MATHEMATICAL PROGRAMMING IN STRUCTURAL MECHANICS – THE PAST AND THE FUTURE

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1. Introduction

During the last phase of the Second World War huge quantities of goods were transported across Atlantic ocean from the United States and Canada to Europe. Diminishing the overall cost of this logistic task even by several percents meant sparing millions of dollars. This demand motivated US-authorities to allocate money for research and many mathematicians started to investigate a problem of transportation: how to organize the flow of goods between given locations in order to minimize the total cost of delivery. G.B. Dantzig proposed very efficient method of solving such problems – the simplex algorithm – and named the domain *Linear Programming(LP)*. He did not realize that the second term of this name will soon collide with the vast area of computer programming.

After Dantzig published his first paper on LP (Dantzig, 1948), this approach attracted much interest in the West. It remained unnoticed that similar results were obtained by L.V. Kantorovich in the Soviet Union already before the Second World War (Kantorovich, 1939). At the beginning of the 1950-ties a general theory of *Mathematical Programming (MP)* was developed, with major contribution given by H.W. Kuhn and A.W. Tucker (Kuhn, Tucker, 1951). The subject of this theory is a *Non-Linear Programming Problem (NLP-problem)*:

$$\min_{\mathbf{x}\geq\mathbf{0}} \{\, f(\mathbf{x}) \mid g_i(\mathbf{x}) \leq 0,\ \mathbf{x}\in R^n,\ i=1,2,\ldots,m \,\} \tag{1}$$

Here $\mathbf{x}\in R^n$ is a column matrix of unknowns, $f = f(\mathbf{x})$ is a cost function and $g_i = g_i(\mathbf{x})$ are constraints with $i = 1,2,\ldots,m$. Usually it is assumed that functions f and g_i are convex. In particular, the cost function may be quadratic and the constraints may be linear. This leads to a particular form of the NLP-problem called *Quadratic Programming Problem (QP-problem)*:

$$\min_{\mathbf{x}\geq\mathbf{0}} \{\, \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{D}\,\mathbf{x} + \mathbf{c}^{\mathbf{T}}\mathbf{x} \mid \mathbf{A}\,\mathbf{x} \geq \mathbf{b} \,\} \tag{2}$$

Matrices $\mathbf{A}\in R^{m\times n}$, $\mathbf{b}\in R^m$, $\mathbf{c}\in R^n$ and $\mathbf{D}\in R^{n\times n}$ are given. In order to assure convexity of the cost function, matrix $\mathbf{D}$ must be positive definite.

Finally, taking $\mathbf{D}=\mathbf{0}$ in (2), we obtain the simplest form of MP-problems, namely a *Linear Programming Problem (LP-problem)*:

$$\min_{\mathbf{x}\geq\mathbf{0}} \{\, \mathbf{c}^{\mathbf{T}}\mathbf{x} \mid \mathbf{A}\mathbf{x} \geq \mathbf{b} \,\} \tag{3}$$

It turns out that each minimization problem in LP has its maximization counterpart:

$$\max_{\mathbf{y}\geq\mathbf{0}} \{\, \mathbf{b}^{\mathbf{T}}\mathbf{y} \mid \mathbf{A}^{\mathbf{T}}\mathbf{y} \leq \mathbf{c} \,\} \tag{4}$$

Problems (3) and (4) are said to be mutually *dual* and the entries of $\mathbf{y}\in R^m$ are called *dual variables*. For the sake of simplicity, we quote MP-problems in their canonical form: the problems (1) to (4)

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contain only non-negative variables and inequality constraints. In general, free variables and equality constraints can be present as well.

At the beginning Linear Programming was used only in management and economics. The remnants of this period are still present in the terminology (the cost function, the shadow prices, etc.). A typical application of the model (4) would be maximizing the total production of a factory that uses different technological processes and different resources. The unknowns y_i are then the time slots allocated for each process, the entries of $\mathbf{c}$ represent given efficiency of each process, the entries of $\mathbf{A}$ tell us how much of each resource is consumed by a particular process and the entries of $\mathbf{b}$ describe available amount of each resource.

In parallel to things happening in Mathematical Programming, revolutionary changes occurred in Structural Analysis. Instead of relying on linear elasticity and on admissible stresses, a concept of safety factors against possible *ultimate states* was introduced. Again the Cold War precluded the exchange of ideas and the pioneering work of A.A. Gvozdev (Gvozdev, 1949) remained unknown in the Western hemisphere.

It was proved soon that the safety factor against plastic collapse does not depend on elastic properties of the structural material and that such factor can be found by maximizing the load multiplier over all statically admissible stress fields. For skeletal structures with a single dominant internal force (e.g. the axial force or the bending moment) a stress state $\mathbf{s}$ is statically admissible if each s_j remains less or equal to the yield stress s_{0j} and if $\mathbf{s}$ is equilibrated with a load $\mathbf{p}$. Assuming strains $\mathbf{q}$ and displacements $\mathbf{w}$ to remain small prior to the plastic collapse, we can write the equilibrium equation as $\mathbf{C}^{\mathrm{T}}\,\mathbf{s} = \mathbf{p}$, where $\mathbf{C}$ is the matrix of kinematics: $\mathbf{q} = \mathbf{C}\,\mathbf{w}$. Let loading be proportional: $\mathbf{p} = \mu\,\mathbf{p}_0$, where μ is an unknown multiplier and $\mathbf{p_0}$ is a given reference load. According to the static theorem, the ultimate value μ_* of the load factor can be found solving the problem:

$$\max_{\mu,\,\mathbf{s}} \{ \mu \mid \mathbf{s} \le \mathbf{s_0},\, \mathbf{C}^{\mathrm{T}}\,\mathbf{s} - \mu\,\mathbf{p_0} = \mathbf{0} \} \tag{5}$$

Looking at this model today we see at once that it is a LP-problem. However, the pioneers of the MP-based modeling of structural behavior had to overcome the barrier between economics and mechanics. Having accomplished that, they could enjoy the power of mathematics: the semantics of the production planning problems and the ultimate load problem is completely different but the formal structure of both problems is identical. We believe that the following papers, cited in the alphabetic order after the first author, were important in providing impetus to the MP-oriented approach: Biron & Hodge, 1968; Brown & Ang, 1964; Ceradini & Gavarini, 1965; Hodge, 1966; Koopman & Lance, 1965; Sacchi & Buzzi-Ferraris, 1966; Wolfensberger, 1964.

