GRADIENT BASED OPTIMIZATION OF ADDED VISCOUS DAMPING IN SEISMIC APPLICATIONS

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Abstract

This paper presents a consistent approach for the optimal seismic design of added viscous damping in framed structures. The approach presented is appropriate for use in elastic as well as yielding frames. The sum of added damping is chosen as the objective function and the performance of the structure, under the excitation of an ensemble of deterministic ground motion records, is constrained. The performance of the structure is measured by the maximal inter-story drifts in both the linear and nonlinear cases. The nonlinear case however, uses an additional performance measure of the normalized hysteretic energy of the plastic hinges

Gradients of the performance measures are first derived to enable the use of an appropriate first order optimization scheme. Moreover, an efficient selection scheme enables the consideration of only a few records rather than the whole ensemble, hence making the optimization process efficient in terms of the computational effort.

Introduction

The problem of seismic retrofitting of existing structures has gained much attention lately due to the new *performance-based-design* approach, which allows engineers to design structures for a desired level of seismic performance. Installation of viscous dampers is an effective means for this seismic retrofitting, hence, the problem of optimal design of these dampers is of paramount importance. This problem was tackled by several researchers with limited results for the particular class of regular buildings (see for example Zhang and Soong 1992; Inaudi *et al.* 1993; Gluck *et al.* 1996; Takewaki 1997 to name only few). Since most existing buildings are irregular, available methodologies remain academic.

For an efficient and computationally effective solution of the optimization problem of dynamic systems subjected to time varying loads, first order schemes that require constraints' gradients are preferred. Zero order optimization schemes, e.g. genetic algorithms, require a large number of function and constraints evaluations, that is to say time history analyses, making them less attractive to use.

Several approaches for the gradient computation have been introduced in the literature. Hsieh and Arora (1985) derived the gradients of point-wise as well as integral type constraints for linear elastic systems by deriving the first variations of these constraints which depend on the variation on the displacements of the degrees of freedom. They further used a direct differentiation method of the equations of motion, and alternatively an adjoint variables method, to evaluate these variations on the displacements. Another approach for the gradient computation uses the finite difference method (see for example Falco *et al*., 2004). Here the derivative of the constraint with respect to each design variable is approximated by the forward or backward finite difference approximation. This method actually requires an additional analysis for each design variable. Conte *et al*. (2003) distinguished two methods for computing the response sensitivities considering plastic behavior of the structure. The first method uses the differentiation of the response equations with respect to each of the design variables, and then discretizes the resulting response sensitivity equations in time. The second method discretizes the response equations in time, and then differentiates the resulting discrete response

M. Pandey et al. (eds), Advances in Engineering Structures, Mechanics & Construction, 663–671.

equations with respect to each of the design variables. It should be noted that both methods require an additional analysis for each design variable.

The present research proposes a gradient based approach for the optimal design of viscous dampers for the seismic retrofitting of existing, regular as well as irregular, structures. This approach uses a first order optimization scheme whose success lies in the ability to derive the gradients of the constraints with respect to the damping coefficients of the dampers. Thus, the main effort in this paper is the gradient derivation of constraints in linear as well as nonlinear dynamic optimization problems under earthquake excitations. The relatively small computational effort associated with their evaluation using the proposed scheme makes these gradients highly desirable.

Problem Formulation

The formulation of the optimization problem is comprised of the total added damping as an objective function, and an inequality constraint on the upper bound of each of the local performance indices which are computed based on the behavior of the structure, i.e., satisfying the equations of motion of the damped structure. These constraints are repeated for each ground motion record. The damping coefficients which are the design variables are required to be nonnegative and are assigned an upper bound.

Equations of motion

The general equations of motion of a yielding structure, retrofitted by added damping, and excited by an earthquake can be given by:

$$\begin{aligned}&\mathbf{M}\ddot{\mathbf{x}}(t)+\left[\mathbf{C}+\mathbf{C}_{\mathrm{d}}(\mathbf{c}_{\mathrm{d}})\right]\cdot\dot{\mathbf{x}}(t)+\mathbf{K}^{\alpha}\mathbf{x}(t)+\mathbf{B}_{\mathrm{fx}}\mathbf{f}_{\mathrm{h}}(t)=-\mathbf{M}\cdot\mathbf{e}\cdot\mathbf{a}_{\mathrm{g}}(t)\ ;\mathbf{x}(0)=\mathbf{0},\dot{\mathbf{x}}(0)=\mathbf{0}\\&\dot{\mathbf{f}}_{\mathrm{h}}(t)=\mathbf{f}(\dot{\mathbf{x}}(t),\mathbf{f}_{\mathrm{h}}(t))\ ;\mathbf{f}_{\mathrm{h}}(0)=\mathbf{0}\end{aligned}\tag{1}$$

where $\mathbf{x}$ = displacements vector of the degrees of freedom (DOFs); $\mathbf{M}$ = mass matrix; $\mathbf{C}$ = inherent damping matrix; $\mathbf{c}_{\mathrm{d}}$ = added damping vector; $\mathbf{C}_{\mathrm{d}}(\mathbf{c}_{\mathrm{d}})$ = supplemental damping matrix; $\mathbf{K}^{\alpha}$ =secondary stiffness matrix; $\mathbf{f}_{\mathrm{h}}(t)$=hysteretic forces/moments vector in local coordinates of the plastic hinges with zero secondary stiffness; $\mathbf{B}_{\mathrm{fx}}$ =transformation matrix that transforms the restoring forces/moments from the local coordinates of the plastic hinges to the global coordinates of the DOFs; $\mathbf{e}$ = excitation direction matrix with zero/one entries; $\mathbf{a}_{\mathrm{g}}(t)$=ground motion acceleration vector, and a dot represents differentiation with respect to the time.

