COLLAPSE ANALYSIS OF REINFORCED CONCRETE SLABS: ARE THE UP AND DOWN ROADS ONE AND THE SAME?

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The road up and the road down are one and the same
(Heraclitus)

O ye'll tak' the high road and I'll tak' the low road,
And I'll be in Scotland afor ye
(Traditional Song)

Abstract

The comparative merits of hand and automated upper and lower bound techniques for the collapse load estimation of reinforced concrete slabs are examined. Examples, drawn from both theoretical and practical design work, are used to show that both hand and automated upper bound yield line techniques can produce significant, unsafe errors. Automated lower bound solutions, however, are shown to consistently provide safe estimates that are not unduly conservative, provided appropriate formulations are adopted. As long as the engineer is willing to dispense with the crutch of a yield line pattern, it is therefore contended that, whilst Heraclitus may be correct in that both the upper and the lower bound roads can lead to one and the same collapse load, the lower bound road gets you there, certainly more safely, and usually quicker, as the Traditional Song suggests.

Introduction

A unique solution for the collapse load of a reinforced concrete slab, as predicted by rigid-plastic theory, requires the simultaneous fulfilment of equilibrium, yield and mechanism conditions. This requirement is normally too onerous to be achieved directly and recourse is therefore commonly made to the use of either upper or lower bound solutions (Wood, 1961). Both approaches satisfy the equilibrium condition, but the upper bound approximation ignores the yield requirement, whilst the lower bound technique does not require the specification of a mechanism. The upper bound method can therefore postulate a collapse mechanism which incorporates "unsafe" violations of the slab yield requirement that moments do not exceed the fully plastic moment of the slab section. Lower bound solutions, on the other hand, can postulate moment distributions that are "safe" (in the sense that the plastic moment is nowhere exceeded), but which do not correspond to a collapse mechanism. For both formulations, either hand or automated approaches are possible and the principal features of a range of these will be outlined prior to a comparative study of the merits of the various techniques, as demonstrated by a range of examples.

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Upper Bound Techniques – The High Road

The traditional hand calculation approach to yield line analysis (Johansen, 1964) requires the initial specification of a potential mechanism for which a collapse load is calculated. This estimate will be an upper bound on the true collapse load. Subsequent trial mechanisms may then be investigated and the lowest collapse load found is taken as the exact value. A refinement is to investigate the effect of geometric variation of a given yield line arrangement, whilst maintaining the same basic topology. This refinement normally results in less significant reductions in the collapse load than is possible from the detection of a more critical mechanism and is therefore sometimes approximated by applying a 10-15% reduction in the collapse load. This reduction is assumed to cover this geometric refinement effect, together with subsidiary variations in collapse mechanisms, such as the presence of "corner levers" (Kennedy *et al*, 2003).

The yield line approach may be automated by casting it as a linear programming formulation (Munro *et al*, 1978). The slab system to be analysed is subdivided by a triangulated grid and the optimisation process then evaluates a critical mechanism, taking the edges of the triangles as potential yield line sections. The process does not necessarily identify either the correct form of mechanism or the exact geometric positioning of the yield lines, since it is limited by the stipulation that yield should only occur along the edges of the specified grid. To improve the accuracy of the estimated collapse load, a two part process has been suggested (Johnson, 1994), in which a "fine" grid is used to identify the likely critical mechanism and a "coarse" grid is then used with geometric optimisation to obtain an improved approximation to the positioning of the yield line pattern.

Lower Bound Techniques – The Low Road

In practice, using hand calculations, lower bound solutions are generally obtained by the "strip" method (Hillerborg, 1975). In its simplest form, the strip method uses lines of assumed zero shear force to allow the determination of equilibrium moment distributions that provide a "safe" solution if used to design the slab's reinforcement pattern. The basic methodology has been extended (Hillerborg, 1996) by a number of approximations and simplifications to allow the design of slabs with complex geometries, loadings, and boundary conditions and also to make it more convenient for practical application.

Lower bound solutions may also be generated by automated approaches, although these are closer to automated upper bound (yield line) techniques rather than to the strip system. As with the automated yield line system, triangulated nets are used, but, rather than taking the net edges as potential yield lines, prescribed distributions of bending moment (Krenk *et al*, 1994) within the triangular regions are presumed. If, as is common practice, the moment within each element is assumed to be a linear function of the values at its nodes, then a lower bound solution may be obtained by optimising the load subject to the nodal moment values not exceeding the slab's yield capacity.

Arrivals and Departures

When using either an upper or a lower bound technique, whether one arrives at the true collapse load or departs significantly from it depends on the approach selected and the slab under investigation. A number of illustrative examples will therefore be studied to show the comparative abilities of the different methods.

