TIME-VARIANT RELIABILITY ANALYSIS FOR SERIES SYSTEMS WITH LOG-NORMAL VECTOR RESPONSE

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Introduction

Failures in randomly vibrating structures are defined to occur if the response exceeds permissible thresholds, within specified time durations. Estimating the failure probability involves characterizing the probability of the exceedance of the structure response, which are modeled as random processes. An elegant approach for addressing this problem lies in expressing the failure probability in terms of the probability distribution function (PDF) of the extreme values associated with the response.

In structural series systems, failure of any of the individual components signals system failure. The system reliability is thus expressible in terms of the joint probability of exceedance of the component response processes. Often, the loads acting on the various components of a system have common source, and hence, the component responses, and in turn, their extremes, are mutually dependent. This emphasizes the need to characterize the joint PDF of these extreme values for estimating the system reliability.

A common approach in characterizing the extreme value distributions for random processes, is to study the associated first passage failures, based on the assumption that level crossings can be modeled as Poisson counting processes. The parameter of the counting process is related to the mean outcrossing rate, which in turn, can be estimated if the joint probability density function (pdf) of the process and its time derivative, is available (Rice, 1956). For Gaussian random processes, this is readily available and closed form expressions for the extreme value distributions have been developed (Lin, 1967; Nigam, 1983). This knowledge is, however, seldom available for non-Gaussian processes. A literature review on the various approximations developed for the extreme value distributions for scalar and vector non-Gaussian processes is available (Manohar and Gupta, 2005). Outcrossing rates of vector random processes have been studied in the context of problems in load combinations and in structural reliability. The focus of many of these problems have been in determining the probability of exceedance of the sum of the component processes, and the outcrossing event has been formulated as a scalar process outcrossing. Some of these results have been used in the geometrical approach (Leira, 1994, 2003) in the studies on development of multivariate extreme value distributions for vector Gaussian/non-Gaussian random processes. Multivariate extreme value distributions associated with a vector of Gaussian random processes have been developed (Gupta and Manohar, 2005), based on the principle that multi-point random processes can be used to model the level crossing statistics associated with the vector Gaussian processes. Similar principles have been applied in developing approximations for the multivariate extreme value distributions associated with a vector of non-Gaussian processes, obtained as nonlinear transformations of vector Gaussian processes (Gupta and van Gelder, 2005).

Here, we extend the above formulation to illustrate its usefulness in estimating the reliability of a randomly vibrating structural system, in series configuration. The response of the structural components have been modeled as a vector of mutually correlated log-normal loads and approximations have been developed for the joint extreme value distribution for the response of the structural components. This is of particular importance in the context of risk analysis of nuclear plants, where the dynamic loads arising from various load effects, are modeled as log-normal random processes.

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Problem Statement

We consider a linear structural system consisting of m components in series configuration. We assume that the structural system is excited by a n-dimensional vector of mutually correlated, stationary, log-normal loads $\{Y_k(t)\}_{k=1}^n$. The structure response of the j^{th} component is given by

$$Z_j(t) = \tilde{g}_j[Y_1(t), \ldots, Y_n(t)] = g_j[X_1(t), \ldots, X_n(t)], \tag{1}$$

where $Y_j(t) = \mathrm{e}^{X_j(t)}$, $(j = 1, \ldots, n)$ and $\tilde{g}_j[\mathrm{e}^{X_1(t)}, \ldots, \mathrm{e}^{X_n(t)}] = g_j[X_1(t), \ldots, X_n(t)]$. Here, $\{X_j(t)\}_{j=1}^n$ constitutes a vector of mutually correlated Gaussian random processes, $g_j[\cdot]$ is a deterministic nonlinear function which relates the random processes $X_j(t)$ to the component response $Z_j(t)$ and t is time. It is clear that $Z_j(t)$ is a non-Gaussian process whose probabilistic characteristics are difficult to estimate. A component failure is defined to occur when $Z_j(t)$ exceeds specified threshold levels and is given by

$$P_{f_j} = 1 - P[Z_j(t) \leq \alpha_j; \forall t \in (0, T)], \tag{2}$$

where α_j denotes the threshold level, T is the duration of interest and $P[\cdot]$ is the probability measure. Equation (2) can be recast into the following time invariant format

$$P_{f_j} = 1 - P[Z_{m_j} \leq \alpha_j] = 1 - P_{Z_{m_j}}(\alpha_j), \tag{3}$$

where $Z_{m_j} = \max_{0 \leq t \leq T} Z_j(t)$ is a random variable denoting the extreme value of $Z_j(t)$ in $[0, T]$ and $P_{Z_{m_j}}(\cdot)$ is the corresponding PDF. We assume that α_j, $(j = 1, \ldots, m)$ to be high and the spectral bandwidth ratio (Vanmarcke, 1972) of the processes $Z_j(t)$, $(j = 1, \ldots, m)$ to be such that the outcrossings of $Z_j(t)$ can be modeled as a Poisson point process. This leads to the following expression for $P_{Z_{m_j}}(\alpha_j)$:

$$P_{Z_{m_j}}(\alpha_j) = \exp[-\nu_j^+(\alpha_j)T]. \tag{4}$$

Here, $\nu_j^+(\alpha_j)$ is the mean outcrossing rate of $Z_j(t)$ across level α_j. An estimate of $\nu_j^+(\alpha_j)$ can be determined from the well known expression (Rice, 1956)

$$\nu_j^+(\alpha_j) = \int_0^\infty \dot{z} p_{Z_j \dot{Z}_j}(\alpha_j, \dot{z}; t, t) \mathrm{d}\dot{z}, \tag{5}$$

where $p_{Z_j \dot{Z}_j}(z, \dot{z}; t, t)$ is the joint pdf of the process $Z_j(t)$ and its instantaneous time derivative $\dot{Z}_j(t)$, at time t. A crucial step in this formulation lies in determining the joint pdf $p_{Z_j \dot{Z}_j}(z, \dot{z})$.

