GALERKIN METHOD FOR STOCHASTIC ALGEBRAIC EQUATIONS AND PLATES ON RANDOM ELASTIC FOUNDATION

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Abstract

A new perspective is presented on the Galerkin solution for linear stochastic algebraic equations, that is, linear algebraic equations with random coefficients. It is shown that (1) a stochastic algebraic equation has an optimal Galerkin solution, that is, a Galerkin solution that is best in the mean square sense, and (2) the optimal Galerkin solution is equal to the conditional expectation of the exact solution with respect to a σ-field coarser than the σ-field relative to which this solution is measurable. Galerkin solutions that are not optimal are called sub-optimal. Both optimal and sub-optimal Galerkin solutions are defined and constructed. Optimal and sub-optimal Galerkin solutions are used to calculate statistics of the displacement of a simply supported plate sitting on a random elastic foundation. The accuracy of these Galerkin solutions is assessed by Monte Carlo simulation.

1. Introduction

Ordinary or partial differential equations with random coefficients, input and/or boundary conditions, referred to as stochastic differential equations, are used to formulate a broad range of mechanics problems. It is common to approximate the solution of stochastic differential equations by that of algebraic equations with random coefficients, called stochastic algebraic equations. There are no general and efficient methods for finding the probability law of the solution of a stochastic algebraic equation. Taylor series, perturbation, Neumann series, decomposition, equivalent linearization, iteration, and other approximate techniques can be used to solve these equations (Deb et al. 2001; Ghanem and Spanos, 1991; Grigoriu, 2002: section 8.3.1). Monte Carlo simulation is the only available method capable of providing estimates for the probability law of the solution of general stochastic equations, but can be numerically prohibitive if applied to solve realistic problems.

Our objectives are to (1) present an alternative interpretation of the Galerkin method for solving stochastic algebraic equations and (2) apply this method to calculate statistics for the displacement of a plate supported by a random elastic foundation. It is shown that there is an optimal Galerkin solution for a stochastic algebraic equation, that is, a Galerkin solution minimizing the mean square error. The optimal Galerkin solution is equal to the conditional expectation of the exact solution of a stochastic algebraic equation with respect to a σ-field coarser than the σ-field with respect to which this solution is measurable. Galerkin solutions that are not optimal are said to be sub-optimal. Second-moment properties, distributions, and other statistics are developed for both optimal and sub-optimal Galerkin solutions. Statistics are calculated for optimal and sub-optimal Galerkin solutions for a simply supported plate sitting on a random elastic foundation. The accuracy of the Galerkin solutions is evaluated by Monte Carlo simulation.

2. Stochastic Algebraic Equations

Consider the stochastic algebraic equation

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$$\big(\boldsymbol{a} + \boldsymbol{r}(\boldsymbol{Z})\big)\, \boldsymbol{X} = \boldsymbol{Y}, \tag{1}$$

where $\boldsymbol{a}$ denotes an (d, d) real-valued deterministic matrix, $\boldsymbol{Z}$ and $\boldsymbol{Y}$ are $\mathbb{R}^n$-valued and $\mathbb{R}^d$-valued random variables defined on some probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$, respectively, that may or may not coincide, and $\boldsymbol{r}$ is an (d, d) real-valued matrix whose entries depend on $\boldsymbol{Z}$. It is assumed that (1) the random variables $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are independent and are in L_2, (2) the function $\boldsymbol{r}(\cdot)$ is measurable, so that $\boldsymbol{r}(\boldsymbol{Z})$ and $\boldsymbol{a} + \boldsymbol{r}(\boldsymbol{Z})$ are random variables on $(\Omega_1, \mathcal{F}_1, P_1)$, and (3) the operator $\boldsymbol{a} + \boldsymbol{r}(\boldsymbol{Z})$ is bounded almost surely (a.s.), that is, $\boldsymbol{a} + \boldsymbol{r}(\boldsymbol{Z}(\omega_1))$ is bounded for all $\omega_1 \in \Omega_1 \setminus N_1$, where $N_1 \in \mathcal{F}_1$ and $P_1(N_1) = 0$.

If $\det(\boldsymbol{a} + \boldsymbol{r}(\boldsymbol{Z})) \neq 0$ a.s., then Equation (1) has the unique solution

$$\boldsymbol{X}(\omega_1, \omega_2) = (\boldsymbol{a} + \boldsymbol{r}(\boldsymbol{Z}(\omega_1)))^{-1}\, \boldsymbol{Y}(\omega_2) = \boldsymbol{\beta}\big(\boldsymbol{Z}(\omega_1)\big)\, \boldsymbol{Y}(\omega_2) \quad \text{a.s.}, \tag{2}$$

which is a random variable on the product probability space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, P_1 \otimes P_2)$, where $\Omega_1 \times \Omega_2$, $\mathcal{F}_1 \otimes \mathcal{F}_2$, and $P_1 \otimes P_2$ denote the product sample space, the σ-field generated by the measurable rectangles $\{A_1 \times A_2\}$, $A_1 \in \mathcal{F}_1$, $A_2 \in \mathcal{F}_2$, and the extension of the set function $R(A_1 \times A_2) = P_1(A_1)\, P_2(A_2)$ defined for $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$ to $\mathcal{F}_1 \otimes \mathcal{F}_2$, respectively.