Special tribute should be given to two persons: Mircea Z. Cohn and Aleksandras Čyras. Already in 1956 Cohn published in Romania his first paper on the plastic structural analysis (Cohn, 1956). In 1972 an inspiring paper on the unified theory of plastic analysis appeared (Cohn et al., 1972). During the NATO Advanced Study Institute that took place in Waterloo in 1977 he was invited to deliver a keynote lecture (Cohn, 1979). The celebration of his 65th birthday in 1991 gathered over 60 contributors from 14 countries. The results of this meeting were published in a book edited by Cohn's former students D.E. Grierson, A. Franchi and P. Riva (Grierson et al., 1991). Their contribution to the considered domain is substantial (Franchi & Cohn, 1980), (Grierson & Gladwell, 1971), (Grierson, 1972), (Riva & Cohn, 1990).

Since Lithuania was a part of the Soviet Union at the time of his scientific carrier, Čyras was for a long time isolated from the Western scientific community. Most of his early papers were written in Russian (Čyras, 1963) and published in a local Lithuanian journal. His first paper in English, written with the present author, appeared in Poland in 1968 (Čyras & Borkauskas, 1968). Already well known in the Soviet Union and in other countries of the Eastern block, he was invited in 1974 by Wacław Olszak to present his results at the CISM-course (Čyras, 1974).

In 1969 Čyras published a book that contained many fundamental results on the applications of Linear Programming in the analysis and design of structures made of rigid-perfectly plastic material (Čyras, 1969). Two further books (Čyras, 1971), (Čyras et al., 1974) were also written in Russian language. The first English edition appeared in 1983 – this was the translation of the book (Čyras, 1982). In 2002 the 75th birthday of Aleksandras Čyras was celebrated in Vilnius. This motivated his former students R. Karkauskas and the present author to prepare new edition of the book (Čyras et al., 1974). Substantially updated and rewritten it appeared in 2004 (Čyras et al., 2004).

Being a founder and a long time Rector of the Institute of Civil Engineering in Vilnius (VISI), Čyras inspired many researchers to work on the MP-applications in Structural Analysis and Optimum Design. This group contributed substantially to progress in such areas as the ultimate state under constrained strains (Čyras & Čižas, 1966), the evaluation of displacements prior to collapse (Čyras & Baronas, 1971), the ultimate state of shells (Čyras & Karkauskas, 1971), (Čyras & Kalanta, 1974), the plastic shakedown problem (Čyras & Atkočiūnas, 1984). The present author took part in the development of general concept of the dual approach (Čyras & Borkauskas, 1969). In 1988 he published a book in Polish that was translated three years later into English (Borkowski, 1988).

The above overview is by no means complete. It reflects personal experience of the author who was involved in this fascinating scientific adventure.

2. Dual View of Mechanics

It seems that the first new insight brought by the MP-approach to Structural Mechanics was discovering the equivalence of kinematic and static formulations. In Mathematical Programming this property is known as duality: under certain premises each problem of constrained extremum has its dual and the values of cost functions for such problems attained at the solutions coincide. We already quoted the dual LP-problems (1) and (2). If $\mathbf{x}_*$ is the solution of (1) and if the constrained minimum $f'(\mathbf{x}_*) = \mathbf{c}^{\mathbf{T}}\mathbf{x}_*$ is finite, then there exists a solution $\mathbf{y}_*$ of the dual problem (2) and the optimum values of the cost functions coincide: $f''(\mathbf{y}_*) = \mathbf{b}^{\mathbf{T}}\mathbf{y}_* = f'(\mathbf{x}_*)$.

Let us expand slightly the QP-problem (4) by introducing additional variables $\mathbf{y} \in R^n$:

$$\min_{\mathbf{x}\geq\mathbf{0},\,\mathbf{y}\geq\mathbf{0}} \{ \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{A}_{\mathbf{xx}}\,\mathbf{x} - \frac{1}{2}\mathbf{y}^{\mathbf{T}}\mathbf{A}_{\mathbf{yy}}\,\mathbf{y} + \mathbf{b}_{\mathbf{x}}^{\mathbf{T}}\mathbf{x} \mid \mathbf{A}_{\mathbf{yx}}\,\mathbf{x} + \mathbf{A}_{\mathbf{yy}}\mathbf{y} + \mathbf{b}_{\mathbf{y}} \leq \mathbf{0}\} \tag{6}$$

Here $\mathbf{b}_{\mathbf{x}} \in R^m$, $\mathbf{b}_{\mathbf{y}} \in R^n$, $\mathbf{A}_{\mathbf{xx}} \in R^{m\times m}$, $\mathbf{A}_{\mathbf{yy}} \in R^{n\times n}$, $\mathbf{A}_{\mathbf{yx}} \in R^{n\times m}$. Moreover, $\mathbf{A}_{\mathbf{xx}}$ is positive definite and $\mathbf{A}_{\mathbf{yy}}$ is negative definite. The dual of (6) reads

$$\max_{\mathbf{x}\geq\mathbf{0},\,\mathbf{y}\geq\mathbf{0}} \{ -\frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{A}_{\mathbf{xx}}\,\mathbf{x} + \frac{1}{2}\mathbf{y}^{\mathbf{T}}\mathbf{A}_{\mathbf{yy}}\,\mathbf{y} + \mathbf{b}_{\mathbf{y}}^{\mathbf{T}}\mathbf{y} \mid \mathbf{A}_{\mathbf{xx}}\,\mathbf{x} + \mathbf{A}_{\mathbf{xy}}\mathbf{y} + \mathbf{b}_{\mathbf{x}} \geq \mathbf{0}\} \tag{7}$$