For a structural system comprising of m components, the structure is deemed to have failed if any of the constituent m components fail. Thus, the system failure, denoted by P_{f_s}, is expressed as

$$P_{f_s} = 1 - P[\cap_{j=1}^m \{Z_{m_j} \leq \alpha_j\}] = 1 - P_{Z_{m_1} \ldots Z_{m_m}}(\alpha_1, \ldots, \alpha_m). \tag{6}$$

Here, $P_{Z_{m_1} \ldots Z_{m_m}}(\cdot)$ is the m-dimensional joint PDF for the vector of extreme value random variables $\{Z_{m_j}\}_{j=1}^m$. Assuming that the respective thresholds, α_j, corresponding to each component process $Z_j(t)$, are sufficiently high for the respective outcrossings to be rare, the level crossings, denoted by $\{N_j(\alpha_j)\}_{j=1}^m$, can be modeled as Poisson random variables. Since the different components have common source of excitations, $\{Z_j(t)\}_{j=1}^m$, and, in turn, $\{N_j(\alpha_j)\}_{j=1}^m$, are mutually correlated. Consequently, $\{Z_{m_j}\}_{j=1}^m$ are also expected to be mutually dependent. This implies the need for developing approximations for the joint multivariate PDF for the level crossings. Based on recent studies (Gupta and Manohar, 2005; Gupta and van Gelder, 2005) we construct the multivariate PDF for the

extreme values of the vector of non-Gaussian random processes $\{Z_j(t)\}_{j=1}^m$. We first illustrate the proposed method for the case when $m = 2$ and then extend it to the more general multi-dimensional situation.

Bivariate Vector

We first consider the case when $m = 2$ and $Z_1(t)$ and $Z_2(t)$ constitute a vector of mutually dependent non-Gaussian random variables, given by

$$Z_1(t) = g[X_1(t), \ldots, X_n(t)], \tag{7}$$
$$Z_2(t) = h[X_1(t), \ldots, X_n(t)], \tag{8}$$

where $g[\cdot]$ and $h[\cdot]$ are deterministic nonlinear functions. Let $N_1(\alpha_1)$ and $N_2(\alpha_2)$ be the number of level crossings for $Z_1(t)$ and $Z_2(t)$, across thresholds α_1 and α_2, in time duration $[0, T]$. For high thresholds, $N_1(\alpha_1)$ and $N_2(\alpha_2)$ can be modeled as mutually dependent Poisson random variables. Introducing the transformations,

$$\begin{aligned} N_1(\alpha_1) &= U_1 + U_3, \\ N_2(\alpha_2) &= U_2 + U_3, \end{aligned} \tag{9}$$

where $\{U_j\}_{j=1}^3$ are mutually independent Poisson random variables with parameters $\{\lambda_j\}_{j=1}^3$, it can be shown that $N_1(\alpha_1)$ and $N_2(\alpha_2)$ are Poisson random variables with parameters $(\lambda_1 + \lambda_3)$ and $(\lambda_2 + \lambda_3)$ respectively and covariance equal to λ_3. This construct for multivariate Poisson random variables has been discussed in the literature (Johnson and Kotz, 1969). The parameters $\{\lambda_j\}_{j=1}^3$ are, as of yet, unknowns.

Taking expectation on both sides of Equation (9), it can be shown that

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{Bmatrix} = \begin{Bmatrix} \langle N_1(\alpha_1) \rangle \\ \langle N_2(\alpha_2) \rangle \\ \mathrm{Cov}[N_1(\alpha_1), N_2(\alpha_2)] \end{Bmatrix}. \tag{10}$$

Here, $\mathrm{Cov}[N_1(\alpha_1), N_2(\alpha_2)] = \langle N_1(\alpha_1) N_2(\alpha_2) \rangle - \langle N_1(\alpha_1) \rangle - \langle N_2(\alpha_2) \rangle$ and if $Z_j(t)$ are stationary random processes,

$$\langle N_j(\alpha_j) \rangle = T \int_0^\infty \dot{z} p_{Z_j \dot{Z}_j}(\alpha_j, \dot{z}) \mathrm{d}\dot{z}, \quad j = 1, 2, \tag{11}$$

$$\langle N_1(\alpha_1) N_2(\alpha_2) \rangle = \int_{-T}^{T} (T - |\tau|) \left\{ \int_0^\infty \int_0^\infty \dot{z}_1 \dot{z}_2 p_{Z_1 Z_2 \dot{Z}_1 \dot{Z}_2}(\alpha_1, \alpha_2, \dot{z}_1, \dot{z}_2; \tau) \mathrm{d}\dot{z}_1 \mathrm{d}z_2 \right\} \mathrm{d}\tau, \tag{12}$$

where $\tau = t_2 - t_1$. Details of the derivation for Equation (12) is available (Gupta and Manohar, 2005). Thus, a solution for $\{\lambda_j\}_{j=1}^3$ can be obtained from Equation (10). Furthermore, it has been shown (Gupta and Manohar, 2005) that the joint PDF for the extreme values are related to $\{\lambda_j\}_{j=1}^3$ through the relation

$$P_{Z_{m_1} Z_{m_2}}(\alpha_1, \alpha_2) = \exp\left[-\sum_{j=1}^{3} \lambda_j \right]. \tag{13}$$

A crucial step in this formulation, however, lies in evaluating the expressions $\langle N_1(\alpha_1)\rangle$, $\langle N_2(\alpha_2)\rangle$ and $\langle N_1(\alpha_1)N_2(\alpha_2)\rangle$, for which, a knowledge of the pdfs $p_{Z_1\dot{Z}_1}(\cdot)$, $p_{Z_2\dot{Z}_2}(\cdot)$ and $p_{Z_1Z_2\dot{Z}_1\dot{Z}_2}(\cdot)$, is essential. This, however, is seldom available, especially when $Z_j(t)$ are non-Gaussian. In the following section, a methodology has been presented for developing approximate models for these pdfs.

Approximation for Joint PDFs

Here, we first illustrate the development of the joint pdf, of the form, $p_{Z_1\dot{Z}_1}(\cdot)$. The method is based on the formulation developed by Naess (1985). Next, based on an earlier study (Gupta and van Gelder, 2005), we show the development of an approximation for $p_{Z_1Z_2\dot{Z}_1\dot{Z}_2}(\cdot)$.