Performance indices

Normalized hysteretic energy: Following Uang and Bertero, (1990), hysteretic energy accumulated at the plastic hinge *i*, (a measure of the structural damage in yielding frames), normalized by an allowable value, is given by:

$$E_{h,i}(t_f)=\left(\int_0^{t_f} f_{h,i}(t)\cdot v_i(t)dt\right)\Big/E_{h,i}^{all}\tag{2}$$

where t_f = the final time of the excitation\computation; $E_{h,i}(t_f)$=normalized hysteretic energy at the plastic hinge *i* at the time t_f ; $f_{h,i}(t)$=hysteretic force/moment in the plastic hinge *i*; $v_i(t)$=velocity of the plastic hinge *i*, and $E_{h,i}^{all}$ =allowable value of the hysteretic energy at the plastic hinge *i* which is usually taken proportional to the elastic energy at yielding of the plastic hinge. In matrix notation, the hysteretic energy in the plastic hinges as depicted by Eq. 2 can be written as:

$$\mathbf{E}_{\mathrm{h}}\left(t_f\right)=\mathsf{D}^{-1}\left(\mathbf{E}_{\mathrm{h}}^{\mathrm{all}}\right)\cdot\int_0^{t_f}\mathsf{D}(\mathbf{f}_{\mathrm{h}}(t))\cdot(\mathbf{B}_{\mathrm{xf}}\dot{\mathbf{x}}(t))dt \tag{3}$$

where $\mathsf{D}(\mathbf{z})$= operator that forms a diagonal matrix whose diagonal elements are the elements of a given vector $\mathbf{z}$, and $\mathbf{B}_{\mathrm{xf}}$=transformation matrix that transforms the velocities from the global coordinates of the DOFs to the local coordinates of the plastic hinges.

Normalized maximal inter-story drifts can be written as:

$$\mathbf{d}_m=\max_t\left(\mathrm{abs}\left(\mathsf{D}^{-1}\left(\mathbf{d}^{\mathrm{all}}\right)\cdot\mathbf{H}_{\mathrm{x}}\mathbf{x}(t)\right)\right) \tag{4}$$

where $\mathbf{d}_m$=vector of normalized maximal inter-story drifts; $\mathbf{d}^{\mathrm{all}}$= vector of allowable maximal inter-story drifts; $\mathbf{H}_{\mathrm{x}}$ =transformation matrix that transforms the displacements from the global coordinates of the DOFs to the coordinates of inter-story drifts, and the "abs" stands for the absolute function as it acts on each of the vector components separately.

Formal optimization problem
The formal optimization problem may now be written as:

$$\begin{aligned}
&\text{mininimize:}\quad J=\mathbf{c}_{\mathrm{d}}^{\mathrm{T}}\cdot\mathbf{1}\\
&\text{subject to:}\\
&\left.\begin{aligned}
&\mathbf{E}_{\mathrm{h}}\left(t_f\right)=\mathsf{D}^{-1}\left(\mathbf{E}_{\mathrm{h}}^{\mathrm{all}}\right)\cdot\int_{t_o}^{t_f}\mathsf{D}(\mathbf{f}_{\mathrm{h}}(t))\cdot(\mathbf{B}_{\mathrm{xf}}\dot{\mathbf{x}}(t))dt\le\mathbf{1}\\
&\mathbf{d}_m=\max_t\left(\mathrm{abs}\left(\mathsf{D}^{-1}\left(\mathbf{d}^{\mathrm{all}}\right)\cdot\mathbf{H}_{\mathrm{x}}\mathbf{x}(t)\right)\right)\le\mathbf{1}\\
&\text{where } \mathbf{x}(t),\dot{\mathbf{x}}(t) \text{ and } \mathbf{f}_{\mathrm{h}}(t) \text{ satisfy the equations of motion}\\
&\mathbf{M}\ddot{\mathbf{x}}(t)+\left[\mathbf{C}+\mathbf{C}_{\mathrm{d}}(\mathbf{c}_{\mathrm{d}})\right]\cdot\dot{\mathbf{x}}(t)+\mathbf{K}^{\alpha}\mathbf{x}(t)+\mathbf{B}_{\mathrm{fx}}\mathbf{f}_{\mathrm{h}}(t)=-\mathbf{M}\cdot\mathbf{e}\cdot\mathbf{a}_{\mathrm{g}}(t)\ ;\begin{cases}\mathbf{x}(0)=\mathbf{0}\\ \dot{\mathbf{x}}(0)=\mathbf{0}\end{cases}\\
&\dot{\mathbf{f}}_{\mathrm{h}}(t)=\mathbf{f}(\dot{\mathbf{x}}(t),\mathbf{f}_{\mathrm{h}}(t))\ ;\mathbf{f}_{\mathrm{h}}(0)=\mathbf{0}
\end{aligned}\right\}\forall\mathbf{a}_{\mathrm{g}}\in\text{ensemble } i\\
&\mathbf{0}\le\mathbf{c}_{\mathrm{d}}\le\mathbf{c}_{\mathrm{d,max}}
\end{aligned} \tag{5}$$