Duality has simple geometrical interpretation. Solving a pair of dual problems (6), (7) is equivalent to finding the *saddle point*

$$L(\mathbf{x}_*,\mathbf{y}_*) = \min_{\mathbf{x}\geq\mathbf{0}}\ \max_{\mathbf{y}\geq\mathbf{0}} L(\mathbf{x},\mathbf{y}) \tag{8}$$

of the *Lagrange function*

$$L(\mathbf{x},\mathbf{y}) = \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{A}_{\mathbf{xx}}\,\mathbf{x} + \frac{1}{2}\mathbf{y}^{\mathbf{T}}\mathbf{A}_{\mathbf{yy}}\,\mathbf{y} + \mathbf{x}^{\mathbf{T}}\mathbf{A}_{\mathbf{xy}}\mathbf{y} + \mathbf{b}_{\mathbf{x}}^{\mathbf{T}}\mathbf{x} + \mathbf{b}_{\mathbf{y}}^{\mathbf{T}}\mathbf{y} \tag{9}$$

Such point can be reached in two ways. One can first establish the parabola that contains all maxima with respect to y-variables and then find the minimum on this curve. This sequence corresponds to the problem (6). Alternatively, one can begin with finding the parabola that corresponds to all minima with respect to the x-variables and then look for the maximum of this concave function. This way leads to the dual problem (7).

Point $(\mathbf{x}_*, \mathbf{y}_*)$ is the saddle point of L if it satisfies *Kuhn-Tucker conditions (KT-conditions)*:

$$\nabla \mathbf{L}_\mathbf{x} \geq \mathbf{0}\,,\, \nabla \mathbf{L}_\mathbf{y} \leq \mathbf{0},\, \mathbf{x} \geq \mathbf{0},\, \mathbf{y} \geq \mathbf{0} \tag{10}$$

$$\mathbf{x}^\mathbf{T} \nabla \mathbf{L}_\mathbf{x} = 0\,,\, \mathbf{y}^\mathbf{T} \nabla \mathbf{L}_\mathbf{y} = 0 \tag{11}$$

Here $\nabla \mathbf{L}_\mathbf{x} \in R^n$ and $\nabla \mathbf{L}_\mathbf{y} \in R^m$ are gradients of L with respect to $\mathbf{x}$ and $\mathbf{y}$. If all variables were free, then the KT-conditions would reduce to the common stationarity conditions:

$$\nabla \mathbf{L}_\mathbf{x} = \mathbf{0}\,,\, \nabla \mathbf{L}_\mathbf{y} = \mathbf{0} \tag{12}$$

or, explicitly, to the set of linear algebraic equations

$$\begin{aligned} \mathbf{A}_{\mathbf{xx}}\mathbf{x} + \mathbf{A}_{\mathbf{xy}}\mathbf{y} + \mathbf{b}_\mathbf{x} &= \mathbf{0} \\ \mathbf{A}_{\mathbf{yx}}\mathbf{x} + \mathbf{A}_{\mathbf{yy}}\mathbf{y} + \mathbf{b}_\mathbf{y} &= \mathbf{0} \end{aligned} \tag{13}$$

Note two features that distinguish this set: a) its matrix of coefficients is symmetric; b) the sub-matrices situated along the diagonal have special properties – $\mathbf{A}_{\mathbf{xx}}$ is positive definite, $\mathbf{A}_{\mathbf{yy}}$ is negative definite

It is easy to check by inspecting the Table 1 that the set of equations governing linear static analysis of elastic structures follows exactly the template (13). The goal of such analysis is to find displacements[1] $\mathbf{w}$, stresses $\mathbf{s}$ and reactions $\mathbf{r}$ of elastic structure caused by a given static load $\mathbf{p_0}$ and by a given kinematic load $\mathbf{w_0}$. The structure is represented by a common discrete model, where $\mathbf{E}$ is the matrix of elasticity, $\mathbf{C}$ is the matrix of compatibility, subscript p refers to the degrees of freedom with prescribed external forces and subscript w refers to the degrees of freedom with prescribed displacements.

It is seen from the Table 1 that displacements play the role of x-variables, whereas stresses and reactions correspond to y-variables in the model (13). The first two rows of the Table 1 contain the equilibrium equations. The third row comes from substituting strains $\mathbf{q} = \mathbf{C}\mathbf{w} = \mathbf{C_p}\mathbf{w_p} + \mathbf{C_w}\mathbf{w_w}$ into the constitutive equation $\mathbf{q} = \mathbf{E}^{-1}\mathbf{s}$. The last row merely says that $\mathbf{w_w} = \mathbf{w_0}$. A matrix with zero entries can be treated either as positive semi-definite or as negative semi-definite. The inverse of matrix of elasticity is strictly positive definite. Hence, $-\mathbf{E}^{-1}$ is strictly negative definite.

Note that there are no inequalities in the Table 1 and that all variables are free with respect to sign. Hence, we don’t need to take the KT-conditions into account.

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[1] In the sequel we write „displacements”, „strains”, „stresses” and „loads” having in mind generalized variables taken usually in Structural Analysis.

