SYMMETRIZATION OF SOME LINEAR CONSERVATIVE NONSELF-ADJOINT SYSTEMS

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Abstract

We derive here equivalent self-adjoint systems for conservative systems of the second kind. Existence of the symmetrized systems confirms that certain conservative systems of the second kind behave as a true conservative system. In this way, study of stability can be carried out on the symmetrized system. In general, it is easier to study a self-adjoint system than a nonself-adjoint system. For the conservative system of the second kind, including the Pflüger column, we also presented a lower bound self-adjoint system. For a linear conservative gyroscopic system, we gave a zero parameter sufficient condition for instability and one for stability. The criteria depend only on the characteristics of the system. For a simple 2-DOF system, the present criteria yield the exact solutions.

1. Introduction

Two types of conservative nonself-adjoint systems are studied. One is the so-called conservative system of the second kind. The other one is a gyroscopic system.

Certain nonself-adjoint systems have only divergent type of instability, despite the presence of a polygenic force. Pflüger's column and Greenhill's shaft are two such systems. Leipholz (1974a, 1974b) called a true divergent nonself-adjoint system which has dynamic properties very similar to those of a self-adjoint system a conservative system of the second kind. He showed that such a system is self-adjoint with respect to an assigned self-adjoint operator, hence it is self-adjoint in a generalized sense. For such a system there exists a Lyapunov for predicting stability (Walker, 1972; Leipholz, 1974a) and a generalized Rayleigh quotient for determining the buckling load (Leipholz, 1974a). Inman and Olsen (1988) included velocity dependent forces in conservative systems of the second kind and proved the generalized self-adjointness and the existence of the eigenfunctions. In this way, the solution can be obtained by a modal analysis.

For certain asymmetric discrete systems, Inman (1983) demonstrated that there exists a similarity transformation that transforms the asymmetric system into an equivalent symmetric one, one that has the same eigenvalues.

Here we show for certain conservative systems of the second kind, an equivalent self-adjoint system can be derived. In this way, a conservative system of the second kind is symmetrized, similar to the symmetrization of an asymmetric discrete system. Existence of the symmetrized system confirms a conservative system of the second kind behaves like a self-adjoint system.

A gyroscopic system is nonself-adjoint because of the presence of a skew-symmetric operator. Walker (1991) presented a Lyapunov containing undetermined parameters for the study of the stability. By determining the parameters by trial and error, he was able to optimize the stability region. Because a zero-parameter stability criterion is easier to use, we derive a zero-parameter sufficient condition for instability and one for stability, all via symmetrization. For the example considered by Walker, the present method also yields the exact solutions.

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2. The Pflüger Column

Pflüger's column is a pinned column under a uniform tangential follower force. Walker (1972) included damping and non-uniform follower force. Here, we include guided end in addition to pinned end. Stability of Pflüger's column is governed by the differential equation

$$-\Omega^2 w + w'''' + pf(x)w'' = 0, \quad x \in [0, 1] \tag{1}$$

and the boundary conditions:

$$\begin{aligned} w = w'' = 0; &\quad \text{for a pinned end,} \\ w = w''' = 0; &\quad \text{for a guided end.} \end{aligned} \tag{2}$$

$w(x)$ is the lateral deflection, Ω is a frequency parameter, $p > 0$ is a load parameter, and $f(x)$ is a non-negative bounded function related to the distribution of the follower force. The operator $f\partial^2$ is nonself-adjoint with respect to the boundary conditions. The operator $-\Omega^2 + \partial^4 + pf(x)\partial^2$, however, is self-adjoint in a generalized sense (Leipholz, 1974a) with respect to the operator ∂^2 under the boundary conditions in Equation (2). The generalized self-adjointness implies

$$\int_0^1 (-\Omega^2 u + u'''' + pfu'')v'' \, \mathrm{d}x = \int (-\Omega^2 v + v'''' + pfv'')u'' \, \mathrm{d}x. \tag{3}$$

$u(x)$ and $v(x)$ are any two admissible functions satisfying all the boundary conditions for $w(x)$.

We will show that the Pflüger column is a conservative system by symmetrizing it. We will also show that Leipholz's generalized self-adjointness can be reduced to the classical self-adjointness.