where $\mathbf{1}$ = unity vector, and $\mathbf{c}_{\mathrm{d,max}}$ =upper bound on $\mathbf{c}_{\mathrm{d}}$.

Optimization Scheme

Gradient derivation
The evaluation of the gradient of the objective function is trivial since this function depends on the design variables explicitly, and is given by $\nabla_{\mathbf{c}_{\mathrm{d}}}J=\mathbf{1}$. The evaluation of the gradients of the constraints, however, is not an easy matter since the constraints depend on the design variables through differential equations. It is achieved indirectly by formulating the problem in *state-space* notation and using optimal control theory.

The optimization problem (Eq. 5) is reformulated in terms of a single constraint on maximal values as:

$$\begin{aligned}&\text{mininimize:} \quad J = \mathbf{c}_\mathrm{d}^{\mathrm{T}} \cdot \mathbf{1} \\ &\text{subject to:} \\ &pi = \max\left(\max\left(\mathbf{E}_\mathrm{h}\left(t_f\right)\right), \max\left(\mathbf{d}_m\right)\right) \le 1.0, \text{equations of motion and } \mathbf{0} \le \mathbf{c}_\mathrm{d} \le \mathbf{c}_{\mathrm{d,max}}\end{aligned} \tag{6}$$

where pi = performance index.

A differentiable equivalent of the constraint: Before proceeding formally with the gradient formulation it is necessary, since use is made of variational approach, to replace the *max* function on *t* in Eq. 6 by a differentiable function.

Differentiable equivalent of $\mathbf{d}_m$. It is proposed to use a norm of the *p*-type differentiable function as an equivalent to $\mathbf{d}_m = \max_t\left(\mathrm{abs}\left(\mathsf{D}^{-1}\left(\mathbf{d}^{\mathrm{all}}\right)\cdot \mathbf{H}_\mathrm{x}\mathbf{x}(t)\right)\right)$.Thus, $\mathbf{d}_m$ takes the form:

$$\mathbf{d}_m\left(t_f\right) = \left(\frac{1}{t_f}\int_0^{t_f}\left(\mathsf{D}^{-1}\left(\mathbf{d}^{\mathrm{all}}\right)\cdot \mathsf{D}\left(\mathbf{H}_\mathrm{x}\cdot \mathbf{x}(t)\right)\right)^p dt\right)^{\frac{1}{p}} \cdot \mathbf{1} = \left(\mathsf{D}\left(\mathbf{d}_{m,p}\left(t_f\right)\right)\right)^{\frac{1}{p}} \cdot \mathbf{1} \tag{7}$$

where p = a large positive even number. It follows that:

$$\mathbf{d}_{m,p}\left(t_f\right) = \left(\mathsf{D}\left(\mathbf{d}_m\right)\right)^p \cdot \mathbf{1} = \frac{1}{t_f}\int_0^{t_f}\left(\mathsf{D}^{-1}\left(\mathbf{d}^{\mathrm{all}}\right)\cdot \mathsf{D}\left(\mathbf{H}_\mathrm{x}\cdot \mathbf{x}(t)\right)\right)^p dt \cdot \mathbf{1} \tag{8}$$

Differentiable equivalent of pi. The maximal component of a vector with non-negative entries, $\mathbf{z}$, can be evaluated using a differentiable weighted average of the form:

$$z_{\max} = \frac{\sum w_i^q z_i}{\sum w_i^q} \tag{9}$$

where w_i =weight of z_i, and q = an index. When q is large, say $q=p$, and the components of $\mathbf{z}$ are used as their own weights, i.e. $w_i = z_i$, this weighted average approaches the value of the maximum component of $\mathbf{z}$. Since $\mathbf{E}_\mathrm{h}\left(t_f\right)$ and $\mathbf{d}_m$ are normalized quantities pi can be written as $pi = \max\left(\mathbf{E}_\mathrm{h}\left(t_f\right), \mathbf{d}_m\right)$ and reformulated Eq. 9 as:

$$pi = \frac{\mathbf{1}^{\mathrm{T}}\cdot \mathsf{D}^{q+1}\left(\mathbf{E}_\mathrm{h}\left(t_f\right)\right)\cdot \mathbf{1} + \mathbf{1}^{\mathrm{T}}\cdot \mathsf{D}^{q+1}\left(\mathbf{d}_m\right)\cdot \mathbf{1}}{\mathbf{1}^{\mathrm{T}}\cdot \mathsf{D}^{q}\left(\mathbf{E}_\mathrm{h}\left(t_f\right)\right)\cdot \mathbf{1} + \mathbf{1}^{\mathrm{T}}\cdot \mathsf{D}^{q}\left(\mathbf{d}_m\right)\cdot \mathbf{1}} \tag{10}$$