Scalar Case

We rewrite $p_{Z_1\dot{Z}_1}(\cdot)$ in the form

$$p_{Z_1\dot{Z}_1}(z,\dot{z};t,t)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}p_{X_2\ldots X_nZ_1\dot{Z}_1}(x_2,\ldots,x_n,z,\dot{z};t,t)\mathrm{d}x_2\ldots\mathrm{d}x_n, \tag{14}$$

where $p_{X_2\ldots X_nZ_1\dot{Z}_1}(\cdot)$ is the joint pdf of random variables X_2,...,X_n, Z_1 and $\dot{Z}_1$, at time t. Using the standard technique of transformation of random variables, we seek the transformation between the joint pdf $p_{X_2\ldots X_nZ_1\dot{Z}_1}(\cdot)$ and $p_{X_1\ldots X_n\dot{Z}_1}(\cdot)$. In order to achieve this, we assume that at time t, Z_1 in Equation (7), is a function of X_1 with all other random variables being fixed. We assume that there are k solutions for X_1 for the equation $z_1=g[X_1,x_2\ldots,x_n]$, for a given set of values for $Z_1=z_1$, $X_2=x_2$,...,$X_n=x_n$. This leads to the expression

$$p_{X_2\ldots X_nZ_1\dot{Z}_1}(x_2,\ldots,x_n,z,\dot{z})=\sum_{j=1}^{k}\left|\frac{\partial Z}{\partial X_1}\right|_j^{-1}p_{X_1\ldots X_n\dot{Z}_1}(x_1^{(j)},\ldots,x_n,\dot{z}). \tag{15}$$

Here, k depends on the form of the function $g[\cdot]$. The joint pdf $p_{X_1\ldots X_n\dot{Z}_1}(\cdot)$ can now be written as

$$p_{X_1\ldots X_n\dot{Z}_1}(x_1,\ldots,x_n,\dot{z})=p_{\dot{Z}_1|X_1\ldots X_n}(\dot{z}|X_1=x_1,\ldots,X_n=x_n)p_{X_1\ldots X_n}(x_1,\ldots,x_n). \tag{16}$$

Here, $p_{X_1\ldots X_n}(x_1,\ldots,x_n)$ is the n-dimensional joint Gaussian pdf and is completely specified if the mean and the covariance matrix of the vector Gaussian process is known. To determine the conditional pdf $p_{\dot{Z}_1|X_1\ldots X_n}(\cdot)$, we first write the time derivative of $\dot{Z}_1(t)$ from Equation (7), and when conditioned on $\{X_j=x_j\}_{j=1}^n$, is given by

$$\dot{Z}_1|_{\mathbf{X}}=\sum_{j=1}^{n}\left|\frac{\partial Z_1}{\partial X_j}\right|_{\mathbf{X}}\dot{X}_j=\sum_{j=1}^{n}g_j\dot{X}_j=\mathbf{G}\dot{\mathbf{X}}. \tag{17}$$

Here, $\mathbf{G}=[g_1,\ldots,g_n]$, $\dot{\mathbf{X}}=[\dot{X}_1,\ldots,\dot{X}_n]'$, the superscript ($'$) denoting transpose and $g_j=\partial Z_1/\partial X_j$, and when conditioned on $\mathbf{X}$, is a constant. $\dot{X}_j(t)$ are the time derivatives of $X_j(t)$ and are thus, zero-mean, stationary, Gaussian random process. Since $\dot{Z}_1|_{\mathbf{X}}$ is a linear sum of Gaussian random variables, $\dot{Z}_1|_{\mathbf{X}}$ is Gaussian, with parameters

$$\begin{aligned}\mu_{\dot{Z}_1|\mathbf{X}}&=\mathbf{G}\langle\dot{\mathbf{X}}\rangle=0,\\ \sigma^2_{\dot{Z}_1|\mathbf{X}}&=\mathbf{G}\langle\dot{\mathbf{X}}\dot{\mathbf{X}}^*\rangle\mathbf{G}'=\mathbf{G}\mathbf{C}_{\dot{\mathbf{X}}\dot{\mathbf{X}}}\mathbf{G}'.\end{aligned} \tag{18}$$

Here, $*$ denotes complex conjugation. Substituting Eqs. (14-16) to Equation (11), we get

$$\langle N_1(\alpha_1)\rangle = T\sum_{j=1}^{k}\int \ddot{\Omega}_j \int |g_1^{(j)}|^{-1} \times \left\{\int_0^\infty \dot{z}\, p_{\dot{Z}_1|\mathbf{X}}(\dot{z};\mathbf{x}^{(j)},t)\mathrm{d}\dot{z}\right\} p_{X_1\ldots X_n}(x_1^{(j)},x_2,\ldots,x_n)\mathrm{d}x_2\ldots\mathrm{d}x_n. \tag{19}$$

Here, Ω_j denotes the domain of integration determined by the permissible set of values $x_2,\ldots,x_n$ for each solution of $x_1^{(j)}$. Since $p_{\dot{Z}_1|\mathbf{X}}(\cdot)$ is Gaussian, it can be shown that (Naess, 1985)

$$\int_0^\infty \dot{z}\, p_{\dot{Z}_1|\mathbf{X}}(\dot{z};\mathbf{x},t)\mathrm{d}\dot{z} = \sigma_{\dot{Z}_1|\mathbf{X}}\Psi\left(\frac{\mu_{\dot{Z}_1|\mathbf{X}}}{\sigma_{\dot{Z}_1|\mathbf{X}}}\right), \tag{20}$$

where $\Psi(x) = \phi(x) + x\Phi(x)$, $\phi(x)$ and $\Phi(x)$ are respectively, the standard normal pdf and PDF. Without loss of generality, $\mathbf{X}$ can be assumed to be a vector of mutually independent Gaussian random variables. When $\mathbf{X}$ are correlated, appropriate linear transformations can be applied to make $\mathbf{X}$ mutually independent. These linear transformations, however, result in a new definition for the function $g[\cdot]$.

