ON THE MONTE CARLO SIMULATION OF MOMENT LYAPUNOV EXPONENTS

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Abstract

The moment Lyapunov exponents are important characteristic numbers for determining the dynamical stability of stochastic systems. Monte Carlo simulations are complement to the approximate analytical methods in the determination of the moment Lyapunov exponents. They also provide criteria on assessing how accurate the approximate analytical methods are. For stochastic dynamical systems described by Itô stochastic differential equations, the solutions are diffusion processes and their variances may increase with time of simulation. Due to the large variances of the solutions and round-off errors, bias errors in the simulation of momemt Lyapunov exponents are significant in the cases of improper numerical approaches. The improved estimation for some systems is presented in this paper.

1 Introduction

The moment Lyapunov exponents, which are define by

$$\Lambda(p) = \lim_{t\to\infty} \frac{1}{t} \log E\left[\|\mathbf{X}(t)\|^p\right], \tag{1}$$

characterize the moment stability of a stochastic dynamical system with state vector $\mathbf{X}(t)$, where $E[\cdot]$ denotes expectation and $\|\cdot\|$ denotes a suitable vector norm. The pth moment of the response of the system is asymptotically stable if $\Lambda(p) < 0$. Moreover, $\Lambda'(0)$ is equal to the largest Lyapunov exponent λ, which is defined by

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log \|\mathbf{X}(t)\| \tag{2}$$

and describes the almost-sure or sample stability of the system.

Even if the solution of a system is almost-sure stable, i.e. $\|\mathbf{X}(t)\| \to 0$ as $t \to \infty$ w.p.1 at a negative exponential rate, the chance that $\|\mathbf{X}(t)\|$ takes large values may lead to the instability of the pth moment. Hence, it is important to obtain the moment Lyapunov exponents such that the complete properties of dynamic stability of stochastic systems can be described.

Although moment Lyapunov exponents may be obtained by approximate analytical methods, such as stochastic averaging or perturbation, in some cases, Monte Carlo simulations have to be applied when it is difficult to do so. On the other hand, the accuracy of the approximate analytical methods needs to be verifed by numerical simulations.

A numerical algorithm for determining the moment Lyapunov exponents using Monte Carlo simulation developed by Xie (Xie, 2005) is described briefly below.

Consider a dynamical system whose state vector satisfies the following Itô stochastic differential equation

$$\mathrm{d}\mathbf{X}(t) = \mathbf{m}(\mathbf{X}, t)\mathrm{d}t + \boldsymbol{\sigma}(\mathbf{X}, t)\mathrm{d}\mathbf{W}(t), \tag{3}$$

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where $\mathbf{W}(t)$ is a vector of standard Weiner processes. Define

$$\rho_m(p) = \frac{E\left[\|\mathbf{X}(mT_N)\|^p\right]}{E\left[\|\mathbf{X}((m-1)T_N)\|^p\right]}, \quad m = 1, 2, \cdots, M, \tag{4}$$

where $T_N = K\Delta$ is the time interval for normalization, Δ is the time step of iteration for an appropriate discrete scheme used to solve equation (3) numerically. In order to avoid data overflow or underflow, when system (3) is linear, at time mT_N the sth sample of response is normalized using

$$\mathbf{X}^s(mT_N) \triangleq \frac{\mathbf{X}^s(mT_N)}{\|\mathbf{X}^s(mT_N)\|} \tag{5}$$

before the iteration continues. Then for a large iteration time $T = MT_N$, by setting $\|\mathbf{X}^s(0)\| = 1$, the approximate moment Lyapunov exponents at time T are given by

$$\begin{aligned} \Lambda(p) &= \frac{1}{T}\log E\left[\|\mathbf{X}(T)\|^p\right] = \frac{1}{MT_N}\log E\left[\|\mathbf{X}(MT_N)\|^p\right] \\ &= \frac{1}{MT_N}\log\left\{\frac{E\left[\|\mathbf{X}(MT_N)\|^p\right]}{E\left[\|\mathbf{X}((M-1)T_N)\|^p\right]}\cdots\frac{E\left[\|\mathbf{X}(T_N)\|^p\right]}{E\left[\|\mathbf{X}(0)\|^p\right]}\right\} \\ &= \frac{1}{MT_N}\sum_{m=1}^{M}\log\rho_m(p). \end{aligned} \tag{6}$$

The expectation in (6) is determined by the sample average

$$E\left[\|\mathbf{X}(mT_N)\|^p\right] = \frac{1}{S}\sum_{s=1}^{S}\|\mathbf{X}_s(mT_N)\|^p, \tag{7}$$

where S is the total sample size for simulation.

