sinh(\xi + \bar{\xi}) + 2A_3(\sinh \xi + \sinh \bar{\xi})}{\sinh \xi \sinh \bar{\xi}}.$$

A yield criterion must be used to obtain deviatoric stress components. For a Mises solid in plane stress for example, the yield criterion can be rearranged to obtain

$$\frac{(\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy})}{2}\,\frac{(\sigma_{yy}-\sigma_{xx}-2i\sigma_{xy})}{2}=\frac{1}{3}\left(\sigma_e^2-\left(\frac{\sigma_{yy}+\sigma_{xx}}{2}\right)^2\right).$$

To find the values of the constants A_1, A_2 and A_3, it is necessary to impose the condition that the stresses must be continuous across the elastic-plastic interface. Moreover, for a non-hardening material, stress along the crack surface in the plastic domain must remain at the yield value σ_Y. The second condition can be satisfied easily by choosing $A_1=\sigma_Y/8$ and

$$A_2\cosh\alpha_f+A_3\cos\beta=0.$$

Both α_f and β must therefore vary in the plastic domain. At the elastic-plastic interface, $\alpha_f=\alpha_e$ and $\beta=\beta_Y$ and hence

$$A_3=-\frac{\cosh\alpha_e}{\cos\beta}A_2.$$

The value $\alpha=\alpha_t$ at the crack tip can be obtained from

$$\cosh\alpha_t=-\frac{A_3}{A_2}=\frac{\cosh\alpha_e}{\cos\beta_Y}.$$

To find the value of A_2, use the expression for stresses on the line $\beta=0$ on which

$$\sigma_{yy}+\sigma_{xx}=\sigma_Y+8\frac{A_2\cosh\alpha+A_3}{\sinh\varepsilon}=\sigma_Y+8A_2\frac{\cosh\alpha-\cosh\alpha_t}{\cosh\alpha}.$$

For continuity across the elastic-plastic boundary $\alpha=\alpha_Y$, the above equation must yield a value equal to that obtained from the elastic solution. Therefore,

$$\sigma_Y+8A_2\frac{\cosh\alpha_Y-\cosh\alpha_t}{\sinh\alpha_Y}=S(-e^{2\alpha_e}+(e^{2\alpha_e}+1)\coth\alpha_Y).$$

The above equation can be solved for

$$8A_2=\frac{S(-e^{2\alpha_e}+(e^{2\alpha_e}+1)\coth\alpha_Y)-\sigma_Y}{\cosh\alpha_Y-\cosh\alpha_t}\sinh\alpha_Y.$$

Hence on $\beta=0$,

$$\sigma_{yy}+\sigma_{xx}=\sigma_Y+\left(\frac{S(-e^{2\alpha_e}+(e^{2\alpha_e}+1)\coth\alpha_Y)-\sigma_Y}{\cosh\alpha_Y-\cosh\alpha_t}-\sinh\alpha_Y\right)\frac{\cosh\alpha-\cosh\alpha-t}{\cosh\alpha}.$$

The difference in stresses is obtained with the help of the yield criterion and it can be expressed in the form

$$\sigma_{yy}-\sigma_{xx}=\sqrt{\frac{4\sigma_Y^2-(\sigma_{yy}+\sigma_{xx})^2}{3}}.$$

The above two equations can be used to find stresses σ_{xx} and σ_{yy} between the crack tip and the yield point on $\beta=0$.

Appendix

Uniaxial stress-strain equation for a bilinear solid can be expressed in the form

$$\varepsilon = \frac{\sigma_Y}{E} + \frac{\sigma - \sigma_Y}{E_n},$$

where E is Young's modulus (slope of the stress strain line of the first segment), E_n is the slope of the second line and σ_Y is the translation stress between the two linear segments. It can be identified with the yield stress. The transverse strain is

$$\varepsilon_{\text{tr}} = -\upsilon\frac{\sigma_Y}{E} - \upsilon_n\frac{\sigma - \sigma_Y}{E_n},$$

where υ_n is the Poisson ratio for the second segment.

Suppose the strain in the second segment consists of a sum of elastic and plastic deformation such that

$$\frac{1}{E_n} = \frac{1}{E} + \frac{1}{E_{pn}}.$$

Therefore, $\varepsilon = (\sigma/E) + (\sigma - \sigma_Y)/E_{pn}$. Assuming plastic incompressibility, we obtain

$$\varepsilon_{\text{tr}} = -\upsilon\frac{\sigma_Y}{E} - \upsilon_n\frac{\sigma - \sigma_Y}{E_n} = -\upsilon\frac{\sigma}{E} - \frac{\sigma - \sigma_Y}{2E_{pn}}.$$

For a work hardening solid in plastic deformation, $\sigma > \sigma_Y$ and hence the above equation can be rearranged to yield

$$\frac{\upsilon_n}{E_n} = \frac{\upsilon}{E} + \frac{1}{2E_{pn}}.$$

Suppose the crack surface open in the form used in LEFM solution or as predicted by the solution of Singh et al. (1994).