Equation (19) can now be expressed as

$$\langle N_1(\alpha_1)\rangle = T\sum_{j=1}^{k}\int \ddot{\Omega}_j \int f(x_1^{(j)},x_2,\ldots,x_n)\, p_{X_2\ldots X_n}(x_2,\ldots,x_n)\mathrm{d}x_2\ldots\mathrm{d}x_n, \tag{21}$$

where

$$f(x_1^{(j)},x_2,\ldots,x_n) = |g_1^{(j)}|^{-1}\sigma_{\dot{Z}_1|\mathbf{X}}\Psi\left(\frac{\mu_{\dot{Z}_1|\mathbf{X}}}{\sigma_{\dot{Z}_1|\mathbf{X}}}\right) p_{X_1}(x_1^{(j)}).$$

The difficulties involved in evaluating Equation (21) are: (a) in determining the domain of integration Ω_j, defined by the possible set of solutions for $X_1^{(j)}$, and (b) in evaluating the multidimensional integral. A recently developed numerical algorithm is used to overcome these difficulties. This has been discussed later in this paper.

Vector Case

We now focus on developing models for $p_{Z_1Z_2\dot{Z}_1\dot{Z}_2}(\cdot)$. As in the scalar case, we rewrite

$$p_{Z_1Z_2\dot{Z}_1\dot{Z}_2}(z_1,z_2,\dot{z}_1,\dot{z}_2) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p_{X_3\ldots X_nZ_1Z_2\dot{Z}_1\dot{Z}_2}(x_3,\ldots,x_n,z_1,z_2,\dot{z}_1,\dot{z}_2)\mathrm{d}x_3\ldots\mathrm{d}x_n, \tag{22}$$

where the dimension of the integrals is $(n-2)$. The joint pdf $p_{X_3\ldots X_nZ_1Z_2\dot{Z}_1\dot{Z}_2}$ is rewritten as

$$p_{X_3\ldots X_nZ_1Z_2\dot{Z}_1\dot{Z}_2} = \sum_{j=1}^{k}|\mathbf{J}|_j^{-1}\, p_{X_1\ldots X_n\dot{Z}_1\dot{Z}_2}(x_1^{(j)},x_2^{(j)},x_3,\ldots,x_n,\dot{z}_1,\dot{z}_2), \tag{23}$$

where, for fixed values of $X_3,\ldots,X_n$, Z_1 and Z_2, there exist k solutions for X_1 and X_2, and $\mathbf{J}_j$ denotes the Jacobian matrix

$$\mathbf{J}_j = \begin{bmatrix} \partial Z_1/\partial X_1 & \partial Z_1/\partial X_2 \\ \partial Z_2/\partial X_1 & \partial Z_2/\partial X_2 \end{bmatrix}, \tag{24}$$

evaluated at $(x_1^{(j)}, x_2^{(j)})$. As before, we now rewrite

$$p_{X_1 \ldots X_n \dot{Z}_1 \dot{Z}_2}(x_1^{(j)}, x_2^{(j)}, x_3, \ldots, x_n, \dot{z}_1, \dot{z}_2; t_1, t_2) = p_{\dot{Z}_1 \dot{Z}_2 | \mathbf{X}}(\dot{z}_1, \dot{z}_2 | \mathbf{x}; t_1, t_2) p_{\mathbf{X}}(\mathbf{x}), \tag{25}$$

where $p_{\mathbf{X}}(\mathbf{x})$ is the n-dimensional Gaussian pdf. The time derivatives for $Z_1(t)$ and $Z_2(t)$, conditioned on $\mathbf{X}$, is expressed as

$$\dot{Z}_1(t_1) = \sum_{j=1}^{n} \left| \frac{\partial g}{\partial X_j} \right|_{\mathbf{X}} \dot{X}_j(t_1) = \mathbf{G}\dot{\mathbf{X}}(t_1), \tag{26}$$

$$\dot{Z}_2(t_2) = \sum_{j=1}^{n} \left| \frac{\partial h}{\partial X_j} \right|_{\mathbf{X}} \dot{X}_j(t_2) = \mathbf{H}\dot{\mathbf{X}}(t_2). \tag{27}$$

Here, $g_j = \partial g / \partial X_j$, $h_j = \partial h / \partial X_j$, evaluated at $\mathbf{X} = \mathbf{x}$ and $\mathbf{G} = [g_1, \ldots, g_n]$, $\mathbf{H} = [h_1, \ldots, h_n]$. Since $\mathbf{G}$ and $\mathbf{H}$ are constants and $\dot{\mathbf{X}}(t)$ constitutes a vector of zero-mean stationary, Gaussian random processes, $\dot{Z}_1(t)$ and $\dot{Z}_2(t)$, when conditioned on $\mathbf{X}$, are zero-mean, stationary Gaussian processes. The joint conditional pdf $p_{\dot{Z}_1 \dot{Z}_2 | \mathbf{X}}(\dot{z}_1, \dot{z}_2 | \mathbf{x}; t_1, t_2)$ is therefore jointly Gaussian and is of the form

$$p_{\dot{Z}_1, \dot{Z}_2 | \mathbf{X}}(\dot{z}_1, \dot{z}_2; t_1, t_2) = \frac{1}{4\pi^2 |\Delta|^{0.5}} \exp\left[-\frac{1}{2} \mathbf{w} \Delta^{-1} \mathbf{w}' \right]. \tag{28}$$

Here, $\mathbf{w} = [\dot{z}_1, \dot{z}_2]'$, $\Delta \equiv \Delta(t_1, t_2) = \mathbf{T}\mathbf{C}_{\dot{\mathbf{X}}}(t_1, t_2)\mathbf{T}'$, the operator $|\cdot|$ denotes the determinant of a matrix, $\mathbf{T} = [\mathbf{G}, \mathbf{H}]'$ and $\mathbf{C}_{\dot{\mathbf{X}}}(t_1, t_2)$ is the covariance matrix $\langle \dot{\mathbf{X}}(t_1) \dot{\mathbf{X}}(t_2)^* \rangle$. Without loss of generality, it can be assumed that $\dot{\mathbf{X}}(t)$ constitutes a vector of mutually independent, stationary, Gaussian random processes. This leads to $\mathbf{C}_{\dot{\mathbf{X}}}(t_1, t_2) = \mathbf{C}_{\dot{\mathbf{X}}}(\tau)$ being a diagonal matrix, where $\tau = t_2 - t_1$.