However, notice that after the normalization operation $\|\mathbf{X}_s(mT_N)\| = 1$ is always satisfied according to equation (5). Actually equation (6) gives the approximate moment Lyapunov exponents at time T_N but not at time T, since $\rho_m(p)$ is always the expectation of response at time T_N in this case and

$$\Lambda(p) = \frac{1}{MT_N}\sum_{m=1}^{M}\log\rho_m(p) = \frac{1}{M}\sum_{m=1}^{M}\frac{1}{T_N}\log\rho_m(p). \tag{8}$$

One revision to correct the insufficiency of the above algorithm is to normalize the response by their expectation. For a linear system, by defining

$$\mathbf{Y}_m^s = \frac{\mathbf{X}^s(mT_N)}{E\left[\|\mathbf{X}((m-1)T_N)\|\right]}, \quad \hat{\rho}_m^s = \|\mathbf{Y}_m^s\|, \tag{9}$$

it is easy to obtain the approximate moment Lyapunov exponents at time T through

$$\begin{aligned} \Lambda(p) &= \frac{1}{MT_N}\log\frac{E\left[\|\mathbf{X}(MT_N)\|^p\right]}{E\left[\|\mathbf{X}((M-1)T_N)\|\right]^p}\,\frac{E\left[\|\mathbf{X}((M-1)T_N)\|\right]^p}{E\left[\|\mathbf{X}((M-2)T_N)\|\right]^p}\cdots\frac{E\left[\|\mathbf{X}(T_N)\|\right]^p}{E\left[\|\mathbf{X}(0)\|\right]^p} \\ &= \frac{1}{MT_N}\left\{\log E\left[\left\|\frac{\mathbf{X}(MT_N)}{E\left[\|\mathbf{X}((M-1)T_N)\|\right]}\right\|^p\right] + \sum_{m=1}^{M-1}p\log E\left[\left\|\frac{\mathbf{X}(mT_N)}{E\left[\|\mathbf{X}((m-1)T_N)\|\right]}\right\|\right]\right\} \\ &= \frac{1}{MT_N}\left\{\log E\left[(\hat{\rho}_M)^p\right] + p\sum_{m=1}^{M-1}\log E\left[\hat{\rho}_m\right]\right\}. \end{aligned} \tag{10}$$

The solution of equation (3) is a diffusion process and its variance may increase significantly with time. Although equation (10) is exact theoretically when M is large enough, there are two main reasons which will lead to significant numerical errors. According to the Central Limit Theorem (Feller, 1965), for independent and identically distributed (i.i.d.) random variables $x_1, x_2, \cdots$ with the same mean μ and variance σ^2, the sample average $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ will tend to the normal distribution $\mathcal{N}(\mu, \sigma^2/n)$. This means that equation (7) will not give acceptable results of the expected values in the cases when the variance of response is so large that it is impossible to reduce the error of estimation with a finite sample size. On the other hand, due to the finite lengths of floating-point representations in computer, when two numbers are summed up, the smaller one will be neglected if the difference of their exponent bits exceeds the limit. Thus this truncated error in estimating the expectations will be dominant for simulations with large variances even if the chance that the responses take extremely large values is rare.

A simple example is the first-order linear stochastic system

$$\mathrm{d}x(t) = ax(t)\mathrm{d}t + \sigma x(t)\mathrm{d}W(t), \tag{11}$$

where a and σ are real constants. Its solution with the initial condition $x(0) = 1$ is

$$x(t) = e^{\left(a-\frac{1}{2}\sigma^2\right)t+\sigma W(t)}. \tag{12}$$

The moment Lyapunov exponents of this system are given by

$$\Lambda(p) = \frac{p}{2}\left[(p-1)\sigma^2 + 2a\right], \tag{13}$$

and the variance of norm is

$$\mathrm{Var}\left[x(t)\right] = e^{2at}(e^{\sigma^2 t} - 1). \tag{14}$$

It can be seen that the variance will increase exponentially with time when $\sigma \neq 0$.