Substituting Eq. 7 yields:

$$pi = \frac{\mathbf{1}^{\mathrm{T}} \cdot \mathsf{D}^{q+1}\left(\mathbf{E}_{\mathrm{h}}\left(t_f\right)\right) \cdot \mathbf{1} + \mathbf{1}^{\mathrm{T}} \cdot \mathsf{D}^{\frac{q+1}{p}}\left(\mathbf{d}_{m,p}\left(t_f\right)\right) \cdot \mathbf{1}}{\mathbf{1}^{\mathrm{T}} \cdot \mathsf{D}^{q}\left(\mathbf{E}_{\mathrm{h}}\left(t_f\right)\right) \cdot \mathbf{1} + \mathbf{1}^{\mathrm{T}} \cdot \mathsf{D}^{\frac{q}{p}}\left(\mathbf{d}_{m,p}\left(t_f\right)\right) \cdot \mathbf{1}} \tag{11}$$

The *state space* formulation (a set of first order differential equations) of the optimization problem thus becomes:

$$\begin{aligned}
&\text{mininimize:} \quad J = \mathbf{c}_{\mathrm{d}}^{\mathrm{T}} \cdot \mathbf{1} \\
&\text{subject to:} \\
&\left.\begin{aligned}
&pi = \frac{\mathbf{1}^{\mathrm{T}} \cdot \mathsf{D}^{q+1}\left(\mathbf{E}_{\mathrm{h}}\left(t_f\right)\right) \cdot \mathbf{1} + \mathbf{1}^{\mathrm{T}} \cdot \mathsf{D}^{\frac{q+1}{p}}\left(\mathbf{d}_{m,p}\left(t_f\right)\right) \cdot \mathbf{1}}{\mathbf{1}^{\mathrm{T}} \cdot \mathsf{D}^{q}\left(\mathbf{E}_{\mathrm{h}}\left(t_f\right)\right) \cdot \mathbf{1} + \mathbf{1}^{\mathrm{T}} \cdot \mathsf{D}^{\frac{q}{p}}\left(\mathbf{d}_{m,p}\left(t_f\right)\right) \cdot \mathbf{1}} \le 1.0 \\
&\text{where} \\
&\dot{\mathbf{E}}_{\mathrm{h}}(t) = \mathsf{D}^{-1}\left(\mathbf{E}_{\mathrm{h}}^{\mathrm{all}}\right) \cdot \mathsf{D}\left(\mathbf{f}_{\mathrm{h}}(t)\right) \cdot \left(\mathbf{B}_{\mathrm{xf}}\mathbf{v}(t)\right) \ ; \ \mathbf{E}_{\mathrm{h}}(0) = \mathbf{0} \\
&\dot{\mathbf{d}}_{m,p} = \frac{1}{t_f}\left(\mathsf{D}^{-1}\left(\mathbf{d}^{\mathrm{all}}\right) \cdot \mathsf{D}\left(\mathbf{H}_{\mathrm{x}} \cdot \mathbf{x}(t)\right)\right)^{p} \cdot \mathbf{1} \ ; \ \mathbf{d}_{m,p}(0) = \mathbf{0} \\
&\text{and } \mathbf{x}(t), \mathbf{v}(t) \text{ and } \mathbf{f}_{\mathrm{h}}(t) \text{ satisfy the equations of motion} \\
&\dot{\mathbf{x}}(t) = \mathbf{v}(t) \ ; \ \mathbf{x}(0) = \mathbf{0} \\
&\dot{\mathbf{v}}(t) = \mathbf{M}^{-1}\left(-\left[\mathbf{C} + \mathbf{C}_{\mathrm{d}}\left(\mathbf{c}_{\mathrm{d}}\right)\right] \cdot \mathbf{v}(t) - \mathbf{K}^{\alpha}\mathbf{x}(t) - \mathbf{B}_{\mathrm{fx}}\mathbf{f}_{\mathrm{h}}(t) - \mathbf{M} \cdot \mathbf{e} \cdot \mathbf{a}_{\mathrm{g}}(t)\right); \mathbf{v}(0) = \mathbf{0} \\
&\dot{\mathbf{f}}_{\mathrm{h}}(t) = \mathbf{f}\left(\mathbf{v}(t), \mathbf{f}_{\mathrm{h}}(t)\right) ; \mathbf{f}_{\mathrm{h}}(0) = \mathbf{0}
\end{aligned}\right\} \forall \mathbf{a}_{\mathrm{g}} \in \textit{ensemble } i \\
&\mathbf{0} \le \mathbf{c}_{\mathrm{d}} \le \mathbf{c}_{\mathrm{d,max}}
\end{aligned} \tag{12}$$

where $\mathbf{v}$ = the velocity vector.