Let us differentiate Equation (1) with respect to x and denote w' by $y(x)$. Then Equation (1) becomes

$$-\Omega^2 y + y'''' + p[f(x)y']' = 0. \tag{4}$$

The follower force now assumes the appearance of a unidirectional loading. Whether or not the operator in Equation (4) is self-adjoint also depends on the boundary conditions for $y(x)$.

At a pinned end of the column, $w'' = 0$ implies $y' = 0$. From Equation (1), $w = w'' = 0$ implies $w'''' = 0$, which in turn implies $y''' = 0$. As can be seen, a pinned end for w becomes a guided end for y. At a guided end of the column, $w' = 0$ implies $y = 0$, and $w''' = 0$ implies $y'' = 0$. It is seen that a guided end for w turns out to be a pinned end for y.

The system in Equation (4) is equivalent to the system in Equation (1), because Equation (4) and the associated boundary conditions are derived from Equation (1) via the transformation of variable: $y = w'$. Hence the eigensolutions of Equation (1) are also eigensolutions of Equation (4).

The operator in Equation (4) is self-adjoint. It can be shown in the usual manner that for admissible y and z, the following inner products hold:

$$\int_0^1 [-\Omega^2 y + y'''' + p(fy')']z \, \mathrm{d}x = \int_0^1 [-\Omega^2 z + z'''' + p(fz')']y \, \mathrm{d}x. \tag{5}$$

The expression in Equation (5) can also be derived from the generalized self-adjointness in Equation (3) by integration by parts to obtain

$$\int_0^1 [-\Omega^2 u' + u'''' + f(fu'')']v' \, \mathrm{d}x = \int_0^1 [-\Omega^2 v' + v'''' + p(fv'')']u' \, \mathrm{d}x \tag{6}$$

and then by denoting u' by y and v' by z. We have thus shown that the operator in Equation (4) is self-adjoint. So the Pflüger column can only have divergent type of instability.

The buckling load p_{cr} has a Rayleigh quotient:

$$p_{cr} = \inf_{y(x)} \frac{\int_0^1 (y'')^2\,\mathrm{d}x}{\int_0^1 f(x)(y')^2\,\mathrm{d}x}, \tag{7}$$

or in terms of w:

$$p_{cr} = \inf_{w(x)} \frac{\int_0^1 (w''')^2\,\mathrm{d}x}{\int_0^1 f(x)(w'')^2\,\mathrm{d}x}. \tag{8}$$

The foregoing derivation shows that the Pflüger column can be symmetrized into a self-adjoint system. Stability of the Pflüger column can be studied via its equivalent self-adjoint system. The relationship between the Pflüger column and its equivalent self-adjoint system is as follows.

Systems	Loading	End Conditions			
Pflüger's column	Tangential follower force	P-P	P-G	G-P	G-G
Equivalent self-adjoint	Unidirectional force	G-G	G-P	P-G	P-P

P stands for a pinned end, and G stands for a guided end. The equivalent self-adjoint system may be viewed as a continuum counterpart of the symmetrized discrete system. A G-G column has a rigid body translational mode. We will ignore it, as we are interested in the flexural modes only.

Note the static equation governing the bending moment in the Pflüger column, $M'' + f(x)M = 0$, is also self-adjoint in the classical sense.

As an example, to study the stability of a G-G column under a uniform tangential follower force, we can instead study a P-P column under its uniform self-weight. While the former is a nonself-adjoint problem, the latter is a well-known classical self-adjoint problem. The critical weight of the self-adjoint problem is 18.57, so is the buckling load of the nonself-adjoint problem.

It is interesting to note that for a divergent nonself-adjoint discrete system $Ax - \lambda Bx = 0$, where A is symmetric, positive definite, B is not symmetric, and λ is real, there exists a lower bound self-adjoint system $Ay - \sigma B^T A^{-1} By = 0$ such that $\sigma_1 \le (\lambda_1)^2$ and

$$\sigma(x_1) = \frac{x_1^T A x_1}{x_1^T B^T A^{-1} B x_1} = \lambda_1^2.$$

Likewise, for Pflüger's column, a lower bound self-adjoint system exists:

$$u''(x) = \sigma \int_0^1 K(x,\xi) u''(\xi)\,\mathrm{d}\xi,$$

where $K(x,\xi) = G(x,\xi) f(x) f(\xi)$ is a symmetric, positive definite kernel, such that

$$\sigma_1 = \inf_{u(x)} \frac{\int_0^1 [u''(x)]^2\,\mathrm{d}x}{\int_0^1 \int_0^1 K(x,\xi) u''(x) u''(\xi)\,\mathrm{d}x\,\mathrm{d}\xi} \le (p_1)^2.$$

3. Greenhill's Shaft

Greenhill's shaft is a pin-ended bar in torsion. The system is not self-adjoint except when θ, the angle between the applied torque vector and the tangent to the end of the bar, is equal to 1/2.