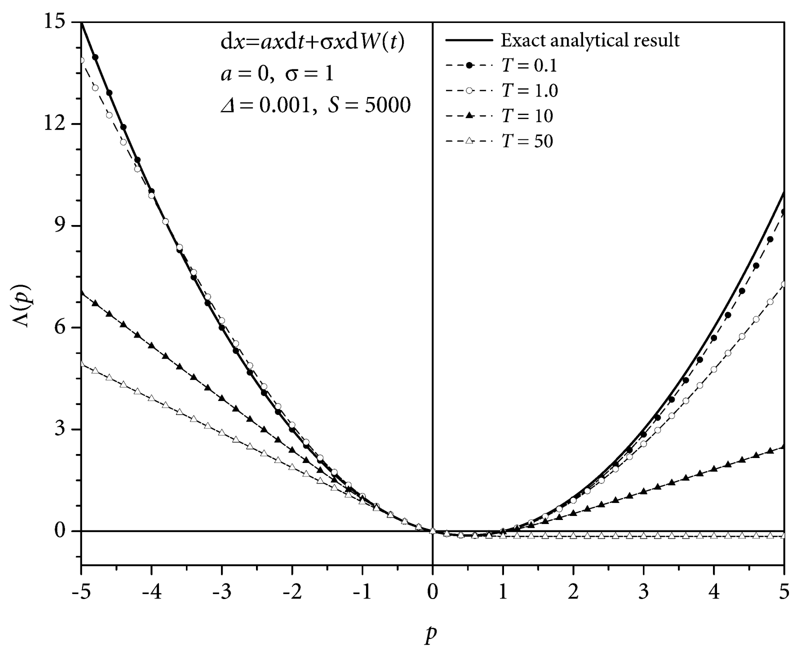

Figure 1: Moment Lyapunov exponents for different times of simulation

Figure 1 shows the numerical results using the explicit Euler scheme for different times T in the case $a = 0$, $\sigma = 1$. The time step for iteration is $\Delta = 0.001$, the sample size is 5000 and equation (10) is used to determine the approximate moment Lyapunov exponents. It is obvious that the longer the time for simulation, the worse the results.

2 Estimation of the Expectation through the Logarithm of Norm

Due to the reasons discussed above, it is required to find a new algorithm to overcome the difficulty in estimating the expectation. Since the error is caused by large variance, it is clear that how to reduce the variance of response in order to obtain a good estimation of expectation using a finite number of samples is important.

Let $\{\mathbf{X}_i\}$ be i.i.d. random vectors with the same distribution as $\mathbf{X}(t)$, then

$$\begin{aligned}\log E\left[\|\mathbf{X}(t)\|^p\right] &= \frac{1}{n}\log\left(E\left[\|\mathbf{X}(t)\|^p\right]\right)^n = \frac{1}{n}\log\left(\prod_{i=1}^{n} E\left[\|\mathbf{X}_i\|^p\right]\right)\\ &= \frac{1}{n}\log E\left[\prod_{i=1}^{n}\|\mathbf{X}_i\|^p\right] = \frac{1}{n}\log E\left[\exp\left(p\sum_{i=1}^{n}\log\|\mathbf{X}_i\|\right)\right].\end{aligned} \tag{15}$$

Defining

$$S_n = \sum_{i=1}^{n}\log\|\mathbf{X}_i\| = \sum_{i=1}^{n}\rho_i, \quad z_n = \frac{1}{n}S_n = \frac{1}{n}\sum_{i=1}^{n}\rho_i, \tag{16}$$

and letting

$$\mu = E\left[\log\|\mathbf{X}(t)\|\right] = E[\rho(t)], \quad \sigma^2 = \text{Var}\left[\log\|\mathbf{X}(t)\|\right] = \text{Var}\left[\rho(t)\right], \tag{17}$$

then

$$\log E\left[\|\mathbf{X}(t)\|^p\right] = \frac{1}{n}\log E\left[e^{npz_n}\right]. \tag{18}$$

With the notation

$$\xi_n = \frac{z_n - \mu}{\sigma/\sqrt{n}}, \tag{19}$$

equation (18) is converted to

$$\log E\left[\|\mathbf{X}(t)\|^p\right] = \frac{1}{n}\log E\left[e^{np\mu}\cdot e^{\sqrt{n}p\sigma\xi_n}\right] = p\mu + \frac{1}{n}\log E\left[e^{\sqrt{n}p\sigma\xi_n}\right]. \tag{20}$$