Substituting Equations (22–28) into Equation (12), and rearranging the order of integrations, we get

$$\begin{aligned} \langle N_1(\alpha_1) N_2(\alpha_2) \rangle = & \sum_{j=1}^{k} \int \ddot{\Omega}_j \int |\mathbf{J}|_j^{-1} \left\{ \int_{-T}^{T} (T - |\tau|) F(\tau) \mathrm{d}\tau \right\} \\ & \times p_{X_1}(x_1^{(j)}) p_{X_2}(x_2^{(j)}) p_{X_3 \ldots X_n}(x_3, \ldots, x_n) \mathrm{d}x_3 \ldots \mathrm{d}x_n, \end{aligned} \tag{29}$$

where

$$F(\tau) = \int_0^{\infty} \int_0^{\infty} \dot{z}_1 \dot{z}_2 p_{\dot{Z}_1 \dot{Z}_2 | \mathbf{X}}(\dot{z}_1, \dot{z}_2; \tau) \mathrm{d}\dot{z}_1 \mathrm{d}\dot{z}_2. \tag{30}$$

The above integral can be evaluated using symbolic software MAPLE or numerically evaluated. Subsequently, the inner integral in Equation (29), with respect to τ, is carried out numerically. The remaining $(n - 2)$ dimensional integrals can be evaluated using the numerical algorithm described later in this paper.

Multivariate Vector

As has been shown earlier (Gupta and Manohar, 2005), the construct for bivariate vector of Poisson random variables can be easily generalized for the case $m > 2$. The number of mutually independent

Poisson random variables can be generalized to be given by $C_1^m + C_2^m$, where C_k^m denotes combination of m variables taken k at a time. Thus, for $m = 3$, consider six mutually independent Poisson random variables, $\{U_i\}_{i=1}^6$, with parameters $\{\lambda_i\}_{i=1}^6$ and define

$$\begin{aligned} N_1(\alpha_1) &= U_1 + U_4 + U_5 \\ N_2(\alpha_2) &= U_2 + U_4 + U_6 \\ N_3(\alpha_3) &= U_3 + U_5 + U_6 \end{aligned} \tag{31}$$

The equations relating $\{\lambda_i\}_{i=1}^6$ to the moments of $\{N_i\}_{i=1}^3$ can be shown to be given by

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{Bmatrix} = \begin{Bmatrix} \langle N_1(\alpha_1) \rangle \\ \langle N_2(\alpha_2) \rangle \\ \langle N_3(\alpha_3) \rangle \\ \mathrm{Cov}[N_1(\alpha_1), N_2(\alpha_2)] \\ \mathrm{Cov}[N_1(\alpha_1), N_3(\alpha_3)] \\ \mathrm{Cov}[N_2(\alpha_2), N_3(\alpha_3)] \end{Bmatrix}. \tag{32}$$

It is to be noted that for $m > 2$, the formulation requires the evaluation of a set of integrals of the form in Equation (12) and at no stage does the order of the integrals becomes greater than that of Equation (12). In general, the number of such integrals that need to be evaluated is C_2^m.

Numerical Algorithm

A crucial step in the above formulation lies in evaluating integrals of the type as in Equation (21). Closed form solutions for the integrals are possible only for a limited class of problems. Here, we propose the use of Monte Carlo methods, in conjunction with importance sampling to increase the efficiency, for evaluating these integrals. The integrals in Equation (21) can be recast as

$$\begin{aligned} \mathit{I}_j &= \int_{-\infty}^{\infty} I[q(\mathbf{X}) \le 0] f(\mathbf{X}) \frac{p_{\mathbf{X}}(\mathbf{x})}{h_{\tilde{\mathbf{X}}}(\mathbf{x})} h_{\tilde{\mathbf{X}}}(\mathbf{x}) \mathrm{d}\mathbf{x} \\ &= \frac{1}{N} \sum_{j=1}^{N} I[q(\mathbf{X}_j) \le 0] f(\mathbf{X}_j) \frac{p_{\mathbf{X}}(\mathbf{x}_j)}{h_{\tilde{\mathbf{X}}}(\mathbf{x}_j)}, \end{aligned} \tag{33}$$

where $h_{\tilde{\mathbf{X}}}(\cdot)$ is the importance sampling pdf and $I[\cdot]$ is an indicator function taking values of unity if $q(\mathbf{X}) \le 0$, indicating that the sample lies within the domain of integration Ω_j, and zero otherwise. Since the problem is formulated into the standard normal space $\mathbf{X}$, $h_{\tilde{\mathbf{X}}}(\cdot)$ can be taken to be Gaussian with unit standard deviation and shifted mean. The difficulty, however, lies in determining where should $h_{\tilde{\mathbf{X}}}(\cdot)$ be centered. An inspection of Equation (33) reveals that the form of the integrals are similar to reliability integrals which are of the form

$$\mathit{I}_j = \int_{-\infty}^{\infty} I[q(\mathbf{X}) \le 0] p_{\mathbf{X}}(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{34}$$

This implies that for efficient computation of the integrals, the importance sampling pdf $h_{\tilde{\mathbf{X}}}(\cdot)$ may be centered around the design point for the function $q(\mathbf{X}) = 0$. If $q(\mathbf{X})$ is available in explicit form, first order reliability methods can be used to determine the design point. If $q(\mathbf{X})$ is not available explicitly, an adaptive importance sampling strategy can be adopted to determine the design point. In certain problems, the domain of integration, characterized by $q(\mathbf{X}) = 0$, may consist of multiple design points or multiple regions which contribute significantly to I_j. This is especially true when

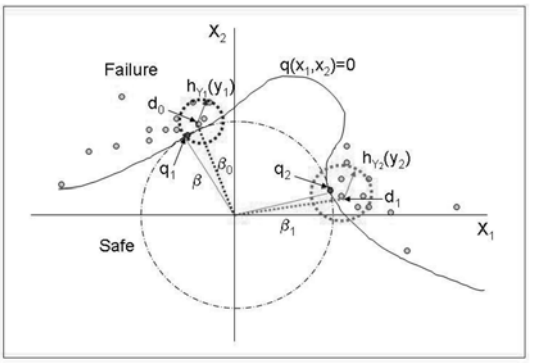

Fig. 1. Schematic diagram for numerical algorithm for evaluating multidimensional integrals; $g(x_1, x_2) = 0$ is the limit surface ion the $X_1 - X_2$ random variable space; $h_{Y_1}(y_1)$ and $h_{Y_2}(y_2)$ are the two importance sampling pdfs; two design points at distance β from the origin.