3. Galerkin Method

The Galerkin solution of Equation (1) requires to discretize the stochastic dimension, that is, the random variables $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$. Polynomials chaos representations and partitions of the sample space $\Omega_1 \times \Omega_2$ or of the range of the random variables $(\boldsymbol{Y}, \boldsymbol{Z})$ can be used to discretize the stochastic dimension of Equation (1). Galerkin method based on polynomial chaos has been applied successfully to solve a broad range of stochastic problems (Ghanem and Spanos, 1991), although there are some theoretical aspects of the method that remain to be clarified. For example, the m.s. convergence of polynomial chaos representations for $\boldsymbol{Y}$; $\boldsymbol{Z}$ to $\boldsymbol{Y}$; $\boldsymbol{Z}$ does not guarantee the m.s. convergence of the corresponding representations for $\boldsymbol{X}$ to the exact solution $\boldsymbol{X}$. Also, moments of order 3 and higher of polynomial chaos representations may not converge to corresponding target moments (Field and Grigoriu, 2004). Galerkin method using partitions of the range of the random variables $(\boldsymbol{Y}, \boldsymbol{Z})$ views the solution $\boldsymbol{X}$ as an unknown function of $(\boldsymbol{Y}, \boldsymbol{Z})$, that can be approximated by polynomials or other functions depending on some unknown coefficients. The solution of Equation (1) is found by solving a deterministic version of this equation obtained by viewing $\boldsymbol{Z}$ as a parameter taking values in $\boldsymbol{Z}(\Omega_1)$. The measure on $\boldsymbol{Z}(\Omega_1)$ is the density of $\boldsymbol{Z}$ rather than the Lebesgue measure (Babuška et al. 2004).

The version of the Galerkin method considered here is based on partitions of the product sample space $\Omega_1 \times \Omega_2$. Let

$$\big(\emptyset, \Omega_k\big) = \mathcal{G}_{k,1} \subset \cdots \subset \mathcal{G}_{k,i} \subset \cdots \subset \mathcal{G}_{k,n_k} = \mathcal{F}_k, \quad k = 1, 2, \tag{3}$$

be two sequences of sub-σ-fields on probability spaces $(\Omega_k, \mathcal{F}_k, P_k)$, $k = 1, 2$, that can be constructed from, for example, finite partitions of the sample spaces Ω_k, $k = 1, 2$. Let $\{\Lambda_q\}$, $q = 1, \ldots, m$, and $\{\Gamma_r\}$, $r = 1, \ldots, m'$, be measurable partitions of the sample spaces Ω_1 and Ω_2, respectively. The σ-fields generated by the sets $\{\Lambda_q\}$, $\{\Gamma_r\}$, and $\{\Lambda_q \times \Gamma_r\}$ are of type $\mathcal{G}_{1,i}$, $\mathcal{G}_{2,j}$, and $\mathcal{G}_{1,i} \otimes \mathcal{G}_{2,j}$ in Equation (3), respectively. In the reminder of this section we define optimal and sub-optimal Galerkin solutions and give some of their properties.

Property 1. *The optimal Galerkin solution corresponding to the information content of a sub-σ-field $\mathcal{G}_{1,i} \otimes \mathcal{G}_{2,j}$ is*

$$\boldsymbol{X}_{i,j} = E\big[\boldsymbol{X} \mid \mathcal{G}_{1,i} \otimes \mathcal{G}_{2,j}\big] = E_1\big[\boldsymbol{\beta}(\boldsymbol{Z}) \mid \mathcal{G}_{1,i}\big]\, E_2\big[\boldsymbol{Y} \mid \mathcal{G}_{2,j}\big] \text{ (a. s.)}, \tag{4}$$

where E_k and E denote expectations with respect to the probability measures P_k, $k = 1, 2$, and $P_1 \otimes P_2$, respectively.

Generally, $\boldsymbol{X} = (\boldsymbol{a} + \boldsymbol{r}(\boldsymbol{Z}))^{-1}\, \boldsymbol{Y} = \boldsymbol{\beta}(\boldsymbol{Z})\, \boldsymbol{Y}$ is not $\mathcal{G}_{1,i} \otimes \mathcal{G}_{2,j}$-measurable for $i < n_1$ and/or $j < n_2$, that is, it is not a random variable with respect to the σ-field $\mathcal{G}_{1,i} \otimes \mathcal{G}_{2,j}$. The first equality in Equation (4) follows from the fact that the conditional expectation is the best mean square estimator for $\boldsymbol{X}$ with respect to the information content of $\mathcal{G}_{1,i} \otimes \mathcal{G}_{2,j}$ (Grigoriu, 2002: section 2.17.2). The validity of the second equality in Equation (4) results from properties of the conditional expectation, properties of σ-fields on product spaces, Fubini's theorem, and a theorem by Dynkin (Resnick, 1998: section 2.2).