Gradient derivation: Equation 12 has the general form of:

$$\begin{aligned}
&\text{minimize } f\left(\mathbf{c}_{\mathrm{d}}\right) \\
&\text{subject to:} \\
&g\left(\mathbf{y}\left(t_f\right)\right) - g_{\max} \le 0 \text{ where } \dot{\mathbf{y}}(t) = \mathbf{a}\left(\mathbf{y}(t), \mathbf{c}_{\mathrm{d}}, t\right); \mathbf{y}(0) = \mathbf{0}
\end{aligned} \tag{13}$$

where $\mathbf{y} = \left\{\mathbf{E}_{\mathrm{h}}^{\mathrm{T}} \quad \mathbf{d}_{m,p}^{\mathrm{T}} \quad \mathbf{x}^{\mathrm{T}} \quad \mathbf{v}^{\mathrm{T}} \quad \mathbf{f}_{\mathrm{h}}^{\mathrm{T}}\right\}^{\mathrm{T}}$. The gradient of $g(\mathbf{y}(t_f))$ is obtained from the following secondary optimization problem:

$$\begin{aligned}
&\text{minimize } g\left(\mathbf{y}\left(t_f\right)\right) \text{or } \int_0^{t_f} \frac{dg\left(\mathbf{y}(t)\right)}{dt} dt + g\left(\mathbf{y}(0)\right) \\
&\text{subject to:} \\
&\dot{\mathbf{y}}(t) = \mathbf{a}\left(\mathbf{y}(t), \mathbf{c}_{\mathrm{d}}, t\right); \mathbf{y}(0) = \mathbf{0}
\end{aligned} \tag{14}$$

The augmented function is given by

$$J_{\mathrm{a}} = \int_{t_o}^{t_f} \left\{ \frac{dg(\mathbf{y}(t))}{dt} + \boldsymbol{\lambda}^T(t)[\mathbf{a}(\mathbf{y}(t),\mathbf{c}_{\mathrm{d}},t) - \dot{\mathbf{y}}(t)] \right\} dt + g(\mathbf{y}(0)) \tag{15}$$

where $\boldsymbol{\lambda} = \{\boldsymbol{\lambda}_{\mathrm{x}}^{\mathrm{T}} \ \ \boldsymbol{\lambda}_{\mathrm{v}}^{\mathrm{T}} \ \ \boldsymbol{\lambda}_{\mathrm{fh}}^{\mathrm{T}} \ \ \boldsymbol{\lambda}_{\mathrm{Eh}}^{\mathrm{T}} \ \ \boldsymbol{\lambda}_{\mathrm{xmp}}^{\mathrm{T}}\}^{\mathrm{T}}$. The variation of the augmented function, with t_f specified results in

$$\begin{aligned} \delta J_{\mathrm{a}}(\mathbf{c}_{\mathrm{d}}) = {} & \left[\left[\frac{\partial g(\mathbf{y}(t))}{\partial \mathbf{y}} - \boldsymbol{\lambda}(t)\right]^T \cdot \delta\mathbf{y}\right|_{t_f} + \int_{t_o}^{t_f} \left\{ \left[\frac{\partial}{\partial \mathbf{y}}\left(\boldsymbol{\lambda}^T(t)\cdot \mathbf{a}(\mathbf{y}(t),\mathbf{c}_{\mathrm{d}},t)\right) + \frac{d\boldsymbol{\lambda}(t)}{dt}\right]^T \cdot \delta\mathbf{y}(t) \right. \\ & \left. + [\mathbf{a}(\mathbf{y}(t),\mathbf{c}_{\mathrm{d}},t) - \dot{\mathbf{y}}(t)]^T \cdot \delta\boldsymbol{\lambda}(t) + \left[\frac{\partial\left(\boldsymbol{\lambda}^T(t)\cdot \mathbf{a}(\mathbf{y}(t),\mathbf{c}_{\mathrm{d}},t)\right)}{\partial \mathbf{c}_{\mathrm{d}}}\right]^T \cdot \delta\mathbf{c}_{\mathrm{d}} \right\} dt \end{aligned} \tag{16}$$

Taking the first three variations as arbitrary results in the following three differential equations and boundary conditions to be satisfied:

$$\begin{aligned} \boldsymbol{\lambda}(t_f) &= \frac{\partial g(\mathbf{y}(t_f))}{\partial \mathbf{y}} \\ \frac{d\boldsymbol{\lambda}(t)}{dt} &= -\frac{\partial}{\partial \mathbf{y}}\left(\boldsymbol{\lambda}^T(t)\cdot \mathbf{a}(\mathbf{y}(t),\mathbf{c}_{\mathrm{d}},t)\right) \\ \dot{\mathbf{y}}(t) &= \mathbf{a}(\mathbf{y}(t),\mathbf{c}_{\mathrm{d}},t) \end{aligned} \tag{17}$$