$q(\mathbf{X}) = 0$ is highly nonlinear, irregular or consists of disjointed regions. In these situations, it is necessary to construct a number of importance sampling functions, with each function centered at the various design points.

The steps for implementing the algorithm for numerical evaluation of integrals of the type in Equation (21), has been developed and discussed (Gupta and van Gelder, 2005). The sequential steps for implementing the algorithm is detailed below, with reference to the schematic diagram in Figure 1.

(1) Carry out pilot Monte Carlo simulations in the standard normal space. If there are too few samples in the failure domain, we carry out Monte Carlo simulations with a Gaussian importance sampling function with mean zero and a higher variance. On the other hand, if there are too few samples in the safe region, the variance of the importance sampling function is taken to be smaller. Repeat this step, till we have a reasonable number of samples in the failure and the safe regions. (2) We sort the samples lying in the failure domain according to their distance from the origin. (3) A Gaussian importance sampling pdf is constructed which is centered at the sample in the failure domain lying closest to the origin. Let this point be denoted by d_0 and its distance from the origin be denoted by β_0. (4) We check for samples in the failure domain, within a hyper-sphere of radius β_1, $\beta_1 - \beta_0 = \epsilon$, where ϵ is a positive number. (5) For samples lying within this hyper-sphere, we check for the sample d_1, which lie closest to the origin but is not located in the vicinity of d_0. This is checked by comparing the direction cosines of d_1 and d_0. (6) By comparing the direction cosines of all samples lying within the hyper-sphere of radius β_1, we can identify the number of design points. We construct importance sampling pdfs at each of these design points. If there exist no samples with direction cosines distinctly different from d_0, there is only one design point and a single importance sampling pdf is sufficient. (7) During importance sampling procedure corresponding to a design point, for each sample realization, we check if x_1 and $\dot{x}_1$ expressed in terms of the random variables $(Z_1, \ldots, Z_{2n-2})$ are real. The indicator function is assigned a value of unity if real, and zero otherwise. (8) An estimate of I_j is obtained from Equation (33).

Numerical Example

For illustrating the proposed formulation, we consider a simple structural system consisting of two components. The component responses, $Z_1(t)$ and $Z_2(t)$, are assumed to be lognormal random processes. We assume that $Z_1(t)$ and $Z_2(t)$ arise from a common source of load effects, and are

expressed as

$$\begin{aligned} Z_1(t) &= \exp[X_1(t) + X_2(t) + X3(t)], \\ Z_2(t) &= \exp[X_1(t) - X_2(t) - X_3(t)], \end{aligned} \tag{35}$$

and hence, are mutually correlated. For the sake of simplicity, we assume that $\{X_j(t)\}_{j=1}^3$ are mutually independent, zero-mean, stationary, Gaussian random processes, with auto-correlation function given by

$$R_{jj}(\tau) = S_j^2 \exp[-\beta_j \tau^2], \quad j = 1, 2, 3, \tag{36}$$

where S_j and β_j are constants. The prescribed safety levels are assumed to be deterministic and constant over time. The time duration considered is $T = 10$ s.

First, we develop the marginal extreme value distributions for $Z_1(t)$ and $Z_2(t)$. Following Equation (21), we get

$$\langle N_j(\alpha_j)\rangle = T \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma_{\dot{Z}_j|\mathbf{X}} \Psi\left(\frac{\mu_{\dot{Z}_j|\mathbf{X}}}{\sigma_{\dot{Z}_j|\mathbf{X}}}\right) \frac{p_{X_1}(\tilde{x}_1)}{\alpha_j} p_{X_2}(x_2) p_{X_3}(x_3) \mathrm{d}x_2 \mathrm{d}x_3, \tag{37}$$

where, for $z_j = \alpha_j$,

$$\tilde{x}_1 = \ln[\alpha_j] - x_2 - x_3, \tag{38}$$

$$\mu_{\dot{z}_j|\mathbf{X}} = \alpha_j(\langle \dot{X}_1\rangle + \langle \dot{X}_2\rangle + \langle \dot{X}_3\rangle) = 0, \tag{39}$$

$$\sigma^2_{\dot{z}|\mathbf{X}} = \alpha_j^2\{\sigma^2_{\dot{X}_1} + \sigma^2_{\dot{X}_2} + \sigma^2_{\dot{X}_3}\} \tag{40}$$

and $\{p_{X_j}(x_j)\}_{j=1}^3$ are Gaussian pdf with mean zero and standard deviation σ_j. It can be shown that Equation (37) can be further simplified to the form

$$\langle N_j(\alpha_j)\rangle = T\sigma_{\dot{Z}_j|\mathbf{X}} \Psi\Big(\frac{\mu_{\dot{Z}_j|\mathbf{X}}}{\sigma_{\dot{Z}_j|\mathbf{X}}}\Big) \frac{1}{\alpha_j} \langle p_{X_1}(\tilde{x_1})\rangle. \tag{41}$$

The analytical predictions for the failure probability for $Z_1(t)$ and $Z_2(t)$, given by Equation (41), for various levels of α_1 and α_2 are compared with those obtained from full scale Monte Carlo simulations in Figures 2 and 3. respectively. The accuracy of the analytical predictions are observed to be acceptable.