Consider the special case in which $\mathcal{G}_{2,n_2} = \mathcal{F}_2$. The corresponding optimal Galerkin solution is

$$\boldsymbol{X}_{i,n_2} = E\big[\boldsymbol{\beta}(\boldsymbol{Z}) \mid \mathcal{G}_{1,i}\big]\, \boldsymbol{Y} \tag{5}$$

since $E[\boldsymbol{Y} \mid \mathcal{G}_{2,n_2}] = \boldsymbol{Y}$ a.s. (Equation (4)). This solution is used extensively in applications (Deb et al., 2001). Once the conditional expectations $E[\boldsymbol{\beta}(\boldsymbol{Z}) \mid \mathcal{G}_{1,i}]$ have been calculated, Equation (5) can be used to calculate statistics of the exact solution $\boldsymbol{X}$ approximately.

Property 2. *The optimal Galerkin solution in Equation (4) ranges from the expectation of the exact solution to the exact solution depending on the information content of the sub-σ-fields $\mathcal{G}_{1,i}$ and $\mathcal{G}_{2,j}$.*

Consider sub-σ-fields of $\mathcal{F}_k$ containing limited or full information on the random variables $\boldsymbol{Y}$ and $\boldsymbol{Z}$. The corresponding optimal Galerkin solutions are (Equation (4))

$$\begin{aligned}
\boldsymbol{X}_{1,1} &= E_1\big[\boldsymbol{\beta}(\boldsymbol{Z})\big]\, E_2\big[\boldsymbol{Y}\big] = E\big[\boldsymbol{X}\big] \\
\boldsymbol{X}_{1,n_2} &= E_1\big[\boldsymbol{\beta}(\boldsymbol{Z})\big]\, \boldsymbol{Y} \\
\boldsymbol{X}_{n_1,1} &= \boldsymbol{\beta}(\boldsymbol{Z})\, E_2\big[\boldsymbol{Y}\big] \\
\boldsymbol{X}_{n_1,n_2} &= \boldsymbol{\beta}(\boldsymbol{Z})\, \boldsymbol{Y} = \boldsymbol{X},
\end{aligned} \tag{6}$$

where the above equalities hold almost surely with respect to the product probability measure $P_1 \otimes P_2$. The above results follow from properties of the conditional expectation (Grigoriu, 2002: section 2.7.2). For example, $E_1[\boldsymbol{\beta}(\boldsymbol{Z}) \mid \mathcal{G}_{1,i}]$ is equal to $E_1[\boldsymbol{\beta}(\boldsymbol{Z})]$ and $\boldsymbol{\beta}(\boldsymbol{Z})$ a.s. for $i = 1$ and $i = n_1$, respectively. Also, $\boldsymbol{X}_{n_1,n_2}$ is equal to the exact solution $\boldsymbol{X}$ a.s. since the sub-σ-fields $\mathcal{G}_{k,n_k}$ coincide with the σ-fields $\mathcal{F}_k$, so that we have $E_1[\boldsymbol{\beta}(\boldsymbol{Z}) \mid \mathcal{G}_{1,n_1}] = \boldsymbol{\beta}(\boldsymbol{Z})$ and $E_2[\boldsymbol{Y} \mid \mathcal{G}_{2,n_2}] = \boldsymbol{Y}$ a.s.

Property 3. *The second-moment properties of the optimal Galerkin solution are*

$$\begin{aligned}
\boldsymbol{\mu}_{i,j} &= E\big[\boldsymbol{X}_{i,j}\big] = E\big[\boldsymbol{X}\big] \\
\boldsymbol{\gamma}_{i,j} &= E\big[\big(\boldsymbol{X}_{i,j} - E[\boldsymbol{X}_{i,j}]\big)\big(\boldsymbol{X}_{i,j} - E[\boldsymbol{X}_{i,j}]\big)^T\big] \\
&= E_1\Big[E_1\big[\boldsymbol{\beta}(\boldsymbol{Z}) \mid \mathcal{G}_{1,i}\big]\, E_2\big[\hat{\boldsymbol{Y}}\, \hat{\boldsymbol{Y}}^T\big]\, E_1\big[\boldsymbol{\beta}(\boldsymbol{Z}) \mid \mathcal{G}_{1,i}\big]^T\Big] \\
&\quad + E_1\Big[\hat{\boldsymbol{\beta}}(\boldsymbol{Z})\, E_2\big[\boldsymbol{Y}\big]\, E_2\big[\boldsymbol{Y}\big]^T\, \boldsymbol{\beta}(\boldsymbol{Z})^T\Big]
\end{aligned} \tag{7}$$

with the notation

$$\begin{aligned}
\hat{\boldsymbol{Y}} &= E_2\big[\boldsymbol{Y} \mid \mathcal{G}_{2,j}\big] - E_2\big[\boldsymbol{Y}\big] \\
\hat{\boldsymbol{\beta}}(\boldsymbol{Z}) &= E_1\big[\boldsymbol{\beta}(\boldsymbol{Z}) \mid \mathcal{G}_{1,i}\big] - E_1\big[\boldsymbol{\beta}(\boldsymbol{Z})\big].
\end{aligned} \tag{8}$$

The expectation of $\boldsymbol{X}_{i,j}$ is $E_1\{E_1[\boldsymbol{\beta}(\boldsymbol{Z}) \mid \mathcal{G}_{1,i}]\}\, E_2\{E_2[\boldsymbol{Y} \mid \mathcal{G}_{2,j}]\}$, and these expectations are equal to $E_1[\boldsymbol{\beta}(\boldsymbol{Z})]\, E_2[\boldsymbol{Y}] = E[\boldsymbol{\beta}(\boldsymbol{Z})\, \boldsymbol{Y}] = E[\boldsymbol{X}]$ by properties of the conditional expectation and the independence of $\boldsymbol{Z}$ and $\boldsymbol{Y}$. Hence, the optimal Galerkin solution $\boldsymbol{X}_{i,j}$ in Equation (4) is an unbiased approximation for the exact solution $\boldsymbol{X} = \boldsymbol{\beta}(\boldsymbol{Z})\, \boldsymbol{Y}$.