Let $F(\xi)$ be the distribution function of ξ_n, then $F(\xi)$ tends to $\mathcal{N}(0,1)$ as $n\to\infty$ according to the Central Limit Theorem, i.e. $F(\xi)\to\Phi(\xi)$ uniformly, where $\Phi(\xi)$ is the probability distribution function of the standard normal variable

$$\Phi(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi} e^{-\frac{1}{2}x^2}\,\mathrm{d}x. \tag{21}$$

Using the Edgeworth expansion theorem for distribution (Gnedenko and Kolmogorov, 1954), $F(\xi)$ can be written as

$$F(\xi) = \Phi(\xi) + \sum_{k=3}^{\infty} c_k \Phi^{(k)}(\xi), \tag{22}$$

where the coefficients c_k are determined by the equivalence of moments on both sides.

The "tail effect" of distribution $F(\xi)$ is important since it is required to solve the expectation of $e^{\sqrt{n}p\sigma\xi_n}$ in equation (20). This means that accurate higher-order moments of ξ_n are needed in order to obtain a good approximation of $E\left[e^{\sqrt{n}p\sigma\xi_n}\right]$. However, it is rather difficult to do so for the solution of a general system (3) in practice. Therefore, in this paper, special systems are considered.

For a linear system with constant coefficients, i.e. the coefficient matrices in equation (3) take the form

$$\mathbf{m}(\mathbf{X}, t) = \mathbf{m}\mathbf{X}, \quad \boldsymbol{\sigma}(\mathbf{X}, t) = \boldsymbol{\sigma}\mathbf{X}, \tag{23}$$

where **m** and $\boldsymbol{\sigma}$ are constant matrice, it has been shown that the limiting distribution of $\rho(t) = \log \|\mathbf{X}(t)\|$ is normal as $t \to \infty$ if there is a constant h such that, for any vector **Y**,

$$\langle \boldsymbol{\sigma}\mathbf{X}, \mathbf{Y} \rangle = \mathbf{Y}^T \boldsymbol{\sigma}\mathbf{X} \geqslant h \|\mathbf{X}\|^2 \|\mathbf{Y}\|^2 \tag{24}$$

is satisfied (Arnold, 1974, Khasminskii, 1980). In this case $F(\xi)$ will be normal since the distribution of sum of independent normal distributed random variables is also normal, i.e. $F(\xi) = \Phi(\xi)$. Thus

$$\begin{aligned} \log E\left[\|\mathbf{X}(t)\|^p\right] &= p\mu + \frac{1}{n} \log \int_{-\infty}^{\infty} e^{\sqrt{n} p \sigma \xi_n} \mathrm{d}\Phi(\xi) \\ &= p\mu + \frac{1}{n} \log e^{\frac{1}{2} n p^2 \sigma^2} = p\mu + \frac{1}{2} p^2 \sigma^2. \end{aligned} \tag{25}$$

Hence for the linear system with constant coefficients, by estimating the mean and variance of logarithm of norm, the moment Lyapunov exponents will be given by

$$\Lambda(p) = \frac{1}{T} \left(pE\left[\log \|\mathbf{X}(T)\|\right] + \frac{1}{2} p^2 \mathrm{Var}\left[\log \|\mathbf{X}(T)\|\right] \right). \tag{26}$$

It is obvious that the variance of $\log \|\mathbf{X}(T)\|$ will be much less than the variance of $\|\mathbf{X}(T)\|$; therefore obtaining a good estimation through the sample average is possible.

3 Algorithm for Linear Systems with Constant Coefficients

According to equation (26), the algorithm of simulating the moment Lyapunov exponents for linear system with constant coefficients

$$\mathrm{d}\mathbf{X}(t) = \mathbf{m}\mathbf{X}(t)\mathrm{d}t + \boldsymbol{\sigma}\mathbf{X}(t)\mathrm{d}\mathbf{W}(t), \tag{27}$$

can be described below. The time step of iteration is Δ, time interval of normalization $T_N = K\Delta$, and the sample size is S.

Step 1. Set the initial conditions of state vector $\mathbf{X}(t)$ by

$$\|\mathbf{X}^s(0)\| = 1, \quad s = 1, 2, \cdots, S, \tag{28}$$

where $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{x}}$.