The multiplier of the variation $\delta\mathbf{c}_d$ will yield the expression for the evaluation of the gradient $\nabla_{\mathbf{c}_d} g(\mathbf{y}(t_f))$. This expression becomes:

$$\delta J_{\mathrm{a}} = \frac{\partial J_{\mathrm{a}}}{\partial \mathbf{c}_{\mathrm{d}}} \cdot \delta\mathbf{c}_{\mathrm{d}} \Rightarrow \frac{\partial J_{\mathrm{a}}}{\partial \mathbf{c}_{\mathrm{d}}} = \int_{t_o}^{t_f} \frac{\partial\left(\boldsymbol{\lambda}^T(t)\cdot \mathbf{a}(\mathbf{y}(t),\mathbf{c}_{\mathrm{d}},t)\right)}{\partial \mathbf{c}_{\mathrm{d}}} dt \tag{18}$$

which is the desired gradient since

$$\begin{aligned} \nabla_{\mathbf{c}_{\mathrm{d}}} g(\mathbf{y}(t_{\mathrm{f}})) &= \frac{\partial}{\partial \mathbf{c}_{\mathrm{d}}}[g(\mathbf{y}(t_{\mathrm{f}}))] = \\ &= \frac{\partial}{\partial \mathbf{c}_{\mathrm{d}}}\left[\int_{t_o}^{t_f}\left\{\left[\frac{dg}{d\mathbf{y}}(\mathbf{y}(t))\right]^T \cdot \dot{\mathbf{y}}\right\} dt + g(\mathbf{y}(t_{\mathrm{o}}))\right] \equiv \left.\frac{\partial J_{\mathrm{a}}}{\partial \mathbf{c}_{\mathrm{d}}}\right|_{\mathbf{a}(\mathbf{y}(t),\mathbf{c}_{\mathrm{d}},t) - \dot{\mathbf{y}}(t) = \mathbf{0}} \end{aligned} \tag{19}$$

The gradient of the constraint in Eq. 12, can now be evaluated from:

$$\nabla_{\mathbf{c}_{\mathrm{d}}} pi(\mathbf{c}_{\mathrm{d}}) = -\int_0^{t_f}\left[\mathbf{v}^{\mathrm{T}}(t)\cdot \frac{\partial \mathbf{C}_{\mathrm{d}}^{\mathrm{T}}(\mathbf{c}_{\mathrm{d}})}{\partial \mathbf{c}_{\mathrm{d}}} \cdot \mathbf{M}^{-1} \cdot \boldsymbol{\lambda}_{\mathrm{v}}\right] dt \tag{20}$$

and the following set of differential equations and boundary conditions:

$$\begin{aligned}
&\boldsymbol{\lambda}_{\mathrm{x}}(t_f)=\mathbf{0};\boldsymbol{\lambda}_{\mathrm{v}}(t_f)=\mathbf{0};\boldsymbol{\lambda}_{\mathrm{fh}}(t_f)=\mathbf{0}\\
&\boldsymbol{\lambda}_{\mathrm{Eh}}(t_f)=\frac{1}{den^2}\left(den\cdot(q+1)\cdot \mathsf{D}^{q}\left(\mathbf{E}_{\mathrm{h}}\right)\cdot\mathbf{1}-num\cdot(q)\cdot\mathsf{D}^{q-1}\left(\mathbf{E}_{\mathrm{h}}\right)\cdot\mathbf{1}\right)\\
&\boldsymbol{\lambda}_{\mathrm{xmp}}(t_f)=\frac{1}{den^2}\left(den\cdot\left(\frac{q+1}{p}\right)\cdot\mathsf{D}^{\left(\frac{q+1}{p}-1\right)}\left(\mathbf{d}_{m,p}\right)\cdot\mathbf{1}-num\cdot\left(\frac{q}{p}\right)\cdot\mathsf{D}^{\left(\frac{q}{p}-1\right)}\left(\mathbf{d}_{m,p}\right)\cdot\mathbf{1}\right)
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\dot{\boldsymbol{\lambda}}_{\mathrm{x}}(t)=\left(\mathbf{M}^{-1}\cdot\mathbf{K}^{\alpha}\right)^{\mathrm{T}}\cdot\boldsymbol{\lambda}_{\mathrm{v}}(t)+\left(-p\mathbf{H}_{\mathrm{x}}^{\mathrm{T}}\cdot\mathsf{D}^{p-1}(\mathbf{H}_{\mathrm{x}}\cdot\mathbf{x}(t))\mathsf{D}^{-p}\left(\mathbf{d}^{\mathrm{all}}\right)\cdot\boldsymbol{\lambda}_{\mathrm{xmp}}(t)\right)\\
&\dot{\boldsymbol{\lambda}}_{\mathrm{v}}(t)=-\boldsymbol{\lambda}_{\mathrm{x}}(t)+\left(\mathbf{M}^{-1}\cdot[\mathbf{C}+\mathbf{C}_{\mathrm{d}}(\mathbf{c}_{\mathrm{d}})]\right)^{\mathrm{T}}\cdot\boldsymbol{\lambda}_{\mathrm{v}}(t)-\left(\frac{\partial\mathbf{f}(\mathbf{v}(t),\mathbf{f}_{\mathrm{h}}(t))}{\partial\mathbf{v}}\right)^{\mathrm{T}}\cdot\boldsymbol{\lambda}_{\mathrm{fh}}(t)+\\
&\qquad+\left(-\mathbf{B}_{\mathrm{xf}}^{\mathrm{T}}\cdot\mathsf{D}(\mathbf{f}_{\mathrm{h}}(t))\cdot\mathsf{D}^{-1}\left(\mathbf{E}_{\mathrm{h}}^{\mathrm{all}}\right)\cdot\boldsymbol{\lambda}_{\mathrm{Eh}}(t)\right)\\
&\dot{\boldsymbol{\lambda}}_{\mathrm{fh}}(t)=\left(\mathbf{M}^{-1}\cdot\mathbf{B}_{\mathrm{fx}}\right)^{\mathrm{T}}\cdot\boldsymbol{\lambda}_{\mathrm{v}}(t)-\left(\frac{\partial\mathbf{f}(\mathbf{v}(t),\mathbf{f}_{\mathrm{h}}(t))}{\partial\mathbf{f}_{\mathrm{h}}}\right)^{\mathrm{T}}\cdot\boldsymbol{\lambda}_{\mathrm{fh}}(t)+\\
&\qquad+\left(-\mathsf{D}\left(\mathbf{B}_{\mathrm{xf}}\cdot\mathbf{v}(t)\right)\cdot\mathsf{D}^{-1}\left(\mathbf{E}_{\mathrm{h}}^{\mathrm{all}}\right)\cdot\boldsymbol{\lambda}_{\mathrm{Eh}}(t)\right)\\
&\dot{\boldsymbol{\lambda}}_{\mathrm{Eh}}(t)=\mathbf{0};\dot{\boldsymbol{\lambda}}_{\mathrm{xmp}}(t)=\mathbf{0}
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\dot{\mathbf{x}}(t)=\mathbf{v}(t)\ ;\ \mathbf{x}(0)=\mathbf{0}\\
&\dot{\mathbf{v}}(t)=\mathbf{M}^{-1}\left(-[\mathbf{C}+\mathbf{C}_{\mathrm{d}}(\mathbf{c}_{\mathrm{d}})]\cdot\mathbf{v}(t)-\mathbf{K}^{\alpha}\mathbf{x}(t)-\mathbf{B}_{\mathrm{fx}}\mathbf{f}_{\mathrm{h}}(t)-\mathbf{M}\cdot\mathbf{e}\cdot\mathbf{a}_{\mathrm{g}}(t)\right)\ ;\ \mathbf{v}(0)=\mathbf{0}\\
&\dot{\mathbf{f}}_{\mathrm{h}}(t)=\mathbf{f}(\mathbf{v}(t),\mathbf{f}_{\mathrm{h}}(t))\ ;\mathbf{f}_{\mathrm{h}}(0)=\mathbf{0}\\
&\dot{\mathbf{E}}_{\mathrm{h}}(t)=\mathsf{D}^{-1}\left(\mathbf{E}_{\mathrm{h}}^{\mathrm{all}}\right)\cdot\mathsf{D}(\mathbf{f}_{\mathrm{h}}(t))\cdot(\mathbf{B}_{\mathrm{xf}}\mathbf{v}(t))\ ;\ \mathbf{E}_{\mathrm{h}}(0)=\mathbf{0}\\
&\dot{\mathbf{d}}_{m,p}=\left(\mathsf{D}^{-1}(\mathbf{d}_{all})\cdot\mathsf{D}(\mathbf{H}_{\mathrm{x}}\cdot\mathbf{x}(t))\right)^{p}\cdot\mathbf{1}\ ;\ \mathbf{x}_{m,p}(0)=\mathbf{0}
\end{aligned} \tag{23}$$

where $num=\mathbf{1}^{\mathrm{T}}\cdot\mathsf{D}^{q+1}\left(\mathbf{E}_{\mathrm{h}}(t_f)\right)\cdot\mathbf{1}+\mathbf{1}^{\mathrm{T}}\cdot\mathsf{D}^{\frac{q+1}{p}}\left(\mathbf{d}_{m,p}(t_f)\right)\cdot\mathbf{1}$;

$den=\mathbf{1}^{\mathrm{T}}\cdot\mathsf{D}^{q}\left(\mathbf{E}_{\mathrm{h}}(t_f)\right)\cdot\mathbf{1}+\mathbf{1}^{\mathrm{T}}\cdot\mathsf{D}^{\frac{q}{p}}\left(\mathbf{d}_{m,p}(t_f)\right)\cdot\mathbf{1}$.