Rectangular slab simply supported on three sides

An automated yield line procedure presented by Shoemaker (Shoemaker, 1989) followed the hand calculation approach of examining a number of predetermined collapse modes, the form of which depended on the particular slab under consideration. In the case of a rectangular slab, simply supported on three sides and free on the fourth, the two collapse modes shown in Figure 1a were examined and the critical values of x and y were determined by a grid based search procedure. For a particular example, the solution obtained by this procedure was as shown in Figure 1b.

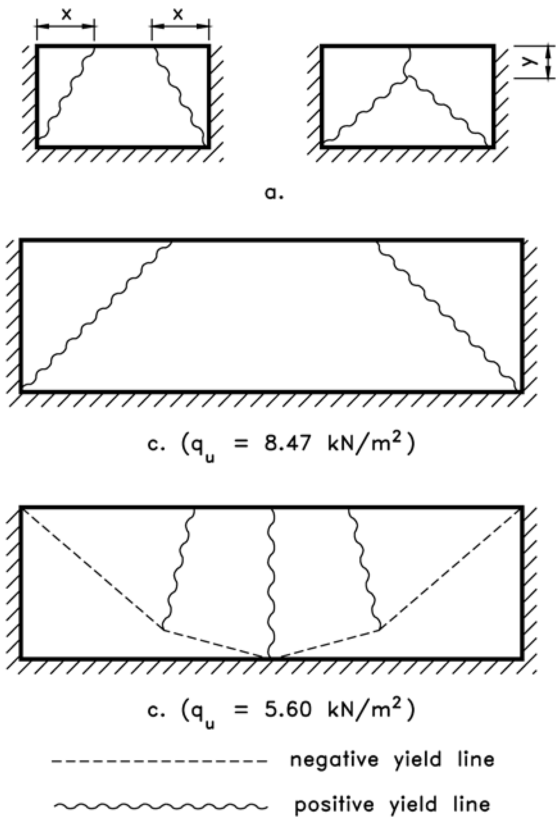


Figure 1: a. Possible modes b. Shoemaker solution c. Hillerborg solution

Subsequently, Hillerborg (Hillerborg, 1991) showed that, in the absence of top reinforcement, the mechanism shown in Figure 1c is critical and that the mechanism employed by Shoemaker had overestimated the collapse loads by at least 50%. This led Hillerborg to conclude that "*In my opinion, it is questionable whether yield line analysis is a suitable basis for the design of slabs, as the risk of getting an unsafe structure is too high.*"

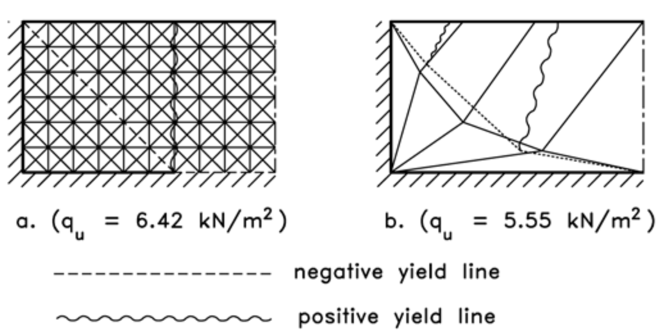


Figure 2: a. Fine mesh solution b. coarse mesh solution

The slab has also been analysed by the automated yield line procedure (Johnson, 1996). In this case, the “fine” net shown in Figure 2a, based on a half-slab model, was used to identify the nature of the collapse mode and the “coarse” net shown in Figure 2b was then used to optimise the geometry of the mode. As may be noted, this procedure produced a solution that corresponded closely to the one suggested by Hillerborg (Figure 1c). In this case, therefore, the automated yield line technique did successfully identify the appropriate mechanism. However, it might well be contended that the two stage solution is somewhat cumbersome since it is dependent on manual intervention to establish a coarse grid appropriate to the identified mode.