We have $\boldsymbol{X}_{i,j} - E[\boldsymbol{X}_{i,j}] = E_1[\boldsymbol{\beta}(\boldsymbol{Z}) \mid \mathcal{G}_{1,i}]\, \hat{\boldsymbol{Y}} + \hat{\boldsymbol{\beta}}(\boldsymbol{Z})\, E_2[\boldsymbol{Y}]$ with the notation in Equation (8). The definition of the covariance matrix $\boldsymbol{\gamma}_{i,j}$, the independence of $\boldsymbol{Z}$ and $\boldsymbol{Y}$, and properties of the conditional expectation give the second relation in Equation (7). If $\mathcal{G}_{2,j}$ coincides with $\mathcal{F}_2$, then $\hat{\boldsymbol{Y}}$ in Equation (8) becomes $\boldsymbol{Y} - E_2[\boldsymbol{Y}]$.

In the remainder of the paper we denote σ-fields of the type $\mathcal{G}_{1,i}$, $\mathcal{G}_{2,j}$, and $\mathcal{G}_{1,i} \otimes \mathcal{G}_{2,j}$ by $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2$ for simplicity. Generally, we chose the σ-fields $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}$ to be coarser then $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_1 \otimes \mathcal{F}_2$, that is, we have $\mathcal{G}_1 \subset \mathcal{F}_1$, $\mathcal{G}_2 \subset \mathcal{F}_2$, and $\mathcal{G} \subset \mathcal{F}_1 \otimes \mathcal{F}_2$. Accordingly, the optimal Galerkin solution with respect to $\mathcal{G}$ is (Equation (4))

$$\tilde{\boldsymbol{X}} = E\big[\boldsymbol{X} \mid \mathcal{G}\big] = E_1\big[\boldsymbol{\beta}(\boldsymbol{Z}) \mid \mathcal{G}_1\big]\, E_2\big[\boldsymbol{Y} \mid \mathcal{G}_2\big]. \tag{9}$$

We also consider Galerkin solutions that differ from $E[\boldsymbol{X} \mid \mathcal{G}]$. These solutions are referred to as sub-optimal Galerkin solutions, and are also denoted by $\tilde{\boldsymbol{X}}$. Generally, sub-optimal Galerkin solution are biased approximations of the exact solutions. The error of a sub-optimal Galerkin solution $\tilde{\boldsymbol{X}}$ is

$$\tilde{\boldsymbol{X}} - \boldsymbol{X} = \Big(E\big[\boldsymbol{X} \mid \mathcal{G}\big] - \boldsymbol{X}\Big) + \Big(\tilde{\boldsymbol{X}} - E\big[\boldsymbol{X} \mid \mathcal{G}\big]\Big). \tag{10}$$

The first term in Equation (10) is the error of the optimal Galerkin solution, and cannot be reduced for a given $\mathcal{G}$. The second term in Equation (10) corresponds to the difference between the optimal and a sub-optimal Galerkin solutions. This component of the error can be reduced by improving the sub-optimal solution.

Property 4. *If $\{\Lambda_q\}$, $q = 1, \ldots, m$, is a partition of Ω_1, $\mathcal{G}_1 = \sigma(\{\Lambda_q\})$, and $\mathcal{G}_2 = \mathcal{F}_2$, then the mean, the correlation, and the distribution of the corresponding optimal and sub-optimal Galerkin solutions $\tilde{\boldsymbol{X}}$ have the expressions:*

$$\begin{aligned}
E\big[\tilde{\boldsymbol{X}}\big] &= \sum_{q=1}^{m} E_2\big[\boldsymbol{W}_q\big]\, P_1\big(\Lambda_q\big) \\
E\big[\tilde{\boldsymbol{X}}\, \tilde{\boldsymbol{X}}^T\big] &= \sum_{q=1}^{m} E_2\big[\boldsymbol{W}_q\, \boldsymbol{W}_q^T\big]\, P_1\big(\Lambda_q\big) \\
P\big(\tilde{X}_{i_1} \le \xi_1, \ldots, \tilde{X}_{i_s} \le \xi_s\big) &= \sum_{q=1}^{m} P_2\big(W_{q,i_1} \le \xi_1, \ldots, W_{q,i_s} \le \xi_s\big)\, P_1\big(\Lambda_q\big),
\end{aligned} \tag{11}$$

where $\boldsymbol{W} = \boldsymbol{\alpha}_q\, \boldsymbol{Y}$ is a vector in $\mathbb{R}^d$ with coordinates $W_{q,j}$, $j = 1, \ldots, d$, and

$$\boldsymbol{\alpha}_q = \begin{cases} E\big[\boldsymbol{\beta}(\boldsymbol{Z}) \mid \Lambda_q\big], & \text{for optimal Galerkin solution} \\ \boldsymbol{\beta}_q = \boldsymbol{\beta}(z_q), & \text{for sub-optimal Galerkin solution.} \end{cases} \tag{12}$$