Leipholz (1974a) studied a pure tangential torque ($\theta = 0$) and showed that the Greenhill shaft is a conservative system of the second kind. Walker (1973) considered the case $\theta \neq 1/2$ and included an axial compression and a damping force. He developed a Lyapunov functional for stability study. Here we examine a bar in a viscoelastic medium of low density under a pure tangential torque and a pure axial torque ($\theta = 1$), respectively. We will symmetrize the systems and improve the stability boundary obtained by Walker.

The linearized system (Bolotin, 1963) is described by the differential equation

$$\left\{ \begin{array}{c} \ddot{u}_1 \\ \ddot{u}_2 \end{array} \right\} + c \left\{ \begin{array}{c} \dot{u}_1 \\ \dot{u}_2 \end{array} \right\} + \left\{ \begin{array}{cc} \partial^4 + p\partial^2 + k & L\partial^3 \\ -L\partial^3 & \partial^4 + p\partial^2 + k \end{array} \right\} \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \tag{9}$$

$$u_1 = u_2 = 0$$

and the boundary conditions:

$$\left\{ \begin{array}{cc} \partial^2 & \theta L\partial \\ -\theta L\partial & \partial^2 \end{array} \right\} \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \tag{10}$$

with $\theta = 0$ or $\theta = 1$, respectively.

$u_1(x)$ and $u_2(x)$ are deflections in the two principal directions, L is the torque, p is the axial end compression, c and k represent the viscoelastic medium, and ∂ denotes $\partial/\partial x$.

Let us transform variables by denoting $z(x)\,\mathrm{e}^{rt} = u_1 + iu_2$, where $i = \sqrt{-1}$. Then the differential equation and the boundary conditions become, respectively,

$$(r^2 + cr + k)z + z'''' - iLz''' = 0 \tag{11}$$

and

$$z'' - i\theta Lz' = 0. \tag{12}$$

For the case of a pure tangential torque ($\theta = 0$), let us denote $z'' = M$. Then $M = 0$ at either end, and

$$z(x) = \int_0^1 G(x, y)M(y)\,\mathrm{d}y, \tag{13}$$

where

$$G(x, y) = \begin{cases} x(y-1) & \text{for } 0 \le x < y \\ y(x-1) & \text{for } y < x \le 1 \end{cases}$$

is Green's influence function. Let $M = w(x)\,\mathrm{e}^{iLx/2}$. The expression in Equation (11) becomes

$$(r^2 + cr + k)\int_0^1 G(x, y)M(y)\,\mathrm{d}y + M'' + pM - iLM' = 0, \tag{14}$$

$$w(0) = w(1) = 0, \tag{15}$$

and

$$(r^2 + cr + k)\int_0^1 H(x, y)w(y)\,\mathrm{d}y + w'' + \left(p + \frac{L^2}{4}\right)w = 0. \tag{16}$$

$H(x, y) = G(x, y)\,\mathrm{e}^{-iL(x-y)/2}$ is a symmetric kernel. Greenhill's shaft is thus symmetrized.

For a given k and L, the critical load has a Rayleigh quotient

$$p_{cr} = -\frac{L^2}{4} + \inf_{w(x)} \frac{\int_0^1 (w')^2\,\mathrm{d}x - k\int_0^1\int_0^1 H(x, y)w(x)w(y)\,\mathrm{d}x\,\mathrm{d}y}{\int_0^1 w^2\,\mathrm{d}x}. \tag{17}$$

It is noted that

$$\int_0^1\int_0^1 H(x, y)w(x)w(y)\,\mathrm{d}x\,\mathrm{d}y = -\int_0^1 \overline{g'}\,g'\,\mathrm{d}x, \tag{18}$$

where $g(x)$ is the solution of

$$g''(x) = w(x)\,\mathrm{e}^{iLx/2}; \quad g(0) = g(1) = 0. \tag{19}$$

When $k = 0$, one can study the stability by considering, from Equation (16),

$$w'' + \left(p + \frac{L^2}{4}\right) w = 0; \quad w(0) = w(1) = 0, \tag{20}$$

which describes the free vibration of a string. So $p_{cr} = -(L^2/4) + \pi^2$ is the necessary and sufficient condition for stability when $k = 0$.