Next, we construct the joint extreme value distribution for $Z_1(t)$ and $Z_2(t)$. For this example, Equation (29) can be written as

$$\langle N_1(\alpha_1) N_2(\alpha_2)\rangle = \int_{-\infty}^{\infty} \left[|\mathbf{J}|^{-1} \left\{ \int_{-T}^{T} (T - |\tau|) F(\tau) \mathrm{d}\tau \right\} p_{X_1}(\tilde{x}_1) p_{X_2}(\tilde{x}_2) \right] p_{X_3}(x_3) \mathrm{d}x_3, \tag{42}$$

where, for $Z_1 = \alpha_1$ and $Z_2 = \alpha_2$,

$$|\mathbf{J}| = \begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_2 & -\alpha_2 \end{bmatrix} = -2\alpha_1\alpha_2, \tag{43}$$

$$\tilde{x}_1 = \frac{1}{2}\ln[\alpha_1\alpha_2], \tag{44}$$

$$\tilde{x}_2 = \frac{1}{2}\ln[\frac{\alpha_1}{\alpha_2}] - x_3, \tag{45}$$

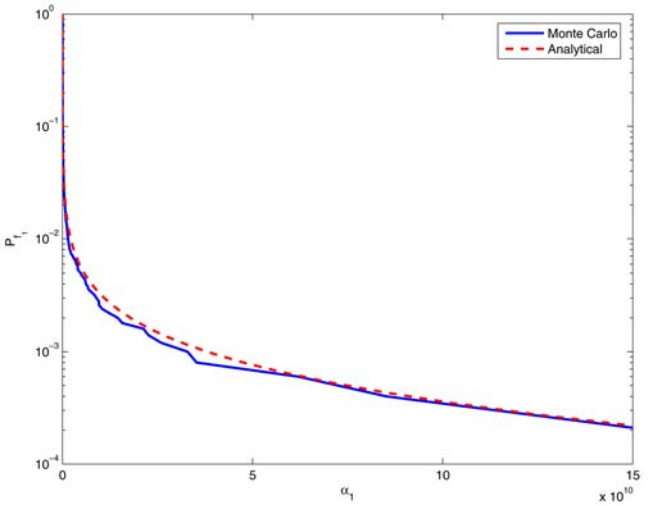

Fig. 2. Probability of exceedance, P_{f_1}, for $Z_1(t)$ across threshold level α_1.

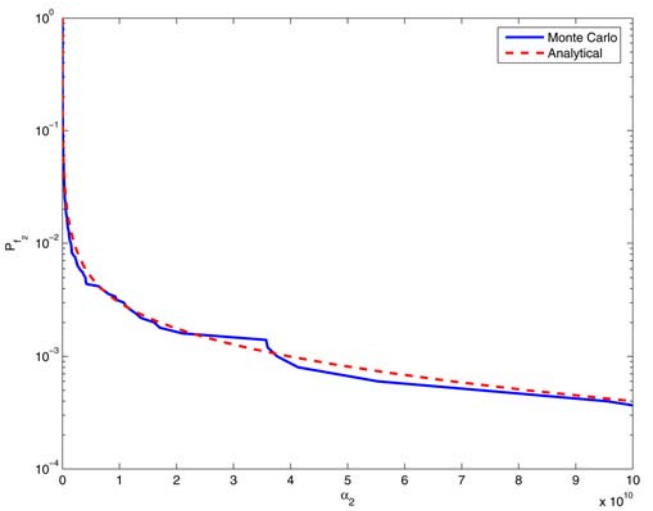

Fig. 3. Probability of exceedance, P_{f_2}, for $Z_2(t)$ across threshold level α_2.

and $F(\tau) = s_0(\tau)\{s_1(\tau) + s_2(\tau) + s_3(\tau) + s_4(\tau)\}$. Here,

$$\begin{aligned}
s_0(\tau) &= 0.25(\pi c_{22})^{-3/2},\\
s_1(\tau) &= \pi c_{12} c_{22}^{3/2},\\
s_2(\tau) &= 2(c_{22}^2 c_{11} - c_{12}^2 c_{22})\sqrt{\pi c_{22}/(c_{11}c_{22} - c_{12}^2)},\\
s_3(\tau) &= 2c_{12}^2(c_{22} - c_{12}^2/c_{11})\sqrt{\pi c_{22}/(c_{11}c_{22} - c_{12}^2)},\\
s_4(\tau) &= 2c_{22}c_{12}\sqrt{\pi}\{2c_{12}/[\sqrt{c_{22}(c_{11}c_{22} - c_{12}^2)}(2 + 2c_{12}^2/(c_{11}c_{22} - c_{12}^2))]\}\\
&\quad + \tan^{-1}[c_{12}/\sqrt{c_{11}c_{22} - c_{12}^2}],
\end{aligned} \tag{46}$$

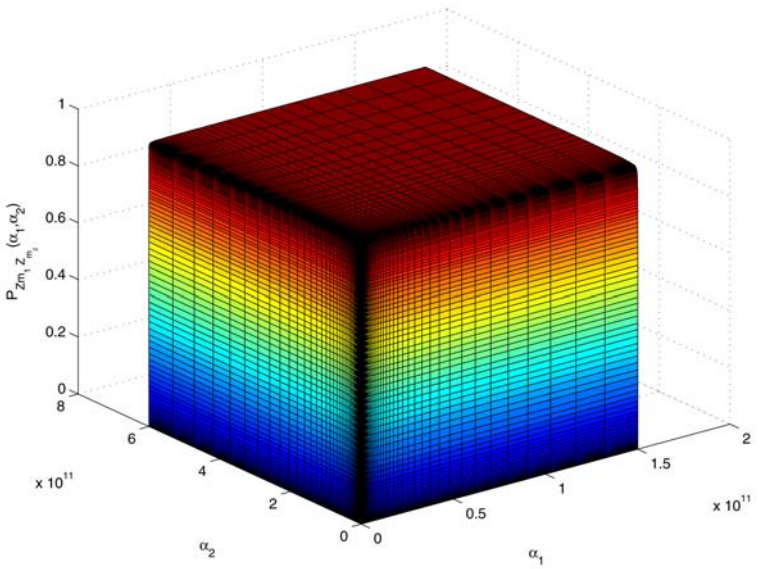

Fig. 4. Joint PDF for the extreme values of $Z_1(t)$ and $Z_2(t)$.