We have

$$\tilde{\boldsymbol{X}} = \Big(\sum_{q=1}^{m} \boldsymbol{\alpha}_q\, 1_{\Lambda_q}\Big), \quad \boldsymbol{Y} = \sum_{q=1}^{m} \boldsymbol{W}_q\, 1_{\Lambda_q} \tag{13}$$

by properties of the conditional expectation and Equation (5). The above sub-optimal Galerkin solution corresponds to an approximate representation of $\boldsymbol{Z}$ setting this variable constant and equal to z_q in each Λ_q, that is, $\boldsymbol{Z}$ is approximated by the simple random variable $\tilde{\boldsymbol{Z}} = \sum_{q=1}^{m} z_q \, 1_{\Lambda_q}$, where 1_{Λ_q} denotes the indicator function for Λ_q defined by $1_{\Lambda_q}(\omega_1) = 1$ and 0 for $\omega_1 \in \Lambda_q$ and $\omega_1 \notin \Lambda_q$, respectively.

The results in Equation (11) follow from Equation (4), by the linearity of the expectation operator and the law of total probability. If $\boldsymbol{Y} = z$ is a deterministic vector, then $E_2[\boldsymbol{W}_q] = \boldsymbol{w}_q = \boldsymbol{\alpha}_q \, z$, $E_2[\boldsymbol{W}_q \, \boldsymbol{W}_q^T] = \boldsymbol{w}_q \, \boldsymbol{w}_q^T$, and $P_2(W_{q,i_1} \leq \xi_1, \ldots, W_{q,i_s} \leq \xi_s) = 1\big(w_{q,i_1} \leq \xi_1, \ldots, w_{q,i_s} \leq \xi_s\big)$. If $\boldsymbol{Y}$ is a random vector, then $E_2[\boldsymbol{W}_q] = \boldsymbol{\alpha}_q \, E_2[\boldsymbol{Y}]$ and $E_2[\boldsymbol{W}_q \, \boldsymbol{W}_q^T] = \boldsymbol{\alpha}_q \, E_2[\boldsymbol{Y} \, \boldsymbol{Y}^T] \, \boldsymbol{\alpha}_q^T$ and the probabilities $P_2(W_{q,i_1} \leq \xi_1, \ldots, W_{q,i_s} \leq \xi_s)$ can be calculated by, for example, Monte Carlo simulation.

4. Plates on Random Elastic Foundation

Consider a simply supported rectangular plate with unit stiffness sitting on a linear elastic foundation with stiffness $K(x, y)$ and subjected to a spatially distributed load $Q(x, y)$. The displacement $W(x, y)$ of the plate is the solution of the partial differential equation

$$\Delta\Delta W(x, y) + K(x, y) \, W(x, y) = Q(x, y), \quad (x, y) \in D = (0, a) \times (0, b), \tag{14}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ denotes the Laplace operator. The plate displacement satisfies the conditions $W = 0$ on the boundary ∂D of D, $\partial^2 W/\partial x^2 = 0$ on $\{0\} \times (0, b)$ and $\{a\} \times (0, b)$, and $\partial^2 W/\partial y^2 = 0$ on $(0, a) \times \{0\}$ and $(0, a) \times \{b\}$. It is assumed that the foundation stiffness K is random and the applied load Q is deterministic.

The foundation stiffness is modeled by the homogeneous translation field

$$K(x, y) = \alpha_1 + \big(\alpha_2 - \alpha_1\big) \, \Phi\big(G(x, y)\big) = \alpha_1 + \big(\alpha_2 - \alpha_1\big) \, \Phi\big(\boldsymbol{\theta}(x, y) \, \boldsymbol{Z}\big), \tag{15}$$

where $0 < \alpha_1 < \alpha_2 < \infty$ are some constants,

$$\begin{aligned} G(x, y) &= \sum_{k=1}^{n/2} \sigma_k \left(A_k \, \cos(\boldsymbol{\nu}_k \cdot (x, y)) + B_k \, \sin(\boldsymbol{\nu}_k \cdot (x, y)) \right) \\ &= \boldsymbol{\theta}(x, y) \, \boldsymbol{Z}, \quad (x, y) \in D = (0, a) \times (0, b), \end{aligned} \tag{16}$$

is a homogeneous Gaussian field, $n/2 \geq 1$ is an integer,

$$\begin{aligned} \boldsymbol{\theta}(x, y) &= \big[\sigma_1 \cos(\boldsymbol{\nu}_1 \cdot (x, y)) \ldots \sigma_n \cos(\boldsymbol{\nu}_n \cdot (x, y)) \;\; \sigma_1 \sin(\boldsymbol{\nu}_1 \cdot (x, y)) \ldots \sigma_n \sin(\boldsymbol{\nu}_n \cdot (x, y))\big], \\ \boldsymbol{Z}^T &= \big[A_1 \ldots A_{n/2} \; B_1 \ldots B_{n/2}\big], \end{aligned} \tag{17}$$