An estimate of p_{cr} in Equation (17) can be obtained by assuming $w(x) = \sin(\pi x)$, resulting in

$$p_{cr} = -\frac{L^2}{4} + \pi^2 + \frac{k\left[\left(\pi^2 - \frac{L^2}{4}\right)\left(\pi^4 - \frac{L^4}{16}\right) - 4\pi^2L^2\left(1 + \cos\frac{L}{2}\right)\right]}{\left(\pi^2 - \frac{L^2}{4}\right)^4}. \tag{21}$$

When $L = 0$, Equation (21) yields $p_{cr} = \pi^2 + (k/\pi^2)$. On the other hand, when $L = \pm 2\pi$, $p_{cr} = k/8((1/3) + (3/\pi^2))$. It is seen that $k > 0$ does increase the stability boundary, directly confirming Walker's conjecture about an increase in the upper bound on $L^2 + 4p$.

The case of a pure axial torque ($\theta = 0$) is considered next. Integrate Equation (11) twice with respect to x to obtain

$$(r^2 + cr + k)\int_0^1 G(x, y)z(y)\,\mathrm{d}y + z'' + pz - iLz' = a_1 + a_2x. \tag{22}$$

The constants of integration a_1 and a_2 are found to be zero, in view of the boundary conditions $z = 0$ and $z'' - iLz' = 0$. Consequently, Equation (22) becomes

$$(r^2 + cr + k)\int_0^1 G(x, y)z(y)\,\mathrm{d}y + z'' + pz - iLz' = 0. \tag{23}$$

The dependent variable needs to satisfy only the boundary conditions $z(0) = z(1) = 0$.

Equation (23) is the same as Equation (14). The symmetrized system in Equation (16) and the stability boundary in Equation (20) also holds true for Greenhill's shaft under an axial torque. Therefore, the shaft under an axial torque and that under a tangential torque behave in a similar manner and both have the same equivalent symmetrized system.

4. A Linear Conservative Gyroscopic System

The deflection of a conservative gyroscopic system being considered here satisfies the equation:

$$\ddot{x} + C\dot{x} + Kx = 0. \tag{24}$$

$C^T = -C$ and $K^T = K$. Walker (1991) gave a Lyapunov with undetermined parameters to study stability. We derive a zero-parameter sufficient condition for instability and one for stability.

A stable conservative gyroscopic system admits periodic solutions. Let $x = (u + iv)\,\mathrm{e}^{i\omega t}$, where u and v are real vectors. After the real and the imaginary terms are collected separately, the following two equations are obtained:

$$\begin{aligned}(K - \omega^2 I)u - \omega C v = 0,\\ \omega C u + (K - \omega^2 I)v = 0.\end{aligned} \tag{25}$$

Elimination of, say, v from the above equations yields

$$[(K - \omega^2 I)C^{-1}(K - \omega^2 I) + \omega^2 C]u = 0. \tag{26}$$

An algebraic equation in ω^2 can be obtained by performing an inner product on Equation (26). Study of stability is easier from the behaviour of ω^2 as revealed by this algebraic equation.

To obtain a nontrivial algebraic equation, we will pre-multiply Equation (26) by $u^T C$, resulting in:

$$(u^T u)\omega^4 - [u^T(CKC^{-1} + K - C^2)u]\omega^2 + u^T CKC^{-1}Ku = 0, \tag{27}$$

which is the desired algebraic equation.

The system is stable (periodic) if ω^2 is positive. On the other hand, the system is unstable if ω^2 is negative or complex. The proposed sufficient conditions, one for stability and one for instability, follow Equation (27) immediately.