Table 1. Governing equations of linear elastic analysis of structures

	$\mathbf{w_p}$	$\mathbf{w_w}$	$\mathbf{s}$	$\mathbf{r}$	1	
$\nabla \mathbf{L}_{\mathbf{w_p}} =$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_p^T}$	$\mathbf{0}$	$\mathbf{p_0}$	$= \mathbf{0}$
$\nabla \mathbf{L}_{\mathbf{w_w}} =$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_w^T}$	$-\mathbf{I}$	$\mathbf{0}$	$= \mathbf{0}$
$\nabla \mathbf{L_s} =$	$\mathbf{C_p}$	$\mathbf{C_w}$	$-\mathbf{E}^{-1}$	$\mathbf{0}$	$\mathbf{0}$	$= \mathbf{0}$
$\nabla \mathbf{L_r} =$	$\mathbf{0}$	$-\mathbf{I}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{w_0}$	$= \mathbf{0}$

What do we gain by using the MP-based approach in elastic analysis? First, having filled the Table 1, we can derive easily the dual energy principles (compare the templates (6) and (7)):

a) kinematic principle –

$$\min_{\mathbf{s},\,\mathbf{w}} \{ \frac{1}{2}\mathbf{s^T E^{-1} s} - \mathbf{w_p^T p_0} \mid \mathbf{C_p w_p} + \mathbf{C_w w_w} - \mathbf{E^{-1} s} = \mathbf{0}, \mathbf{w_w} = \mathbf{w_0} \} \tag{14}$$

b) static principle –

$$\max_{\mathbf{s},\,\mathbf{r}} \{ -\frac{1}{2}\mathbf{s^T E^{-1} s} + \mathbf{w_0^T r} \mid \mathbf{C_p^T s} = \mathbf{p_0}, \mathbf{C_w^T s} - \mathbf{r} = \mathbf{0} \} \tag{15}$$

Second, since the QP-problems (14), (15) contain no inequality constraints or non-negative variables, each of them can be reduced to a set of equations. This leads us very naturally to the fundamental computational tools of elastic analysis: the Stiffness (Force) Method and the Flexibility (Displacement) Method.

Third, the existence and uniqueness of solution for any given loading $\mathbf{p_0}, \mathbf{w_0}$ follows immediately from the convexity of problems (14), (15). Moreover, a generalization of the model (14), (15) to unilateral contact is straightforward. The replacement of $\mathbf{w_w} = \mathbf{w_0}$ in (14) by less restrictive condition $\mathbf{w_w} \geq \mathbf{w_0}$ induces sign constraint on $\mathbf{r}$ in the dual problem:

$$\min_{\mathbf{s},\,\mathbf{w}} \{ \frac{1}{2}\mathbf{s^T E^{-1} s} - \mathbf{w_p^T p_0} \mid \mathbf{C_p w_p} + \mathbf{C_w w_w} - \mathbf{E^{-1} s} = \mathbf{0}, \mathbf{w_w} \geq \mathbf{w_0} \} \tag{16}$$

$$\max_{\mathbf{s},\,\mathbf{r}} \{ -\frac{1}{2}\mathbf{s^T E^{-1} s} + \mathbf{w_0^T r} \mid \mathbf{C_p^T s} = \mathbf{p_0}, \mathbf{C_w^T s} - \mathbf{r} = \mathbf{0}, \mathbf{r} \geq \mathbf{0} \} \tag{17}$$

Seemingly minor, this modification has dramatic consequences: a) the linearity of the problem is lost due to the KT-condition $\mathbf{r^T (w_0 - w_w)} = 0$; b) the energy principles (16), (17) can not be replaced by the sets of equations. Moreover, for certain loads $\mathbf{p_0}, \mathbf{w_0}$ the constraints of the problems (16), (17) might become contradictory. Thus the existence of solution is not warranted any more.

3. Expanding Area of Application

As shown by G. Maier in (Maier, 1970) Quadratic Programming allows us to model a broad range of piecewise-linear structural behaviors. The constitutive laws of such behaviors are given in Fig.2. An exhaustive overview of the QP-based approach, including both continuum and discrete models, can be found in (Borkowski, 2004). Let us recall, for the sake of brevity, only two cases – a cable-strut elastic structure and a structure made of strain-hardening material.

The entries of the stress matrix $\mathbf{s}$ for the cable-strut structure can be split into two sub-matrices: $\mathbf{s}_s$ represents axial forces in struts and $\mathbf{s}_c$ represents axial forces in cables. Obviously, cables work only in tension, while struts can be either extended or compressed. Table 2 shows the set of relations governing elastic behavior of the cable-strut system. In order to simplify things, we assume purely static loading.

The adjoint variable for the axial force in cable is its *slackness* – the axial strain taken with negative sign. Grouping the slackness unknowns into a column matrix $\mathbf{q}_c$, we consider it as a matrix of non-negative variables. Then, the entries of $\mathbf{s}_c$ remain formally unconstrained in sign: they will become non-negative in the solution due to the constraint $\nabla \mathbf{L}_{\mathbf{q}_c} \geq \mathbf{0}$.