Suppose a crack $\alpha = 0$ opens under external load into a surface $\alpha = \alpha_f$. The presence of crack is likely to induce non-homogeneity in the stress field. Hence, to solve for stresses, choose

$$f = cA\cos\xi + cB\sin\xi + C = c(A + B)\cosh\xi - cBe^{-\xi} + C,$$

where A, B, C and c are constants. Since the crack surface remains free of traction,

$$\frac{\partial\Phi}{\partial\bar{z}} = f + z\overline{\left(\frac{\partial f}{\partial z}\right)} + \bar{g} + C = A \text{ constant on } \alpha = \alpha_f.$$

To satisfy the above condition, choose

$$\bar{g} = -2cA\cosh(2\alpha_f - \bar{\xi}) - cB\frac{\cosh 2\alpha_f}{\sinh\bar{\xi}}.$$

In view of these choices,

$$\frac{\partial\Phi}{\partial\bar{z}} = 2cA\{\cosh\xi - \cosh(2\alpha_f - \bar{\xi})\} + cB\frac{\cosh 2\alpha - \cosh 2\alpha_f}{\sinh\bar{\xi}} + C.$$

Stress field can now be obtained as follows:

$$\sum_s \equiv \sigma_{yy} + \sigma_{xx} = 4\frac{\partial^2\Phi}{\partial z\partial \bar{z}} = 8A + 8B\frac{\sinh 2\alpha}{\cosh 2\alpha - \cos 2\beta},$$

$$\sum_d \equiv \sigma_{yy} - \sigma_{xx} - 2i\sigma_{xy} = \frac{\partial^2\Phi}{\partial \bar{z}\partial \bar{z}} = 8A\frac{\sinh(2\alpha_f - \bar{\xi}}{\sinh\bar{\xi}} + 4B\frac{\cosh 2\alpha_f \cosh\bar{\xi} - \cosh\xi}{(\sinh\bar{\xi})^3}.$$

To evaluate the constants A and B, observe that the above stress field must be consistent with the applied far-field stress at large distances from the origin. In other words, they must satisfy the condition that both $\sum_s$ and $\sum_d \to S$ as $\alpha \to \infty$. These conditions lead to

$$A = -Se^{2\alpha_f}/8, \quad B = S(e^{2\alpha_f} + 1)/8.$$

When these constants are substituted in (2), (3) and (4), they lead to

$$f = \frac{cS}{8}\{-e^{2\alpha_f}\cosh\xi + (e^{2\alpha_f} + 1)\sinh\xi\} + C,$$

$$\frac{\partial\Phi}{\partial\bar{z}} = \frac{cS}{8}\left\{-2e^{2\alpha_f}(\cosh\xi - \cosh(2\alpha_f - \bar{\xi})) + (e^{2\alpha_f} + 1)\frac{\cosh 2\alpha - \cosh 2\alpha_f}{\sinh\bar{\xi}}\right\},$$

$$\sum_s \equiv \frac{\sigma_{yy} + \sigma_{xx}}{S} = -2e^{2\alpha_f} + (2e^{2\alpha_f} + 1)\frac{\sinh 2\alpha}{\cosh 2\alpha - \cos 2\beta},$$

$$\sum_d \equiv \frac{\sigma_{yy} - \sigma_{xx} - 2i\sigma_{xy}}{S} = -2e^{2\alpha_f}\frac{\sinh(2\alpha_f - \bar{\xi})}{\sinh\bar{\xi}} + (2e^{2\alpha_f} + 1)\frac{\cosh 2\alpha_f \cosh\bar{\xi} - \cosh\xi}{(\sinh\bar{\xi})^3}.$$

To find stress components, it is necessary to evaluate the real and imaginary parts, Re $\sum_d$ and Im $\sum_d$ of $\sum_d$. Subsequently, stresses can be obtained from

$$\sigma_{xx} = S\frac{\sum_s - \mathrm{Re}\sum_d}{2}, \quad \sigma_{yy} = S\frac{\sum_s + \mathrm{Re}\sum_d}{2}, \quad \sigma_{xy} = -S\frac{\mathrm{Im}\sum_d}{2}.$$

The last two equations in (5) suggest that for a given applied load S, stresses in the plate depend only on the crack opening parameter α_f. It is therefore reasonable to assume that the parameter controls the stress field around the crack. To find its value, consider the displacement of points on the crack surface.

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