$\boldsymbol{\nu}_k = (\nu_{k,x}, \nu_{k,y})$ are wave frequencies, $\nu_{k,x}, \nu_{k,y} > 0$ are some constants, $\boldsymbol{\nu}_k \cdot (x, y) = \nu_{k,x} \, x + \nu_{k,y} \, y$, $\sigma_k > 0$ are some constants such that $\sum_{k=1}^{n/2} \sigma_k^2 = 1$, and (A_k, B_k) are independent $N(0, 1)$ variables. We note that K depends on n independent $N(0, 1)$ random variables, the entries of $\boldsymbol{Z}$, and its marginal distribution is uniform in (α_1, α_2). The Gaussian field G has mean 0, variance 1, and covariance function

$$E[G(x, y) \, G(x', y')] = \sum_{k=1}^{n} \sigma_k^2 \, \cos\big(\boldsymbol{\nu}_k \cdot (x - x', y - y')\big) \tag{18}$$

so that it is homogeneous. The translation field K in Equation (15) with G in Equation (16) is homogeneous and its covariance and correlation functions can be calculated from the probability

law of G. In applications the covariance function of K, rather than that of G, is given, and we need to find a Gaussian field G such that K in Equation (15) has the required properties. The existence of G and the determination of the second-moment properties of G, if it exists, are discussed in (Grigoriu, 1995: section 3.1).

4.1. *Galerkin Solution*

The solution of Equation (14) involves two steps. First, Equation (14) is approximated by a stochastic algebraic equation. Second, optimal and sub-optimal Galerkin solutions are developed for this stochastic algebraic equation and some of their statistics are calculated.

The stochastic differential equation for W in Equation (14) with K in Equation (15) depends on the random variable $\boldsymbol{Z}$ in the definition of the random field K. The finite difference approximation of Equation (14) at an interior node has the form

$$\begin{aligned}&\left(20 + K_{i,j}\right) W_{i,j} + 2\left(W_{i-1,j-1} + W_{i-1,j+1} + W_{i+1,j-1} + W_{i+1,j+1}\right)\\&\quad - 8\left(W_{i,j-1} + W_{i,j+1} + W_{i-1,j} + W_{i+1,j}\right)\\&\quad + W_{i,j-2} + W_{i,j+2} + W_{i-2,j} + W_{i+2,j} = Q_{i,j},\end{aligned} \tag{19}$$

where $W_{i,j}$, $K_{i,j}$, and $Q_{i,j}$ denote the values of $W(x, y)$, $K(x, y)$, and $Q(x, y)$ and the node (i, j) of the finite difference mesh. The above relation written at all nodes with adequate modifications to account for boundary conditions yields a stochastic algebraic equation of the type in Equation (1) with coefficients depending on the random variables $\boldsymbol{Z}$ and solution $\boldsymbol{X}$ with entries the displacements $W_{i,j}$.

Since $\boldsymbol{Z}$ in Equation (17) is a standard Gaussian vector in $\mathbb{R}^n$, $n = 12$, it has the representation

$$\boldsymbol{Z} = \boldsymbol{U}\, R, \tag{20}$$

where $\boldsymbol{U}$ is uniformly distributed on the unit sphere $S_n(1)$ in $\mathbb{R}^n$ centered at the origin of this space and R is a real-valued random variable following a chi distribution with n degrees of freedom. The random variable R is independent of $\boldsymbol{U}$ and has the distribution

$$P_1\left(R \le r\right) = I\left(r^2/2, n/2\right), \tag{21}$$

where $I(x, q) = \int_0^x t^{q-1}\, e^{-t}\, dt/\,\Gamma(q)$ and $\Gamma(q)$ denote the incomplete and the complete gamma functions, respectively.

We begin the construction of a partition $\{\Lambda_q\}$ of Ω_1 by dividing $\mathbb{R}^n$ in rings of radii $0 = r_0 < r_1 < \ldots < r_{n_r} = \infty$ such that each ring has the same probability content, that is,

$$P_1\left(r_{u-1} < R \le r_u\right) = I\left(r_u^2/2, n/2\right) - I\left(r_{u-1}^2/2, n/2\right) = 1/n_r, \quad u = 1, \ldots, n_r. \tag{22}$$

We then divide each ring in subsets of equal volume, and these subsets define the partition $\{\Lambda_q\}$ of Ω_1 used to construct the optimal Galerkin solution. The resulting sets Λ_q have the same probability content, that is, $P_1(\Lambda_q) = 1/(n_r\, n_l)$ for all q, where $n_l \ge 1$ denotes the number of subsets of equal volume in each ring. The conditional expectations $E_1[\boldsymbol{\beta}(\boldsymbol{Z}) \mid \Lambda_q]$ in the definition of the optimal Galerkin solution can be estimated from

$$E_1\left[\boldsymbol{\beta}(\boldsymbol{Z}) \mid \Lambda_q\right] \simeq \frac{1}{n_s} \sum_{u=1}^{n_s} \boldsymbol{\beta}\left(\boldsymbol{Z}(\omega_{1,u})\right) 1_{\Lambda_q}(\omega_{1,u}) \tag{23}$$

where $\{\boldsymbol{Z}(\omega_{1,u})\}$, $u = 1, \ldots, n_s$, are n_s independent samples of $\boldsymbol{Z}$.