Applying the templates (6), (7) to the Table 2, we obtain the following energy principles for the cable-strut system:

$$\min_{\mathbf{s}_s,\, \mathbf{w},\, \mathbf{q}_c \geq \mathbf{0}} \left\{ \frac{1}{2} \mathbf{s}_s^T \mathbf{E}^{-1} \mathbf{s}_s - \mathbf{w}^T \mathbf{p}_0 \mid \mathbf{C}_s \mathbf{w} + \mathbf{C}_c \mathbf{w} - \mathbf{E}^{-1} \mathbf{s}_s = \mathbf{0}, \mathbf{C}_c + \mathbf{q}_c = \mathbf{0} \right\} \tag{18}$$

$$\max_{\mathbf{s}_s,\, \mathbf{s}_c} \left\{ -\frac{1}{2} \mathbf{s}_s \mathbf{E}^{-1} \mathbf{s}_s \mid \mathbf{C}_s^T \mathbf{s}_s + \mathbf{C}_c^T \mathbf{s}_c = \mathbf{p}_0,\ \mathbf{s}_c \geq \mathbf{0} \right\} \tag{19}$$

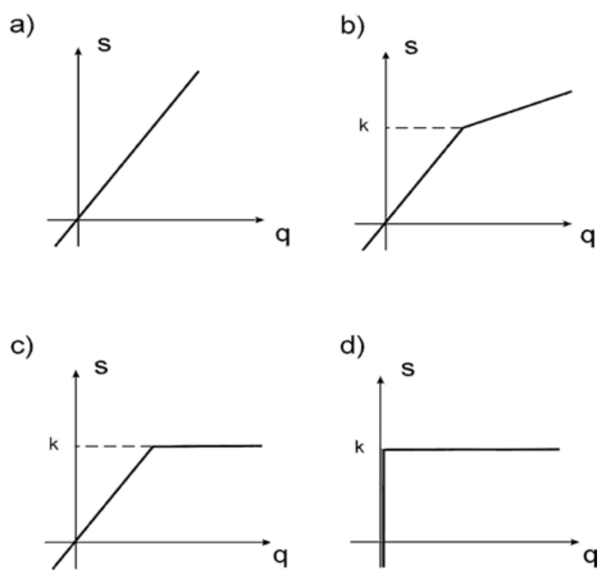

Fig. 2. Piecewise-linear behaviors of material: a) elastic; b) elastic-strain hardening; c) elastic-perfectly plastic; d) rigid-perfectly plastic.

Note that the existence of solution is not warranted: for a certain load $\mathbf{p_0}$ there might be no equilibrated stress state. Then the constraints of the QP-problem (19) become contradictory. On the other hand, if a solution exists, then it is unique due to the convexity of the problems (18), (19).

Table 2. Governing relations for a cable-strut structure

	$\mathbf{w}$	$\mathbf{q_c}$	$\mathbf{s_s}$	$\mathbf{s_c}$	1	
$\nabla \mathbf{L_w}$ =	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C_s^T}$	$\mathbf{C_c^T}$	$\mathbf{p_0}$	$= \mathbf{0}$
$\nabla \mathbf{L_{q_c}}$ =	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{0}$	$\geq \mathbf{0}$
$\nabla \mathbf{L_{s_s}}$ =	$\mathbf{C_s}$	$\mathbf{0}$	$-\mathbf{E}^{-1}$	$\mathbf{0}$	$\mathbf{0}$	$= \mathbf{0}$
$\nabla \mathbf{L_{s_c}}$ =	$\mathbf{C_c}$	$\mathbf{I}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$= \mathbf{0}$
	$\mathbf{q_c} \geq \mathbf{0}$, $\mathbf{q_c^T} \nabla \mathbf{L_c} = 0$					

Similar procedure can be applied to a structure made of elastic-strain hardening material. A complete set of governing relations for such a structure is shown in Table 3. The first row of this table includes linearised yield condition $\mathbf{N^T s} - \mathbf{H\lambda} \leq \mathbf{k_0}$. A positive definite $(r \times r)$-matrix of hardening $\mathbf{H}$ is responsible for a shift of the yield planes caused by the plastic strains. The equation of equilibrium that relates the given load $\mathbf{p_0}$ to the unknown stress $\mathbf{s}$ can be recognized in the second row. The last row ensures the kinematic compatibility of strains $\mathbf{q} = \mathbf{q_e} + \mathbf{q_p}$ and displacements $\mathbf{w}$. Elastic strains obey the Hooke's law $\mathbf{q_e} = \mathbf{E^{-1} s}$. Plastic strains are governed by the associated flow rule $\mathbf{q_p} = \mathbf{N}\,\boldsymbol{\lambda}$. Plastic multipliers represented by a column matrix $\boldsymbol{\lambda} \in R^r$ are supposed to be non-negative.

Table 3. Governing relations for a structure made of elastic-strain hardening material.

	$\boldsymbol{\lambda}$	$\mathbf{w}$	$\mathbf{s}$	1	
$\nabla \mathbf{L}_{\boldsymbol{\lambda}}$ =	$\mathbf{H}$	$\mathbf{0}$	$-\mathbf{N^T}$	$\mathbf{k_0}$	$\geq \mathbf{0}$
$\nabla \mathbf{L_w}$ =	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C^T}$	$-\mathbf{p_0}$	$= \mathbf{0}$
$\nabla \mathbf{L_s}$ =	$-\mathbf{N}$	$\mathbf{C}$	$-\mathbf{E}^{-1}$	$\mathbf{0}$	$= \mathbf{0}$
	$\boldsymbol{\lambda} \geq \mathbf{0}$, $\boldsymbol{\lambda}^\mathbf{T} \nabla \mathbf{L}_{\boldsymbol{\lambda}} = 0$				

Applying the templates (6), (7) to the Table 2, we obtain the following energy principles for elastic-strain hardening structures:

$$\min_{\mathbf{s},\,\mathbf{w},\boldsymbol{\lambda}\geq\mathbf{0}} \{ \frac{1}{2}\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{H}\,\boldsymbol{\lambda} + \frac{1}{2}\mathbf{s}^{\mathrm{T}}\mathbf{E}^{-1}\,\mathbf{s} + \mathbf{k}_0^{\mathrm{T}}\boldsymbol{\lambda} - \mathbf{w}^{\mathrm{T}}\mathbf{p}_0 \mid \mathbf{C}\,\mathbf{w} - \mathbf{N}\,\boldsymbol{\lambda} - \mathbf{E}^{-1}\mathbf{s} = \mathbf{0} \} \tag{20}$$

$$\max_{\mathbf{s}} \{ -\frac{1}{2}\boldsymbol{\lambda}^{\mathrm{T}}\,\mathbf{H}\,\boldsymbol{\lambda} - \frac{1}{2}\mathbf{s}\,\mathbf{E}^{-1}\,\mathbf{s} \mid \mathbf{N}^{\mathrm{T}}\,\mathbf{s} - \mathbf{H}\,\boldsymbol{\lambda} \leq \mathbf{k}_0,\, \mathbf{C}^{\mathrm{T}}\mathbf{s} = \mathbf{p}_0 \} \tag{21}$$