Practical design example

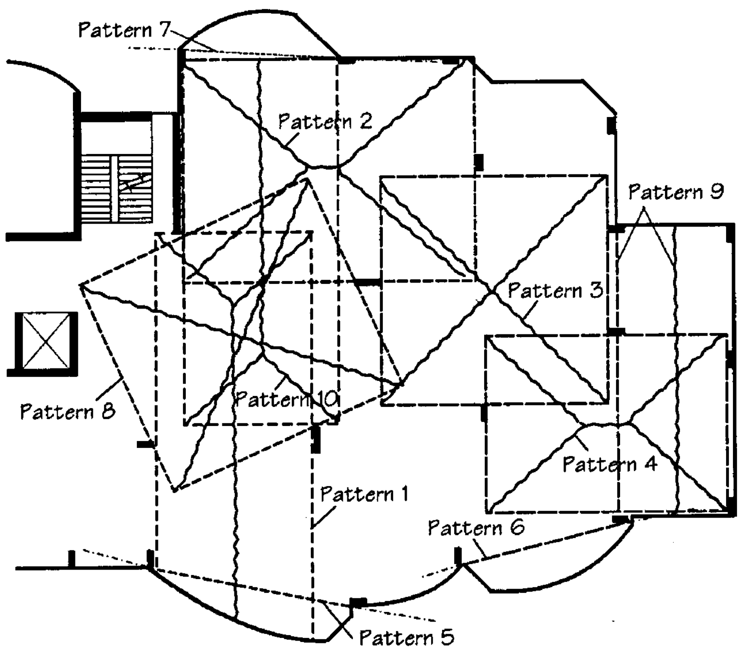
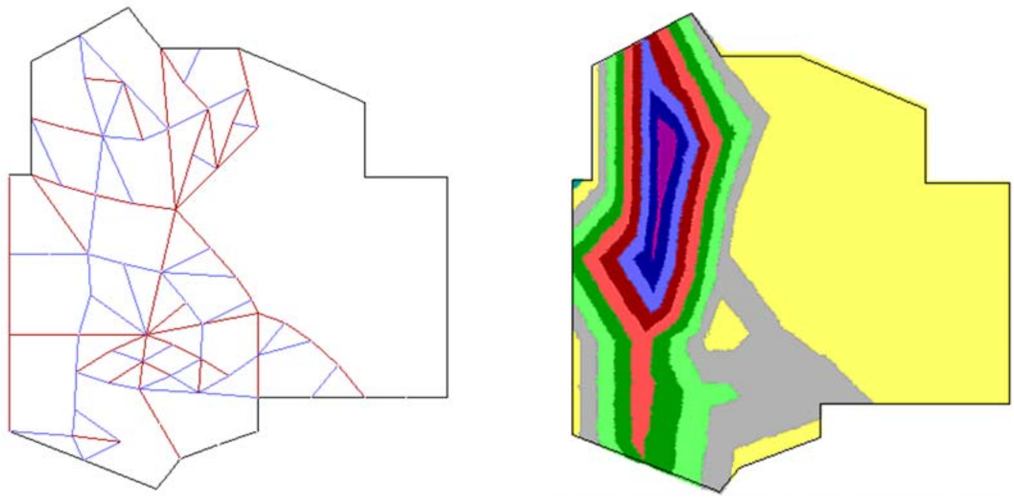


Figure 3: Part-plan of floor slab for block of flats and trial yield line patterns *(Kennedy et al, 2003)*

The part-plan of one storey of a seven-storey block of flats is shown in Figure 3 (Kennedy et al, 2003). The floor slab was initially analysed by hand using no less than ten possible yield line patterns, which are also shown in Figure 3. On the basis of this analysis, a required moment of resistance of 43.3 kNm/m was established for the slab, the critical mechanism being Pattern 2.

The analysis was subsequently checked by the use of the automated yield procedure, which produced the complex yield line pattern shown in Figure 4a, and an ultimate moment requirement of 47.2 kNm/m. The pattern demonstrates a basic problem with the use of the automated yield line method, namely that the form of the collapse mechanism is not readily obvious from Figure 4a. In these circumstances, reference to an associated contour plot of the collapse mode (Figure 4b) is often useful and, in this case, demonstrates that the collapse mode is of the “folded plate” nature. This collapse mode approximates to a combination of the Patterns 1 and 10 considered in the hand analysis (Figure 3) and these two patterns were, in fact, identified as the third and second most critical cases, respectively, by the hand analysis.



a. b.

Figure 4: a. Yield line pattern b. collapse mode contour plot

To obtain a bound on the true collapse load, an automated lower bound solution was also obtained and produced the collapse mode shown in Figure 5. As may be seen from Figure 5, the collapse mode is a variation on that given by the upper bound solution (Figure 4b) and the associated ultimate moment requirement of 64.1 kNm/m is a 36% increase on the 47.2 kNm/m obtained from the upper bound analysis. A subsequent upper bound analysis, based on a finer, more restricted net, gave an ultimate moment of 53.8 kNm/m, which corresponded with a value of 55.8 kNm/m derived from a hand analysis (Kennedy et al, 2003). These revised results, however, are still an underestimate of 15% from the lower bound prediction, suggesting that the safer lower bound should generally be preferred, unless it can be shown to produce unduly conservative results.