Step 2. Between every normalization operation, i.e. $m = 1, 2, \cdots, M$, and for every sample, use appropriate discrete scheme for equation (27) to perform K iterations for the Monte Carlo simulation of $\mathbf{X}^s(t)$.

Step 3. Let

$$\rho^s(mT_N) = \log \|\mathbf{X}^s(mT_N)\|, \tag{29}$$

and then use equation (5) to normalize the value of $\mathbf{X}^s(mT_N)$ such that $\|\mathbf{X}^s(mT_N)\| = 1$.

Step 4. Repeat steps 2 and 3 until $m = M$, i.e. $T = MT_N$. Then by the description in step 3,

$$\begin{aligned} \rho^s(T) = \log \|\mathbf{X}^s(MT_N)\| &= \log \frac{\|\mathbf{X}(MT_N)\|}{\|\mathbf{X}((M-1)T_N)\|} \cdots \frac{\|\mathbf{X}(T_N)\|}{\|\mathbf{X}(0)\|} \\ &= \sum_{m=1}^{M} \log \frac{\|\mathbf{X}(mT_N)\|}{\|\mathbf{X}((m-1)T_N)\|} = \sum_{m=1}^{M} \rho^s(mT_N). \end{aligned} \tag{30}$$

Step 5. Use

$$\begin{aligned} E\left[\log\|\mathbf{X}(T)\|\right] &= E\left[\rho(T)\right] = \frac{1}{S}\sum_{s=1}^{S}\rho^s(T) = \bar{\rho}(T), \\ \mathrm{Var}\left[\log\|\mathbf{X}(T)\|\right] &= \mathrm{Var}\left[\rho(T)\right] = \frac{1}{S-1}\sum_{s=1}^{S}\left[\rho^s(T)^2 - \bar{\rho}(T)^2\right], \end{aligned} \tag{31}$$

to estimate the mean and variance of $\log\|\mathbf{X}(T)\|$.

Step 6. Use equation (26) to calculate the moment Lyapunov exponents for all values of p of interest.

4 Application on the Stability of a Viscoelastic System

Consider a single degree-of-freedom viscoelastic system excited by a zero mean wide-band stationary noise $\xi(t)$,

$$\ddot{q}(t) + 2\varepsilon\beta\dot{q}(t) + \omega^2\left\{\left[1-\varepsilon^{1/2}\xi(t)\right]q(t) - \varepsilon\int_0^t h(t-s)q(s)\mathrm{d}s\right\} = 0, \tag{32}$$

where $h(t)$ is the viscoelastic kernel function, the small parameter ε is introduced to denote that the damping, viscoelastic effect, and the amplitude of noise are small. The method of stochastic averaging (Khasminskii, 1966a,b) and the averaging method for integro-differential equations (Larinov, 1969) are applied to obtain the approximate analytical moment Lyapunov exponents.

By applying the transformation

$$q(t) = a(t)\cos\Phi(t), \quad \dot{q}(t) = -\omega a(t)\sin\Phi(t), \quad \Phi(t) = \omega t + \varphi(t), \tag{33}$$

equation (32) is converted to the form

$$\begin{Bmatrix} \dot{a}(t) \\ \dot{\varphi}(t) \end{Bmatrix} = \varepsilon\mathbf{F}^{(1)}(a,\varphi,t) + \varepsilon^{1/2}\mathbf{F}^{(0)}(a,\varphi,\xi(t),t), \tag{34}$$

$$\begin{aligned} \mathbf{F}^{(1)}(a,\varphi,t) &= \begin{Bmatrix} -2\beta a(t)\sin^2\Phi(t) - \omega\sin\Phi(t)\displaystyle\int_0^t h(t-s)a(s)\cos\Phi(s)\mathrm{d}s \\ -2\beta a(t)\sin\Phi(t)\cos\Phi(t) - \frac{1}{a(t)}\omega\cos\Phi(t)\displaystyle\int_0^t h(t-s)a(s)\cos\Phi(s)\mathrm{d}s \end{Bmatrix} \\ &= \begin{Bmatrix} F_1^{(1)}(a,\varphi,t) \\ F_2^{(1)}(a,\varphi,t) \end{Bmatrix}, \end{aligned} \tag{35}$$

$$\begin{aligned} \mathbf{F}^{(0)}(a,\varphi,\xi(t),t) &= \begin{Bmatrix} -\frac{1}{2}\omega\xi(t)a(t)\sin 2\Phi(t) \\ -\frac{1}{2}\omega\xi(t)a(t)[1+\cos 2\Phi(t)] \end{Bmatrix} \\ &= \begin{Bmatrix} F_1^{(0)}(a,\varphi,\xi(t),t) \\ F_2^{(0)}(a,\varphi,\xi(t),t) \end{Bmatrix}. \end{aligned} \tag{36}$$