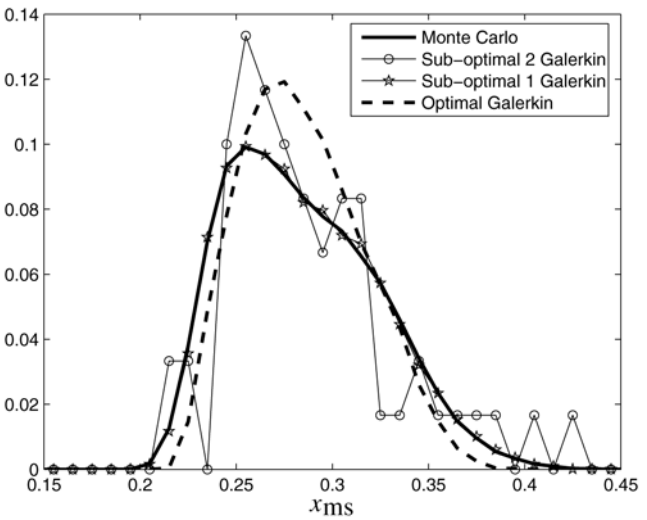

Fig. 1. Probability density function of X_{ms}.

4.2. *Numerical Results*

Numerical results are for $a = 60$, $b = 40$, a finite difference mesh with equal step of 10 in the x and y directions, $Q(x, y) = 1$ in D so that the load vector is $\boldsymbol{Y} = \boldsymbol{y} = \mathbf{1}$, and foundation stiffness K with parameters $\alpha_1 = 1, \alpha_2 = 7, n = 12, \sigma_k = 1/\sqrt{6}$, $\boldsymbol{\nu}_1 = -\boldsymbol{\nu}_4 = (1, 2)$, $\boldsymbol{\nu}_2 = -\boldsymbol{\nu}_5 = (2, 1)$, and $\boldsymbol{\nu}_3 = -\boldsymbol{\nu}_6 = (2, 2)$. The resulting stochastic algebraic equation depends on $n = 12$ independent $N(0, 1)$ variables, the entries of $\boldsymbol{Z}$, and has dimension $d = 15$, so that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are vectors in $\mathbb{R}^{15}$.

We use for the optimal Galerkin solution $n_r = 5$ rings and partition of $S_n(1)$ as follows. The sphere $S_2(1)$ has been divided in 4 equal parts. The image of each of this parts in $S_3(1)$ has been divided in two equal parts. The division in two equal parts has been continued to partition the sphere $S_k(1)$, $k = 4, \ldots, 12$. The resulting number of partitions of $S_{12}(1)$ is 4096. Two partitions of Ω_1 have been used to construct sub-optimal Galerkin solutions, the partition $\{\Lambda_q\}$ defining the optimal Galerkin solution with $z_q \in \Lambda_q$ and a coarser partition. The corresponding solutions are referred to as sub-optimal 1 and sub-optimal 2 Galerkin solutions, respectively. The points z_q for the sub-optimal 2 Galerkin are the points of intersections of spheres of radii $(r_{u-1} + r_u)/2$, $u = 1, \ldots, n_r$, with the coordinates of $\mathbb{R}^n$. Each of these spheres intersects the coordinates of $\mathbb{R}^n$ at $2\,n$ points, so that there are $m = 2\,n\,n_r = 70$ points z_q and subsets Λ_q. The probability of the sets Λ_q is $P_1(\Lambda_q) = 1/(2\,n\,n_r)$ for all q's.

Let X_{ms} and X_{msn} denote the plate displacement at the mid span node of the finite difference mesh and at a node neighboring it, respectively. Figure 1 shows estimates of the probability density function of X_{ms} obtained by Monte Carlo simulation and optimal/sub-optimal Galerkin solutions. The estimate obtained by the sub-optimal 2 Galerkin solution exhibits relatively large fluctuations. The estimates of the density of X_{ms} by the other Galerkin solutions are similar to that obtained by direct Monte Carlo. The density of X_{ms} by the sub-optimal 1 Galerkin solution and Monte Carlo simulation nearly coincide. The Monte Carlo estimates have been calculated from 100,000 independent samples of $\boldsymbol{Z}$ and the Galerkin estimates have been calculated from Equations (11) and (12). Figure 2 shows estimates of the joint probability density of $(X_{\mathrm{ms}}, X_{\mathrm{msn}})$. The estimates of this density by optimal and sub-optimal 1 Galerkin solutions and by Monte Carlo are similar. The sub-optimal 2 Galerkin solution provides a less satisfactory approximation. As in the previous figure, the