Solutions of these QP-problems exist for any $\mathbf{p}_0$ since the admissible domain for stresses adjusts itself automatically to the loading. The convexity of the problems (20), (21) ensures uniqueness of the structural response. The only drawback of the model (20), (21) is its holonomic nature: a possible local unloading is not taken into account.

Assuming $\mathbf{H} = \mathbf{0}$, we obtain structure made of elastic-perfectly plastic material. The relevant dual QP-problems

$$\min_{\mathbf{s},\,\mathbf{w},\boldsymbol{\lambda}\geq\mathbf{0}} \{ \frac{1}{2}\mathbf{s}^{\mathrm{T}}\mathbf{E}^{-1}\,\mathbf{s} + \mathbf{k}_0^{\mathrm{T}}\boldsymbol{\lambda} - \mathbf{w}^{\mathrm{T}}\mathbf{p}_0 \mid \mathbf{C}\,\mathbf{w} - \mathbf{N}\,\boldsymbol{\lambda} - \mathbf{E}^{-1}\mathbf{s} = \mathbf{0} \} \tag{22}$$

$$\max_{\mathbf{s}} \{ \frac{1}{2}\mathbf{s}\,\mathbf{E}^{-1}\,\mathbf{s} \mid \mathbf{N}^{\mathrm{T}}\,\mathbf{s} \leq \mathbf{k}_0,\, \mathbf{C}^{\mathrm{T}}\mathbf{s} = \mathbf{p}_0 \} \tag{23}$$

still retain the uniqueness of solutions but the nice property of the existence of solution for any $\mathbf{p}_0$ is lost. The yield surface $\mathbf{N}^{\mathrm{T}}\mathbf{s} = \mathbf{k}_0$ is now fixed and too high loading would cause the absence of statically admissible field of stresses.

If we would like to neglect elastic strains as well, the static energy principle (23) would loose its cost function (since there would be $\mathbf{E}^{-1} = \mathbf{0}$). This shows clearly that for a structure made of the rigid-perfectly plastic material the problem "find the response to the given load" is ill-posed. The right formulation is "find the load factor that corresponds to the state of plastic collapse". A complete set of relations for this formulation is given in Table 4.

Table 4. Governing relations for ultimate load factor.

	$\boldsymbol{\lambda}$	$\mathbf{w}$	$\mathbf{s}$	μ	1	
$\nabla\mathbf{L}_{\boldsymbol{\lambda}} =$	$\mathbf{0}$	$\mathbf{0}$	$-\mathbf{N}^{\mathrm{T}}$	$\mathbf{0}$	$\mathbf{k}_0$	$\geq \mathbf{0}$
$\nabla\mathbf{L}_{\mathbf{w}} =$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C}^{\mathrm{T}}$	$-\mathbf{p}_0$	$\mathbf{0}$	$= \mathbf{0}$
$\nabla\mathbf{L}_{\mathbf{s}} =$	$-\mathbf{N}$	$\mathbf{C}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$= \mathbf{0}$
$\nabla L_{\mu} =$	$\mathbf{0}$	$-\mathbf{p}_0^{\mathrm{T}}$	$\mathbf{0}$	$\mathbf{0}$	1	$= 0$
	$\boldsymbol{\lambda} \geq \mathbf{0}$, $\boldsymbol{\lambda}^{\mathrm{T}}\,\nabla\mathbf{L}_{\boldsymbol{\lambda}} = 0$					

The primal problem generated by this table is

$$\min_{\boldsymbol{\lambda},\,\mu} \{ \mathbf{k}_0^{\mathrm{T}}\,\boldsymbol{\lambda} \mid -\mathbf{N}\,\boldsymbol{\lambda} + \mathbf{C}\,\mathbf{w} = \mathbf{0},\; \mathbf{p}_0^{\mathrm{T}}\mathbf{w} = 1 \} \tag{24}$$

and the dual one has been already given as Eq. (5). All sub-matrices located at the diagonal of the Table 4 have zero values. This leads to vanishing quadratic terms in the cost functions of the dual problems. On the other hand, we can not expect the solution to be unique in terms of stresses and/or collapse mechanisms, since linear functions are not strictly convex (concave).

4. Explicit and Implicit Optimization

Models based on Mathematical Programming give the user clear insight into the possible ways of structural optimization. Let us begin with the topological optimum design. Changing the layout of structural elements makes the entries of $\mathbf{C}$ variable. Hence, all relations where this matrix appears become non-linear. This circumstance explains why the topological formulation of the optimum design problem is still an open research area.

The search for optimum sizing of structural elements under given topology becomes much easier if we neglect elastic strains. Assuming additionally that a form of each element is given up to certain parameters, we can take the entries of $\mathbf{k}$ as unknowns, retaining the fixed matrices $\mathbf{C}$ and $\mathbf{N}$. Hence, basic relations of the rigid-perfectly plastic model remain linear. Note, that is not possible for elastic structures: in the sizing problem matrix $\mathbf{E}$ becomes variable which destroys linearity of the constitutive relations.