Since $\xi(t)$ is a wide-band noise, according to the method of stochastic averaging, system (34) can be approximated by the following averaged equations

$$\mathrm{d}\begin{Bmatrix} \bar{a}(t) \\ \bar{\varphi}(t) \end{Bmatrix} = \varepsilon\begin{Bmatrix} \bar{m}_a \\ \bar{m}_\varphi \end{Bmatrix}\mathrm{d}t + \varepsilon^{1/2}\bar{\boldsymbol{\sigma}}\mathrm{d}\mathbf{W}(t), \tag{37}$$

where

$$\bar{m}_a = \underset{t}{\mathcal{M}} \left\{ F_1^{(1)}(a,\varphi,t) + \int_{-\infty}^{0} E\left[\frac{\partial F_1^{(0)}}{\partial a} F_{1\tau}^{(0)} + \frac{\partial F_1^{(0)}}{\partial \varphi} F_{2\tau}^{(0)} \right] \mathrm{d}\tau \right\},$$

$$\bar{m}_\varphi = \underset{t}{\mathcal{M}} \left\{ F_2^{(1)}(a,\varphi,t) + \int_{-\infty}^{0} E\left[\frac{\partial F_2^{(0)}}{\partial a} F_{1\tau}^{(0)} + \frac{\partial F_2^{(0)}}{\partial \varphi} F_{2\tau}^{(0)} \right] \mathrm{d}\tau \right\},$$

$$\left[\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^T\right]_{ij} = \underset{t}{\mathcal{M}} \left\{ \int_{-\infty}^{\infty} E\left[F_i^{(0)} F_{j\tau}^{(0)} \right] \mathrm{d}\tau \right\}, \quad i,j = 1,2,$$

$$F_{j\tau}^{(0)} = F_j^{(0)}(a,\varphi,\xi(t+\tau),t+\tau), \quad j = 1,2, \tag{38}$$

and

$$\underset{t}{\mathcal{M}}\{\,\cdot\,\} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \{\,\cdot\,\}\,\mathrm{d}t \tag{39}$$

is the averaging operator.

When applying the averaging operation, $a(t)$ and $\varphi(t)$ are treated as constants and are replaced by $\bar{a}$ and $\bar{\varphi}$, respectively. After some calculations the corresponding terms in the averaged equations are

$$\bar{m}_a = \left[-\beta - \frac{1}{2}\omega\mathcal{H}^s(\omega) + \frac{3}{16}\omega^2 S(2\omega) \right] \bar{a}, \quad \bar{m}_\varphi = -\frac{1}{2}\omega\mathcal{H}^c(\omega) - \frac{1}{8}\omega^2 \Psi(2\omega),$$

$$\left[\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^T\right]_{11} = b_{11} = \frac{1}{8}\omega^2 S(2\omega)\bar{a}^2, \quad \left[\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^T\right]_{12} = \left[\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^T\right]_{21} = 0,$$

$$\left[\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^T\right]_{22} = b_{22} = \frac{1}{8}\omega^2 \left[2S(0) + S(2\omega) \right], \tag{40}$$

where

$$\mathcal{H}^s(\omega) = \int_0^\infty h(\tau)\sin\omega\tau\mathrm{d}\tau, \quad \mathcal{H}^c(\omega) = \int_0^\infty h(\tau)\cos\omega\tau\mathrm{d}\tau \tag{41}$$

are the sine and cosine transformations of the viscoelastic kernel function $h(t)$, $S(\omega)$ and $\Psi(\omega)$ are the cosine and sine power spectral density functions of the wide-band noise $\xi(t)$, respectively. Noting that transition density function for the solution of the averaged equation is the solution of the Fokker-Planck equation, which depends on the diffusion matrix $\bar{\boldsymbol{\sigma}}\bar{\boldsymbol{\sigma}}^T$ but not every single element $\bar{\sigma}_{ij}$, it can be set that

$$\bar{\sigma}_{12} = \bar{\sigma}_{21} = 0, \quad \bar{\sigma}_{11} = \sqrt{b_{11}} = \omega\bar{a}\sqrt{\frac{S(2\omega)}{8}}, \quad \bar{\sigma}_{22} = \sqrt{b_{22}} = \omega\sqrt{\frac{2S(0)+S(2\omega)}{8}}. \tag{42}$$