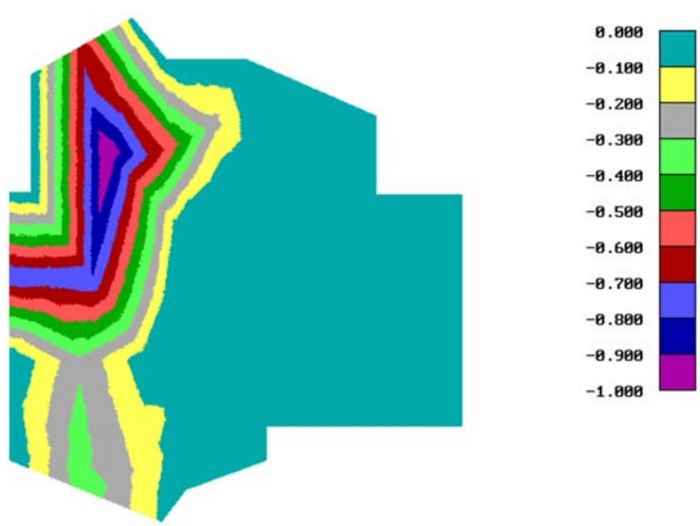
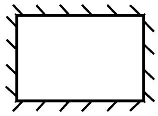
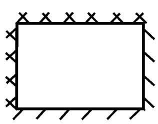
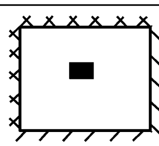
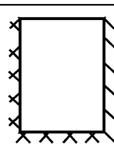
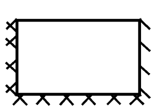
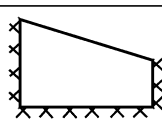
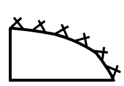
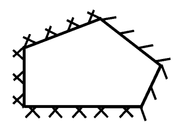
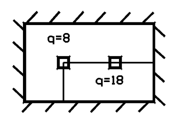
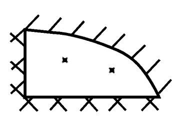
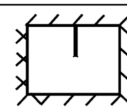
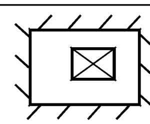
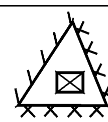
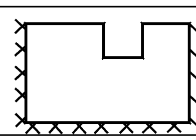


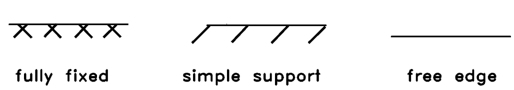
Figure 5: Lower bound solution collapse mode contour plot

Safety of the strip method

Hillerborg (Hillerborg, 1996) introduced a number of approximations and simplifications to the strip method in order to make it more convenient for practical application. Although it was the intention that the modifications should be conservative, no formal guarantees exist that this is the case and it is therefore conceivable that unsafe designs might be generated, although Hillerborg contends that *"the yield line theory of slabs designed according to the recommendations of the book never show that it is on the unsafe side, at least as far as I have found"*. To examine the validity of this claim, the automated yield line method, employing fine meshes to establish probable mechanisms and then coarse meshes to optimise the geometry of the detected mechanism, have been undertaken (Johnson, 2001) for a range of the examples considered by Hillerborg. The results of these analyses are summarised in Table 1.

Table 1: Automated yield line analyses summary

No.	Example	LF (fine) (coarse)
1		1.08 1.06
2		1.06 1.01
3		1.14 1.04
4		1.02 1.03
5		1.03 1.03
6		1.13 0.99
7		1.00 1.00
8		1.36 1.40
9		1.24 1.29
10		1.10 n/a
11		1.13 1.04
12		1.22 1.30
13		1.00 0.99
14		1.06 1.21



In Table 1 the pairs of load factors (LF) given for each example relate to the initial fine mesh automated yield line solution and the subsequent coarse mesh used to optimise the detected mechanism form. In general, it is expected that the coarse mesh should produce a more critical (lower load factor) than the fine mesh. This may be seen not to be the case in all instances, presumably because in some of the examples the fine mesh allows the generation of a somewhat more complex variation of the mechanism taken as the basis for the coarse analysis. The load factors are based on the reinforcement design generated by the strip method and, taking the yield line results as essentially exact, a value above unity therefore represents a conservative strip design and a below unity represents an unsafe design. On this basis, Table 1 supports Hillerborg's contention that the strip method does not generate unsafe designs for the examples but does indicate significant conservatism for some of the more complex examples, where 20-30% overdesigns are encountered on occasion.