Let us assume that the unknown plastic modulae $\mathbf{k}$ are governed by a relatively small number of design variables the : $\mathbf{k} = \mathbf{G}^{\mathrm{T}}\mathbf{z}$, where $\mathbf{G}$ is $(g \times r)$-matrix of configuration. In order to maintain linearity of the problem, we adopt linear cost function $f(\mathbf{z}) = \mathbf{c}_0^{\mathrm{T}}\,\mathbf{z}$, where the entries of column matrix $\mathbf{c}_0 \in R^g$ are given cost coefficients. A problem to be solved reads: "given the structural layout find the optimum sizing $\mathbf{z}_*$ that minimizes f and assures given safety factor $\boldsymbol{\mu}_*$ against plastic collapse". In fact, we know an expected load carrying capacity $\mathbf{p}_* = \mu_*\mathbf{p}_0$ of the optimized structure, since both the safety factor $\boldsymbol{\mu}_*$ and the reference load $\mathbf{p}_0$ are given.

Table 5. Governing relations for optimum plastic design.

	$\boldsymbol{\lambda}$	$\mathbf{w}$	$\mathbf{s}$	$\mathbf{z}$	1	
$\nabla\mathbf{L}_{\lambda} =$	$\mathbf{0}$	$\mathbf{0}$	$-\mathbf{N}^{\mathrm{T}}$	$\mathbf{G}^{\mathrm{T}}$	$\mathbf{0}$	$\geq \mathbf{0}$
$\nabla\mathbf{L}_{\mathbf{w}} =$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C}^{\mathrm{T}}$	$\mathbf{0}$	$-\mathbf{p}_*$	$= \mathbf{0}$
$\nabla\mathbf{L}_{\mathbf{s}} =$	$-\mathbf{N}$	$\mathbf{C}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$= \mathbf{0}$
$\nabla\mathbf{L}_{\mathbf{z}} =$	$\mathbf{G}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$-\mathbf{c}_0$	$\leq \mathbf{0}$
	$\boldsymbol{\lambda} \geq \mathbf{0}$, $\mathbf{z} \geq \mathbf{0}$, $\boldsymbol{\lambda}^{\mathrm{T}}\nabla\mathbf{L}_{\lambda} = 0$, $\mathbf{z}^{\mathrm{T}}\nabla\mathbf{L}_{\mathbf{z}} = 0$					

Table 5 shows the internal structure of the problem of optimum plastic design. This problem is equivalent to the following pair of dual LP-problems:

$$\min_{\mathbf{s},\,\mathbf{z}\geq\mathbf{0}} \{ \mathbf{c}_0^{\mathrm{T}}\,\mathbf{z} \mid \mathbf{G}^{\mathrm{T}}\mathbf{z} - \mathbf{N}^{\mathrm{T}}\mathbf{s} \geq \mathbf{0},\ \mathbf{C}^{\mathrm{T}}\mathbf{s} = \mathbf{p}_* \} \tag{25}$$

$$\max_{\mathbf{w},\,\boldsymbol{\lambda}\geq\mathbf{0}} \{ \mathbf{p}_*^{\mathrm{T}}\,\mathbf{w} \mid -\mathbf{N}\,\boldsymbol{\lambda} + \mathbf{C}\,\mathbf{w} = \mathbf{0}, \mathbf{G}\,\boldsymbol{\lambda} \leq \mathbf{c}_0 \} \tag{26}$$

Duality reveals an interesting role of the cost coefficients: according to the second constraint of the kinematic energy principle (26) these coefficients bound linear combinations of plastic multipliers. If we would take $\mathbf{G} = \mathbf{I}$ and $g = r$ which means that each plastic modulus is treated as an independent design variable, then the second constraint in (26) would reduce to the inequality $\boldsymbol{\lambda} \leq \mathbf{c}$. In those parts of optimum structure that undergo yielding this constraint must be satisfied as equality. Hence, assuming certain cost coefficients we in fact impose certain collapse mechanism on the optimum structure. It can be shown that this mechanism ensures uniform dissipation of energy over the structure.

Sizing is the case of explicitly formulated optimization problem. It is worth noting that the MP-approach allows us to uncover also certain possibilities of optimization hidden in the problems formulated from the analysis point of view. A good example of that is the problem of rigid-perfectly plastic structure brought to the state of plastic collapse by purely kinematic loading. Let us split again the degrees of freedom of the discrete structural model into the parts denoted by the indices p and w, as it was already done in the model (14), (15). Then the considered problem can be formulated in the following way: "given the structural layout and the distribution of plastic modulae $\mathbf{k_0}$ find the state $\mathbf{s}_*, \dot{\boldsymbol{\lambda}}_*, \dot{\mathbf{w}}_*$ of the structure brought to the plastic collapse by a given kinematic load $\dot{\mathbf{w}}_0$"[2].

Table 6. Governing relations for kinematically induced plastic collapse.

	$\dot{\boldsymbol{\lambda}}$	$\dot{\mathbf{w}}_\mathbf{p}$	$\dot{\mathbf{w}}_\mathbf{w}$	$\mathbf{s}$	$\mathbf{r}$	1	
$\nabla\mathbf{L}_\lambda =$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$-\mathbf{N}^\mathbf{T}$	$\mathbf{0}$	$\mathbf{k_0}$	$\geq \mathbf{0}$
$\nabla\mathbf{L}_{\mathbf{w_p}} =$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C}_\mathbf{p}^\mathbf{T}$	$\mathbf{0}$	$\mathbf{0}$	$= \mathbf{0}$
$\nabla\mathbf{L}_{\mathbf{w_w}} =$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C}_\mathbf{w}^\mathbf{T}$	$-\mathbf{I}$	$\mathbf{0}$	$= \mathbf{0}$
$\nabla\mathbf{L}_\mathbf{s} =$	$-\mathbf{N}$	$\mathbf{C}_\mathbf{p}$	$\mathbf{C}_\mathbf{w}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$= \mathbf{0}$
$\nabla\mathbf{L}_\mathbf{r} =$	$\mathbf{0}$	$\mathbf{0}$	$-\mathbf{I}$	$\mathbf{0}$	$\mathbf{0}$	$\dot{\mathbf{w}}_\mathbf{0}$	$= \mathbf{0}$
	$\dot{\boldsymbol{\lambda}} \geq \mathbf{0}$, $\dot{\boldsymbol{\lambda}}^\mathbf{T}\, \nabla\mathbf{L}_\lambda = 0$						