Letting $P = \bar{a}^p$ and applying Itô's Lemma to the first equation of (37) lead to the Itô differential equation for pth norm P,

$$\begin{aligned} \mathrm{d}P = \varepsilon &\left[p\left(-\beta - \frac{1}{2}\omega\mathcal{H}^s(\omega) + \frac{3}{16}\omega^2 S(2\omega) \right) + \frac{p(p-1)}{16}\omega^2 S(2\omega) \right] P\mathrm{d}t \\ &+ \varepsilon^{1/2}\omega p P \sqrt{\frac{S(2\omega)}{8}}\mathrm{d}W_1(t). \end{aligned} \tag{43}$$

Taking the expected value on both sides of eqation (43) results in

$$\mathrm{d}E[P] = \varepsilon\left[p\left(-\beta - \frac{1}{2}\omega\mathcal{H}^s(\omega) + \frac{3}{16}\omega^2 S(2\omega) \right) + \frac{p(p-1)}{16}\omega^2 S(2\omega) \right] E[P]\mathrm{d}t. \tag{44}$$

Hence the moment Lyapunov exponents are given by

$$\Lambda(p) = \lim_{t\to\infty} \frac{\log E[P]}{t} = \varepsilon \left[p \left(-\beta - \frac{1}{2}\omega \mathcal{H}^s(\omega) + \frac{3}{16}\omega^2 S(2\omega) \right) + \frac{p(p-1)}{16}\omega^2 S(2\omega) \right]. \tag{45}$$

It can be seen that the presence of viscoelasticity helps to stabilize the system, and the stronger the noise, the more unstable the system.

In oder to verify the accuracy of the approximate analytical results given by equation (45), the algorithm described in section 3 is used to simulate the moment Lyapunov exponents. The viscoelastic kernel function is assumed to be of the form

$$h(t) = \gamma e^{-\kappa t}, \tag{46}$$

and the wide-band noise is taken as the Gaussian white noise

$$\xi(t) = \sigma \dot{W}(t). \tag{47}$$

Then $S(2\omega) \equiv S_0 = \sigma^2$.

Let

$$x_1(t) = q(t), \quad x_2(t) = \dot{q}(t), \quad x_3(t) = \int_0^t h(t-s)q(s)\mathrm{d}s, \tag{48}$$

system (32) can be converted to the Itô differential equations

$$\mathrm{d}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} x_2 \\ -\omega^2 x_1 - 2\varepsilon\beta x_2 + \varepsilon\omega^2 x_3 \\ \gamma x_1 - \kappa x_3 \end{Bmatrix} \mathrm{d}t + \begin{Bmatrix} 0 \\ -\varepsilon^{1/2}\sigma\omega^2 x_1 \\ 0 \end{Bmatrix} \mathrm{d}W(t). \tag{49}$$

Then the iteration equations for explicit Euler scheme are given by

$$\begin{aligned} x_1^{k+1} &= x_1^k + x_2^k \cdot \Delta, \\ x_2^{k+1} &= x_2^k + (-\omega^2 x_1^k - 2\varepsilon\beta x_2^k + \varepsilon\omega^2 x_3^k) \cdot \Delta - \varepsilon^{1/2}\sigma\omega^2 x_1^k \cdot \Delta W^k, \\ x_3^{k+1} &= x_3^k + (\gamma x_1^k - \kappa x_3^k) \cdot \Delta. \end{aligned} \tag{50}$$

Figure 2 shows typical results of the moment Lyapunov exponents for different values of σ, with the parameters taken as $\gamma = \kappa = \omega = 1.0$, $\varepsilon = 0.1$, $\beta = 0.05$. The sample size for estimating the expected value is $S = 5000$, time step $\Delta = 0.001$, and the total length of time of simulation is $T = 2000$, i.e. the number of iterations is $MK = 2 \times 10^6$. It can be seen that when σ is small, the approximate result from the averaging method agrees very well with the simulation result. When σ becomes larger, i.e. the noise becomes stronger, the discrepancy between the simulation and analytical results increases.