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} = \begin{bmatrix} g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{bmatrix} \begin{bmatrix} \rho_{11}(\tau) & 0 & 0 \\ 0 & \rho_{22}(\tau) & 0 \\ 0 & 0 & \rho_{33}(\tau) \end{bmatrix} \begin{bmatrix} g_1 & h_1 \\ g_2 & h_2 \\ g_3 & h_3 \end{bmatrix}, \tag{47}$$

$g_1 = g_2 = g_3 = \alpha_1$ and $h_1 = \alpha_2$, $h_2 = h_3 = -\alpha_2$, and $\rho_{jj}(\tau) = \langle \dot{X}_j(t)\dot{X}_j(t+\tau)\rangle = -\partial^2 R_{jj}(\tau)/\partial\tau^2$, $(j = 1, 2, 3)$. The inner integral in Equation (42), given by

$$\mathcal{I} = \int_T^T (T - |\tau|)F(\tau)\mathrm{d}\tau, \tag{48}$$

can be evaluated numerically. This leads to the following simplified form for Equation (42):

$$\langle N_1(\alpha_1)N_2(\alpha_2)\rangle = -\frac{\mathcal{I}}{2\alpha_1\alpha_2} p_{X_1}(\tilde{x}_1)\langle p_{X_2}(\tilde{x}_2)\rangle. \tag{49}$$

The joint extreme value distribution function for $Z_1(t)$ and $Z_2(t)$ are computed analytically and is shown in Figure 4.

The marginal distribution for the exceedance probability, for various threshold levels of α_1, have been shown in Figure 5. In the same figure, the corresponding conditional exceedance probability, when conditioned on various threshold levels α_2, have also been shown. The significant levels of difference in the probability levels indicate the importance of the correlations that exist between the extreme values of $Z_1(t)$ and $Z_2(t)$. The results obtained from Monte Carlo simulations are also shown in the same figure and are observed to have close resemblance with the analytical predictions. The corresponding results for $Z_2(t)$ are shown in Figure 6.

The analytical predictions are compared with those obtained from Monte Carlo simulations carried out on an ensemble of 5000 samples of time histories for $Z_1(t)$ and $Z_2(t)$. These results have been shown in Figure 6.

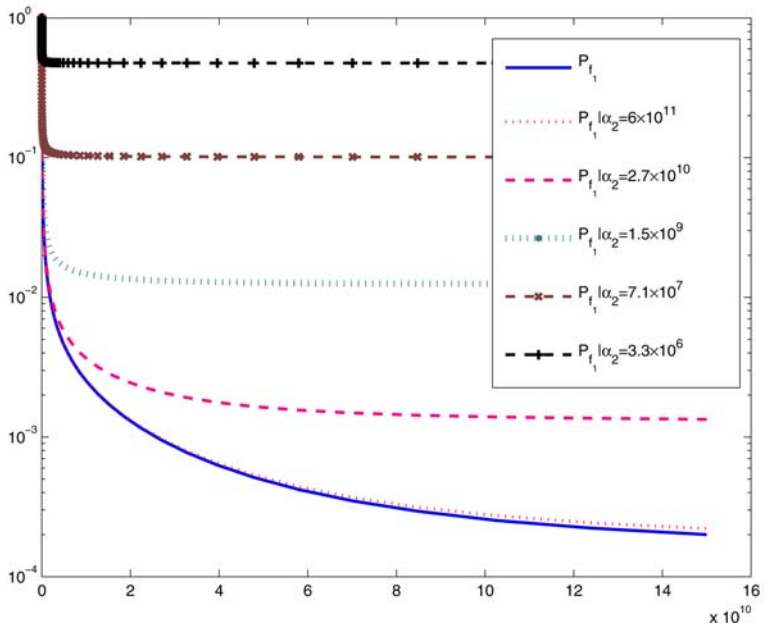

Fig. 5. Probability of exceedance for $Z_1(t)$; the markers indicate the estimates obtained from Monte Carlo simulations.

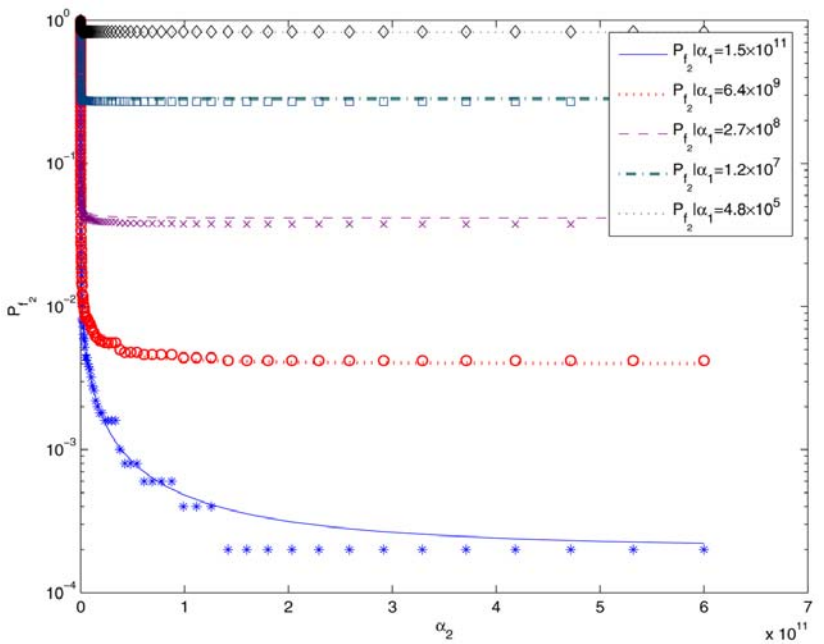

Fig. 6. Probability of exceedance for $Z_2(t)$; the markers indicate the estimates obtained from Monte Carlo simulations.

Concluding Remarks

A methodology has been developed for estimating the time variant reliability of a randomly vibrating series system, when the component response processes constitute a vector of mutually correlated log-normal random processes. A key feature in the development of the proposed method lies in the assumption, that for high thresholds, the number of level crossings of a non-Gaussian process can be modeled as a Poisson point process. The assumption of the outcrossings being Poisson distributed have been proved to be mathematically valid for Gaussian processes when the threshold approaches infinity (Cramer, 1966). However, it has been pointed out that for threshold levels of practical interest, this assumption results in errors whose size and effect depend on the bandwidth of the processes (Vanmarcke, 1972). While it can be heuristically argued that for high thresholds, the outcrossings of non-Gaussian processes can be viewed to be statistically independent and hence can be modeled as a Poisson point process, to the best of the authors' knowledge, studies on the validity of this assumption for non-Gaussian processes, do not exist in structural engineering literature. The multivariate extreme value distributions obtained by the proposed method, is thus expected to inherit the associated inaccuracies and limitations due to this assumption.

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