Highly orthotropic slab

The automated lower bound approach routinely makes use of a linear programming algorithm for optimisation purposes. To accommodate the algorithm, a linearization of the yield criterion (Wolfensberger, 1964) is commonly undertaken, so introducing an approximation that tends to become more significant as slabs become more highly orthotropic. To test the degree of approximation involved and also to test an alternative "maximum twist" yield criterion, the highly orthotropic "bridge" type slab shown in Figure 6 has been examined (Johnson, 1999).

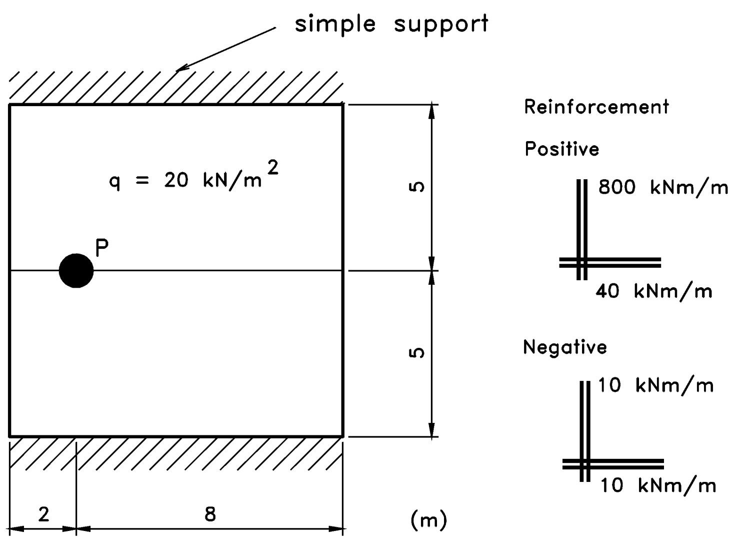


Figure 6: Highly orthotropic slab example

The slab shown in Figure 6 was analysed by the hand yield line approach; the automated yield line method using both a fine and a coarse mesh; and the automated lower bound method using a linearized (Wolfensberger) yield criterion and a revised "maximum twist" approximation to the yield criterion. The results of the various analyses are shown in Table 2, from which it may be seen that the automated yield line method (coarse analysis) and the automated lower bound (maximum twist yield criterion) provide acceptably close bounds on the collapse load. For this case, likely collapse mechanisms are well established and so the hand yield line analysis should also provide an acceptable collapse load estimate. This is indeed the case if the calculated value of 1152 kN is reduced by a conventional 15% to produce a revised value of 979 kN. On the other hand, the automated yield line method (fine analysis) and the automated lower bound (Wolfensberger yield criterion) result in significant over- and under-estimates, respectively, of the collapse load.

Table 2: Comparative analyses for highly orthotropic slab

Analysis	Collapse Load (kN)
Hand yield line analysis	1152
Automated yield line – fine mesh	1743
Automated yield line – coarse mesh	1009
Lower bound analysis (maximum twist yield criteria)	938
Lower bound analysis (Wolfensberger yield criteria)	775

Summary

- The hand approach to yield line (upper bound) analysis can lead to significant unsafe approximations due to the critical mechanism not being examined, as in the Shoemaker and practical design examples. This approach is therefore only reasonably trustworthy in the case of conventional slab layouts and loadings for which the critical yield line patterns have been reliably established.

- The automated yield line analysis will generally be more reliable in identifying critical mechanisms but it involves a somewhat cumbersome two-stage process and can involve uncertainty on the unsafe side. The practical design example, for instance, resulted in a significant discrepancy between automated upper and lower bound solutions.

- The strip (lower bound) hand method will lead to safe designs but can produce conservatism of up to 20-30% if applied to complex configurations and loadings.

- The automated lower bound method has the overriding virtue of guaranteeing a safe solution and is also a single stage analysis, without the need for manual intervention. It is perhaps slightly less intuitive than the automated yield line method, since, by its nature, it does not provide a yield line system. However, a contour plot of the associated collapse mode can be generated. The technique has been shown to produce conservative solutions for highly orthotropically reinforced slabs if the linearized Wolfensberger yield criterion is used, but this can be improved either by using a "maximum twist" linearization of the yield criterion (Johnson, 1999) or by using a formulation that employs the non-linear form of the yield criterion (Krabbenhoft et al, 2002).

Conclusion

Although, in favourable circumstances, the upper and lower bound approaches can be one and the same (just as Heraclitus' roads), in the sense that the same unique collapse load can be found, the chances of going astray and becoming unsafe are substantially greater with the upper bound (yield line) approaches. The lower bound methods are definitely securer and generally quicker (just as the Traditional Song's low road).

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