Equations 23 return the equality constraints; Eqs. 21 and 22 give expressions for the evaluation of the Lagrange multipliers, $\boldsymbol{\lambda}_{\mathrm{x}}$, $\boldsymbol{\lambda}_{\mathrm{v}}$ and $\boldsymbol{\lambda}_{\mathrm{xmp}}$ which are needed for the evaluation of Eq. 20. Now since the elements of $\mathbf{C}_{\mathrm{d}}{}^{\mathrm{T}}(\mathbf{c}_{\mathrm{d}})$ are linear combinations of the elements of $\mathbf{c}_{\mathrm{d}}$, the differentiation of $\mathbf{C}_{\mathrm{d}}{}^{\mathrm{T}}(\mathbf{c}_{\mathrm{d}})$ with respect to $\mathbf{c}_{\mathrm{d,i}}$ (also needed in Eq. 20) is rather simple and easily programmed. The computation of the gradient for a single record is summarized as follows:

Step 1: Solve the equations of motion (Eq. 23).
Step 2: Solve the equations of the Lagrange multipliers (Eq. 21 with conditions for t_f in Eq. 22).
Step 3: Calculate the desired gradient (Eq. 20).

Optimization scheme

The gradients of the objective function and the constraints are needed at each iteration for first order optimization schemes. Thus the solution requires a time history analysis for each record (constraint) at every iteration cycle. In order to reduce the computational effort, optimization is first carried out for one "active" ground motion (loading condition), rather than for the whole ensemble. If the optimal solution for this ground motion violates other records in the ensemble, additional ground motions are added one at a time (*Stage 4* below). Following are the main steps in the methodology that are used for the optimization scheme.

Step 1: **Select the "active" ground motion.** The record with the maximal displacement is selected to begin the process. It is evaluated from a SDOF with the 1st period of the undamped structure within the expected total damping ratio range.
Step 2: **Compute an initial starting value for the damping vector.** The starting point is evaluated by first assuming a distribution of equal dampers for the damping vector. Then this damping vector is factored so as to satisfy $pi = 1.0$ where pi is computed from a time history analysis of the frame excited by the "active" ground motion of *Step1*.
Step 3: **Solve the optimization problem for the active set of records**. An appropriate gradient based optimization scheme is used. The gradients are evaluated as described above. If more than one record is "active", say two, then the gradients are calculated separately for each record and the size of the problem doubles.
Step 4: **Feasibility check.** A time history analysis is performed on the optimally damped structure for each of the remaining records in the ensemble, separately. One new ground motion is added to the active set only if its pi is largest and greater than 1.0. Then *Step 3* is repeated.
Step 5: **Stop.**

Linear Frames –A Particular Case

A class of structures, usually regular, can be brought to behave elastically under an earthquake excitation, by the addition of a reasonable amount of added damping. In this case nonlinear analysis methodologies, which are computationally expensive, are not essential and some of the nonlinear performance indices, such as hysteretic energy, become meaningless. Since the optimization problem of linear structures is a particular case the same procedure presented earlier holds. Minor changes ease the computations. First, the equations of motion are reduced to their linear form by substituting $\mathbf{K}^{\alpha} = \mathbf{K}$ and $\mathbf{f}_{\mathrm{h}}(t) \equiv \mathbf{0}$ (or, equivalently, $\dot{\mathbf{f}}_{\mathrm{h}}(t) \equiv \mathbf{0}$; $\mathbf{f}_{\mathrm{h}}(t_o) = \mathbf{0}$). Substituting these relations into Eqs. 25 and 26 leads to simpler equations for the gradient computation as well. Then, the constraint on the hysteretic energy will be omitted since no hysteretic energy dissipates in the elastic range. This leads to simpler equations for the gradient computation as well since $\mathbf{E}_{\mathrm{h}}(t_f) = \mathbf{0}$. Inter-story drifts remain the only constraints. In 3D structures the inter-story drifts of the peripheral frames are used.

Fundamental Results

- *The optimal design of added damping in 2D frames, assuming linear behavior of the damped structure, and in 2D yielding frames is characterized by assigning damping only in stories that reached the allowable drift.*
- *The optimal design of added damping in 2D yielding shear frames is characterized by assigning damping only in stories that reached the allowable normalized hysteretic energy.*
- *The optimal design of added damping of 3D framed structures is characterized by assigning damping at the peripheral frames only, where the peripheral drift has reached the allowable.*

These observations are strikingly analogous to the classical "fully-stressed-design" behavior of optimal trusses reported in the sixties.

Conclusions

A gradient based methodology for the optimal design of added viscous damping for an ensemble of realistic ground motion records with constraints on the maximum inter-story drifts for linear frames, and additional constraints on maximum energy based local damage indices for nonlinear frames, was presented. This methodology is appropriate for use in linear, as well as nonlinear, frames.

The computational effort is appreciably reduced by first using one "active" ground motion record since experience shows that one or two records dominate the design.

The gradients of the constraints were derived so as to enable the use of an efficient first order optimization scheme for the solution of the optimization problem. The approach for the gradient derivation has several advantages over other approaches. It is appropriate for use when the equations of motion assume nonlinear plastic behavior as well, and it requires a relatively small computational effort, in the form of a single additional solution of a set of differential equations (that is, the equations for the Lagrange multipliers).

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