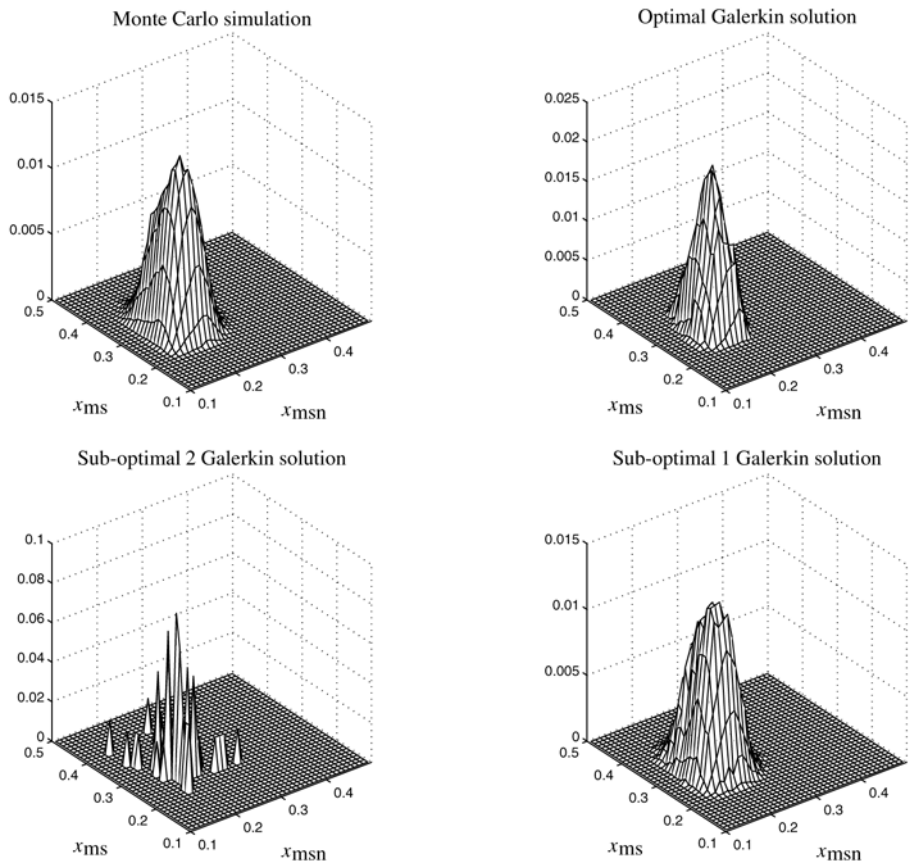

Fig. 2. Joint probability density function of $(X_{\text{ms}}, X_{\text{msn}})$

Monte Carlo estimates are based on 100,000 independent samples of $\boldsymbol{Z}$ and the Galerkin solutions have been calculated from Equations (11) and (12). The differences between the mean displacements by Galerkin solutions and Monte Carlo simulation are less then 0.0958%, 0.1267%, and 1.1056% for optimal, sub-optimal 1, and sub-optimal 2 Galerkin solutions, respectively. These differences and the plots in Figures 1 and 2 show that the optimal Galerkin solution is closer to the Monte Carlo result in the average, consistent with the fact that this solution is unbiased, but the sub-optimal 1 Galerkin solution provides a superior approximation for the probability law of the displacement field.

5. Conclusions

Galerkin solutions have been presented for a class of stochastic algebraic equations, that is, linear algebraic equations with random coefficients. The construction of the Galerkin solution is based on partitions of the sample space associated with the random parameters in the definition of a stochastic algebraic equation. It was shown that there is a Galerkin solution for an arbitrary stochastic algebraic equation that is optimal in the mean square sense. Moreover, the optimal Galerkin solution is equal to the conditional expectation of the exact solution of a stochastic differential equation taken with respect to a σ-field coarser than the σ-field relative to which this solution is measurable.

Galerkin solutions that are not optimal are referred to as sub-optimal. Algorithms are presented for the construction of both optimal and sub-optimal Galerkin solutions.

Optimal and sub-optimal Galerkin solutions have been used to calculate statistics for the displacement of the simply supported plate sitting on a random elastic foundation. The elastic foundation is modeled by a homogeneous translation random field with uniform marginal distribution. The accuracy of the Galerkin solutions depends on the partition of the sample space used in their definition, and was evaluated by Monte Carlo simulation.

References

Babuška, I. M., Temptone, R., and Zouraris, E. (2004). Galerkin finite element approximations of stochastic elliptic partial differential equations. *SIAM Journal of Numerical Analysis*, 42(2), 800–825.

Deb, M. K., Babuška, I. M., and Oden, J. T. (2001). Solution of stochastic partial differential equations using Galerkin finite element techniques. *Computational Methods in Applied Mechanics and Engineering*, 190, 6359–6372.

Field, Jr., R. V. and Grigoriu, M. (2004). On the accuracy of the polynomial chaos approximation. *Probabilistic Engineering Mechanics*, 19(1–2), 65–80.

Ghanem, R. G. and Spanos, P. D. (1991). *Stochastic Finite Elements: A Spectral Approach*. Springer-Verlag, New York.

Grigoriu, M. (1995). *Applied Non-Gaussian Processes: Examples, Theory, Simulation, Linear Random Vibration, and MATLAB Solutions*. Prentice Hall, Englewoods Cliffs, NJ.

Grigoriu, M. (2002). *Stochastic Calculus. Applications in Science and Engineering*. Birkhäuser, Boston.

Resnick, S. I. (1998). *A Probability Path*. Birkhäuser, Boston.