As a comparison, the simulation results using the revised algorithm in section 1 are plotted in Figure 3. It is clear that with the increase of σ, the algorithm in section 3 gives better results for $p < 0$.

5 Conclusion

For linear stochastic dynamical systems with constant coefficients, when the solutions have large variances, the algorithm used to simulate the moment Lyapunov exponents presented in this paper, which uses the mean and variance of logarithm of norm, gives better numerical approximation than the previous method, which uses the direct sample average of norm as the estimation of expectation. Effective algorithms for general stochastic systems are currently being developed.

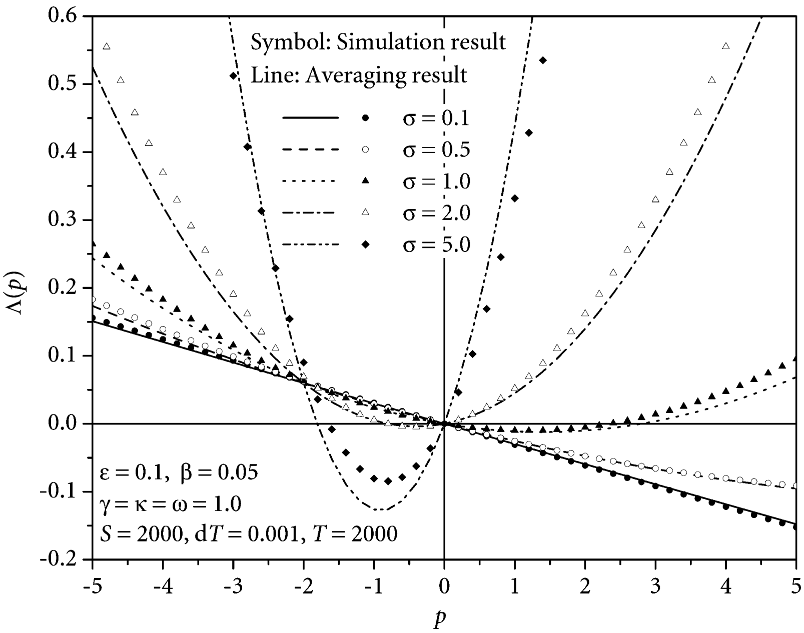

Figure 2: Moment Lyapunov exponents for different σ using algorithm in section 3

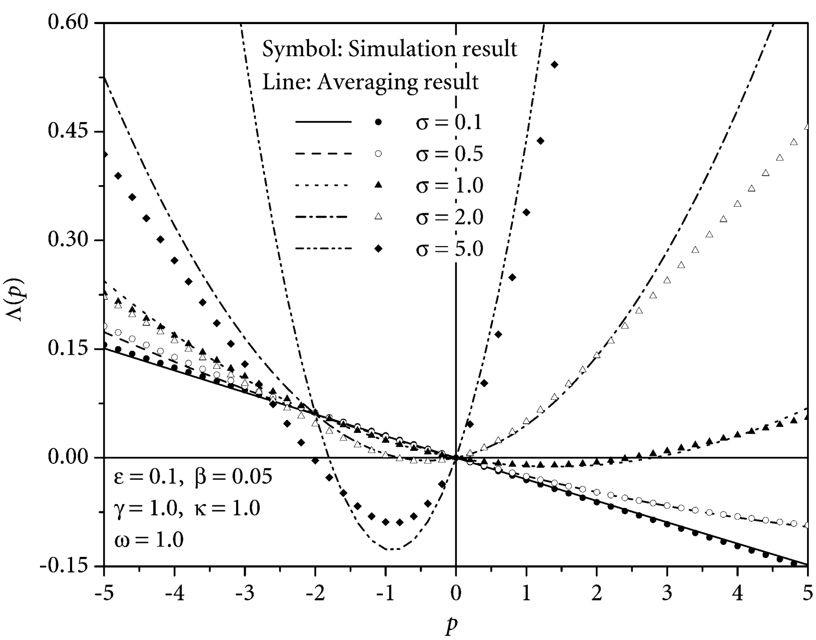

Figure 3: Moment Lyapunov exponents for different σ using algorithm in section 1

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