4.1. *Sufficient Condition for Instability*

The system is unstable if any of the following three conditions is satisfied:

$$u^T(CKC^{-1} + K - C^2)u < 0, \tag{28}$$

or

$$u^T CKC^{-1}Ku < 0, \tag{29}$$

or

$$\left[\frac{u^T(CKC^{-1} + K - C^2)u}{2u^T u}\right]^2 - \frac{u^T CKC^{-1}Ku}{u^T u} < 0. \tag{30}$$

The Cauchy–Schwartz inequality can be used to show that Equation (30) is implied by

$$u^T(C^{-1}KC + K - C^2)(CKC^{-1} + K - C^2)u - 4u^T CKC^1 Ku < 0. \tag{31}$$

With the identity $u^T Au = u^T A_S u$, where $A_s = A + A^T/2$, for any real matrix A, $A_S > 0$ (or $A_S < 0$) is sufficient for $u^T Au > 0$ (or $u^T Au < 0$). Therefore, a sufficient condition in terms of the operators for the system to be unstable is:

(A.1) $2(K - C^2) + C^{-1}KC + CKC^{-1} < 0$, or

(A.2) $CKC^{-1}K + KC^{-1}KC < 0$, or

(A.3) $(C^{-1}KC + K - C^2)(CKC^{-1} + K - C^2) - 2(CKC^{-1}K + KC^{-1}KC) < 0.$

The matrices in the above expressions are symmetric. Their negative-definiteness can be verified by using the Sylvester criteria. The union of the regions defined by Equations (A.1), (A.2) and (A.3) is a domain of instability.

For the two-degree-of-freedom system considered by Walker, the system is unstable if:

$$k_1 + k_2 + 16 < 0;$$

or

$$k_1 k_2 < 0;$$

or

$$(K_1 + k_2 + 16)^2 - 4k_1 k_2 < 0,$$

per (A.1), (A.2), and (A.3), respectively. The result is the same as the exact solution, because all the matrix products here are diagonal. The product of two 2×2 skew matrices is diagonal.

4.2. *Sufficient Condition for Stability*

The solution is stable (periodic) if all the following three conditions are satisfied:

$$u^T(CKC^{-1} + K - C^2)u > 0 \tag{32}$$

or

$$u^T CKC^{-1}Ku > 0 \tag{33}$$

or

$$\left[\frac{u^T(CKC^{-1} + K - C^2)u}{2u^T u}\right]^2 - \frac{u^T CKC^{-1}Ku}{u^T u} > 0. \tag{34}$$

An approximation will be made to simplify Equation (34). Let us require that Equation (34) be true for all real vectors, including w, which is the eigenvector in the eigenvalue problem

$$(CKC^{-1} + K - C^2)w = \lambda w. \tag{35}$$

Then Equation (34) becomes

$$w^T[(CKC^{-1} + K - C^2)^2 - 4CKC^{-1}K]w > 0. \tag{36}$$

Consequently, a sufficient condition for stability is:

(B.1) $2(K - C^2) + C^{-1}KC + CKC^{-1} > 0$, and

(B.2) $CKC^{-1}K + KC^{-1}KC > 0$, and

(B.3) $(C^{-1}KC + K - C^2)^2 + (CKC^{-1} + K - C^2)^2 - 4(CKC^{-1}K + KC^{-1}KC) > 0.$

The intersection of the regions in (B.1), (B.2) and (B.3) is a domain of stability.

According to the present criterion, the same 2-DOF system will be stable if all the conditions

$$k_1 + k_2 + 16 > 0;$$

and

$$k_1 k_2 > 0;$$

and

$$(k_1 + k_2 + 16)^2 - 4k_1 k_2 > 0,$$

are satisfied. Again, the result is the same as the exact solution.

5. Conclusions

Walker, Leipholz, and Inman and Olsen have studied conservative systems of the second kind. Our objective here is to obtain an equivalent self-adjoint system. Existence of the symmetrized systems confirms certain conservative systems of the second kind behave as a true conservative system. In this way, study of stability can be carried out on the symmetrized system. In general, it is easier to study a self-adjoint problem than a nonself-adjoint problem. For the conservative system of the second kind, including the Pflüger column, we also presented a lower bound self-adjoint system. For a linear conservative gyroscopic system, we gave a zero parameter sufficient condition for instability and one for stability. The criteria depend only on the characteristics of the system. For a simple 2-DOF system, the present criteria yield the exact solutions.

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