Table 6 shows the governing relations for this problem. They correspond to the following pair of dual LP-problems:

$$\min_{\mathbf{w_p},\, \mathbf{w_w},\, \lambda \geq 0} \{ \mathbf{k}_0^\mathbf{T}\, \dot{\boldsymbol{\lambda}} \mid -\mathbf{N}\, \dot{\boldsymbol{\lambda}} + \mathbf{C}_\mathbf{p}\, \dot{\mathbf{w}}_\mathbf{p} + \mathbf{C}_\mathbf{w}\, \dot{\mathbf{w}}_\mathbf{w} = \mathbf{0}, \dot{\mathbf{w}}_\mathbf{w} = \dot{\mathbf{w}}_\mathbf{0} \} \tag{27}$$

$$\max_{\mathbf{s},\, \mathbf{r}} \{ \dot{\mathbf{w}}_\mathbf{0}^\mathbf{T}\, \mathbf{r} \mid \mathbf{N}^\mathbf{T}\mathbf{s} \geq \mathbf{k_0},\ \mathbf{C}_\mathbf{p}^\mathbf{T}\, \mathbf{s} = \mathbf{0}, \mathbf{C}_\mathbf{p}^\mathbf{T}\, \mathbf{s} - \mathbf{r} = \mathbf{0} \} \tag{28}$$

The kinematic principle (27) says us that the collapse mechanism $\dot{\boldsymbol{\lambda}}_*, \dot{\mathbf{w}}_*$ corresponds to the minimum dissipated power $D = \mathbf{k}^\mathbf{T}\dot{\boldsymbol{\lambda}}$. According to the static principle (28), the stresses $\mathbf{s}_*$ and the reactions $\mathbf{r}_*$ at the plastic collapse correspond to the maximum power of reactions done on the prescribed displace-

[2] For structures made from the rigid-perfectly plastic material the kinematic unknowns $\boldsymbol{\lambda}, \mathbf{w}$ should be replaced by their rates $\dot{\boldsymbol{\lambda}}, \dot{\mathbf{w}}$.

ment rates $\dot{\mathbf{w}}_0$. Note that if the kinematic loading would be introduced in unilateral manner, which means replacing the last constraint in (27) by $\dot{\mathbf{w}}_w \geq \dot{\mathbf{w}}_0$, then the reactions would become sign constrained: $\mathbf{r} \geq \mathbf{0}$.

A. Čyras and his co-workers looked at the problem (28) from different perspective. They treated $\mathbf{p}$ as unknown loading and introduced linear *quality measure* of load $f = \mathbf{d}_0^{\mathrm{T}}\,\mathbf{p}$. Here $\mathbf{d}_0 \in R^n$ is a given column matrix of weight factors. Then the following load optimization problem was formulated: "given the structural layout and the distribution of plastic modulae $\mathbf{k}_0$ find the ultimate load $\mathbf{p}_*$ that has the highest quality index f_*". Obviously, the dual problem revealed that the entries of $\mathbf{d}_0$ should be treated as prescribed displacement rates.

Approaching the problem from the kinematic side seems to be more natural. The implicit optimization of reaction forces that comes out via duality is probably more interesting in the continuum formulation. It can be shown then that by prescribing displacement rates on a part of the surface of the rigid-perfectly plastic body we obtain the distribution of surface tractions optimal in a certain sense (Borkowski, 2004).

5. Final Remarks

Computational complexity of MP-problems depends heavily upon their degree of non-linearity. It is quite easy to solve large LP-problems. Several simplex codes available on the market are able to solve problems with hundreds of thousands of variables and/or constraints. These codes usually use some version of sparse-matrix technique in order to cope with large matrices. Interestingly enough, the simplex algorithm, discovered over 50 years ago, is still the best solver.

QP-problems are more demanding and one can hardly expect to solve in reasonable time a problem with more than couple of hundreds variables and/or constraints. Despite huge effort spend on developing general purpose Non-Linear Programming solvers, the result is rather unsatisfactory. Most available codes work sufficiently well in the range of several tenths of variables and/or constraints.

Structural analysis and optimization taking into account elastic properties of the material is not reducible to Linear Programming. On the other hand, the efficiency of QP- and NLP-solvers is far below the efficiency of modern solvers of the sets of linear algebraic equations. This explains why expectations that Mathematical Programming will replace Linear Algebra in the domain of computing were not met.

On the other hand, the language of Mathematical Programming is excellent in teaching Structural Analysis and Structural Optimum Design. It discloses common background of the broad class of problems governed by geometrically linear kinematics, allows students to grasp the principal difference between bilaterally and unilaterally constrained problems, trains them in a good custom of looking at each problem from two perspectives – the kinematic one and the static one, simplifies checking of existence and uniqueness of solutions.

Obviously, a prerequisite of teaching the MP-based approach to the theory of structures is the prior knowledge of the Mathematical Programming by the students. A class on this subject should be taught during the first or second year of undergraduate studies, as a supplement to courses on Linear Algebra and Differential Calculus. The knowledge acquired on the MP-theory could be exploited in teaching not only Structural Analysis and Structural Optimum Design but also in the classes on other aspects of Civil Engineering (e.g. road planning, cost optimization, construction planning, etc.).

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